Drawing Reeb Graphs

... is interesting and you should be doing it!

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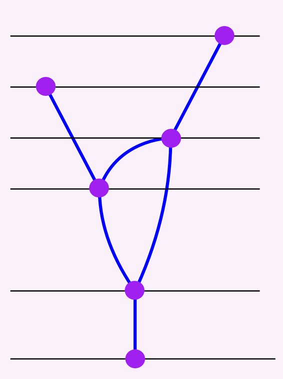
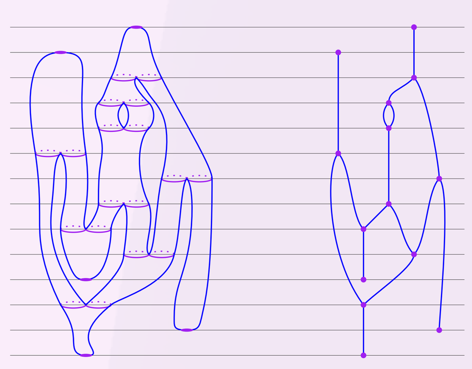
Reeb Graphs

Reeb graphs are used in computational topology to analyze continuous functions on complex spaces. Essentially, these graphs capture how level sets of a function evolve and behave in a topological space; components of the each level set become a vertex, and edges connect vertices on adjacent level sets that are connected.



Drawing Reeb Graphs

Drawing of Reeb Graphs is an important aspect for the purpose of visualizing continuous functions on complex spaces as a simplified structure.



Related Work

Related work on drawing Reeb graphs is surprisingly limited, despite how prevalent the use of Reeb graphs is. The only work in this area we are aware of considers book embeddings of these graphs; while of combinatorial interest, the algorithms seem less practical for easy viewing of larger Reeb graphs.

Challenge

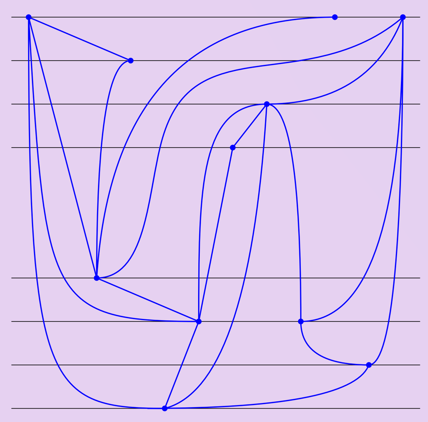
Unfortunately, because drawing layered graphs is NP-hard, existing approaches are mostly heuristic.

Reeb graphs have specific structural properties, which may make drawing them easier than the general case.

- Reeb graphs of *generic* manifolds only have vertices of degree 1 or 3.
- Reeb graphs have a graph-theoretical *genus* that is directly related to the topological genus of the manifold so they contain a small number of (undirected) cycles.

Layered Graphs

Layered graph drawing is a drawing style where vertices are placed on layers and edges are y-monotone curves. Reeb graphs may be seen as a special case of layered graph drawing with preassigned layers, hence, existing algorithms can be used.



Contributions

We present the following results:

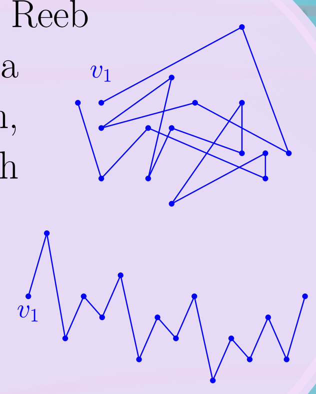
- Prove the decision problem is NP-complete
- We conjecture that our problem may be fixed-parameter tractable (FPT) in the number of (undirected) cycles in the graph.
- We show that finding a crossing-minimal embedding when the graph is a cycle is feasible in polynomial time.

FPT

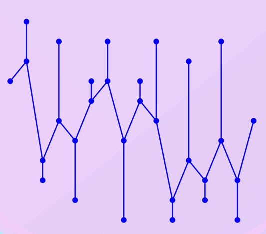
Parameterized graph drawing is a promising and growing field. In the context of layered graph drawing, the number of layers is a natural parameter. However, for Reeb graphs, this parameter is generally large.

On the other hand, the genus of the graph is often small.

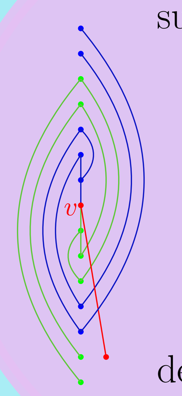
If the Reeb graph is a single path, drawing with no crossing is always possible.



The same applies to caterpillars.



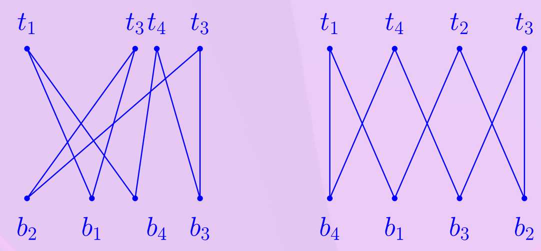
We consider three subtrees rooted at v: the green one, the blue one, and the red one. As shown, it is not possible to draw the red edge to be monotonically decreasing without crossing the others.



However, in general, it is not always feasible to draw trees with no crossings.

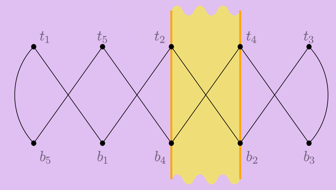
Conjecture

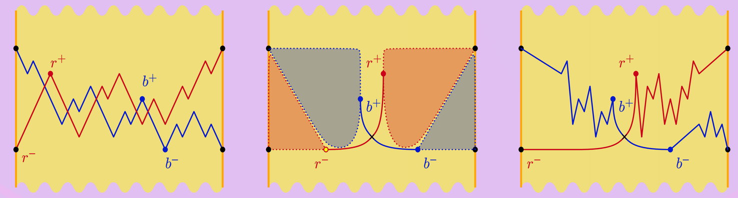
We conjecture that the problem of drawing Reeb graphs may be fixed-parameter tractable (FPT) in the number of (undirected) cycles in the graph.



Lemma

As a first step towards establishing our conjecture, we show that when the graph is a cycle, we can find a crossing-minimal embedding in polynomial time. After arranging the vertices on the top and bottom layers of the graph to make the *bowtie*, we can resolve the crossings in the middle layers as shown below:



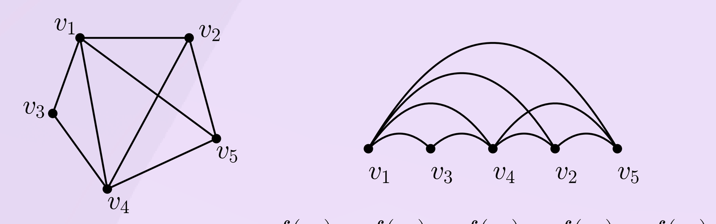


Theorem

In order to prove NP-completeness, we transform a known NP-complete problem, Optimal Linear Arrangement, to the Generic Reeb Graph Crossing Number problem.

We can interpret this problem as arranging the vertices of the graph in a linear order, such that the total length of all the edges of the graph is minimized.

Given $G = (V, E)$ and an integer k, is there a function $f : V \to \{1, 2, ..., |V|\}$ s.t. $\Sigma_{\{v,w\}\in E} |f(v) - f(w)| \leq k$?



$f(v_1) < f(v_3) < f(v_4) < f(v_2) < f(v_5)$

NP-completeness

Many variants of crossing minimization are hard and the same is true for layered graph drawing. This follows from a result by Garey and Johnsson, who show that the problem is already hard with only 2 layers.

We show that the problem of deciding whether a given (generic) Reeb graph admits a drawing with at most k crossings, is NP-complete. Our proof uses only vertices of constant degree, but does require a large number of both layers and cycles.

Proof

We first construct a corresponding instance of Generic Reeb Graph Crossing Number problem using an instance of Optimal Linear Arrangement. Then, we argue that if the instance of Optimal Linear Arrangement has a valid order, the corresponding instance of Generic Reeb Graph Crossing Number also has a valid embedding. Finally, we argue that conversely, a valid embedding of the corresponding instance of Generic Reeb Graph Crossing Number implies that the original instance of Optimal Linear Arrangement had a valid order.

Construction: Duplicate vertices. Replace each vertex with a triangular hexagonal grid. Draw $|E|^2$ edges between corresponding grids. Add a chain of $|degree(v_i) + 1|$ vertices in the mid bottom of each grid. Draw the original edges E (orange) between the said vertices (red) based on the chosen order of the original vertices V. Assume $k' = |E|^2(k - |E|) + (|E|^2 - 1)$.