# Preprocessing Imprecise Points and Splitting Triangulations


Maarten Löffler
Utrecht University
the Netherlands

Joseph S. B. Mitchell
SUNY at Stony Brook
United States

Marc van Kreveld
Utrecht University
the Netherlands

Let's start by defining *imprecise points*.

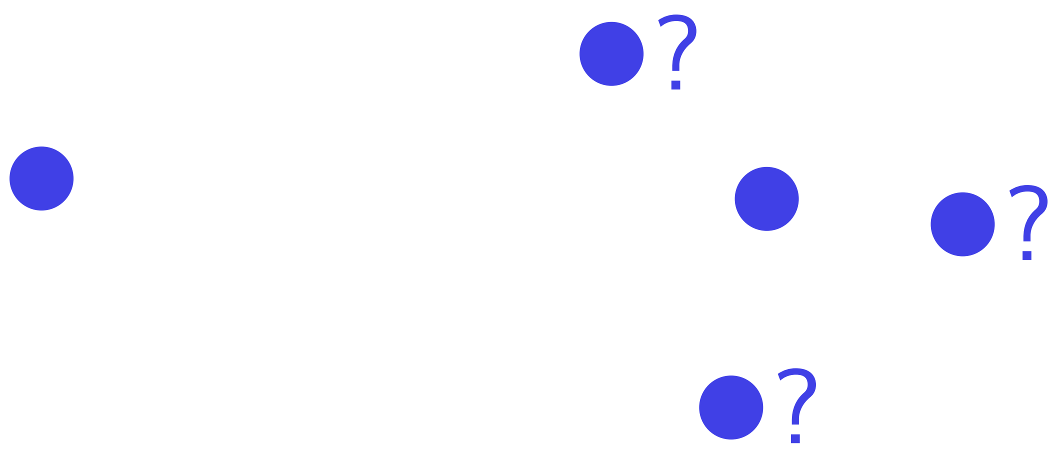
Computational geometry deals with problems on *precisely* specified *points* in $\mathbb{R}^2$.

Computational geometry deals with problems on *precisely* specified *points* in $\mathbb{R}^2$.

However, in many practical applications, locations of input points are *not* precise.

Computational geometry deals with problems on *precisely* specified *points* in $\mathbb{R}^2$.

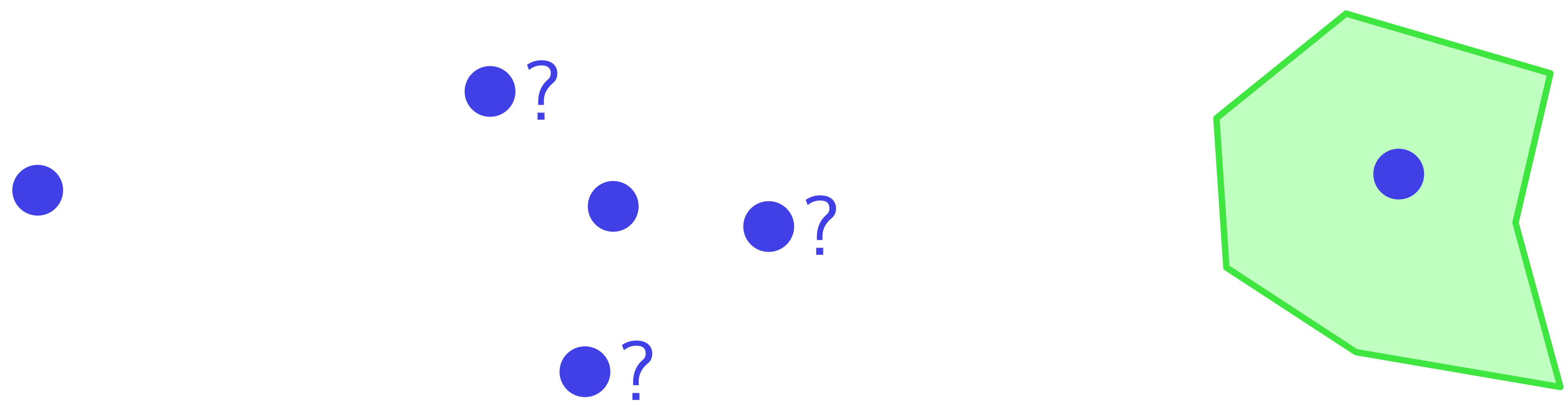
However, in many practical applications, locations of input points are *not* precise.



Computational geometry deals with problems on *precisely* specified *points* in $\mathbb{R}^2$.

However, *in* many practical applications, locations of input points are *not* precise.

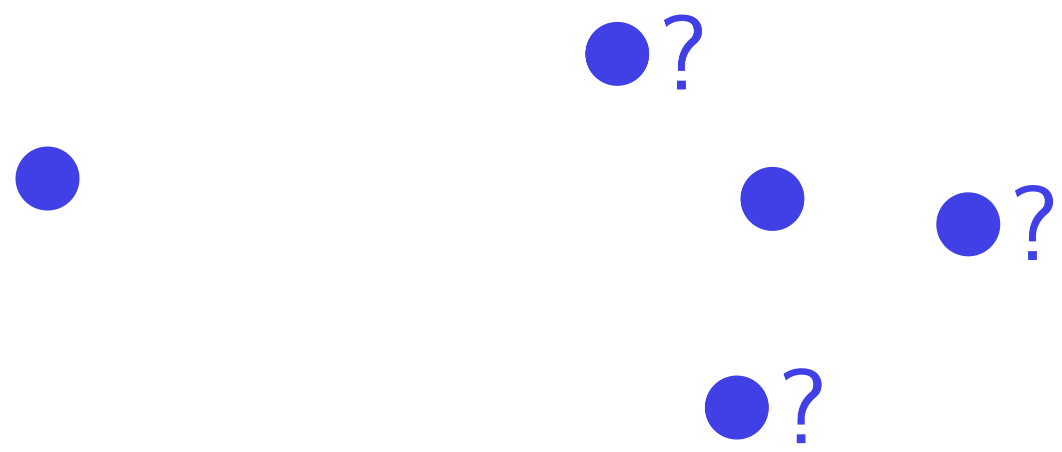
Often, a bound is available: a point is at least certain to be in a given *region*.

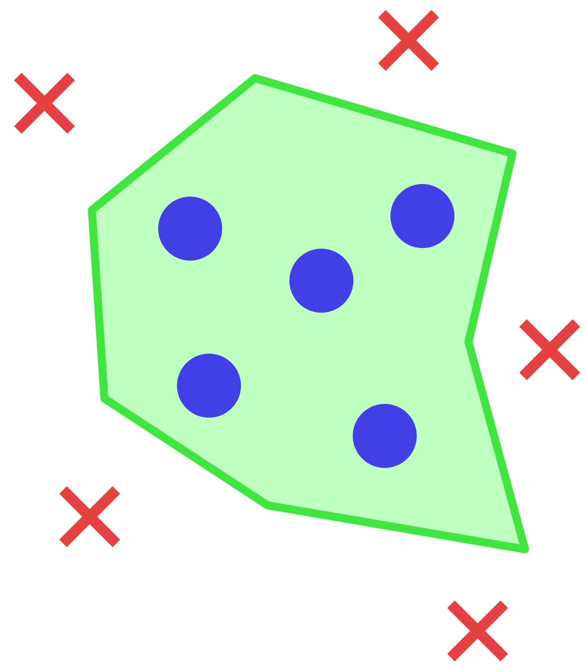


Computational geometry deals with problems on *precisely* specified *points* in $\mathbb{R}^2$.

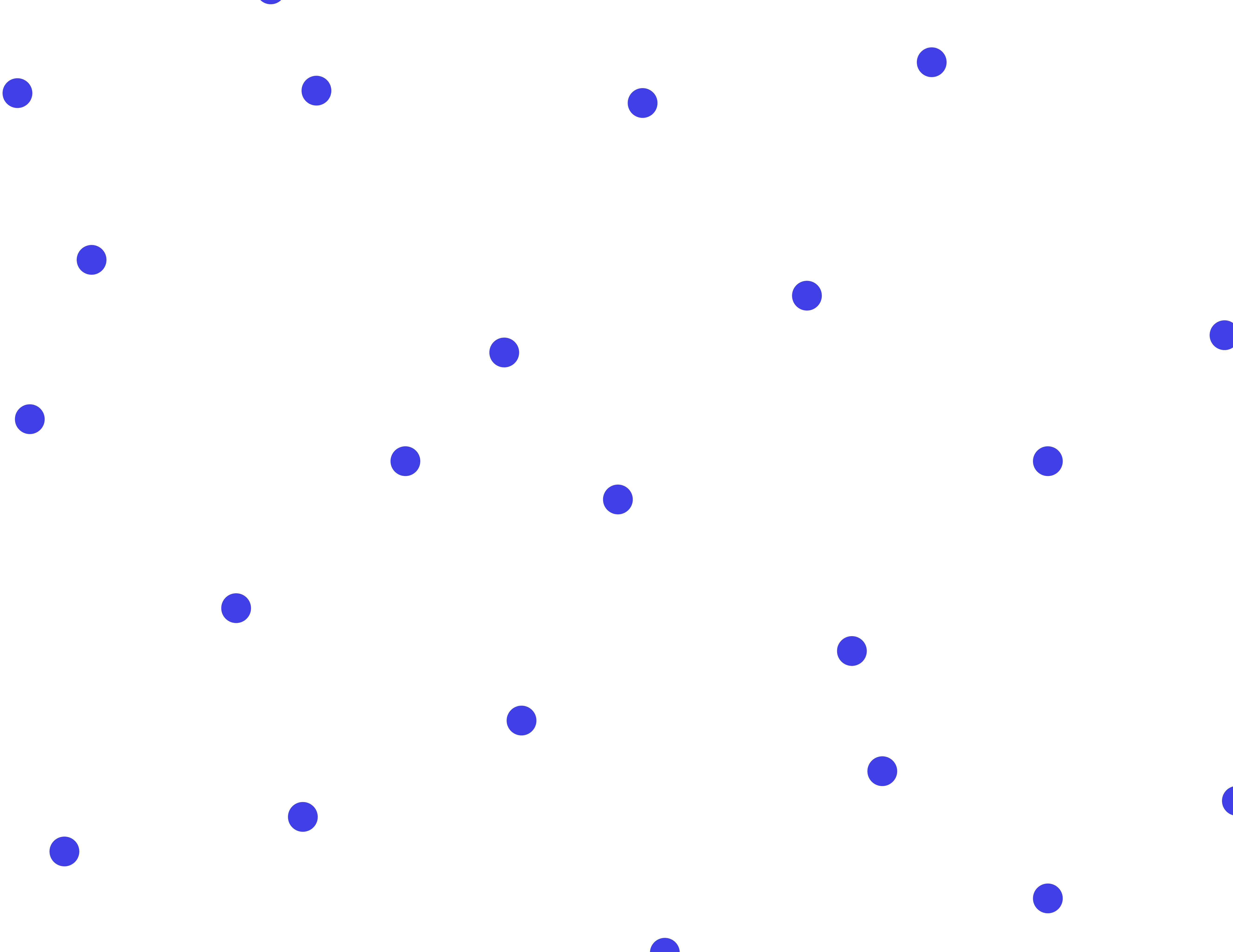
However, *in* many practical applications, locations of input points are *not* precise.

Often, a bound is available: a point is at least certain to be in a given *region*.

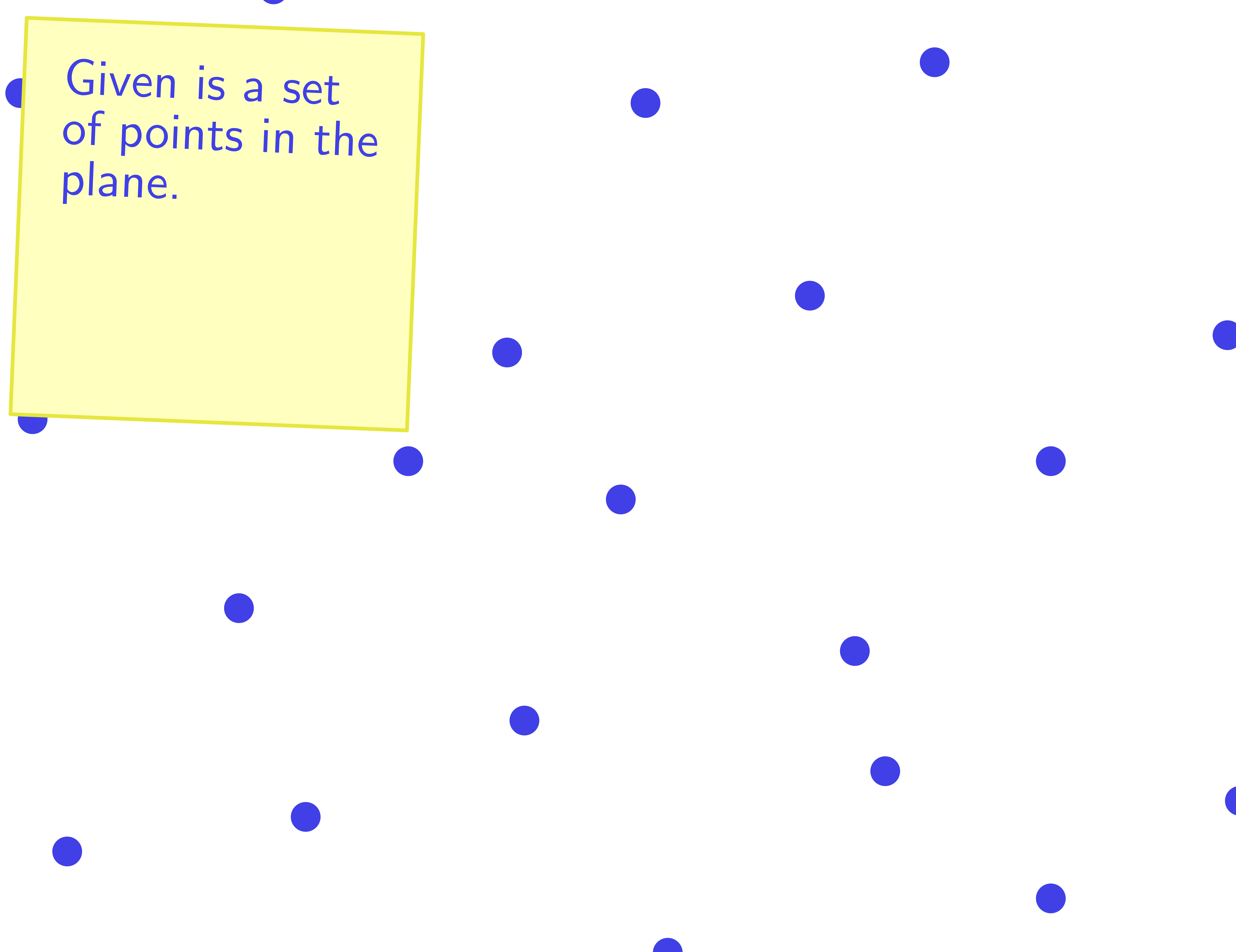


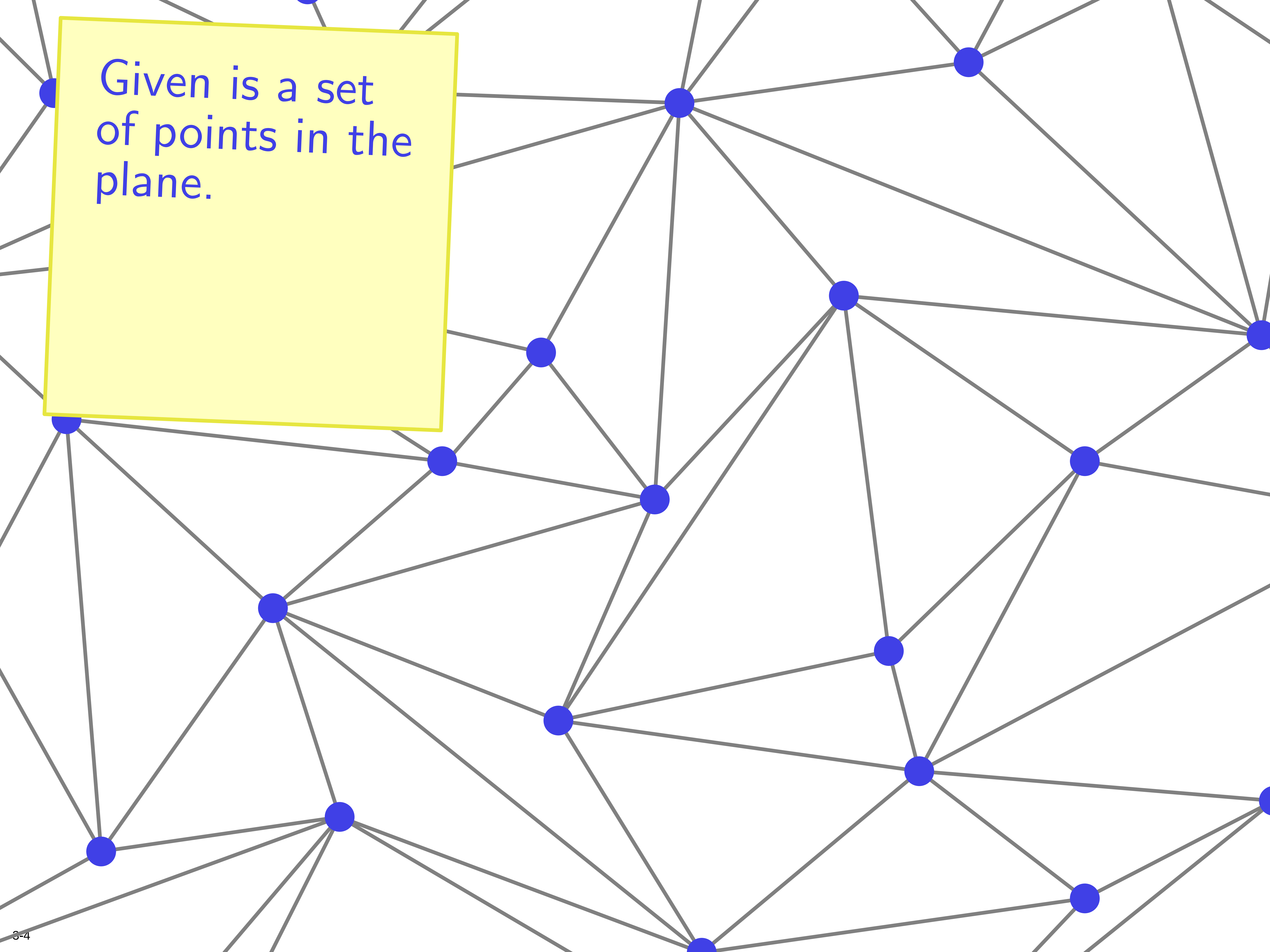
Then, what is a *triangulation*?

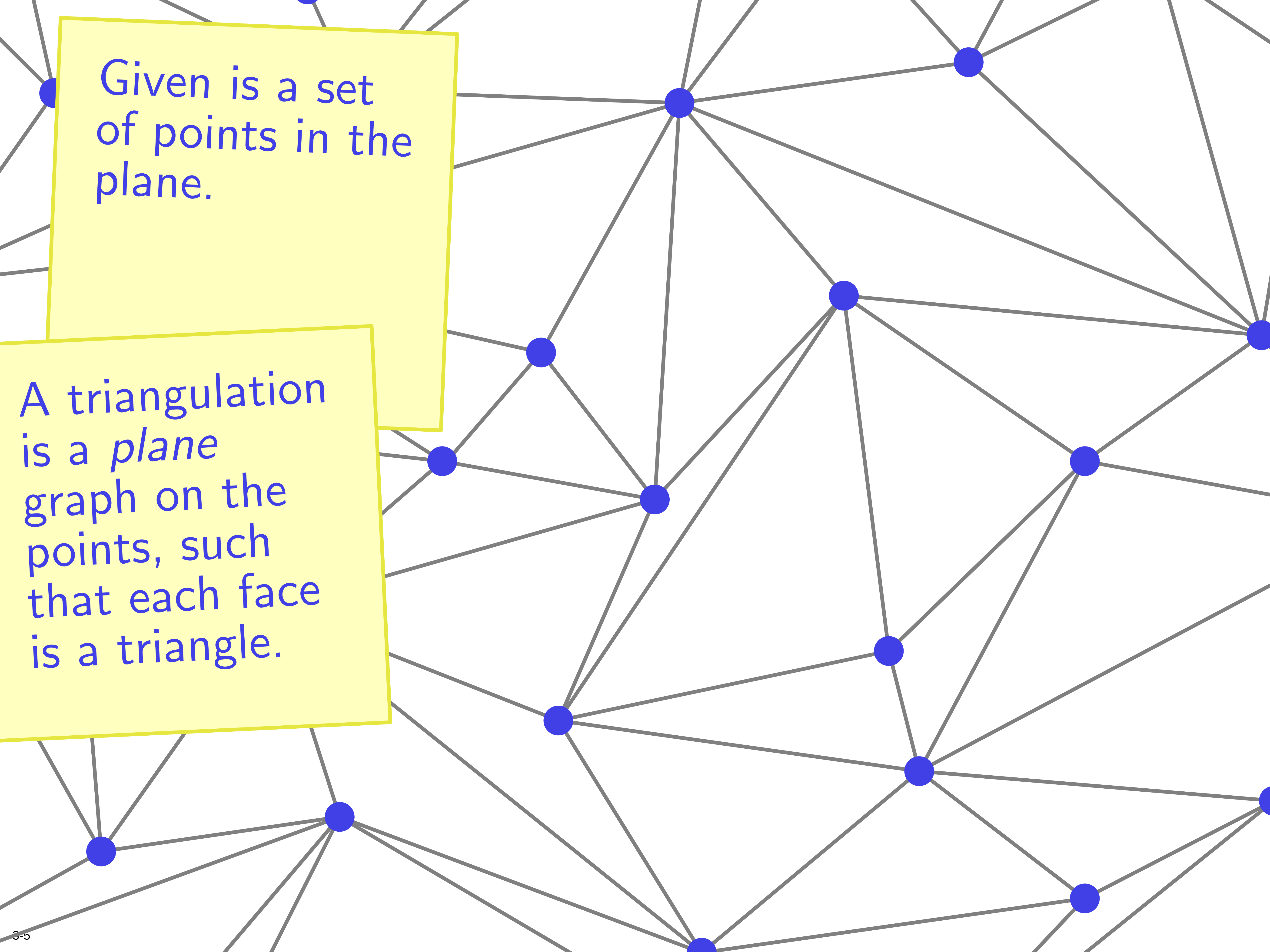


Given is a set of points in the plane.



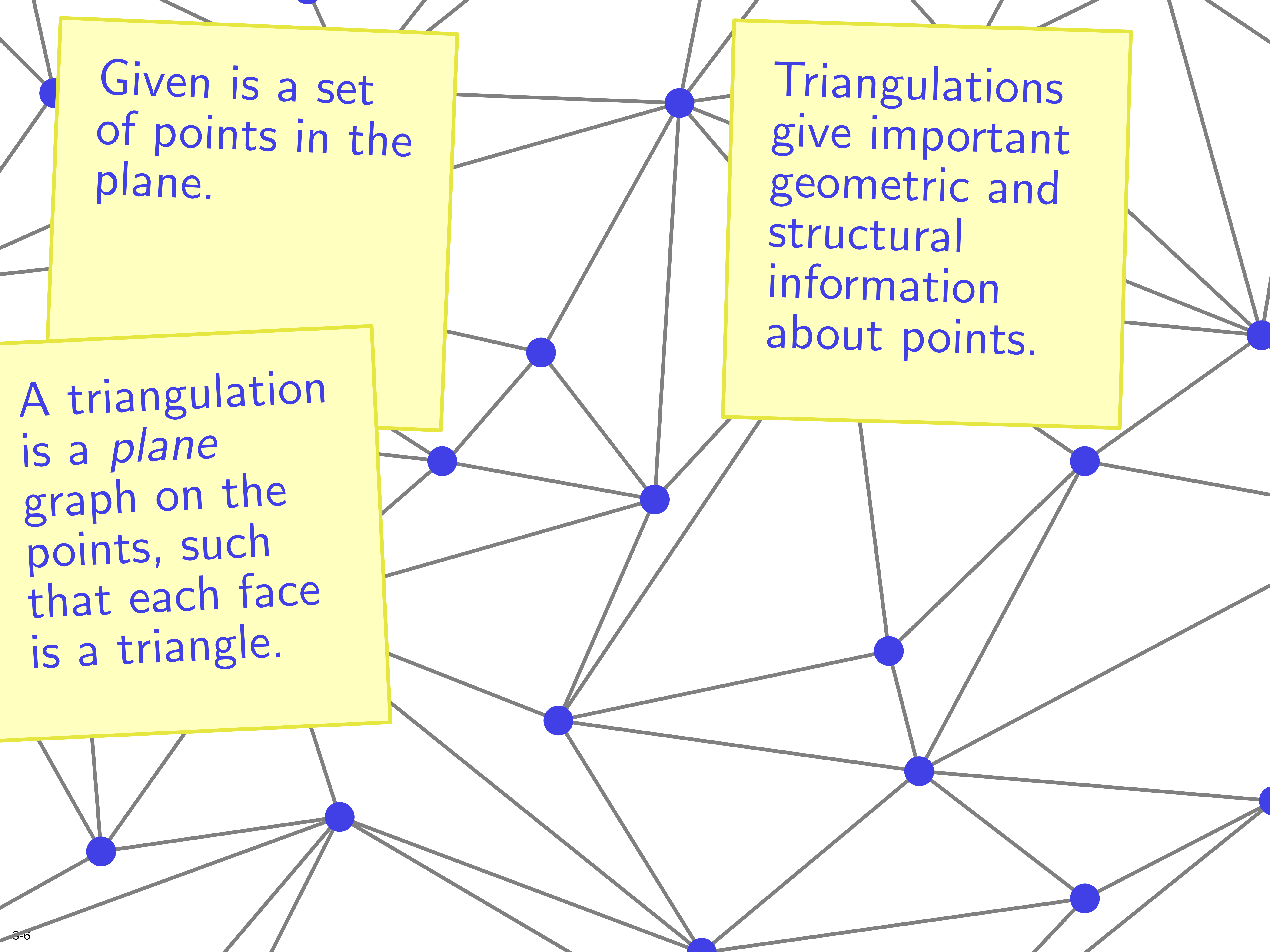
Given is a set of points in the plane.





Given is a set of points in the plane.

A triangulation is a *plane* graph on the points, such that each face is a triangle.



Given is a set of points in the plane.

A triangulation is a *plane* graph on the points, such that each face is a triangle.

Triangulations give important geometric and structural information about points.

Given is a set of points in the plane.

A triangulation is a *plane* graph on the points, such that each face is a triangle.

Triangulations give important geometric and structural information about points.

What happens, though, when the points are imprecise?

Given is a set of points in the plane.

Triangulations give important geometric and structural information about points.

A triangulation is a *plane* graph on the points, such that each face is a triangle.

What happens, though, when the points are imprecise?

Given is a set of points in the plane.

Triangulations give important geometric and structural information about points…

A triangulation is a *plane* graph on the points, such that each face is a triangle.

The true points might be anywhere, so what is a triangulation in this case?

What happens, though, when the points are imprecise?

Given is a set of points in the plane.

Triangulations give important geometric and structural information about points

A triangulation is a *plane* graph on the points, such that each face is a triangle.

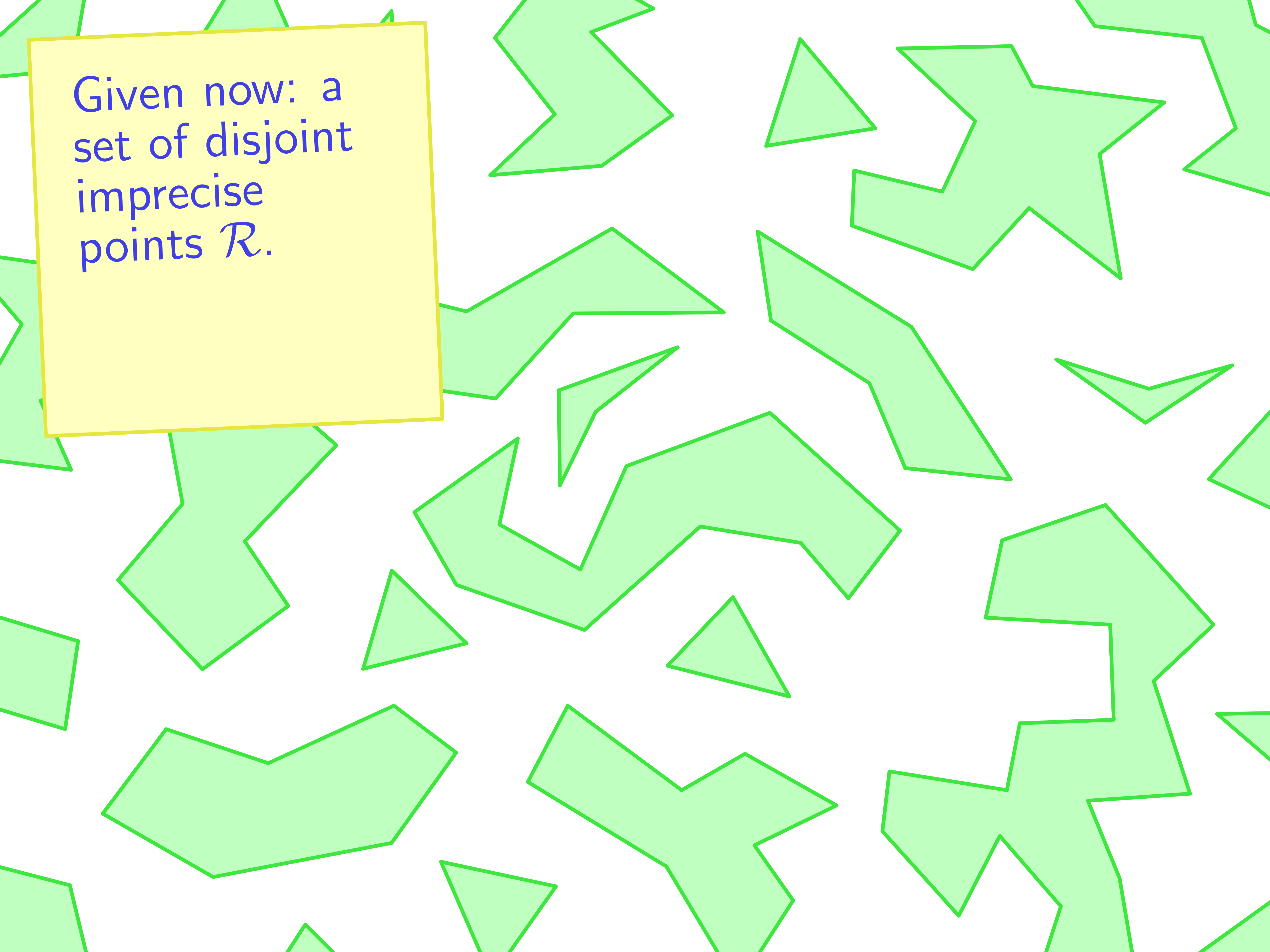
The true points might be anywhere, so what is a triangulation in this case?

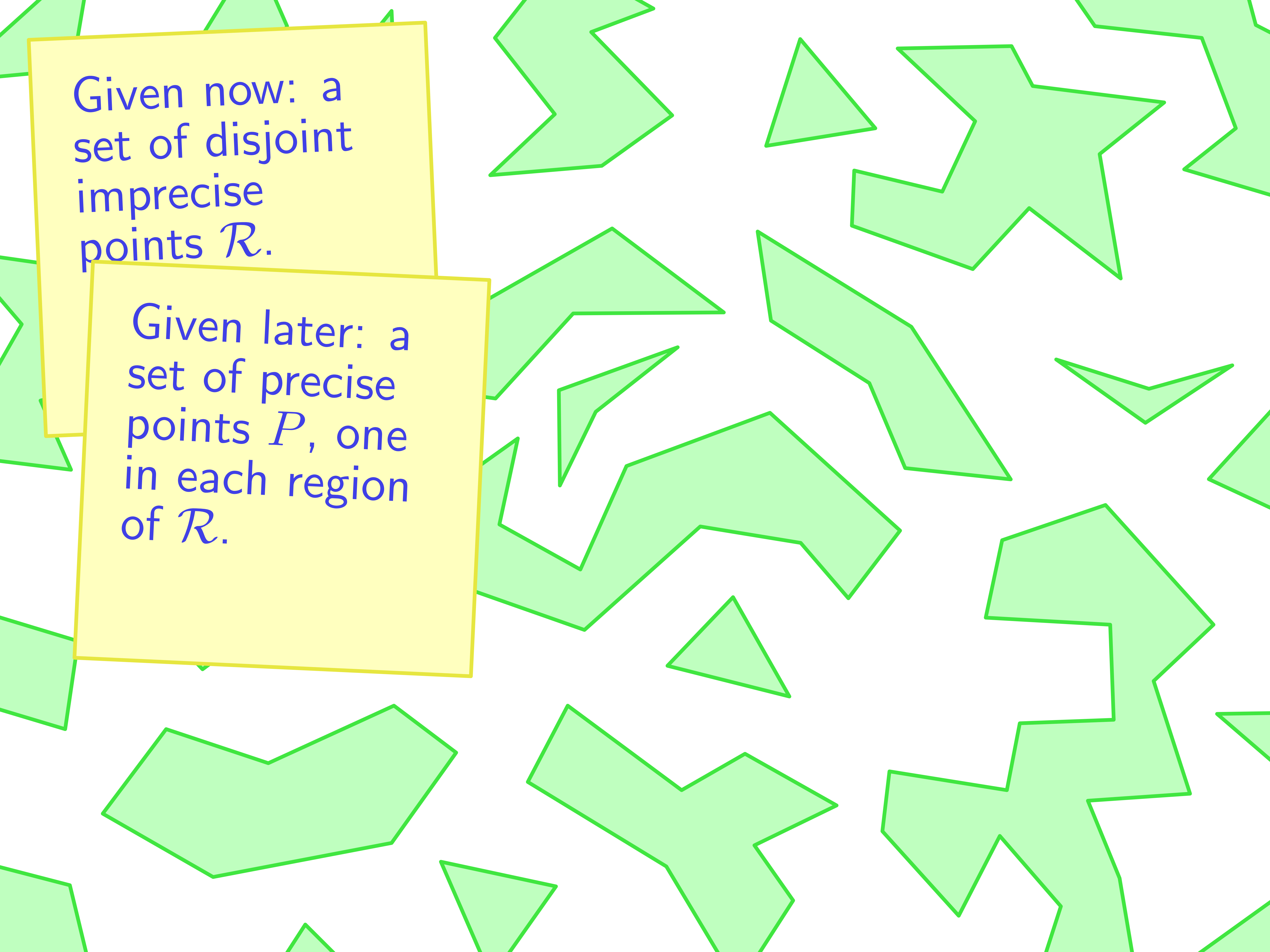
What happens, though, when the points are imprecise?

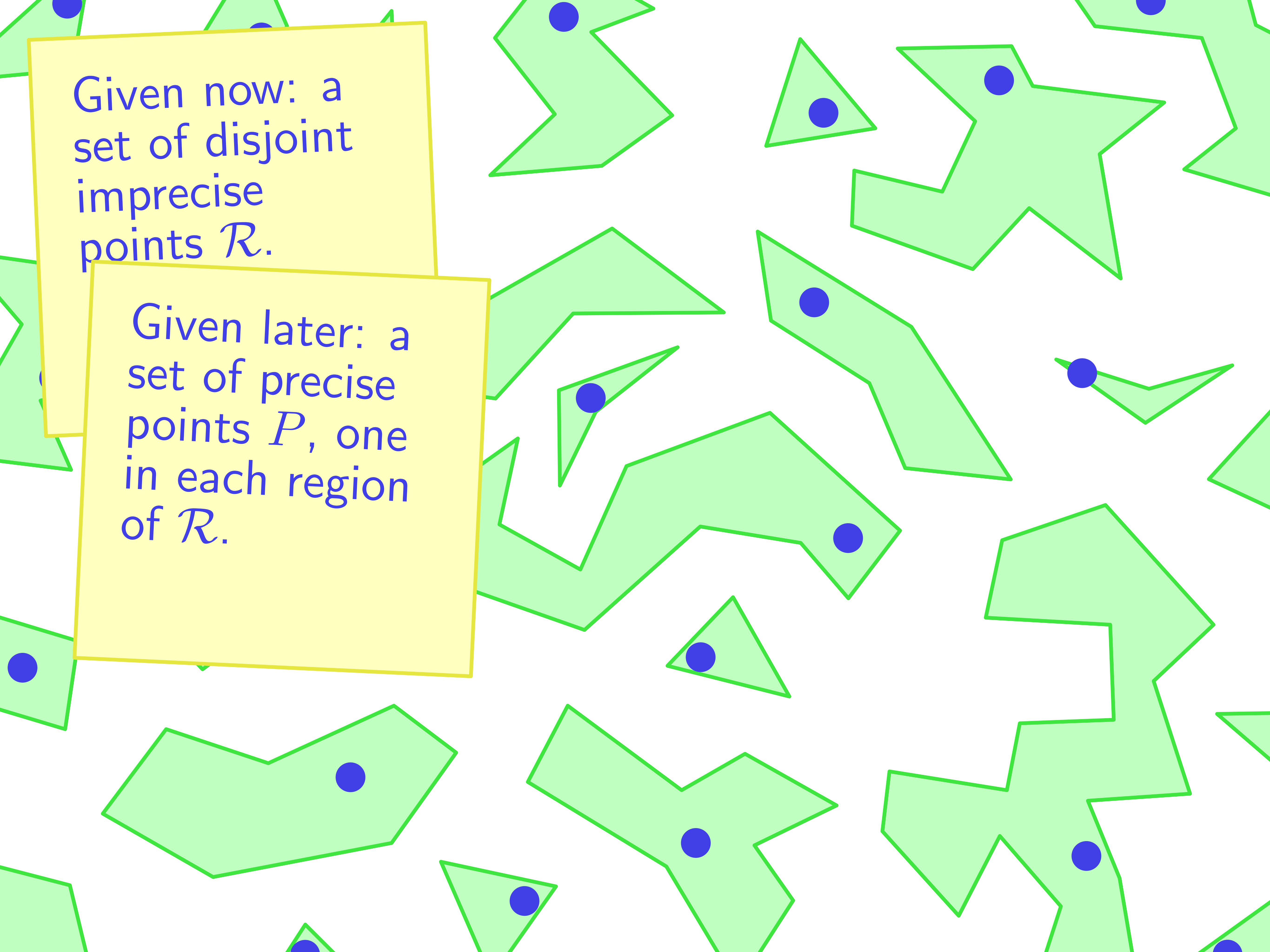
Let's define a *problem statement*.

Given now: a set of disjoint imprecise points $\mathcal{R}$.

Given now: a set of disjoint imprecise points $\mathcal{R}$.




Given now: a set of disjoint imprecise points $\mathcal{R}$.
Given later: a set of precise points $P$, one in each region of $\mathcal{R}$.

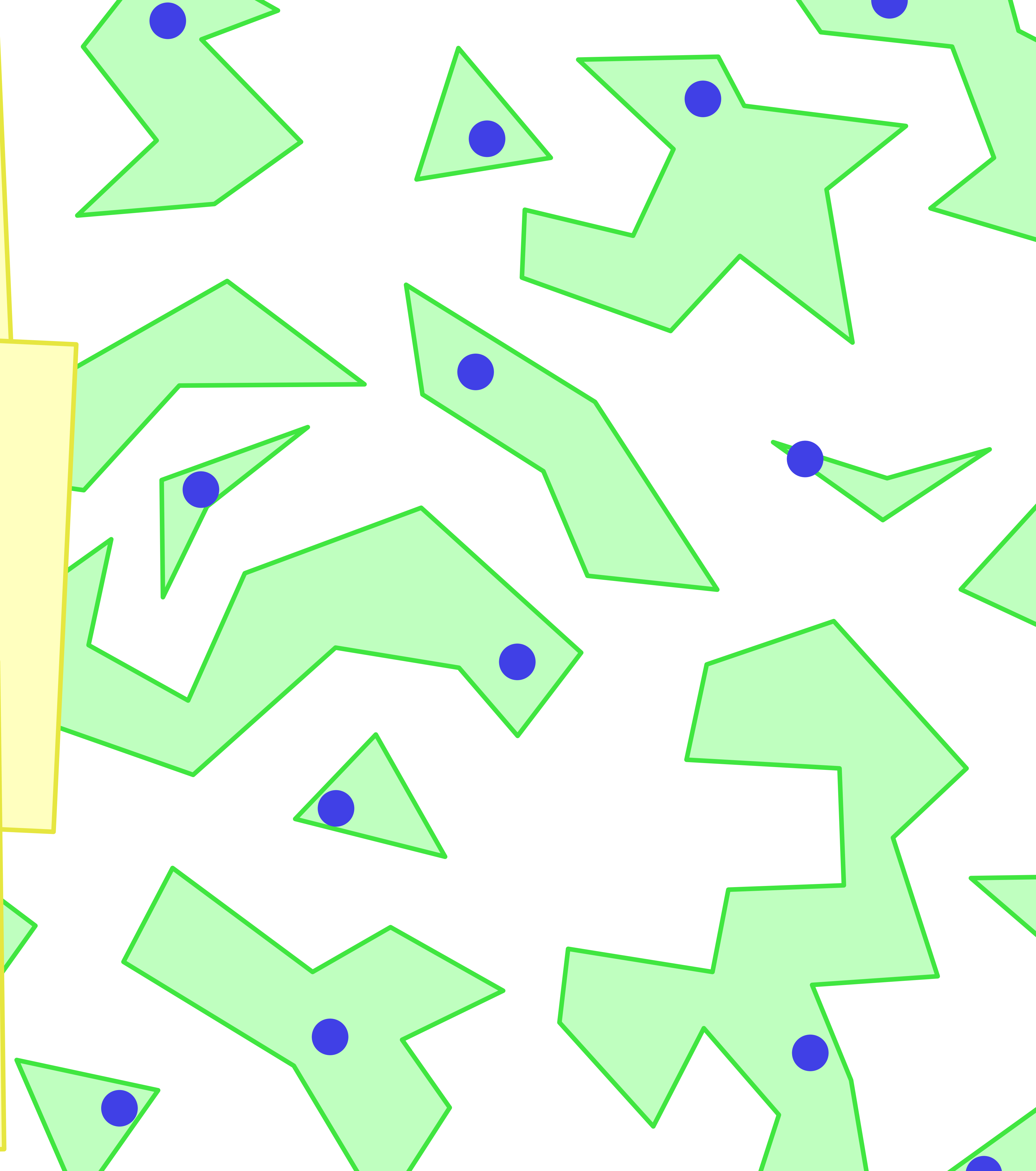

Given now: a set of disjoint imprecise points $\mathcal{R}$.
Given later: a set of precise points $P$, one in each region of $\mathcal{R}$.

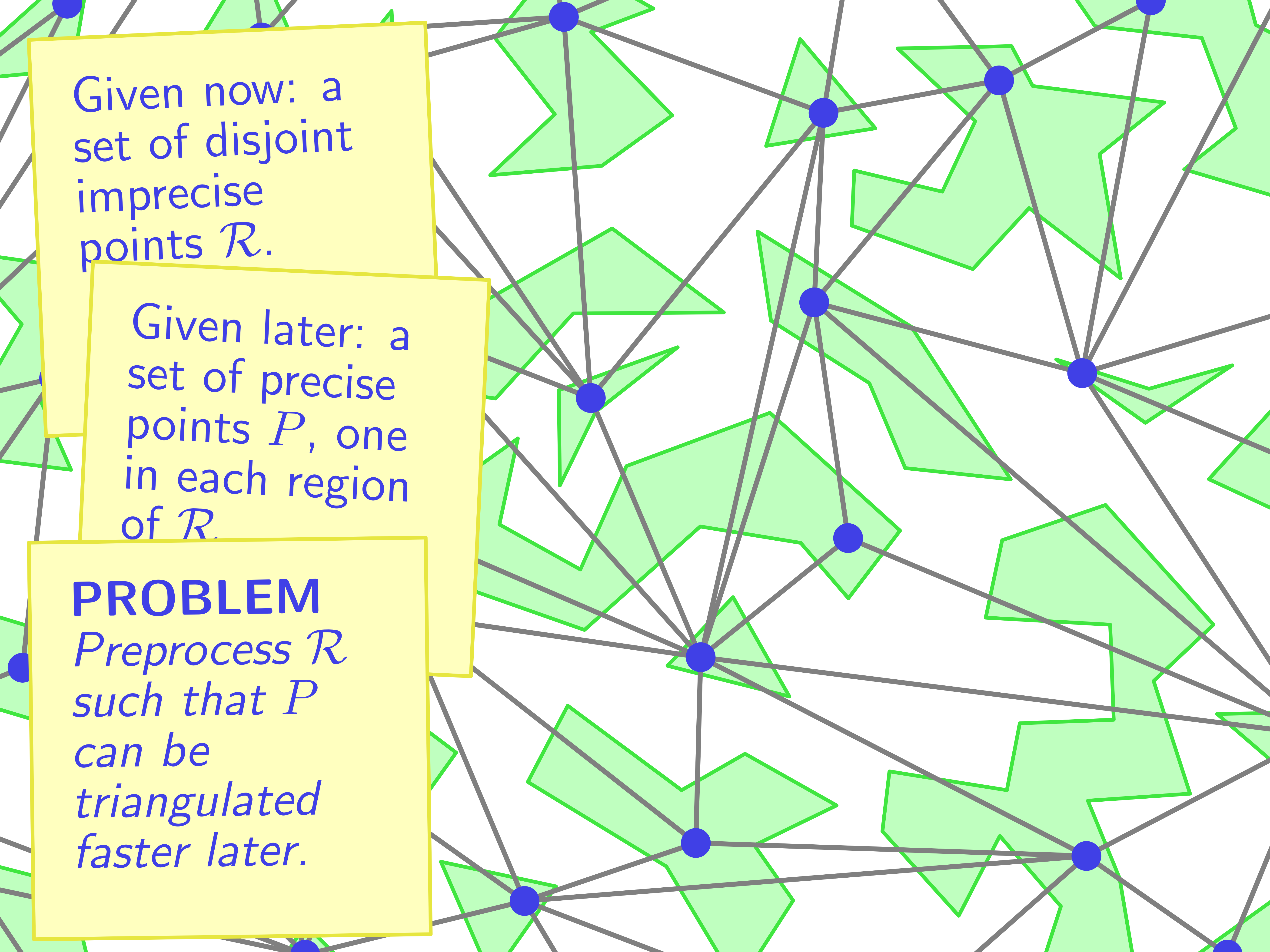
Given now: a set of disjoint imprecise points $\mathcal{R}$.

Given later: a set of precise points $P$, one in each region of $\mathcal{R}$.

**PROBLEM**

*Preprocess $\mathcal{R}$ such that $P$ can be triangulated faster later.*



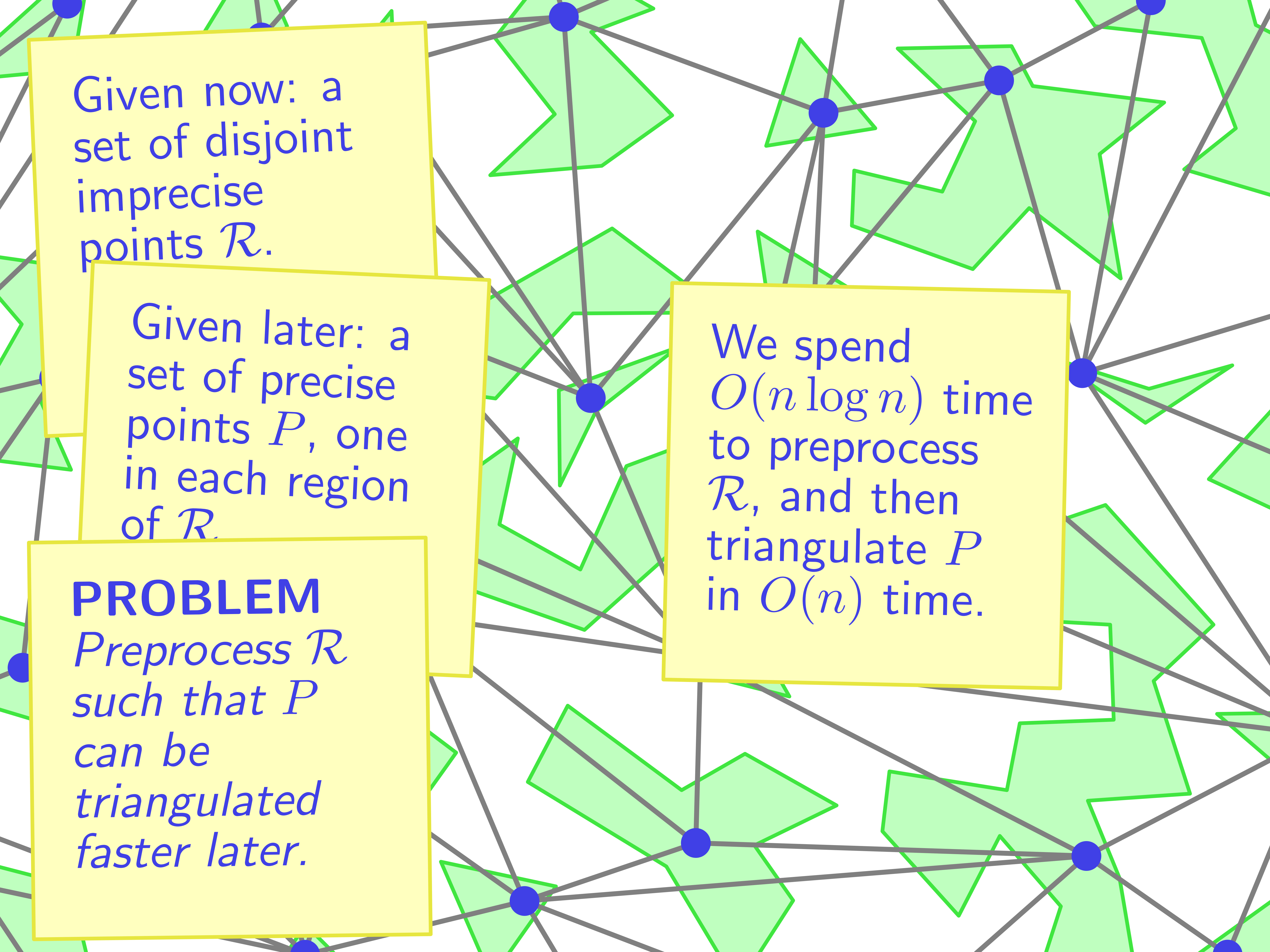


Given now: a set of disjoint imprecise points $\mathcal{R}$.

Given later: a set of precise points $P$, one in each region of $\mathcal{R}$.

**PROBLEM**

*Preprocess $\mathcal{R}$ such that $P$ can be triangulated faster later.*


Given now: a set of disjoint imprecise points $\mathcal{R}$.
Given later: a set of precise points $P$, one in each region of $\mathcal{R}$.
**PROBLEM**
*Preprocess $\mathcal{R}$ such that $P$ can be triangulated faster later.*
We spend $O(n \log n)$ time to preprocess $\mathcal{R}$, and then triangulate $P$ in $O(n)$ time.

A few similar results have recently been obtained.

Given a set of disjoint discs, preprocess them to triangulate the points later.

Given a set of disjoint discs, preprocess them to triangulate the points later.

Preprocessing in $O(n \log n)$, reconstruction in $O(n)$.
[Held & Mitchell, 2008]

Given a set of disjoint discs, preprocess them to triangulate the points later.

Preprocessing in $O(n \log n)$, reconstruction in $O(n)$.
[Held & Mitchell, 2008]

Given a set of disjoint discs, preprocessing them for the *Delaunay* triangulation.

Given a set of disjoint discs, preprocess them to triangulate the points later.

Preprocessing in $O(n \log n)$, reconstruction in $O(n)$.
[Held & Mitchell, 2008]

Given a set of disjoint discs, preprocessing them for the *Delaunay* triangulation.

Preprocessing in $O(n \log n)$, reconstruction in $O(n)$.
[Löffler & Snoeyink, 2008]

Given a set of disjoint discs, preprocess them to triangulate the points later.

Given a set of disjoint discs, preprocessing them for the *elaunay* angulation.

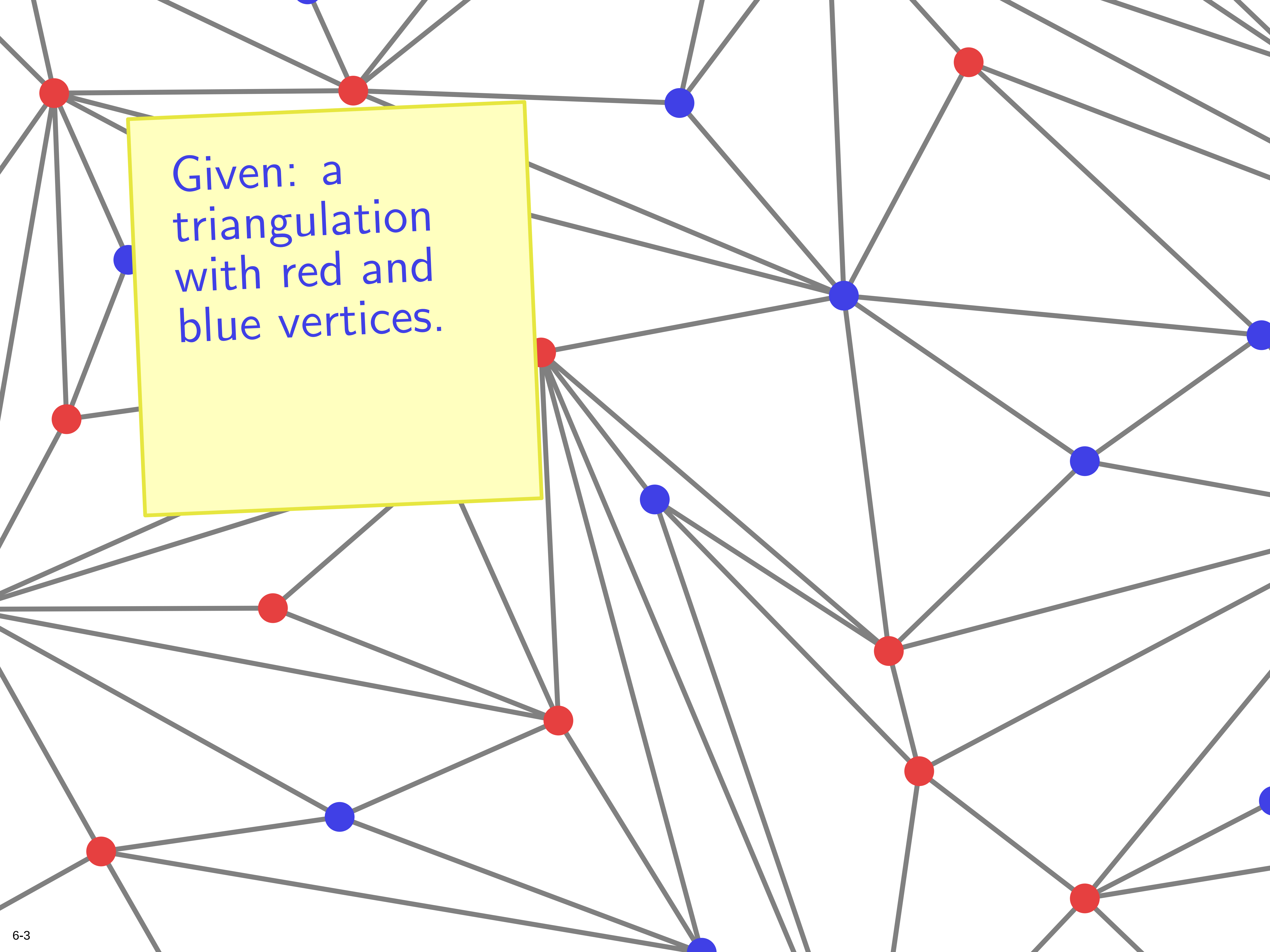
Our improvement: not just discs, but *general* regions.

Preprocessing in $O(n \log n)$, reconstruction in $O(n)$.
[Held & Mitchell, 2008]

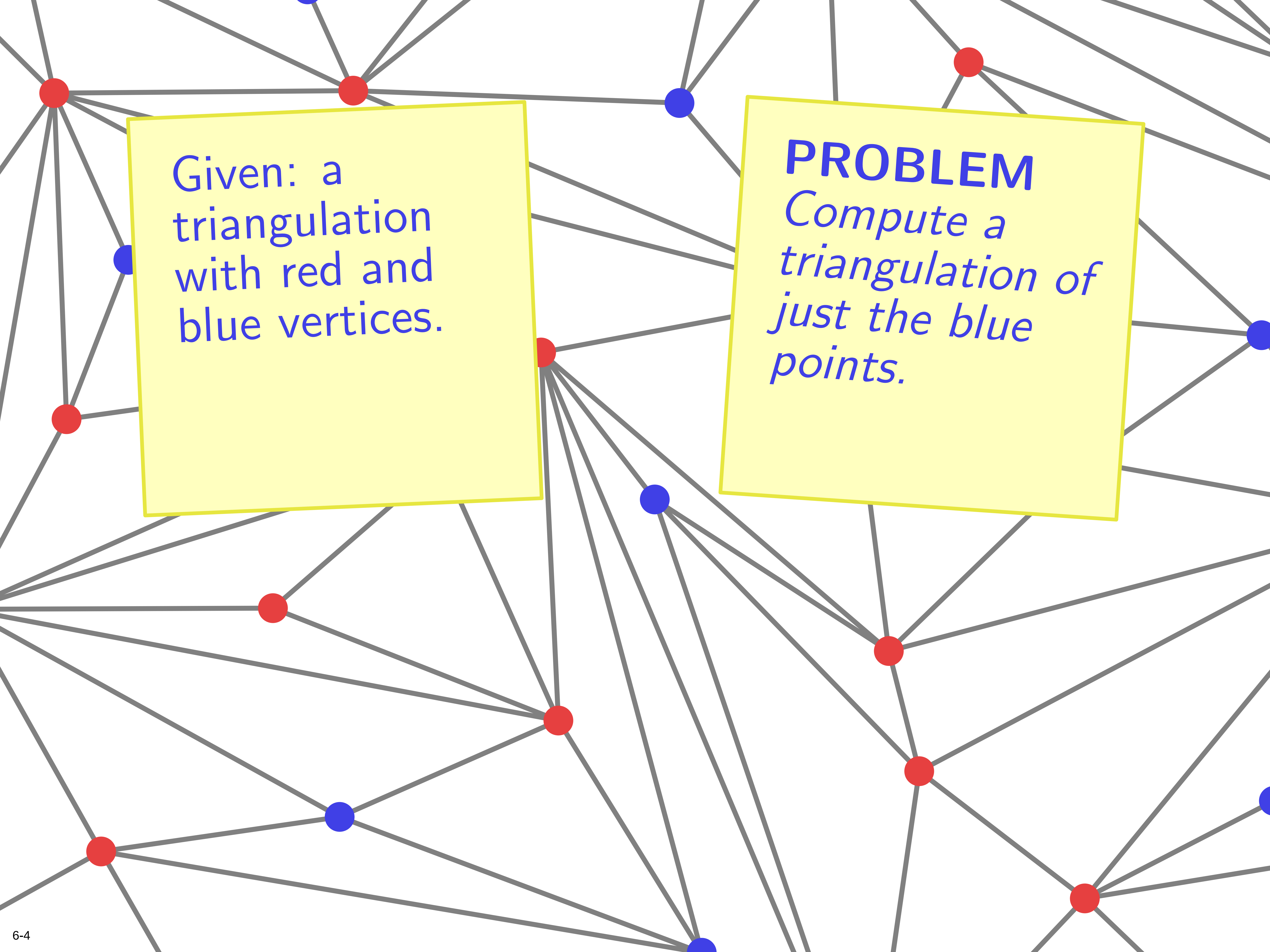
Preprocessing in $O(n \log n)$, reconstruction in $O(n)$.
[Löffler & Snoeyink, 2008]

As the main tool in our solution, we solve a different problem.

Given: a triangulation with red and blue vertices.



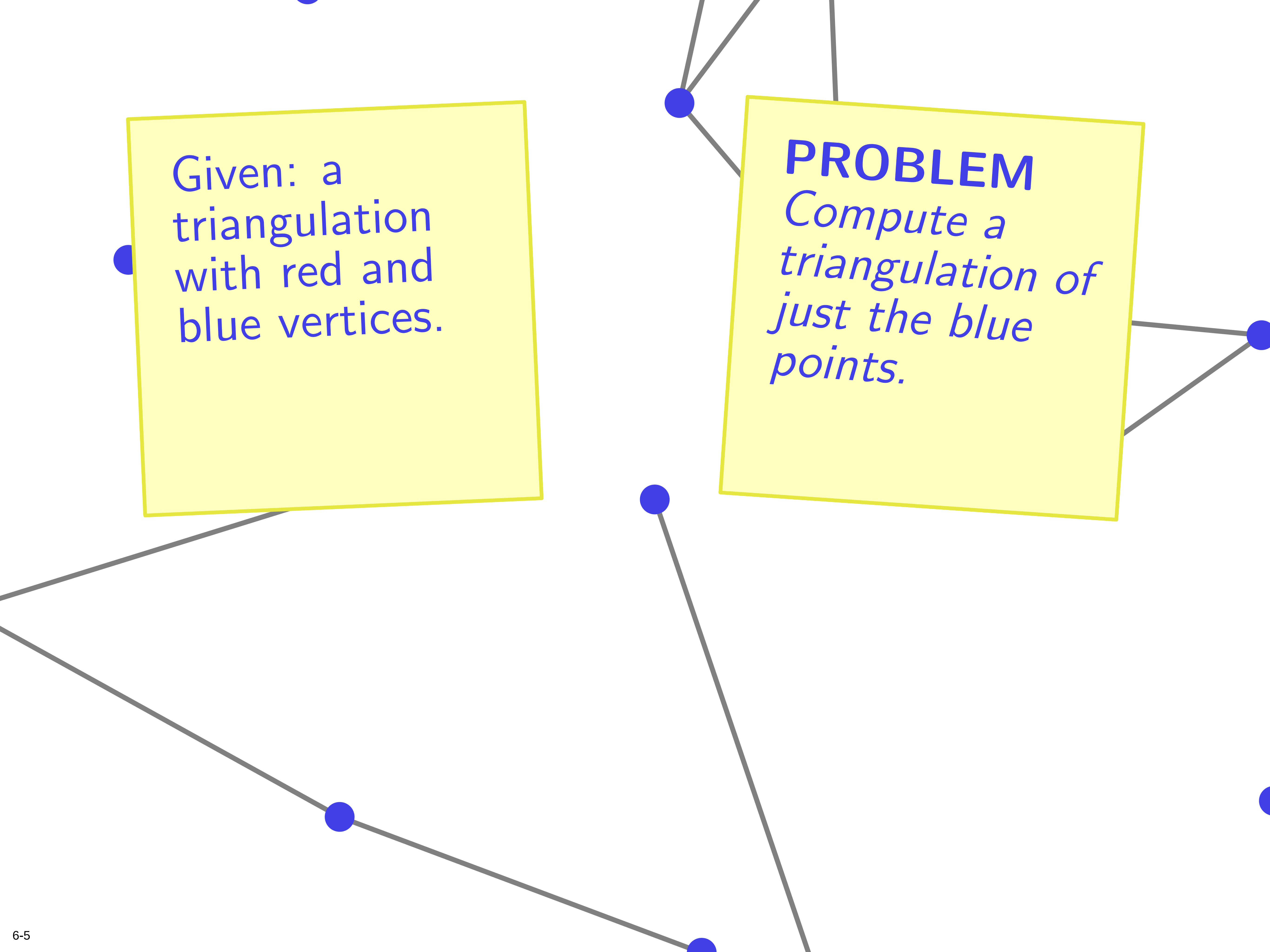
Given: a triangulation with red and blue vertices.



Given: a triangulation with red and blue vertices.

**PROBLEM**

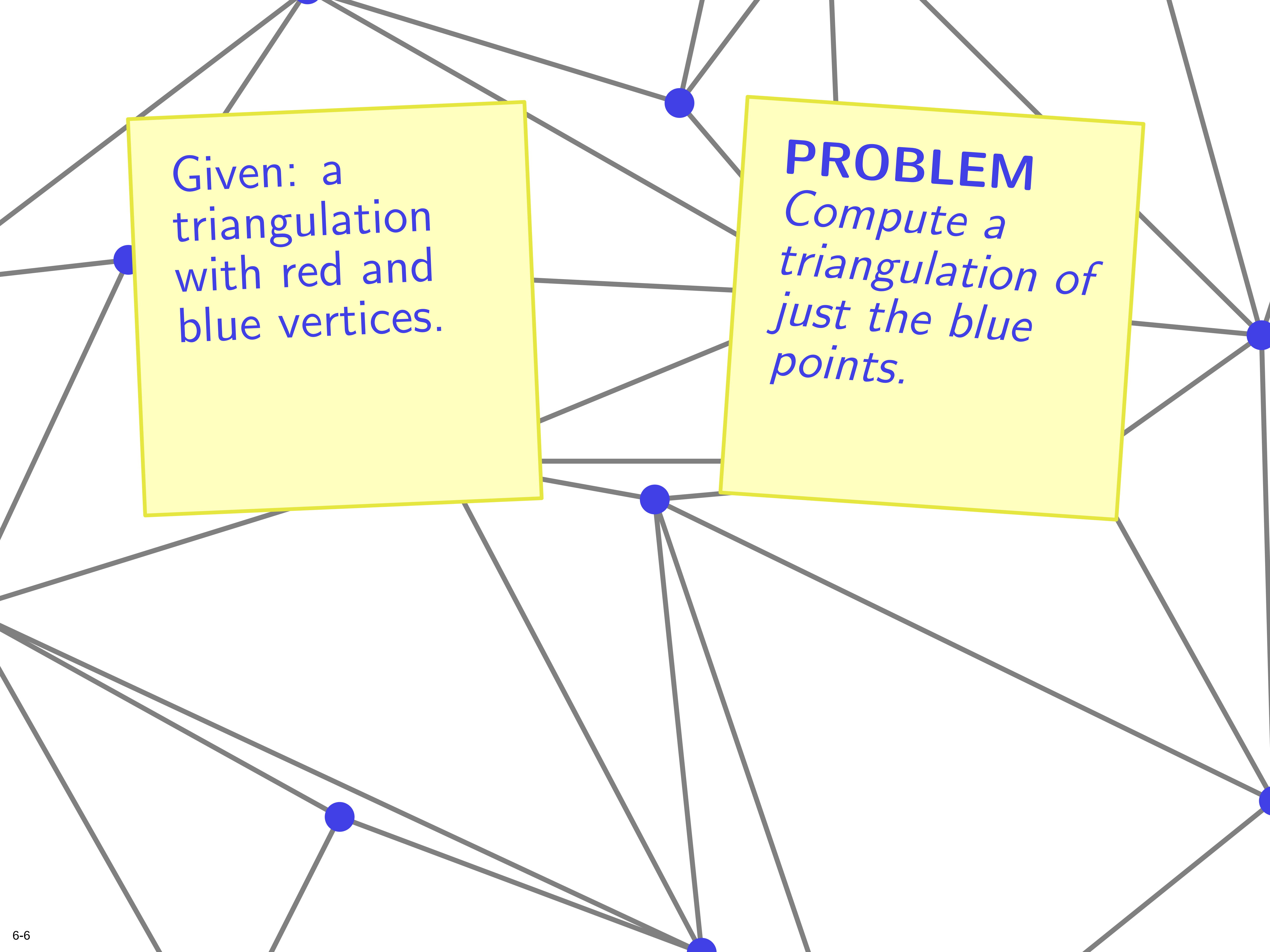
*Compute a triangulation of just the blue points.*

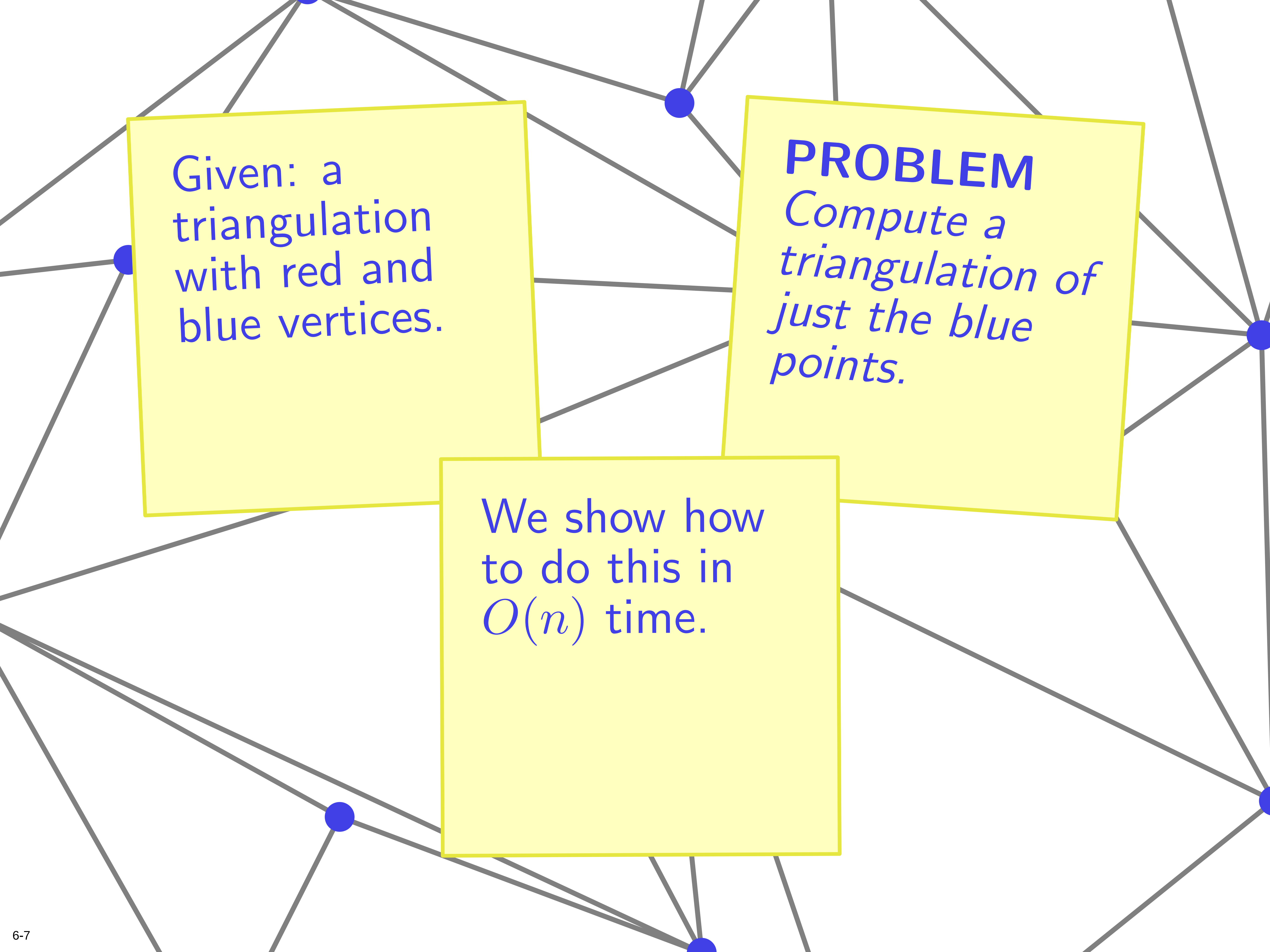


Given: a triangulation with red and blue vertices.

**PROBLEM**

*Compute a triangulation of just the blue points.*


Given: a triangulation with red and blue vertices.
PROBLEM
Compute a triangulation of just the blue points.



Given: a triangulation with red and blue vertices.

**PROBLEM**
*Compute a triangulation of just the blue points.*

We show how to do this in $O(n)$ time.

Idea: remove constant degree red vertices one by one and retriangulate.

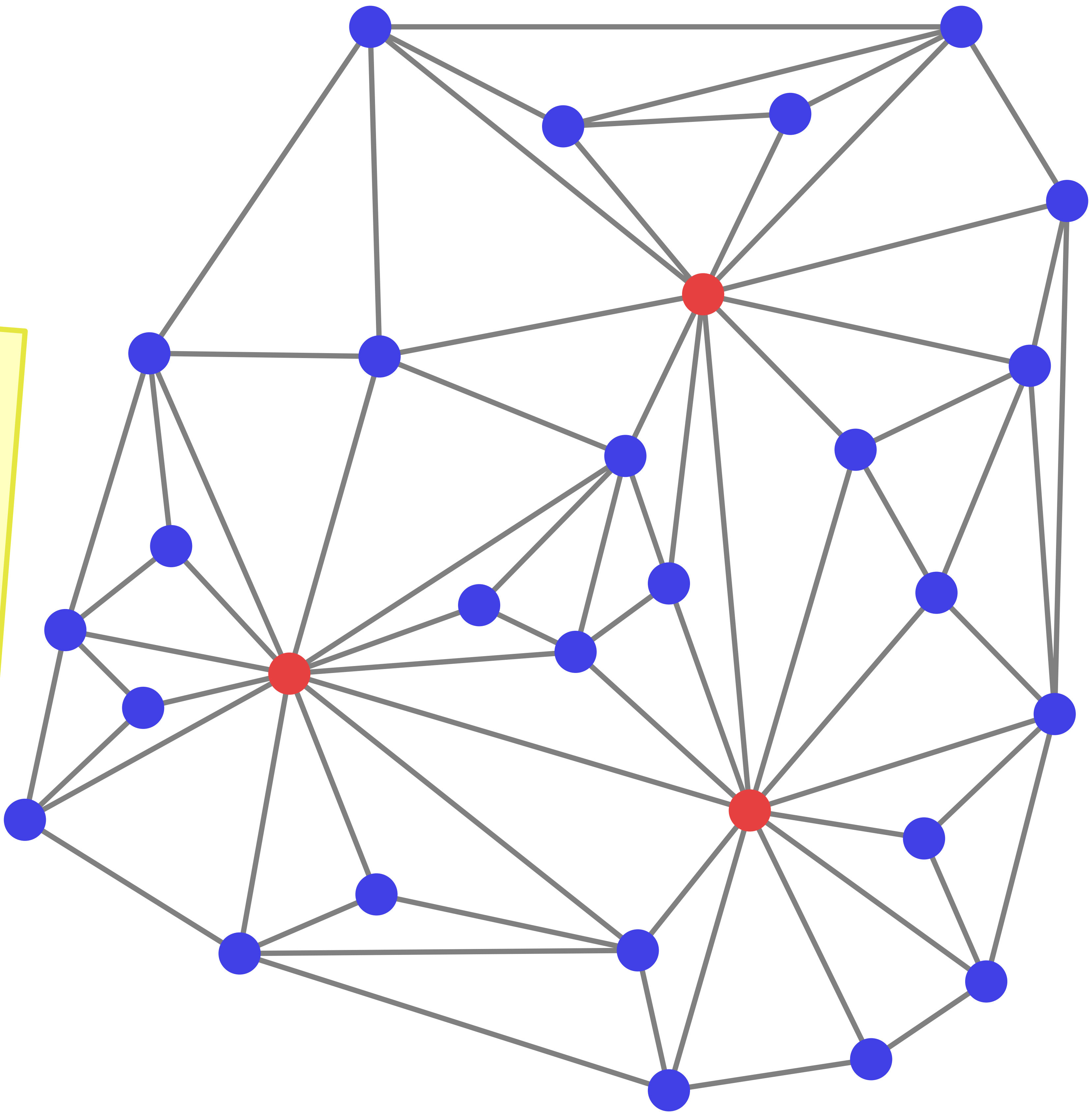
There is always a vertex of constant degree.

There is always a vertex of constant degree.

Slight problem: they might all be blue.

There is always a vertex of constant degree.

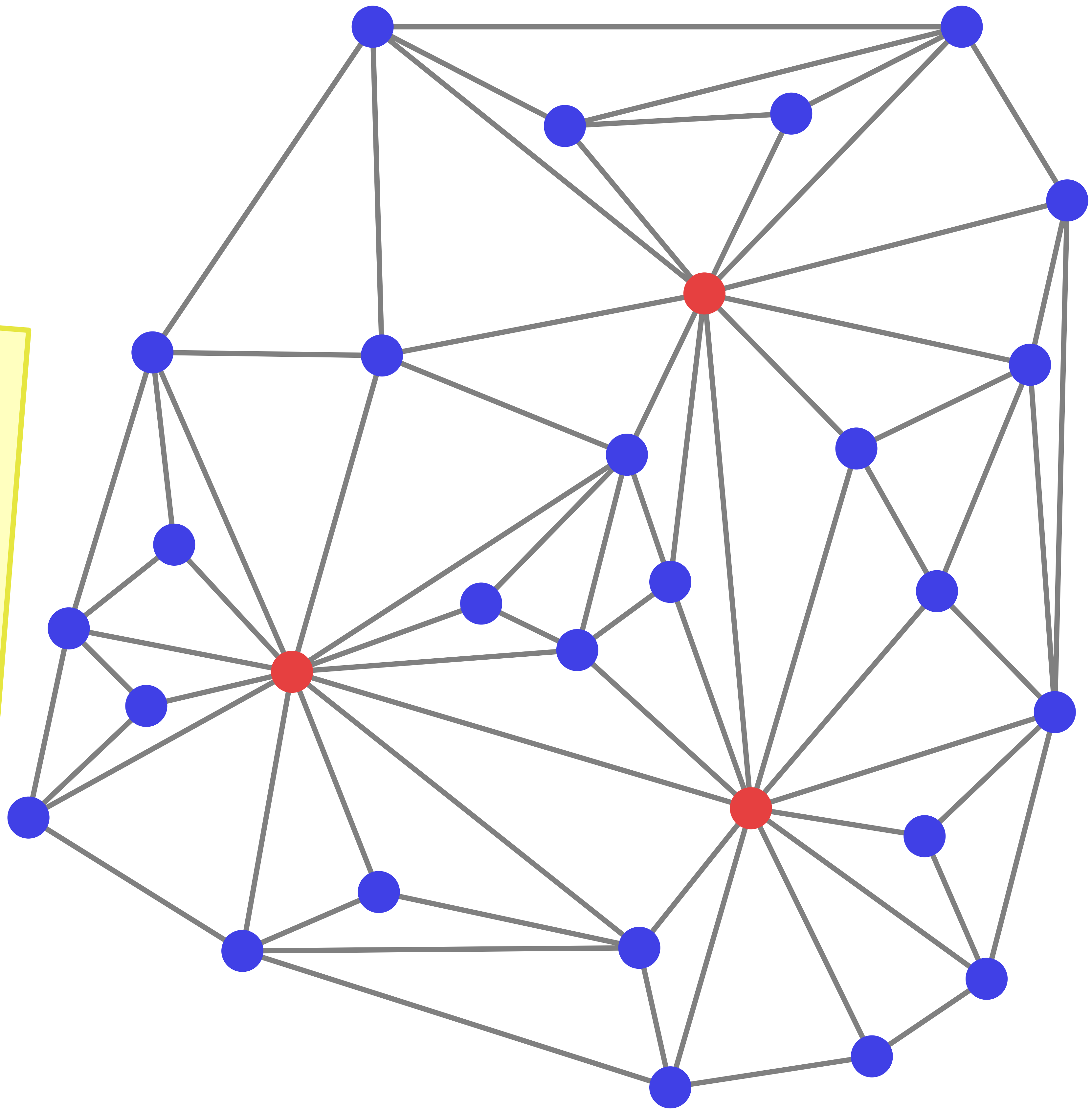
Slight problem: they might all be blue.



There is always a vertex of constant degree.

Slight problem: they might all be blue.

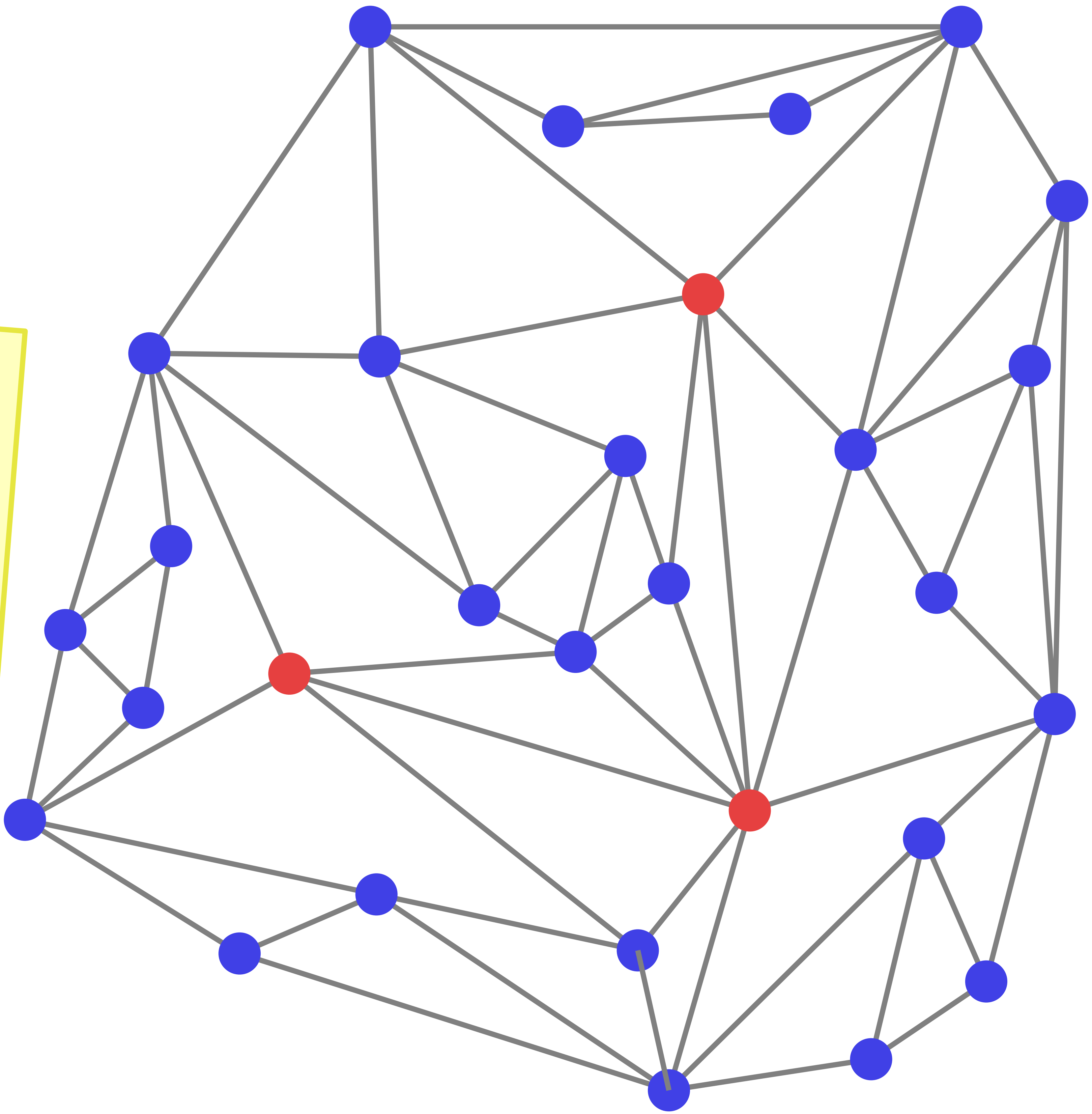
Instead, we maintain a *pseudo-triangulation*.



There is always a vertex of constant degree.

Slight problem: they might all be blue.

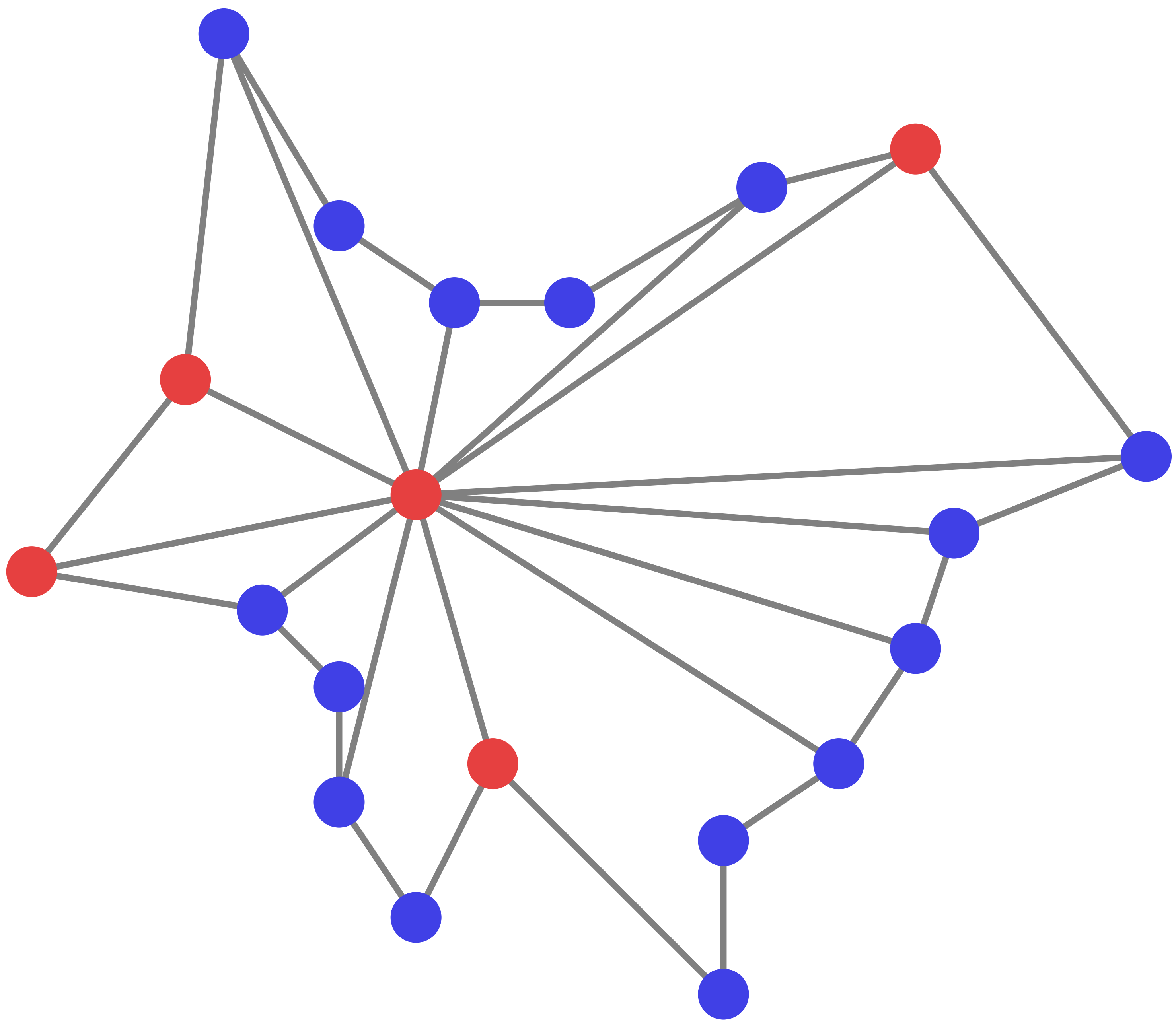
Instead, we maintain a *pseudo-triangulation.*



We need a way to simplify a red point in time linear in its degree.

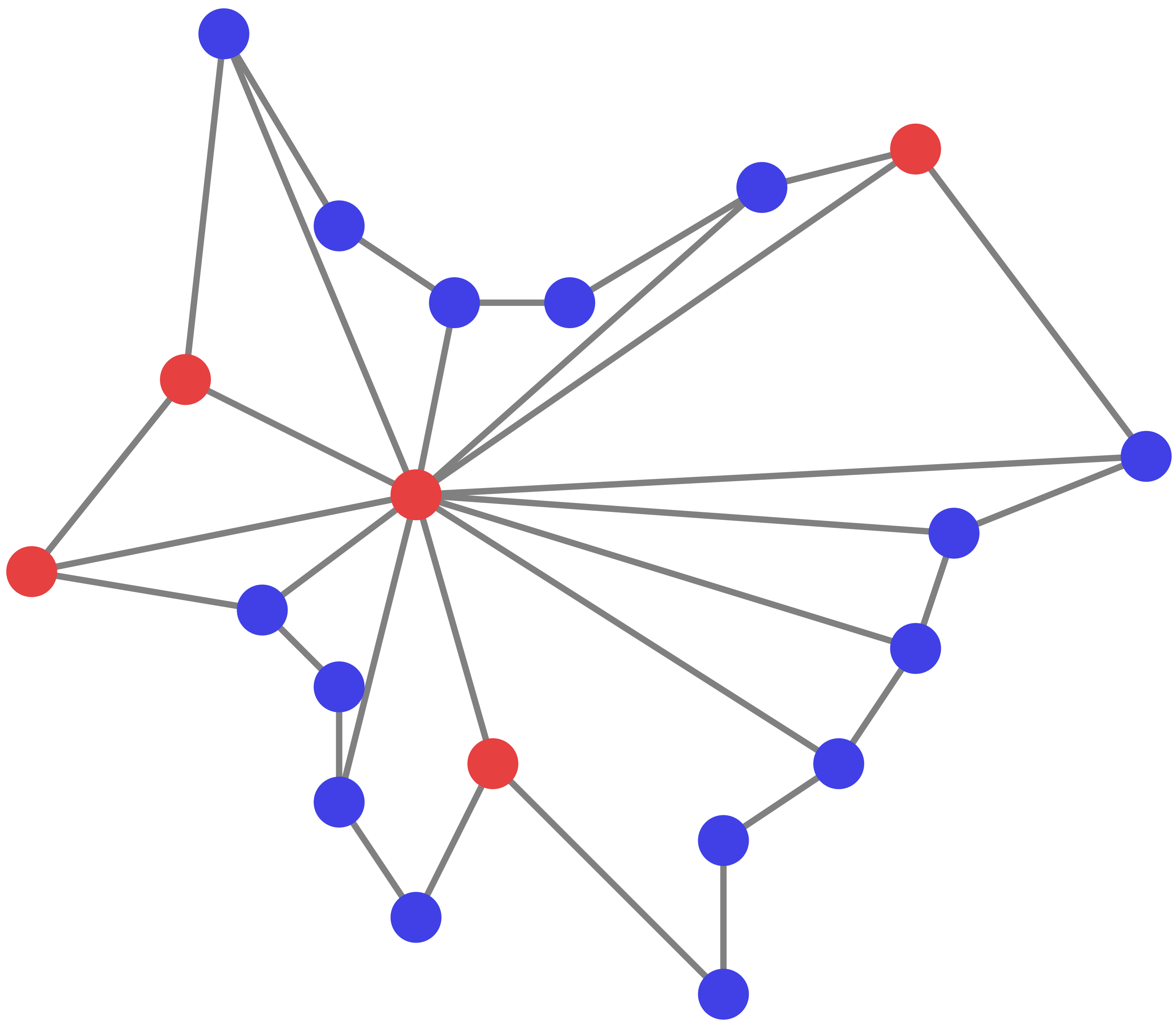
A red vertex $v$ should not have many more blue than red neighbours.

A red vertex $v$ should not have many more blue than red neighbours.



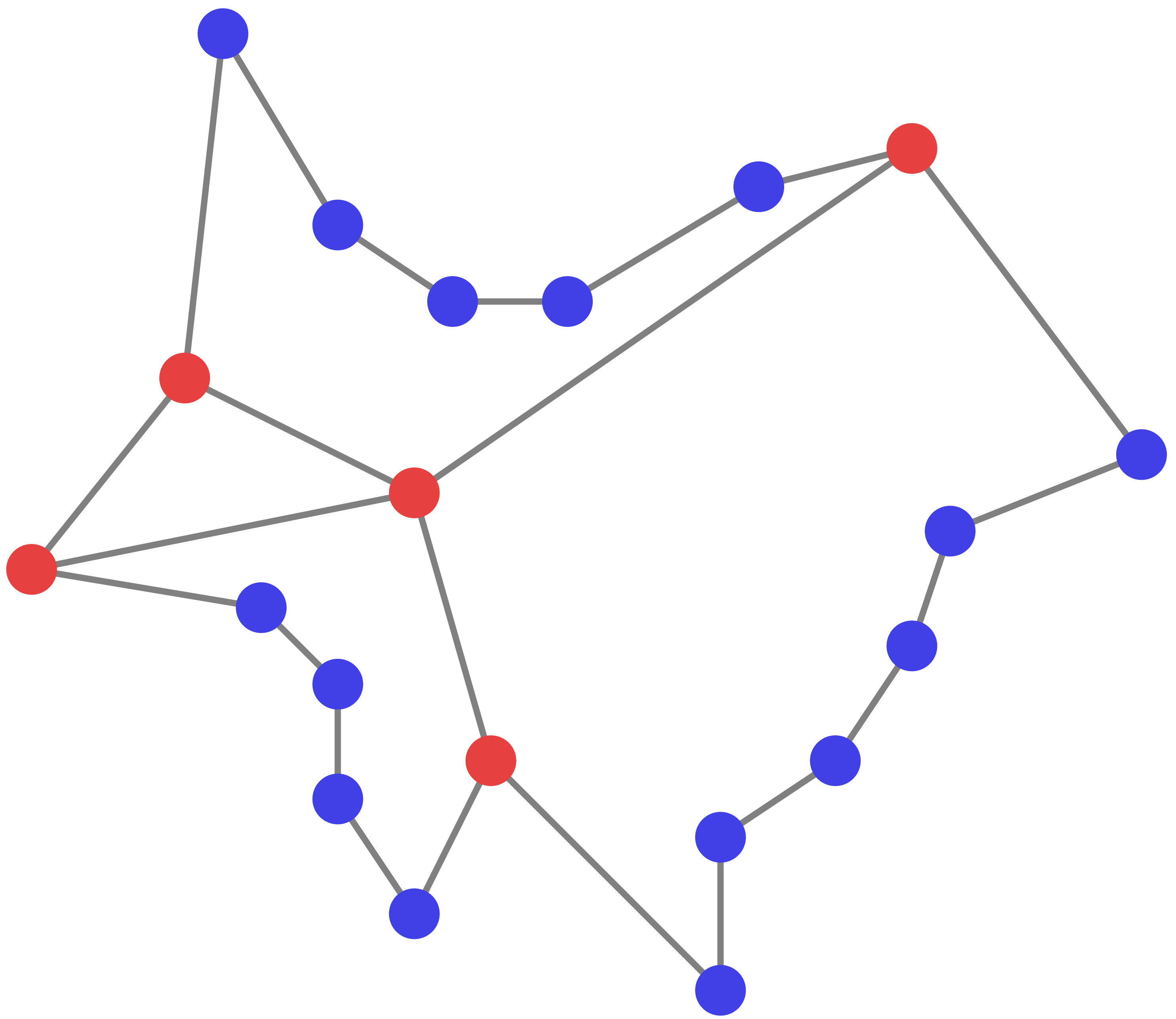
A red vertex $v$ should not have many more blue than red neighbours.

Remove all red-blue edges of $v$.



A red vertex $v$ should not have many more blue than red neighbours.

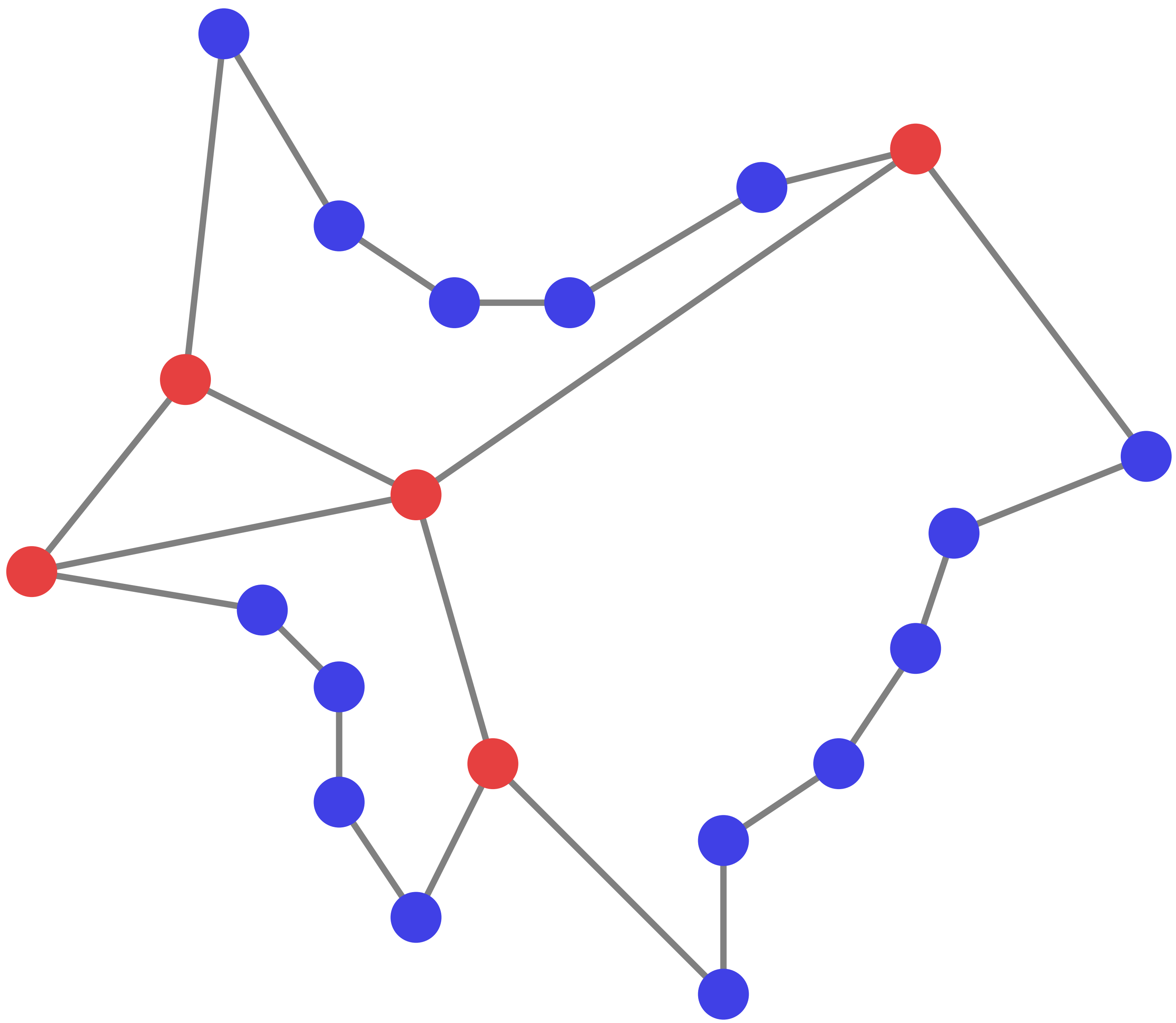
Remove all red-blue edges of $v$.



A red vertex $v$ should not have many more blue than red neighbours.

Remove all red-blue edges of $v$.

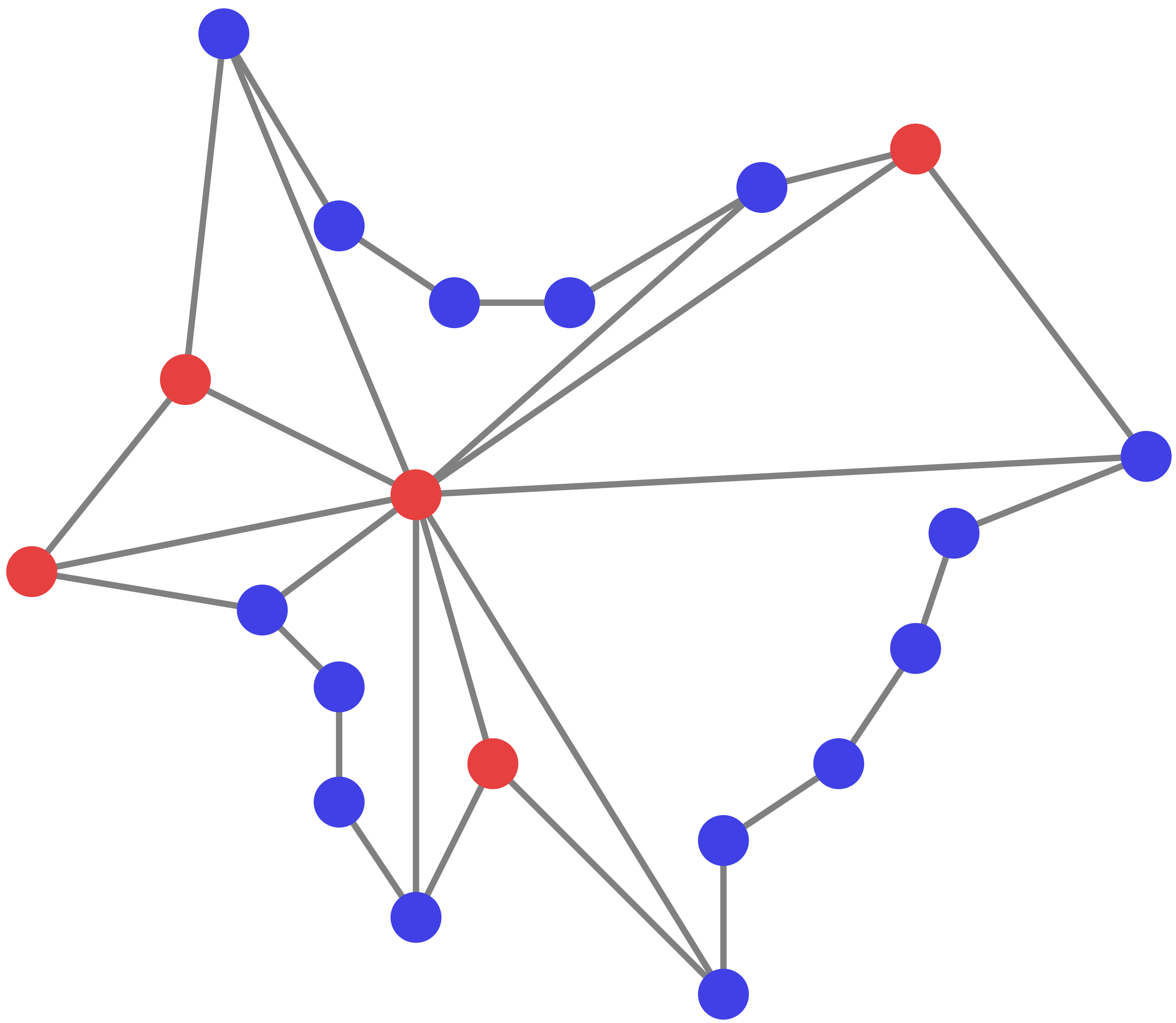
Add edges to the blue neighbours of red-red edges.



A red vertex $v$ should not have many more blue than red neighbours.

Remove all red-blue edges of $v$.

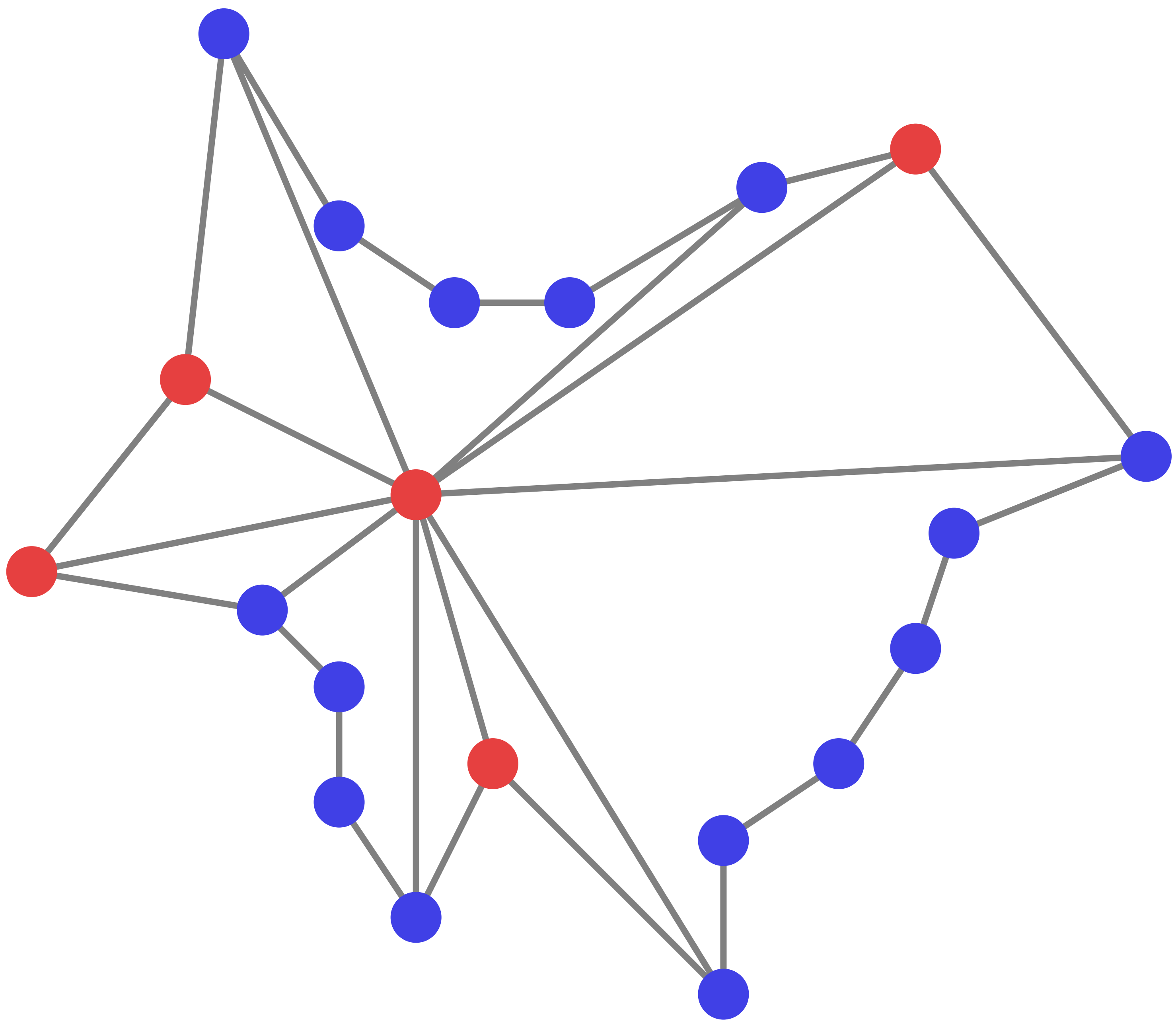
Add edges to the blue neighbours of red-red edges.



A red vertex $v$ should not have many more blue than red neighbours.

Remove all red-blue edges of $v$.

Add edges to the blue neighbours of red-red edges.

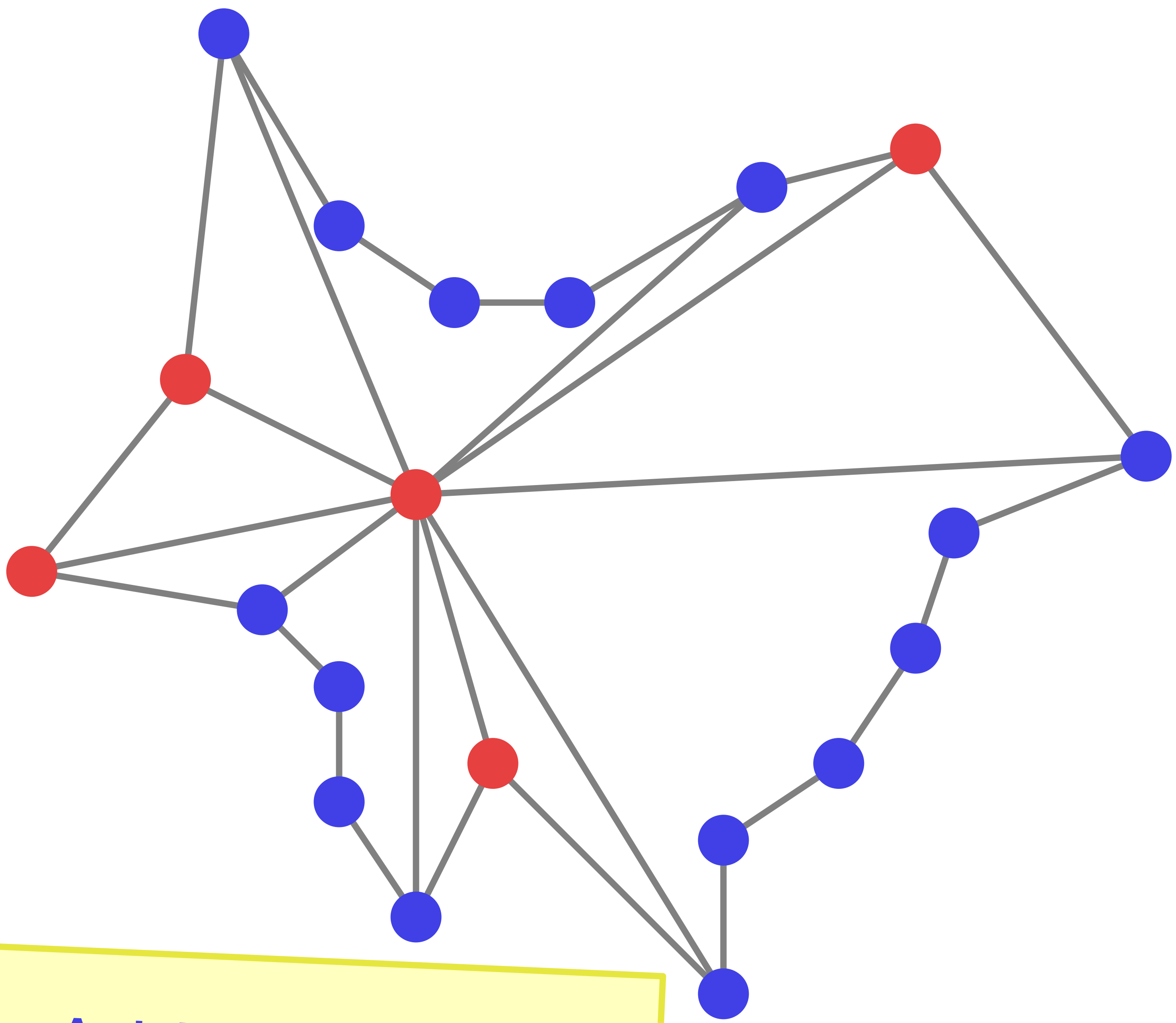


A red vertex $v$ should not have many more blue than red neighbours.

Remove all red-blue edges of $v$.

Add edges to the blue neighbours of red-red edges.

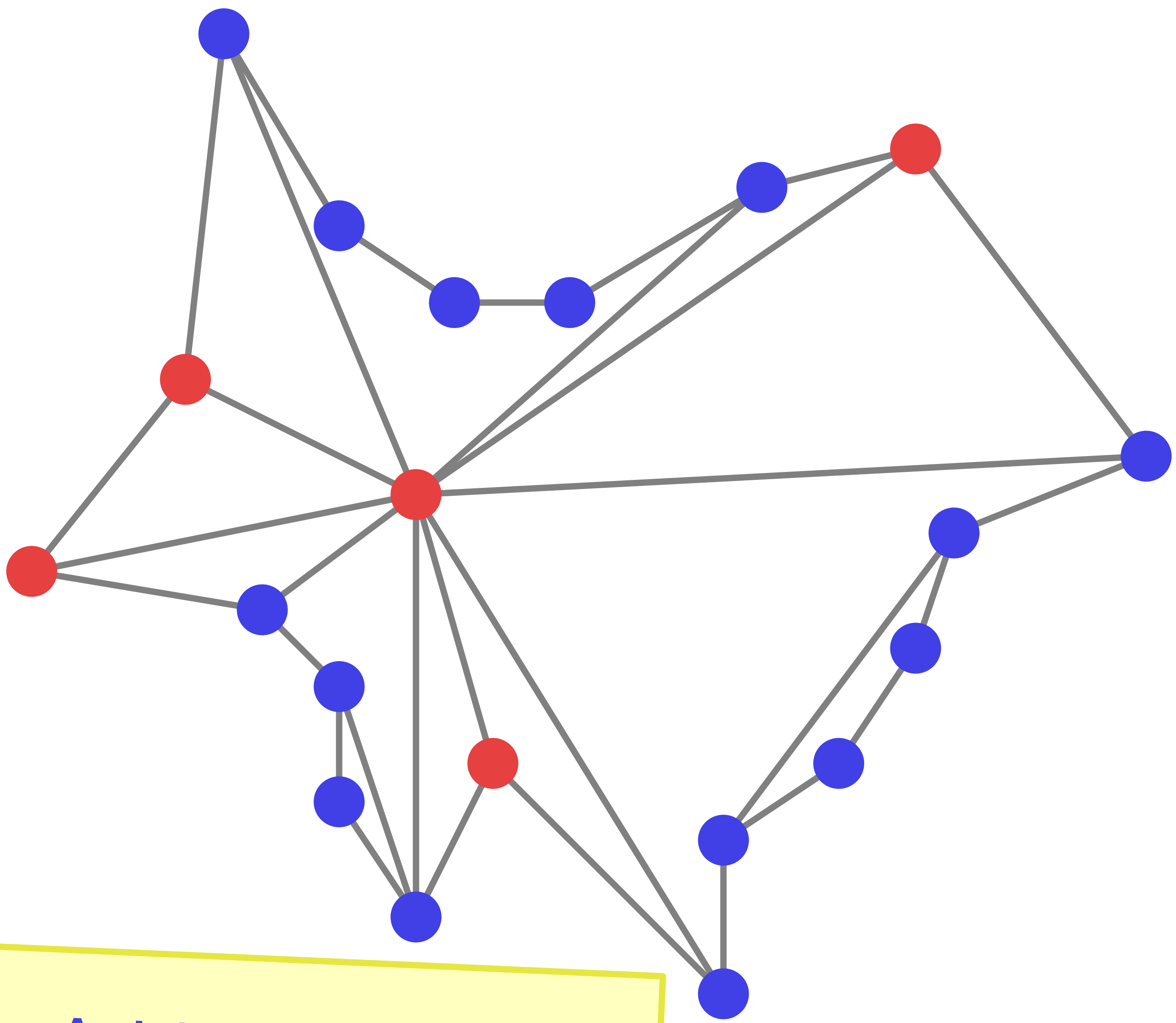
Add convex paths connecting them.



A red vertex $v$ should not have many more blue than red neighbours.

Remove all red-blue edges of $v$.

Add edges to the blue neighbours of red-red edges.



Add convex paths connecting them.

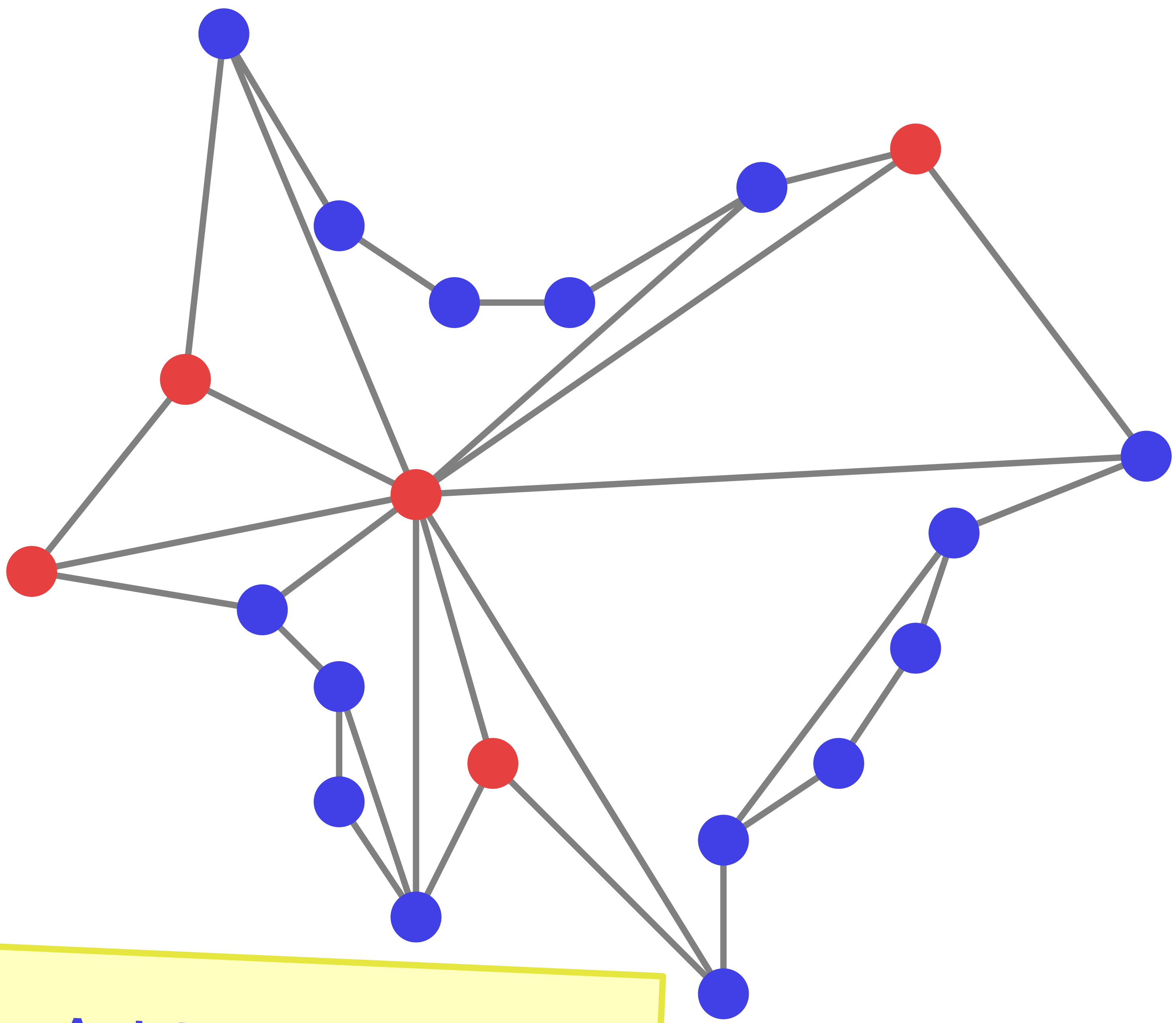
A red vertex $v$ should not have many more blue than red neighbours.

Remove all red-blue edges of $v$.

Add edges to the blue neighbours of red-red edges.

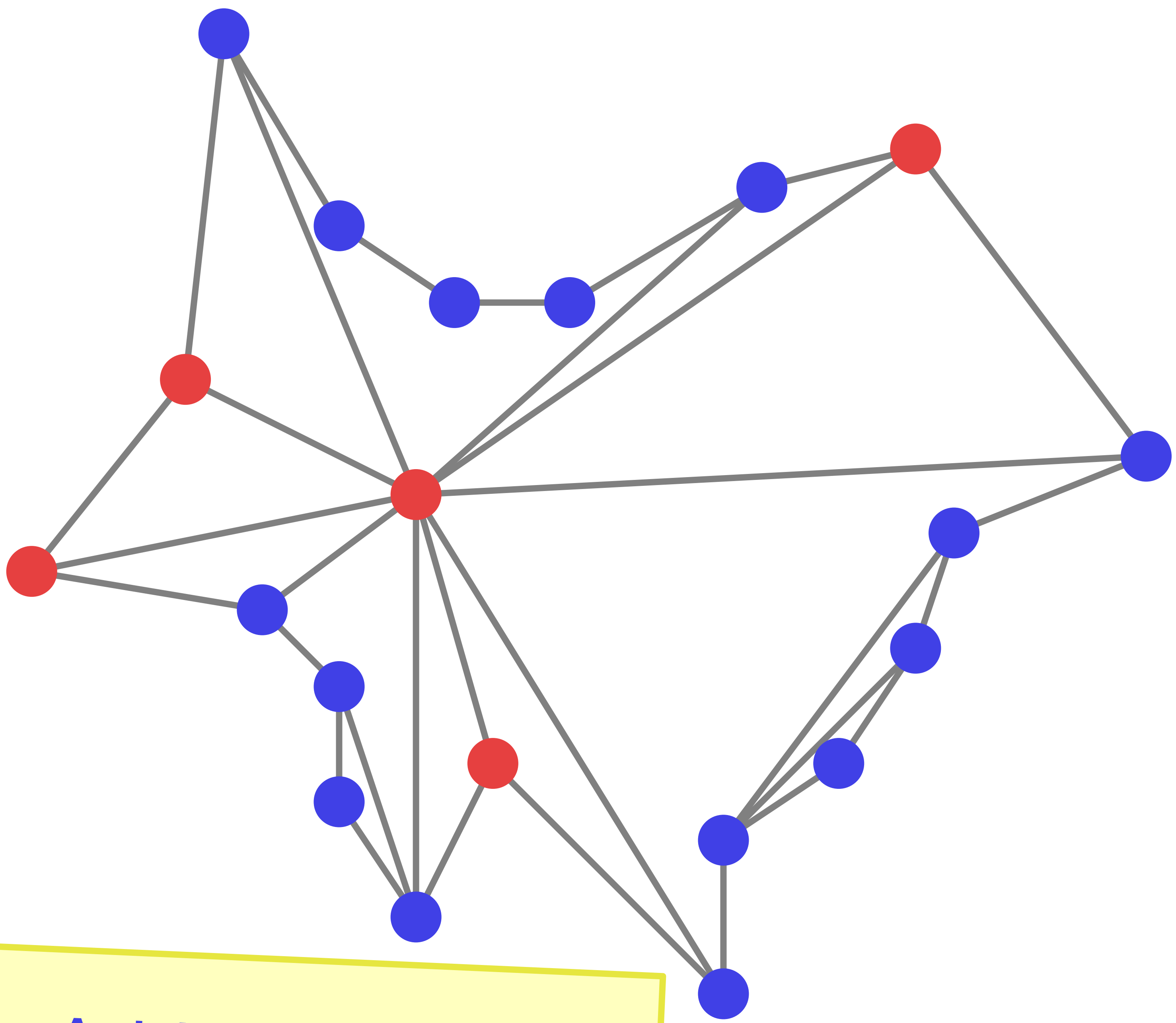
Add convex paths connecting them.

Finally, triangulate the remaining blue regions.



A red vertex $v$ should not have many more blue than red neighbours.

Remove all red-blue edges of $v$.



Add edges to the blue neighbours of red-red edges.

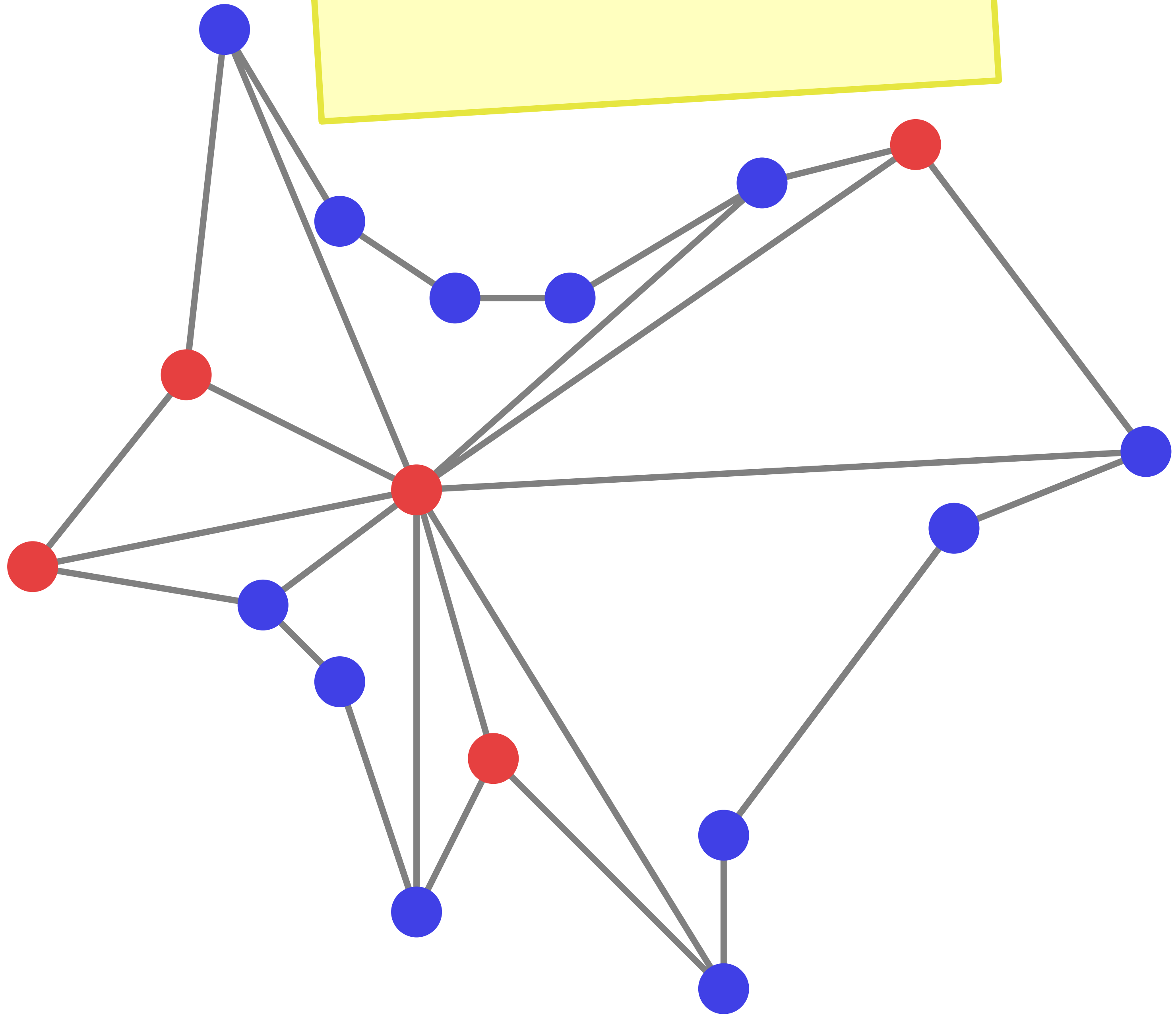
Add convex paths connecting them.

Finally, triangulate the remaining blue regions.

We also need a way to remove a constant degree red point.

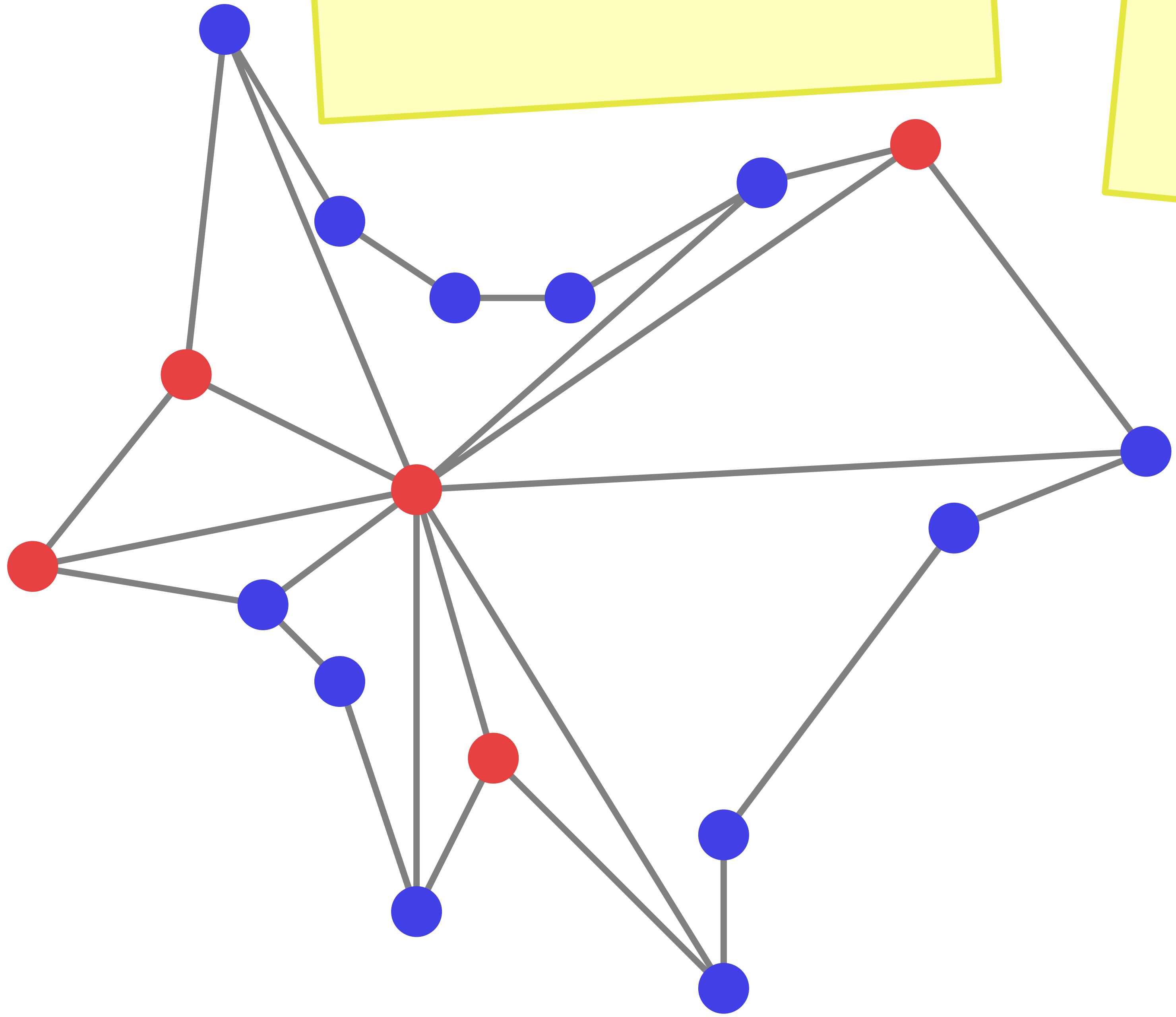
We can remove a constant degree red point in $O(\log b)$ time.

We can remove a constant degree red point in $O(\log b)$ time.



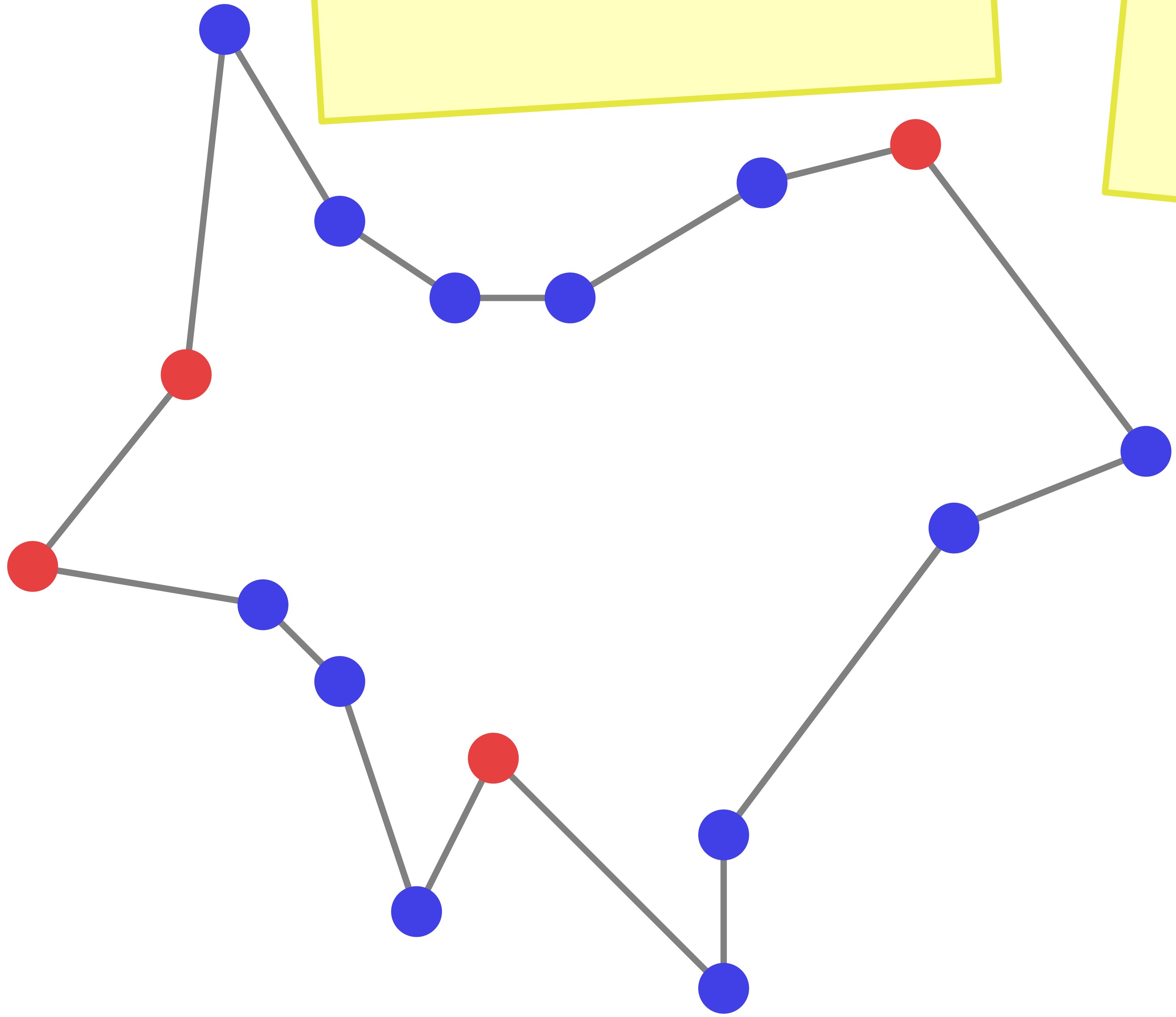
We can remove a constant degree red point in $O(\log b)$ time.

Remove the red point and its $O(1)$ edges.



We can remove a constant degree red point in $O(\log b)$ time.

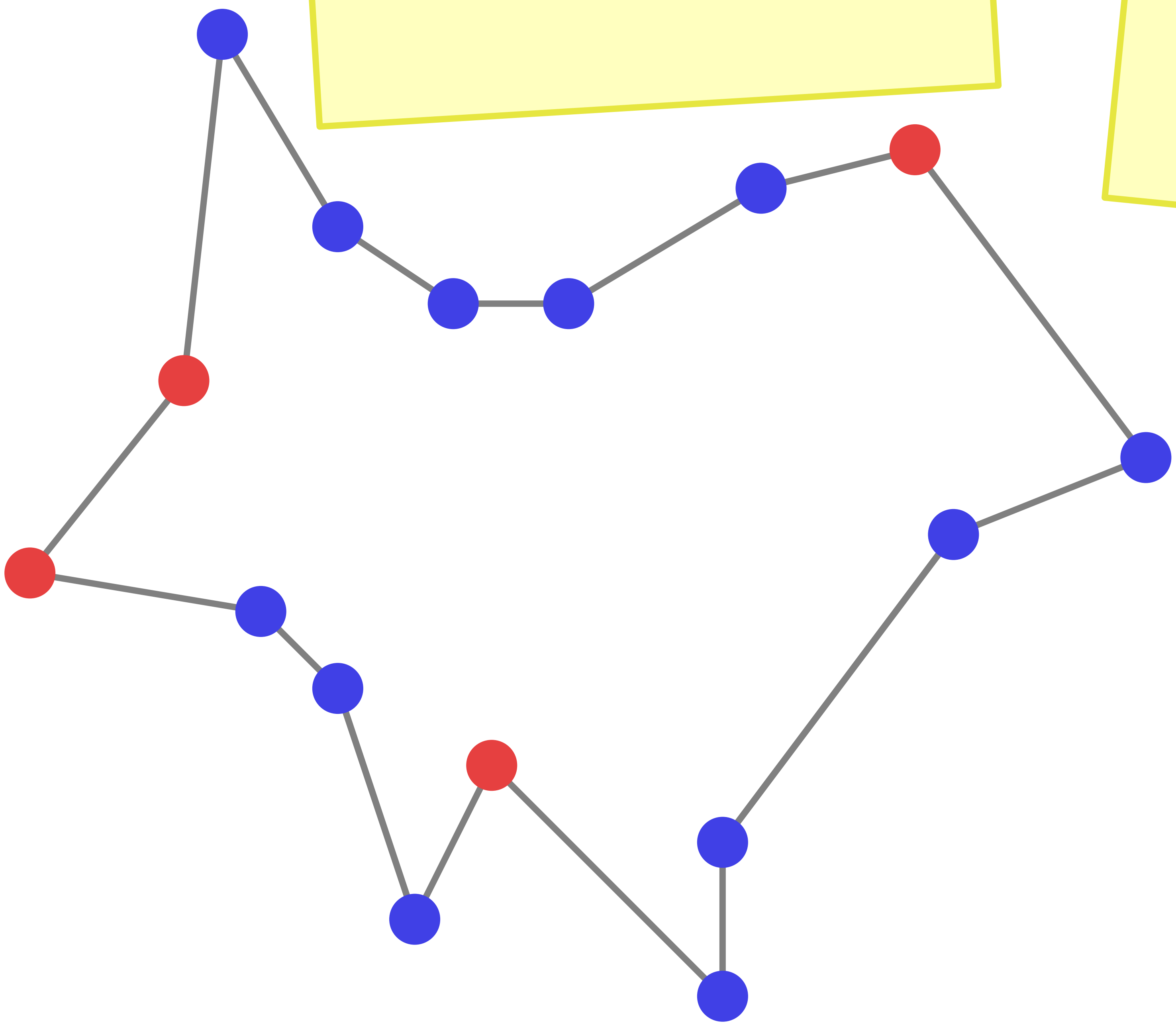
Remove the red point and its $O(1)$ edges.



We can remove a constant degree red point in $O(\log b)$ time.

Remove the red point and its $O(1)$ edges.

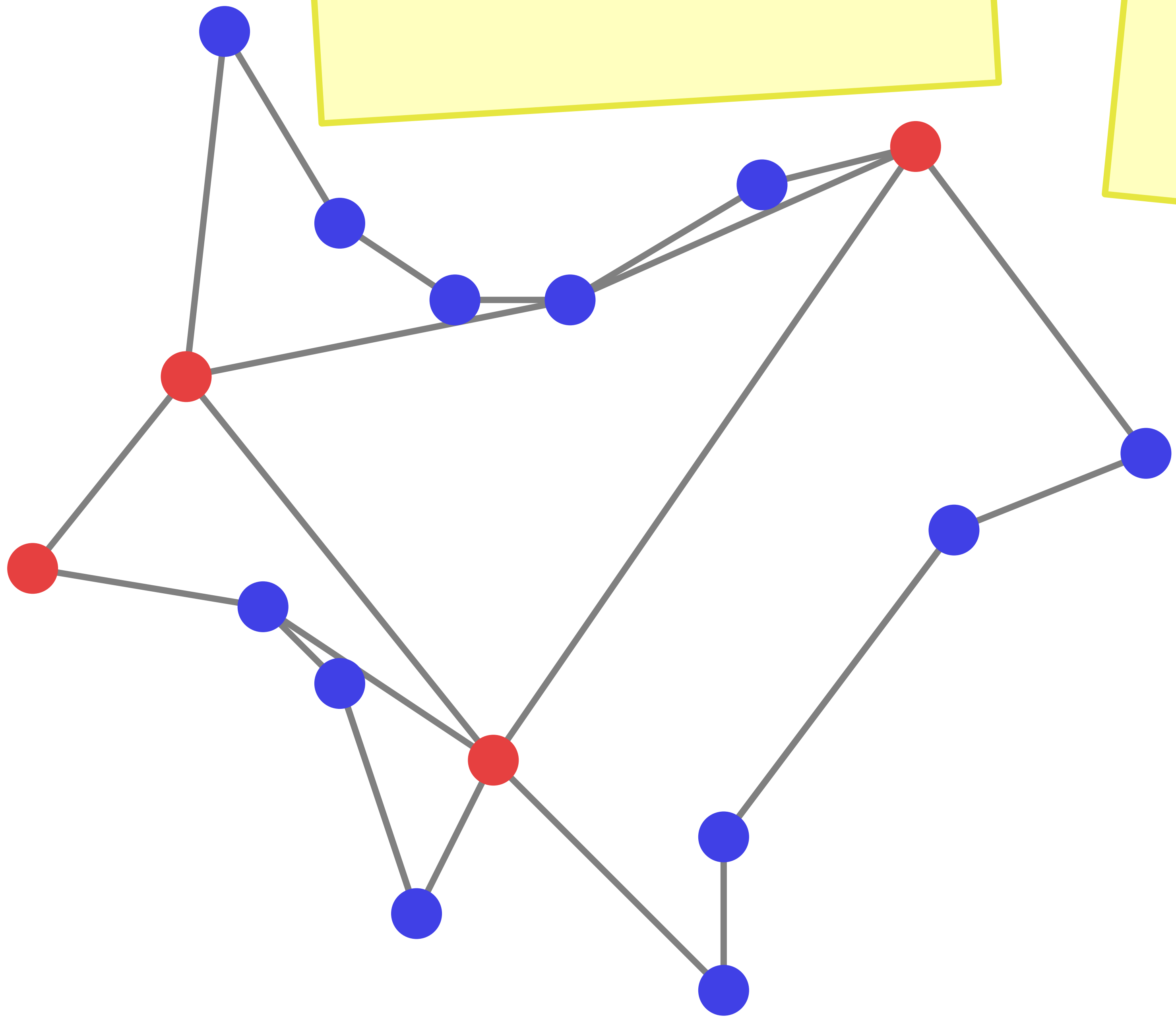
Compute shortest paths between the other red points.



We can remove a constant degree red point in $O(\log b)$ time.

Remove the red point and its $O(1)$ edges.

Compute shortest paths between the other red points.

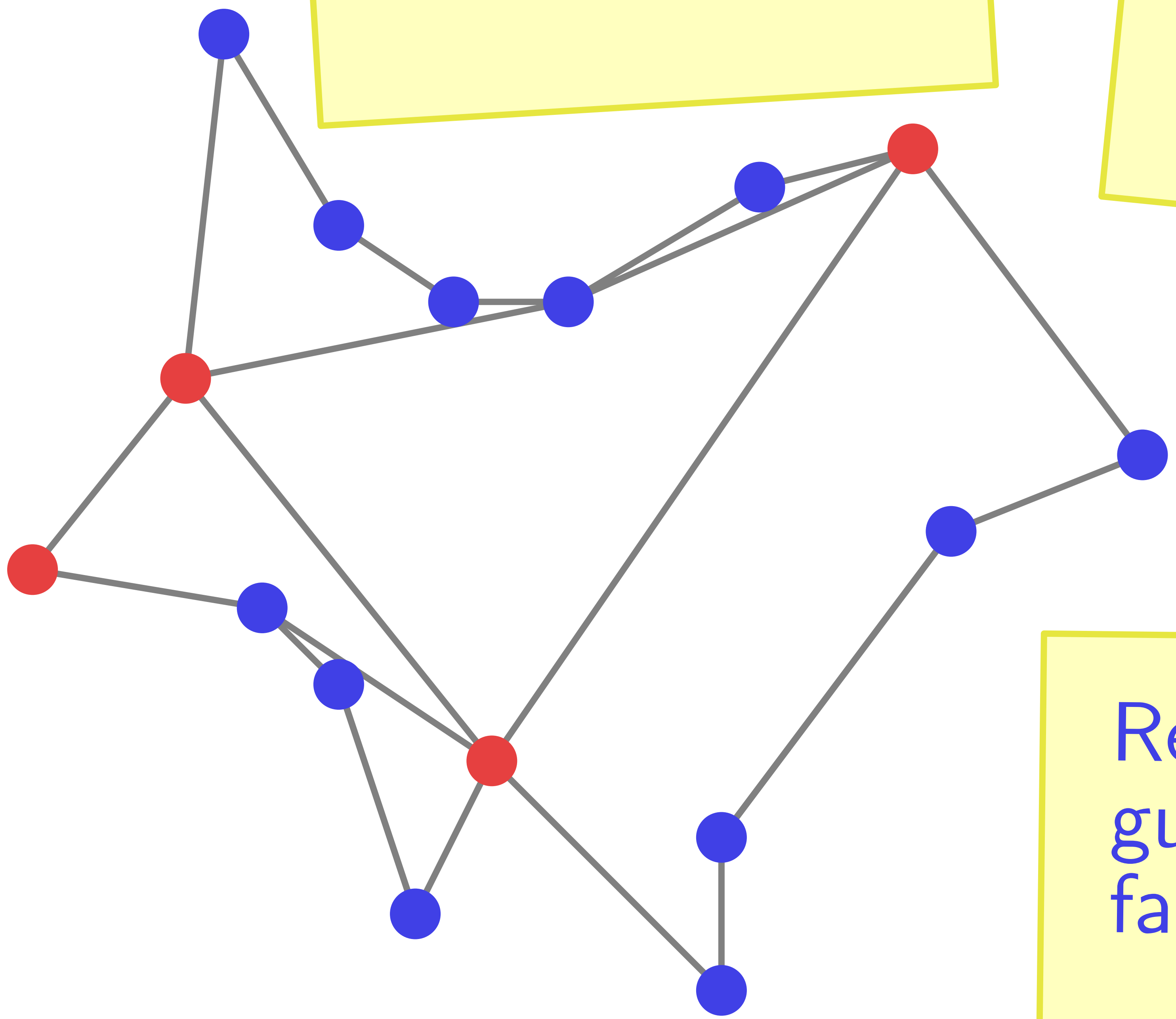


We can remove a constant degree red point in $O(\log b)$ time.

Remove the red point and its $O(1)$ edges.

Compute shortest paths between the other red points.

Repseudotriangulate the new faces.

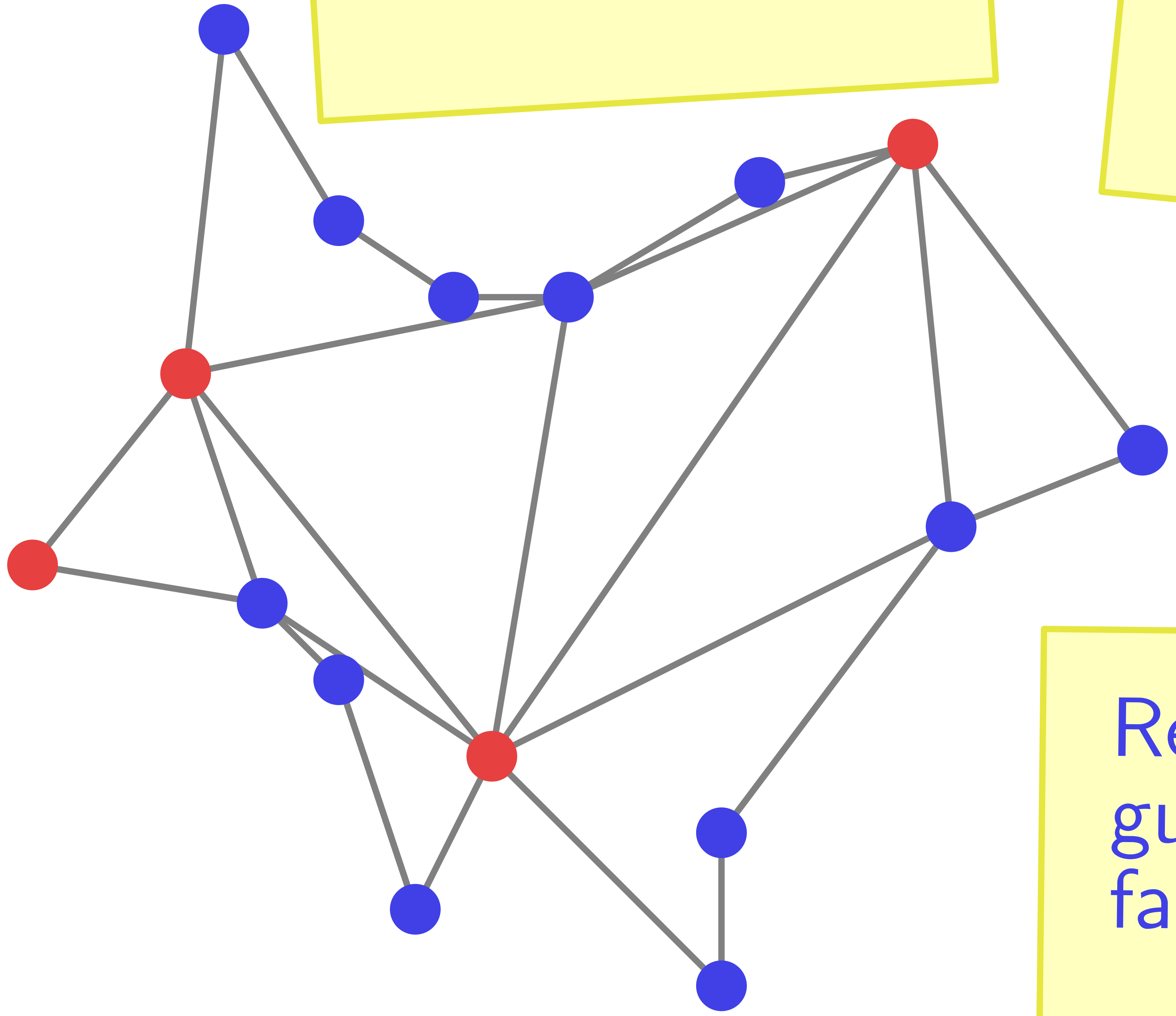


We can remove a constant degree red point in $O(\log b)$ time.

Remove the red point and its $O(1)$ edges.

Compute shortest paths between the other red points.

Repseudotriangulate the new faces.



What is the total running time?

We remove a fraction of $1 - \frac{1}{f}$ red points in each phase.

We remove a fraction of $1 - \frac{1}{f}$ red points in each phase.

In the worst case, this takes $O(k \cdot \log \frac{n}{k})$ time if there are $k$ red points left.

We remove a fraction of $1 - \frac{1}{f}$ red points in each phase.

In the worst case, this takes $O(k \cdot \log \frac{n}{k})$ time if there are $k$ red points left.

There are $\log_f n$ phases, with $k = n/f^i$ red points remaining after the $i$th phase.

We remove a fraction of $1 - \frac{1}{f}$ red points in each phase.

In the worst case, this takes $O(k \cdot \log \frac{n}{k})$ time if there are $k$ red points left.

There are $\log_f n$ phases, with $k = n/f^i$ red points remaining after the $i$th phase.

$$\sum_{i=1}^{\log_f n} O(\frac{n}{f^i} \log f^i)$$

We remove a fraction of $1 - \frac{1}{f}$ red points in each phase.

In the worst case, this takes $O(k \cdot \log \frac{n}{k})$ time if there are $k$ red points left.

There are $\log_f n$ phases, with $k = n/f^i$ red points remaining after the $i$th phase.

$$\sum_{i=1}^{\log_f n} O\left(\frac{n}{f^i} \log f^i\right) = \ldots$$

We remove a fraction of $1 - \frac{1}{f}$ red points in each phase.

In the worst case, this takes $O(k \cdot \log \frac{n}{k})$ time if there are $k$ red points left.

There are $\log_f n$ phases, with $k = n/f^i$ red points remaining after the $i$th phase.

$$\sum_{i=1}^{\log_f n} O(\frac{n}{f^i} \log f^i) = O(n)$$

We remove a fraction of $1 - \frac{1}{f}$ red points in each phase.

In the worst case, this takes $O(k \cdot \log \frac{n}{k})$ time if there are $k$ red points left.

There are $\log_f n$ phases, with $k = n/f^i$ red points remaining after the $i$th phase.

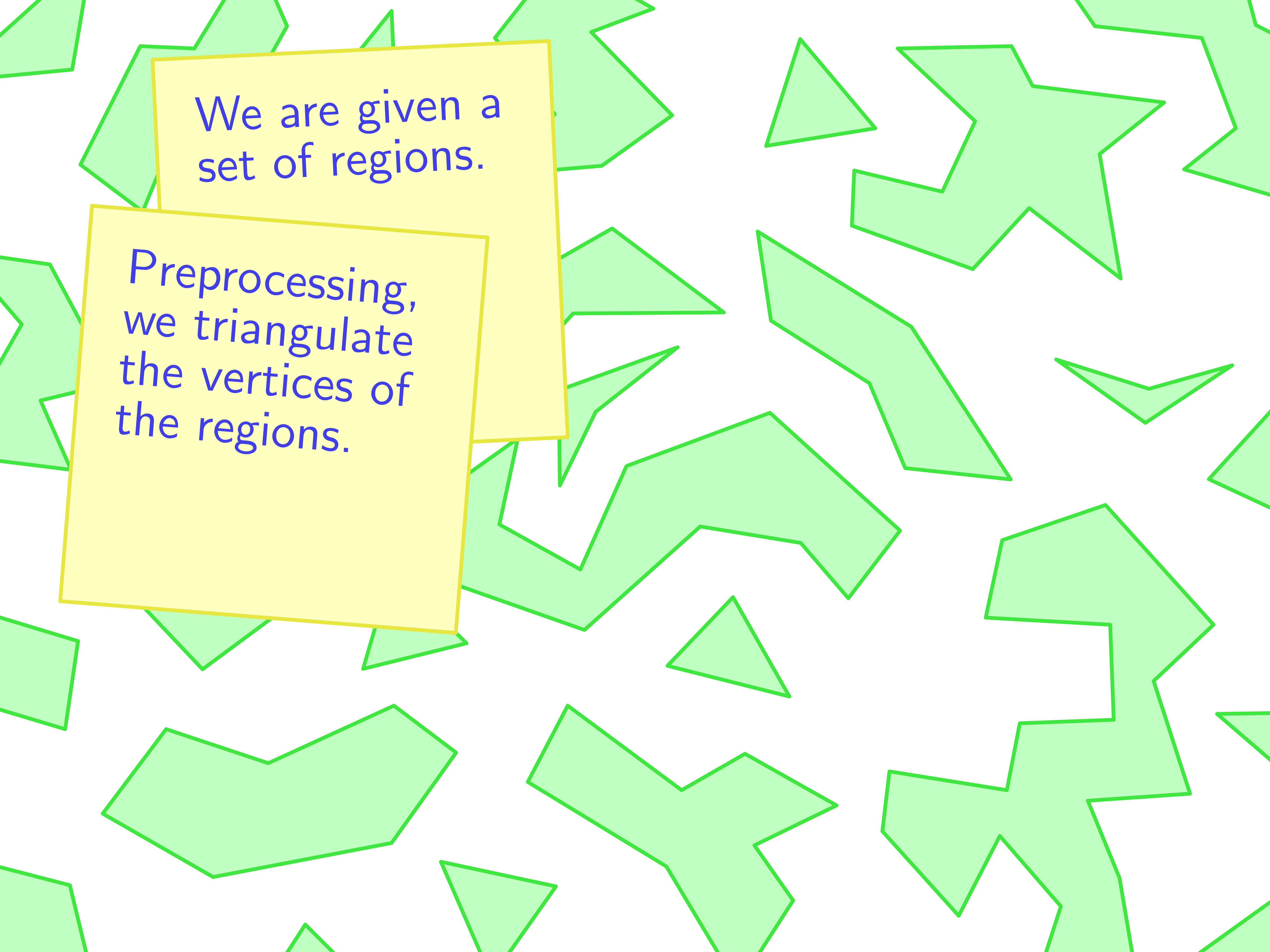
$$\sum_{i=1}^{\log_f n} O(\frac{n}{f^i} \log f^i) = O(n)$$

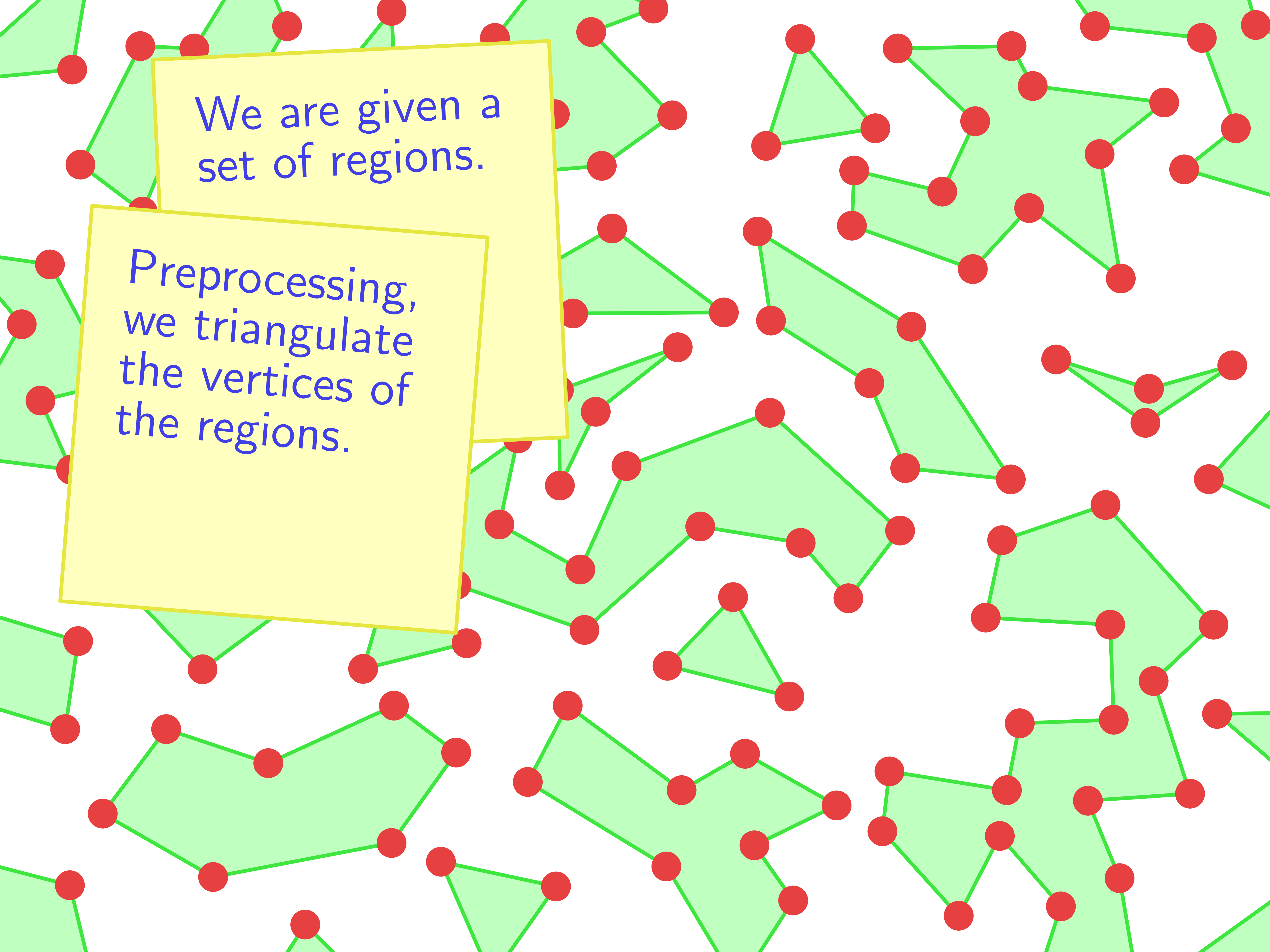
Problem solved!

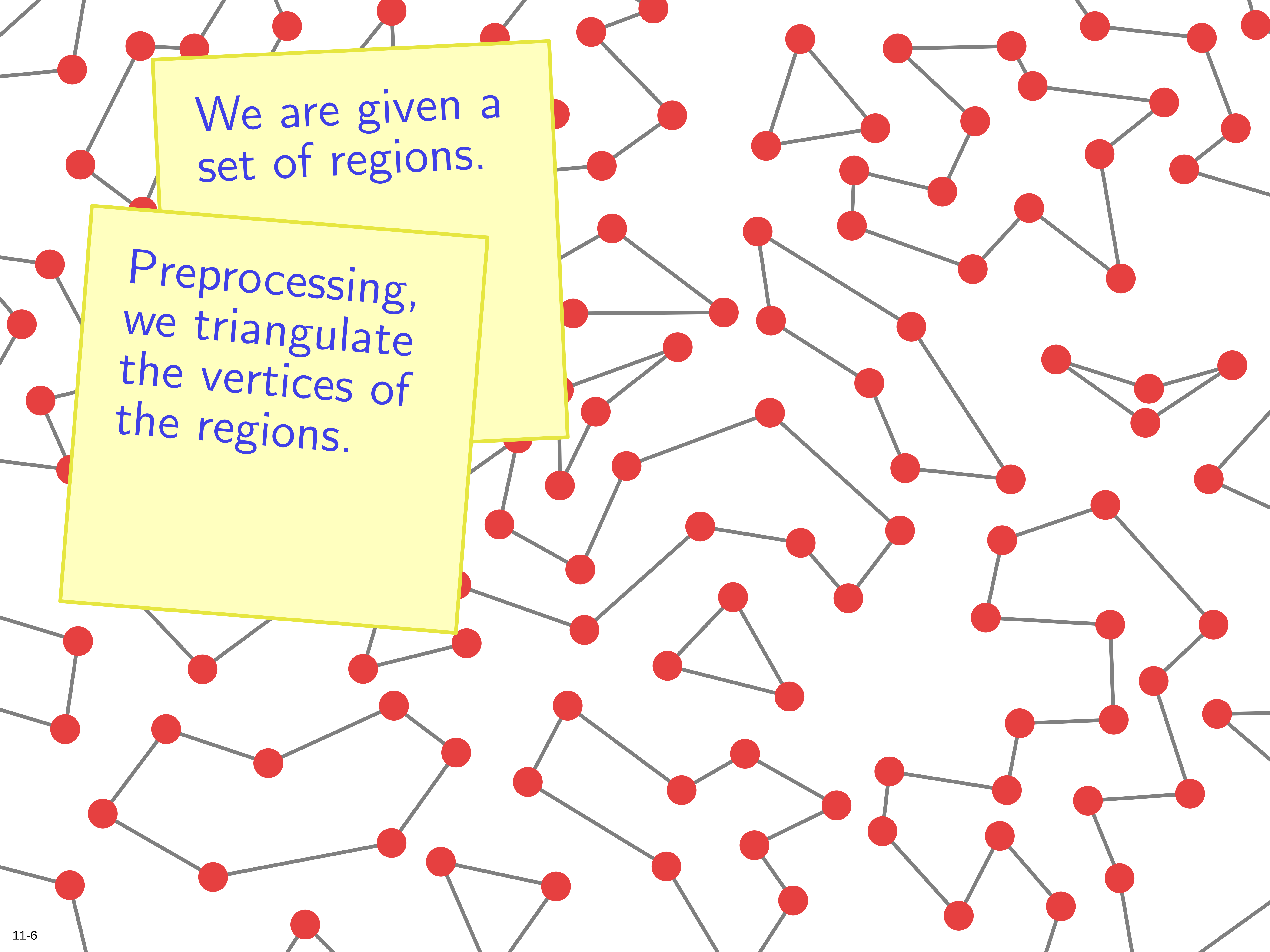
Now, we are ready to return to the original problem.

We are given a set of regions.


We are given a set of regions.

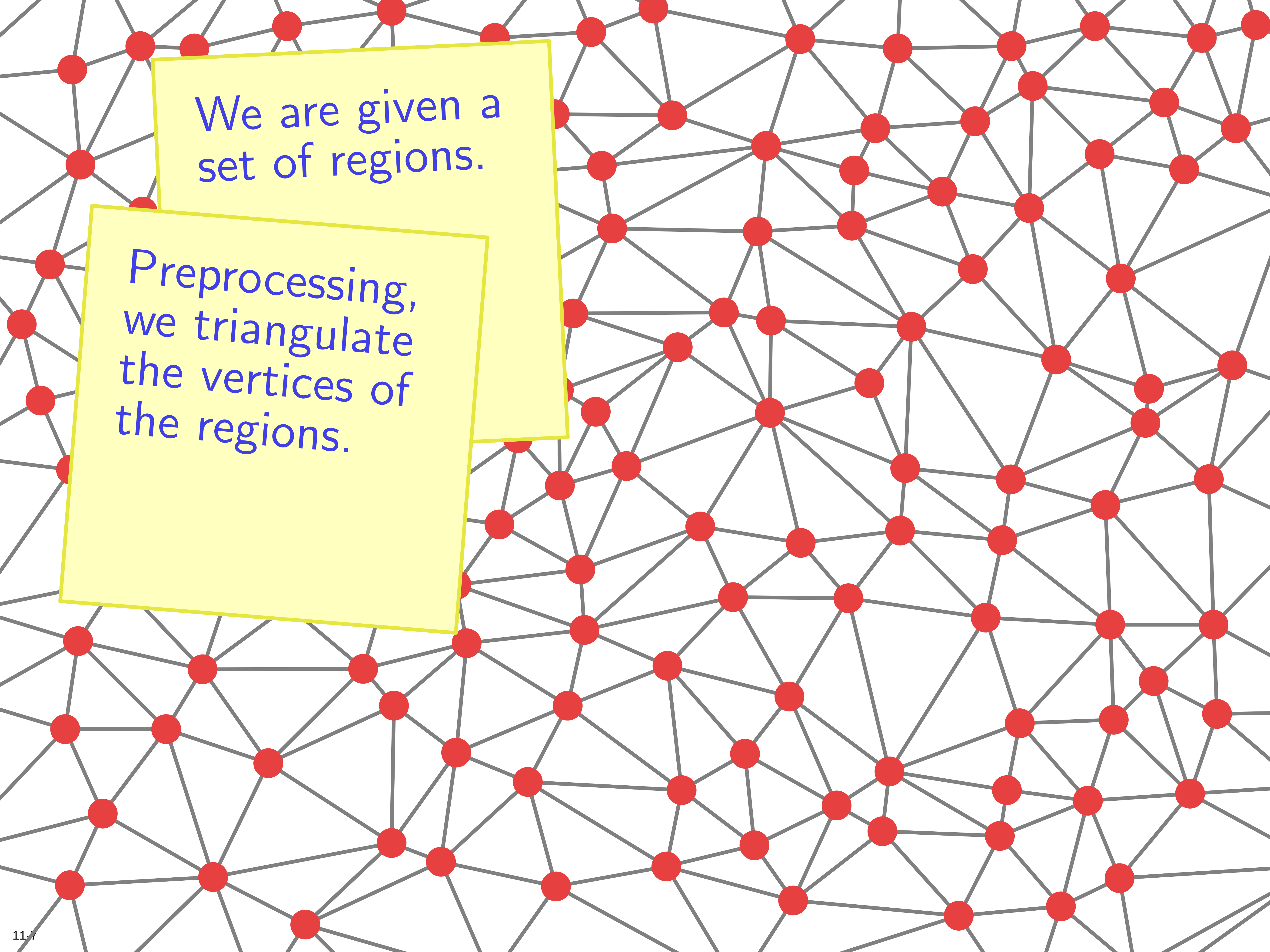

We are given a set of regions.
Preprocessing, we triangulate the vertices of the regions.


We are given a set of regions.
Preprocessing, we triangulate the vertices of the regions.



We are given a set of regions.

Preprocessing, we triangulate the vertices of the regions.



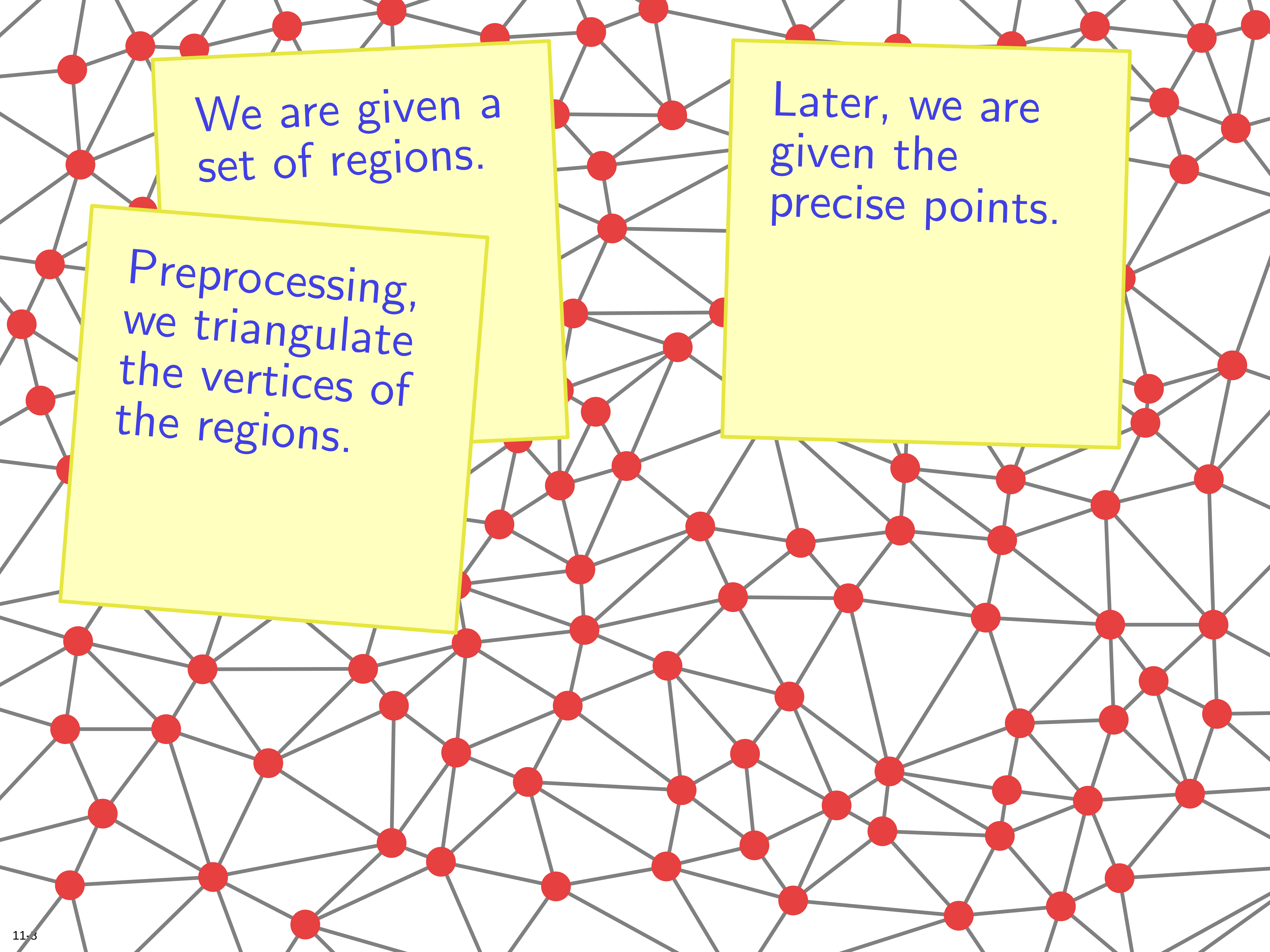
We are given a set of regions.

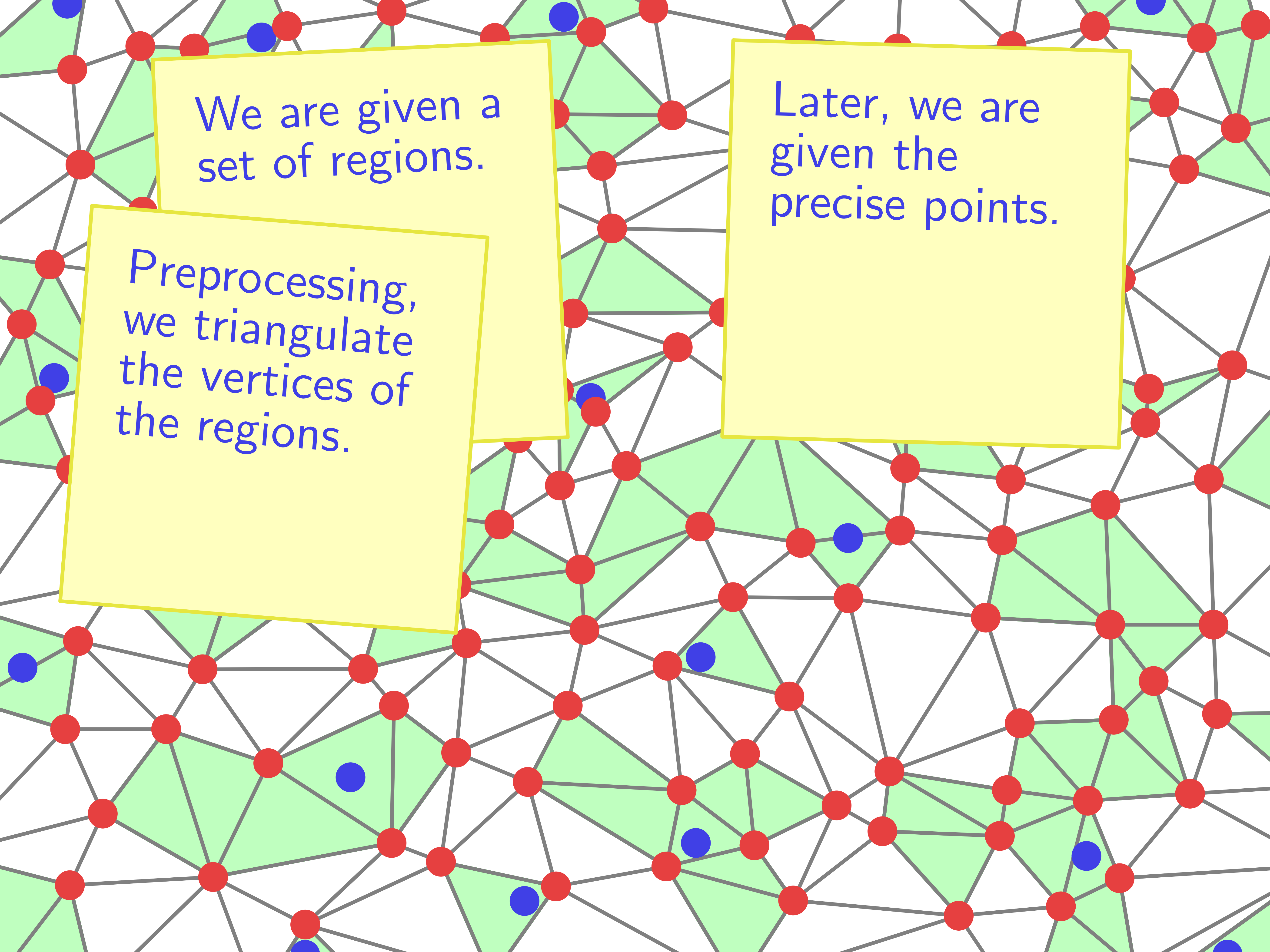
Preprocessing, we triangulate the vertices of the regions.

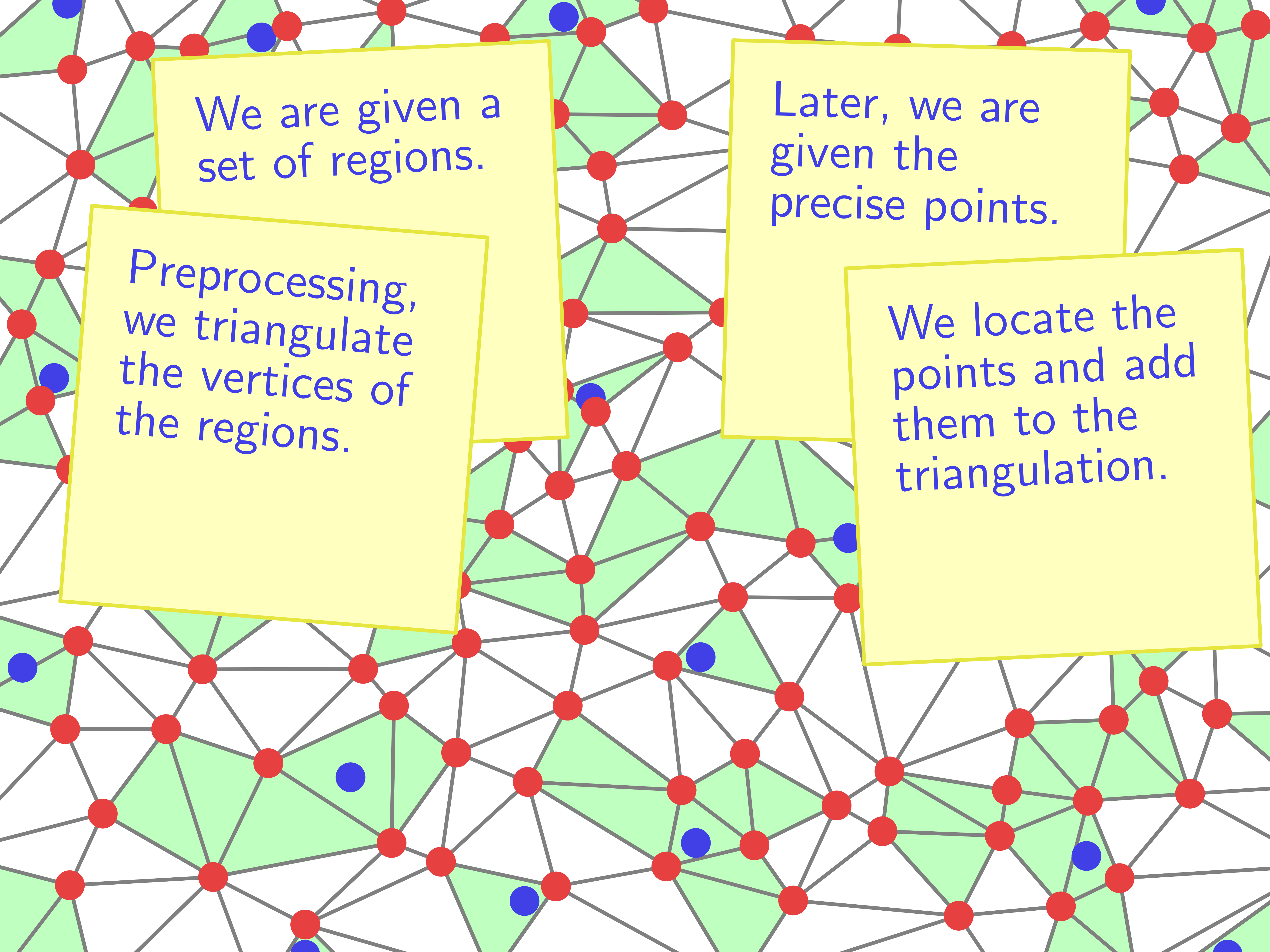
We are given a set of regions.

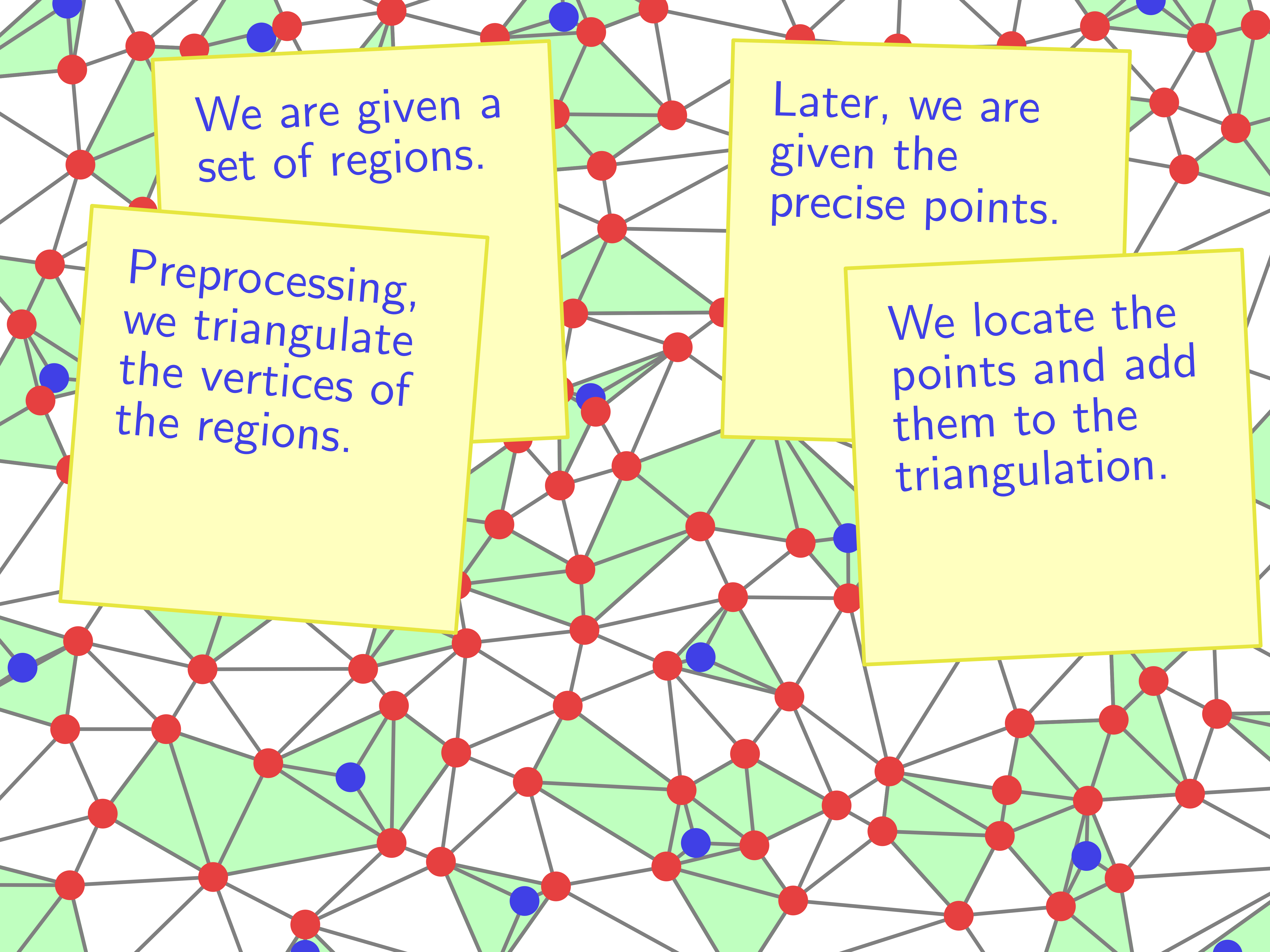
Preprocessing, we triangulate the vertices of the regions.

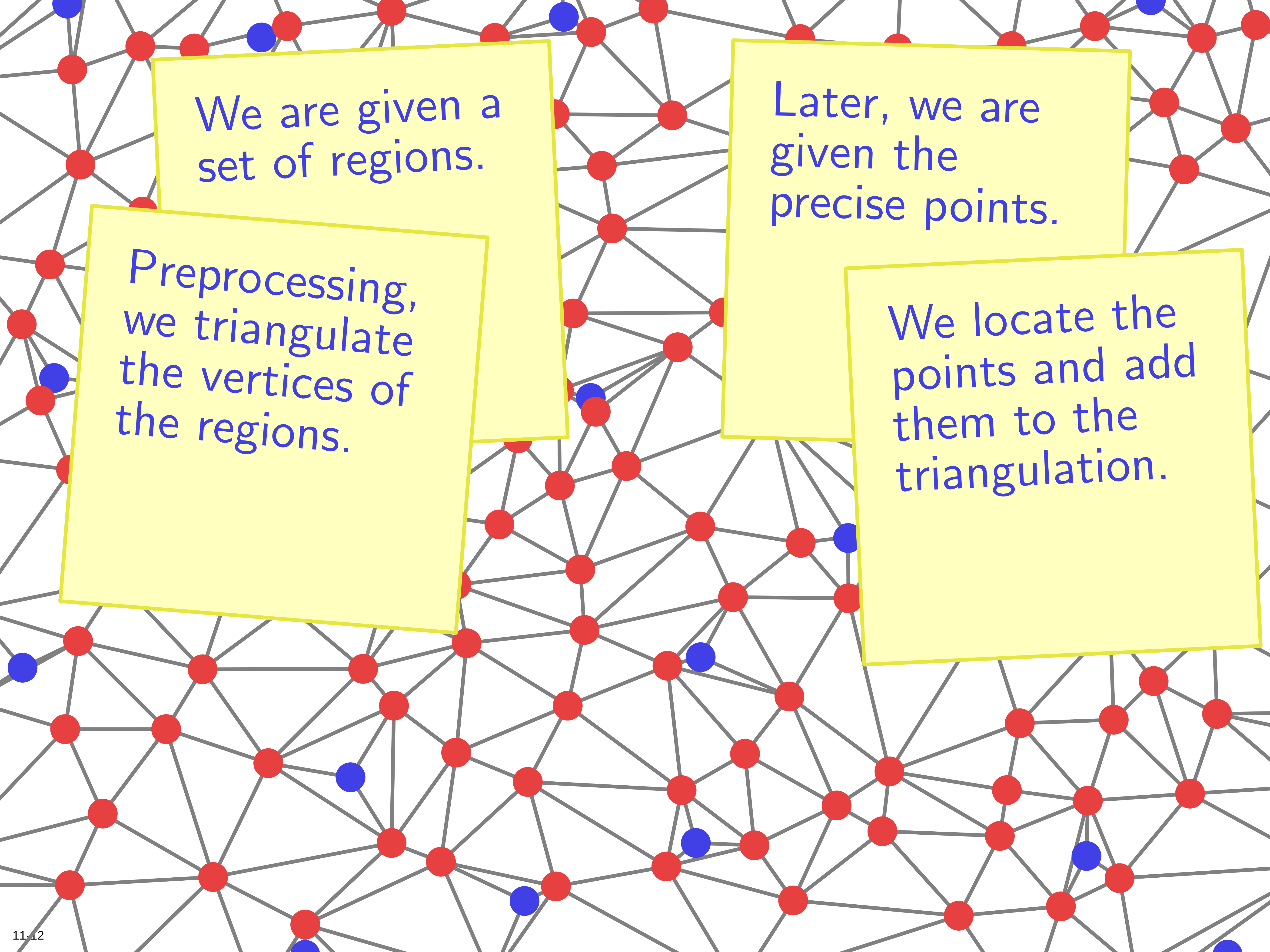
Later, we are given the precise points.

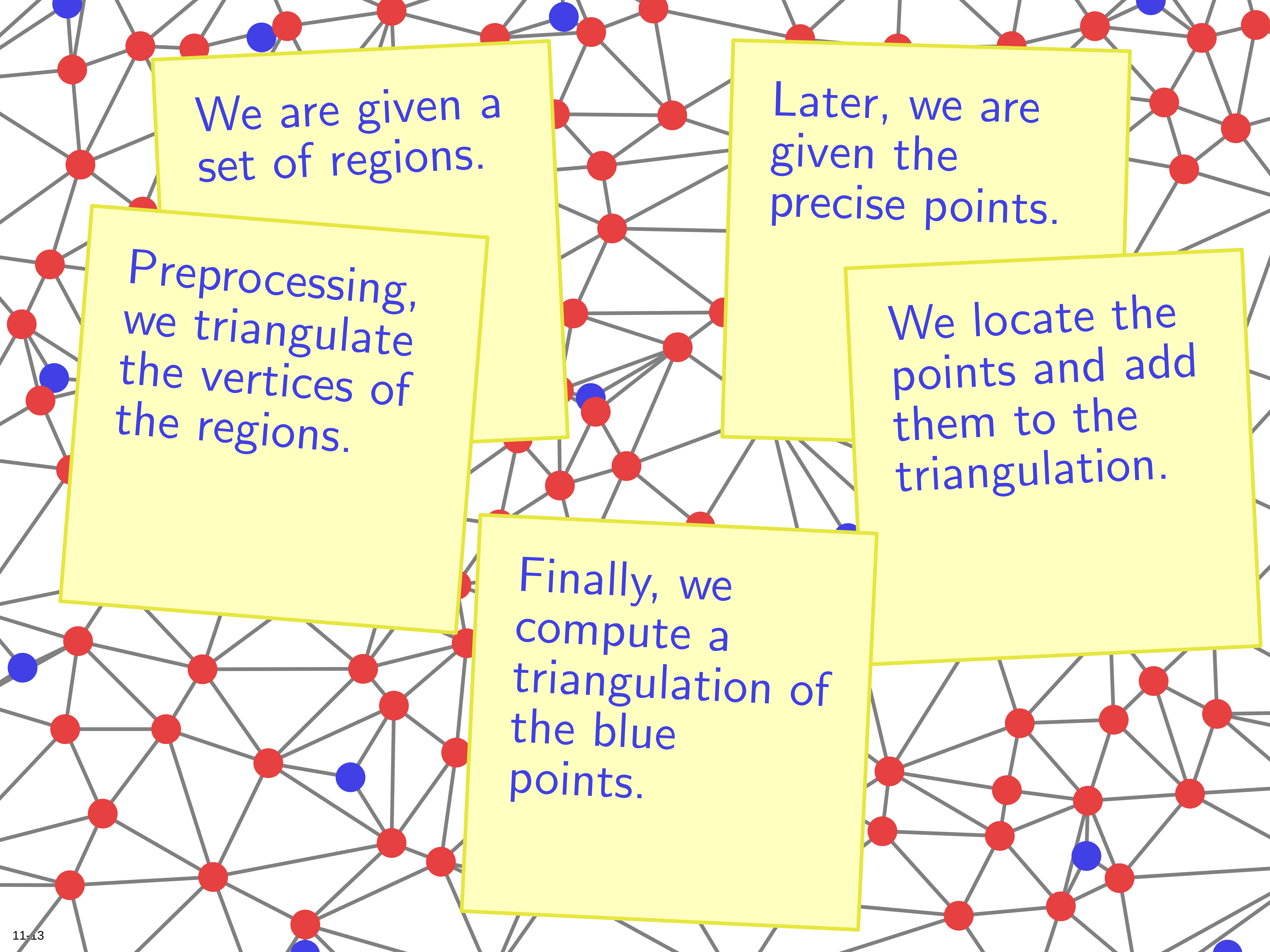



We are given a set of regions.
Preprocessing, we triangulate the vertices of the regions.
Later, we are given the precise points.


We are given a set of regions.
Preprocessing, we triangulate the vertices of the regions.
Later, we are given the precise points.
We locate the points and add them to the triangulation.


We are given a set of regions.
Preprocessing, we triangulate the vertices of the regions.
Later, we are given the precise points.
We locate the points and add them to the triangulation.


We are given a set of regions.
Preprocessing, we triangulate the vertices of the regions.
Later, we are given the precise points.
We locate the points and add them to the triangulation.



We are given a set of regions.

Preprocessing, we triangulate the vertices of the regions.

Later, we are given the precise points.

We locate the points and add them to the triangulation.

Finally, we compute a triangulation of the blue points.

We are given a set of regions.

Preprocessing, we triangulate the vertices of the regions.

Later, we are given the precise points.

We locate the points and add them to the triangulation.

Finally, we compute a triangulation of the blue points.

Time to conclude.

We preprocess a set of regions in the plane in $O(n \log n)$ time.

We preprocess a set of regions in the plane in $O(n \log n)$ time.

Given a point inside each region, we compute a triangulation in $O(n)$ time.

We preprocess a set of regions in the plane in $O(n \log n)$ time.

Given a point inside each region, we compute a triangulation in $O(n)$ time.

Can be extended to partially overlapping regions.

We preprocess a set of regions in the plane in $O(n \log n)$ time.

Given a point inside each region, we compute a triangulation in $O(n)$ time.

Can be extended to partially overlapping regions.

**OPEN PROBLEM**

Preprocess a set of lines in the plane for linear time triangulation.

We preprocess a set of regions in the plane in $O(n \log n)$ time.

Given a point inside each region, we compute a triangulation in $O(n)$ time.

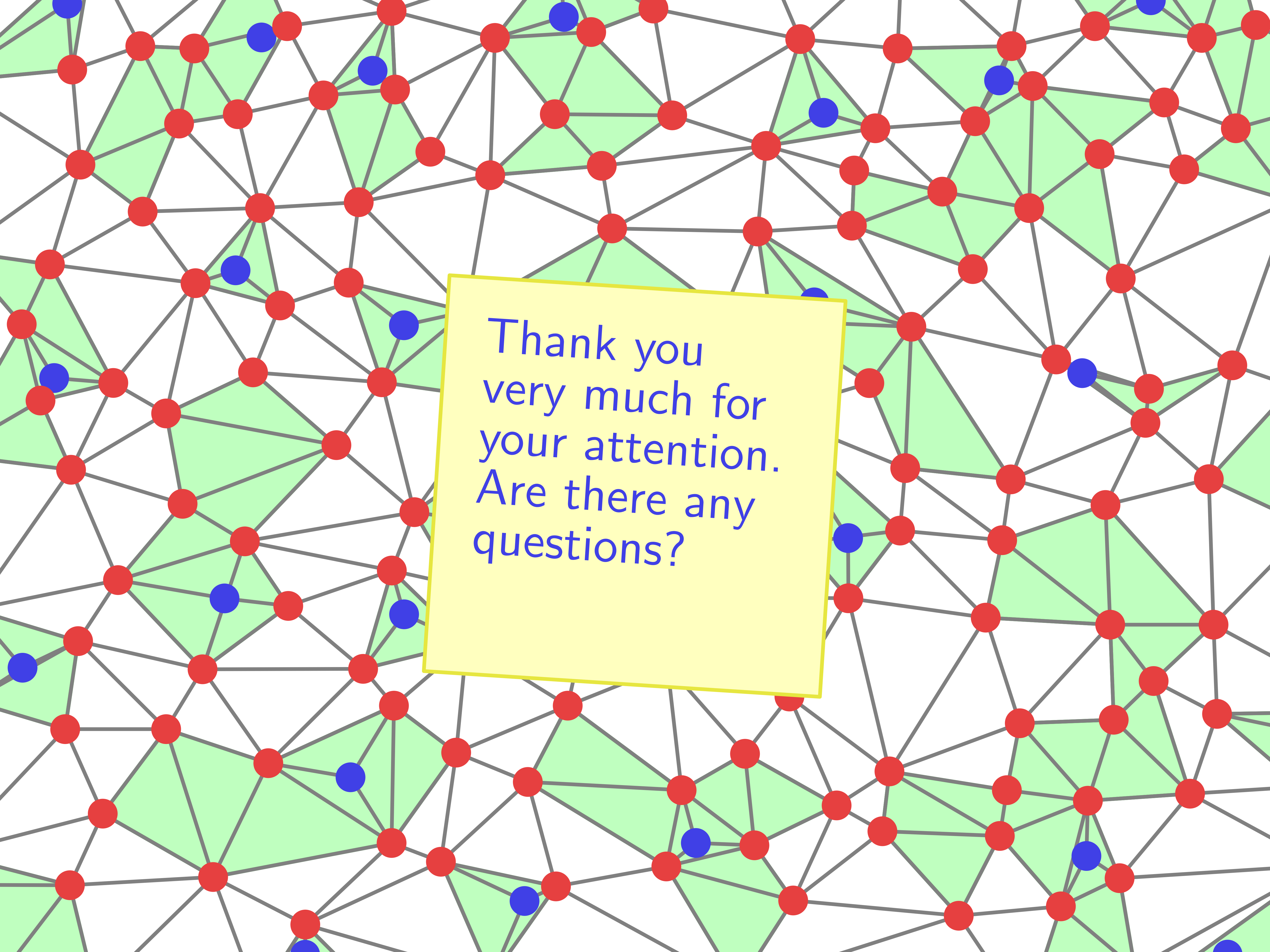
Can be extended to partially overlapping regions.

**OPEN PROBLEM**

Preprocess a set of lines in the plane for linear time triangulation.

**OPEN PROBLEM**

Can similar results be obtained in higher dimensions?


Thank you
very much for
your attention.
Are there any
questions?