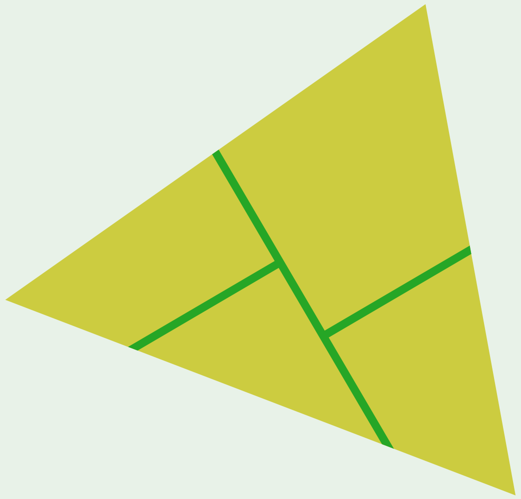
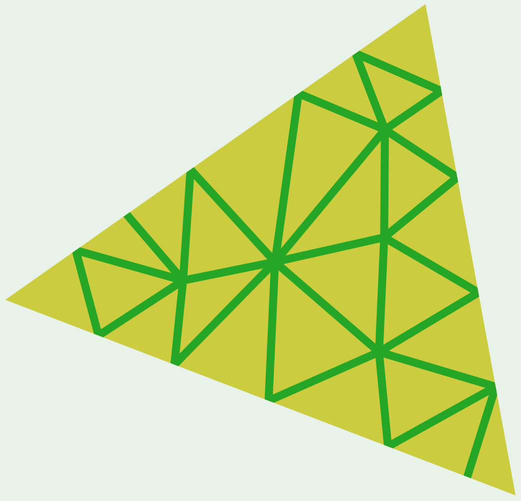
TRIANGULATING THE SQUARE & SQUARING THE TRIANGLE



Maarten Löffler — University of California, Irvine

Wolfgang Mulzer — Freie Universität Berlin

OR, QUADTREES *AND DELAUNAY TRIANGULATIONS* ARE EQUIV*ALENT*



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PART I *INTRODUCTION*

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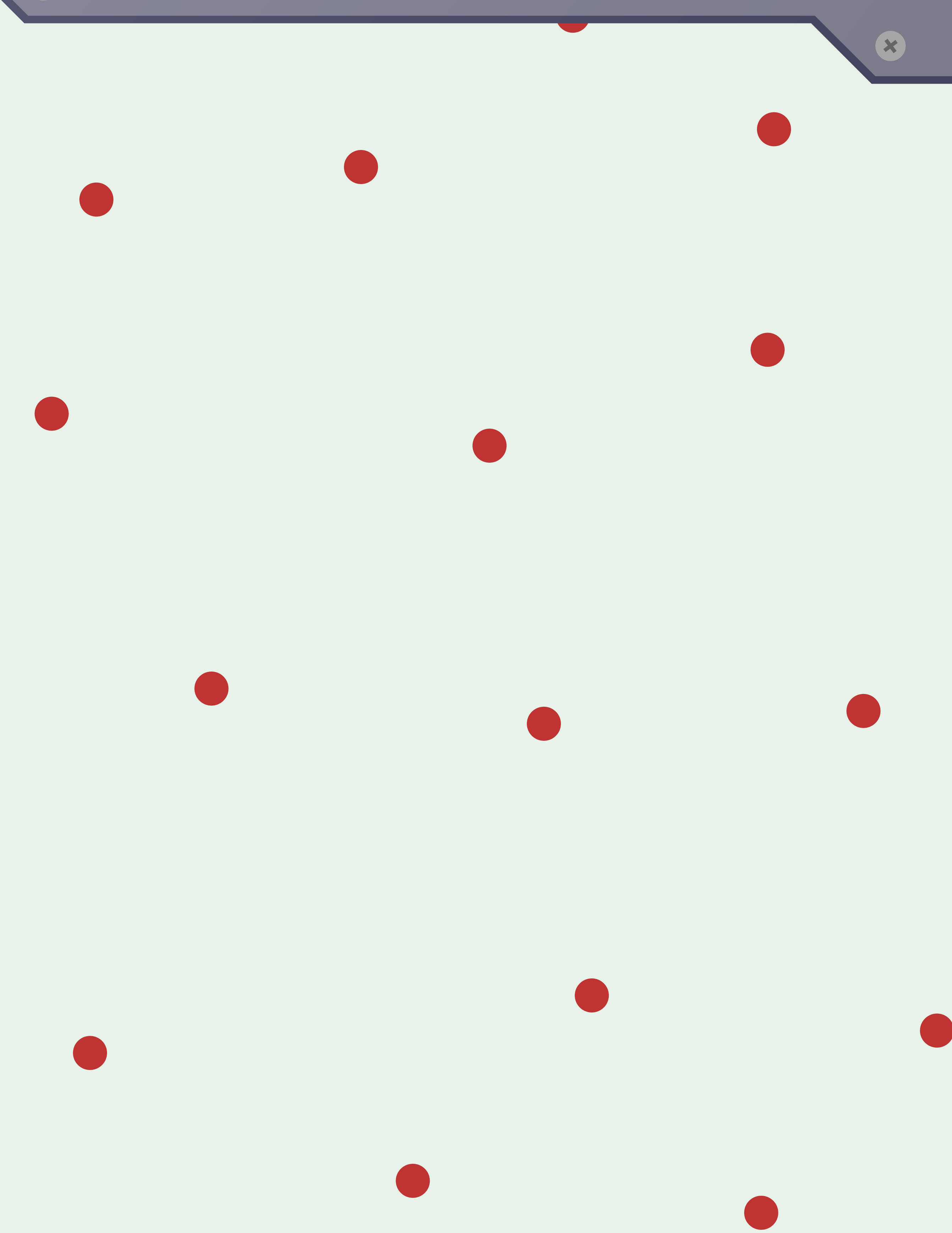
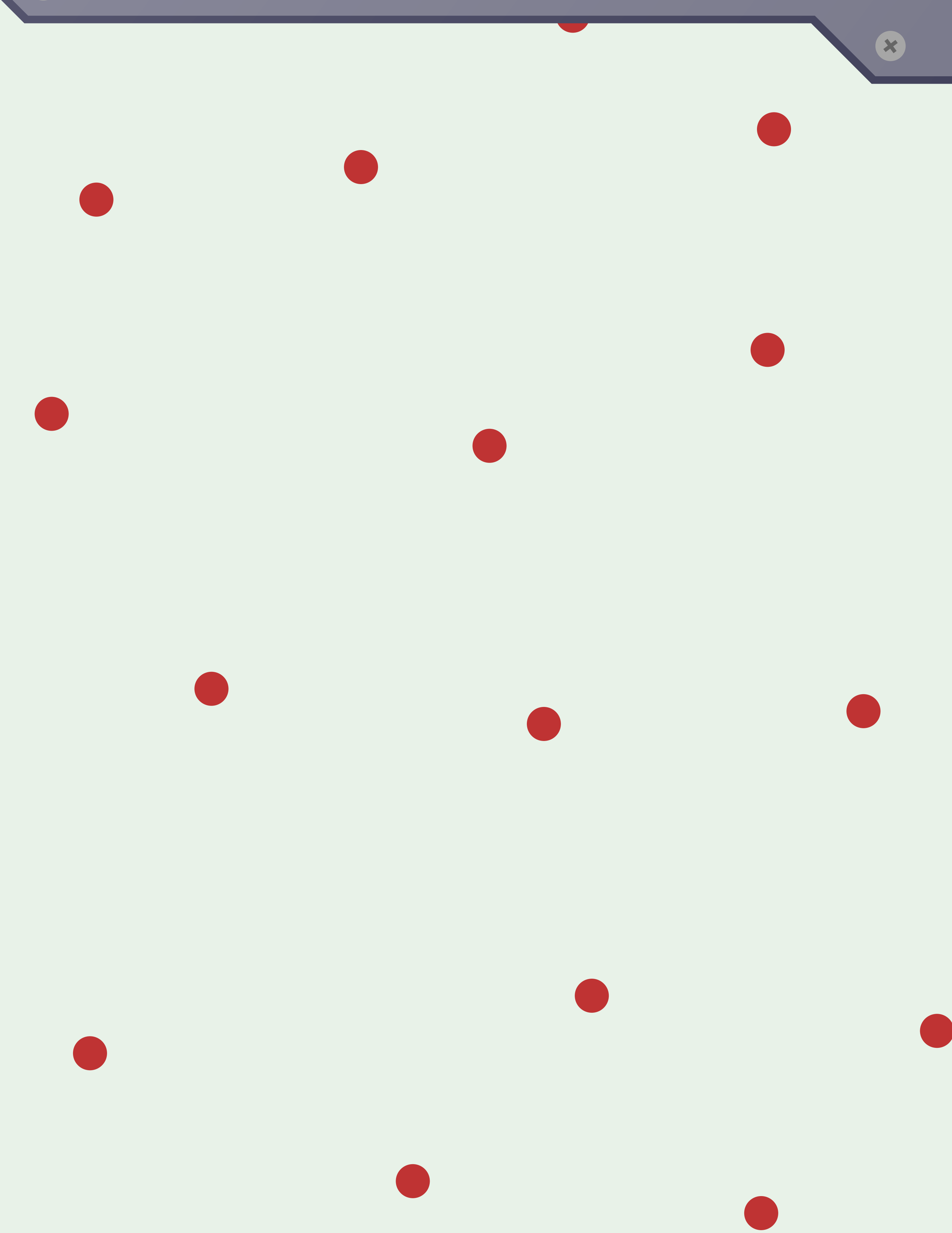


PROXIMITY STRUCTURES

Let P be a set of n points in the plane.

A *proximity structure* on P is ‘a structure that stores some kind of useful local information’.



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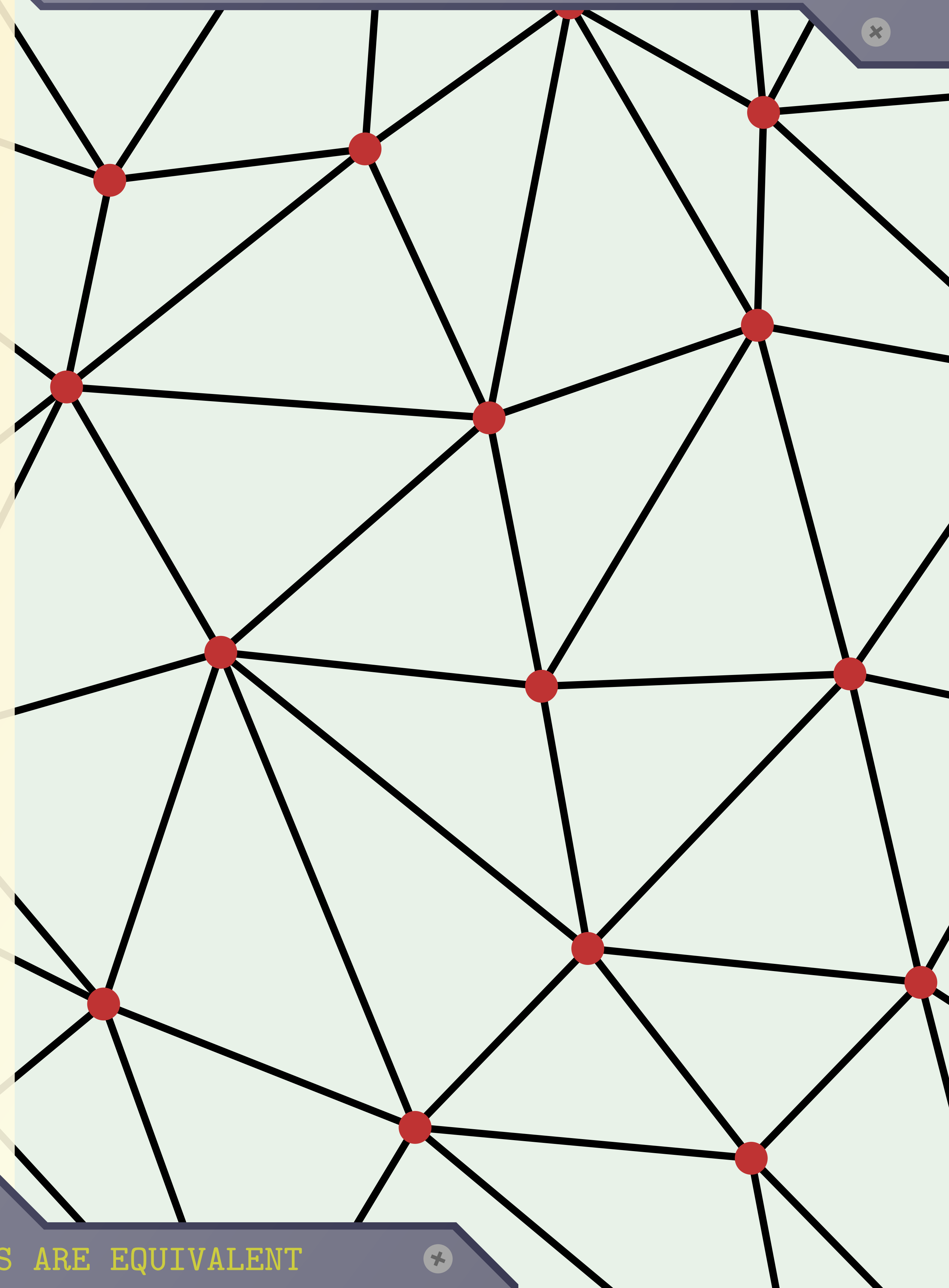

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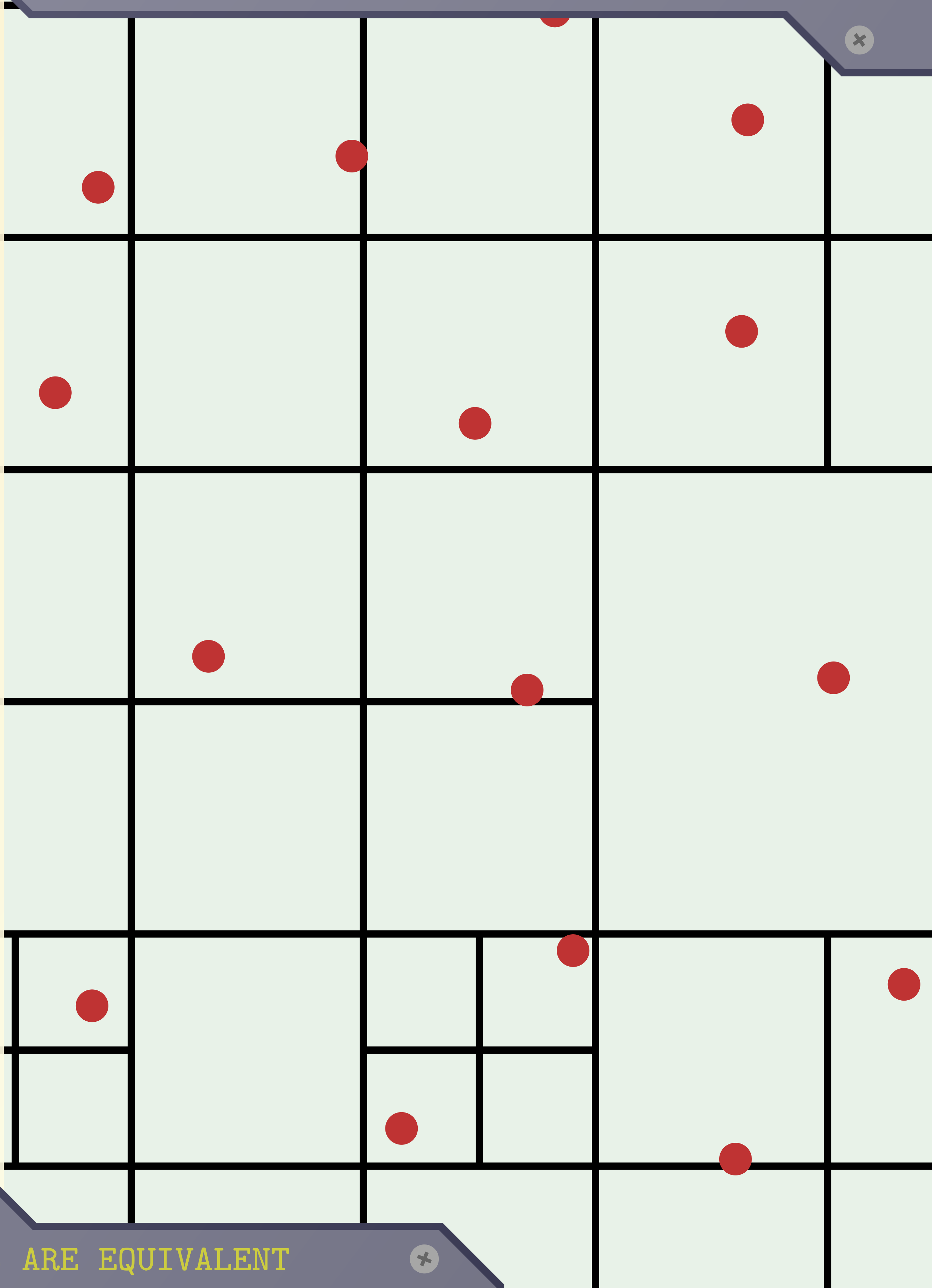

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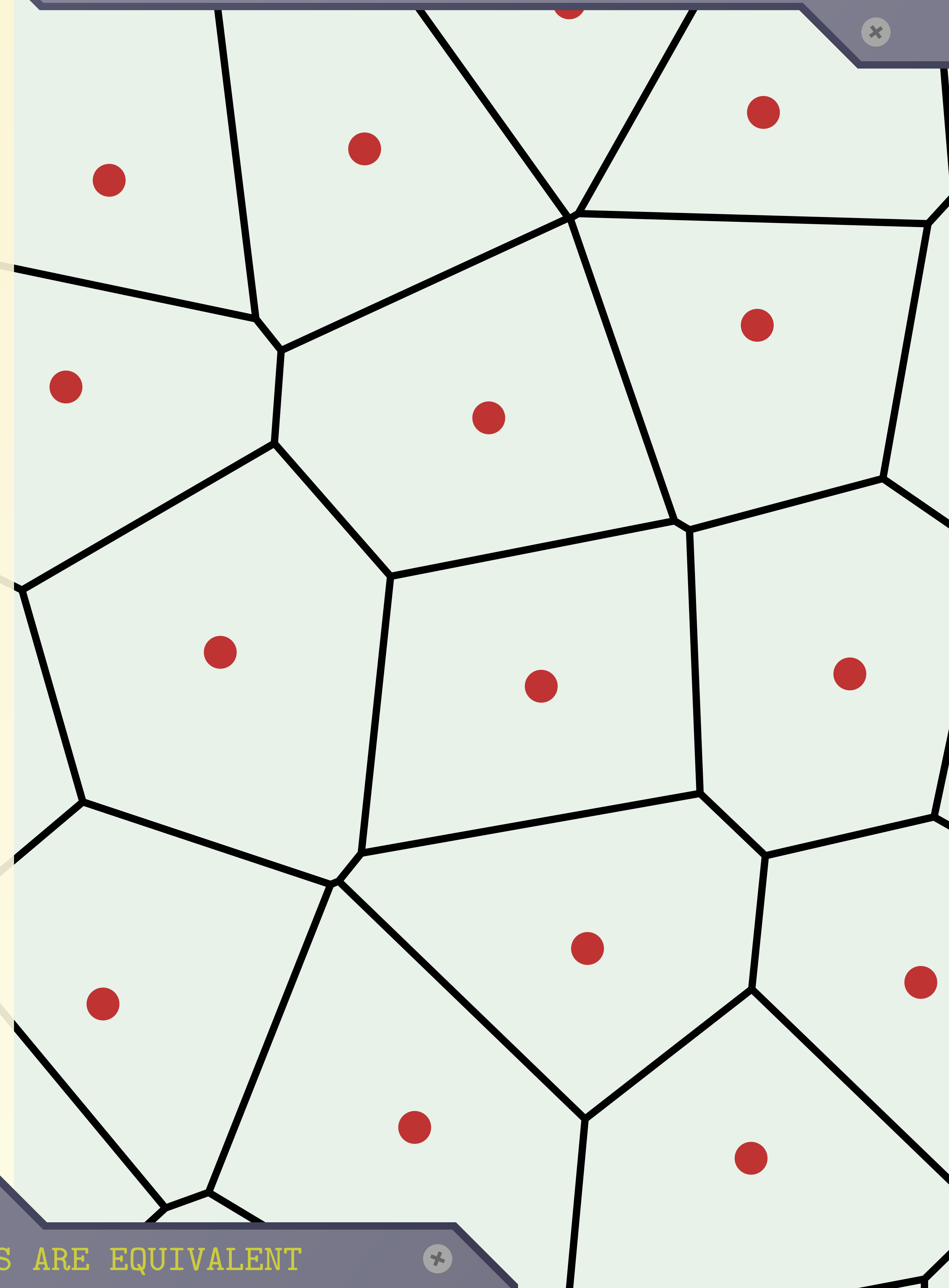

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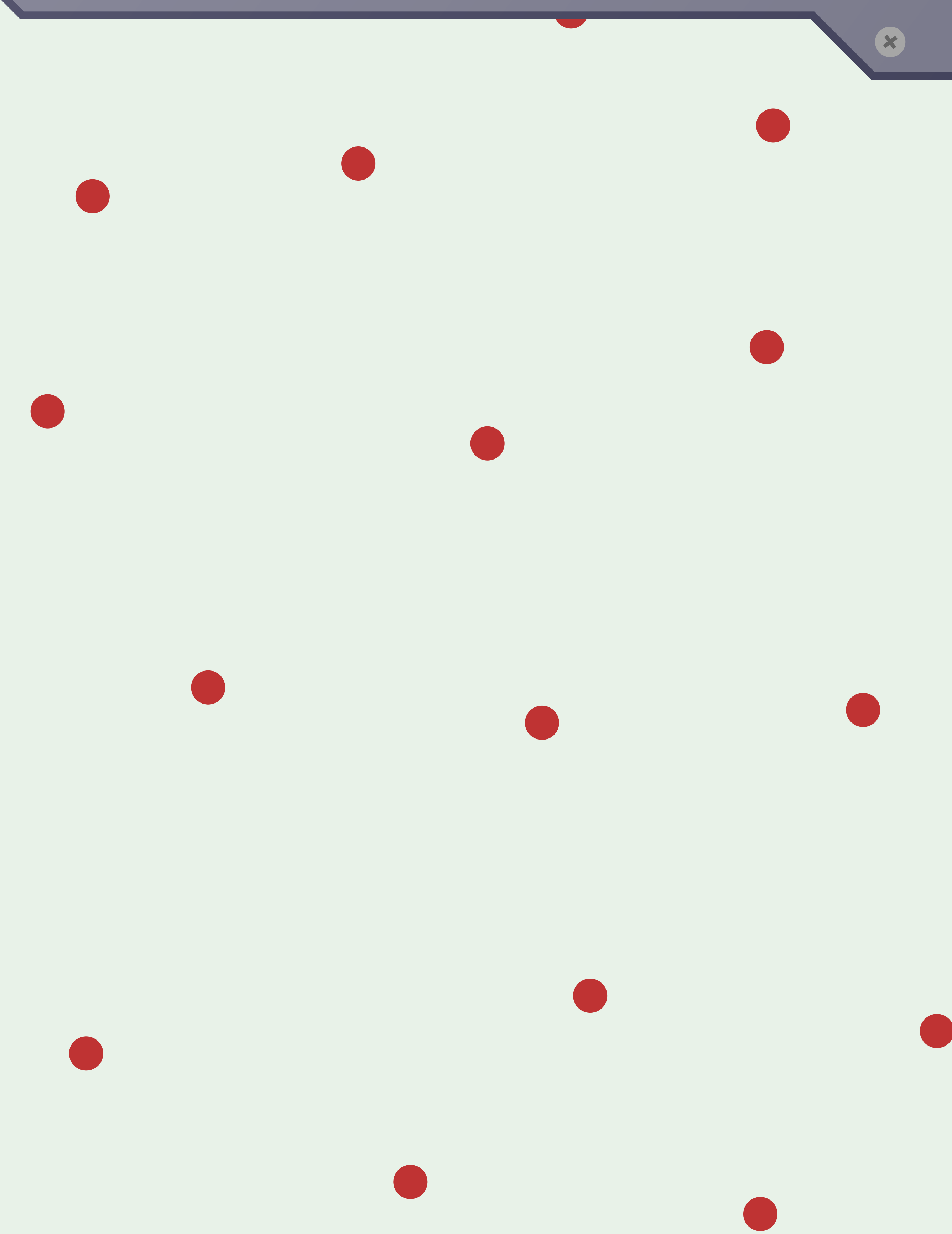

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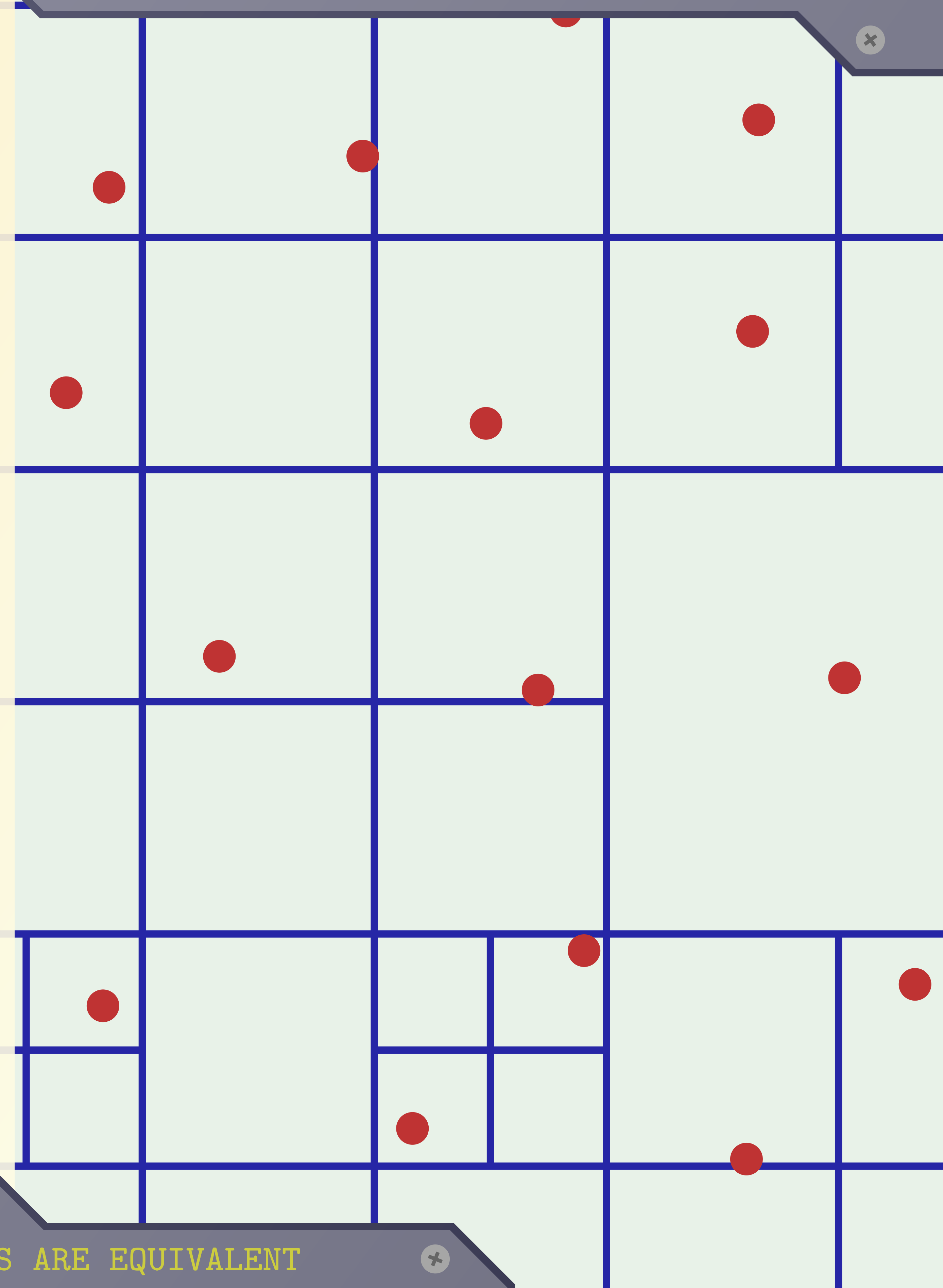

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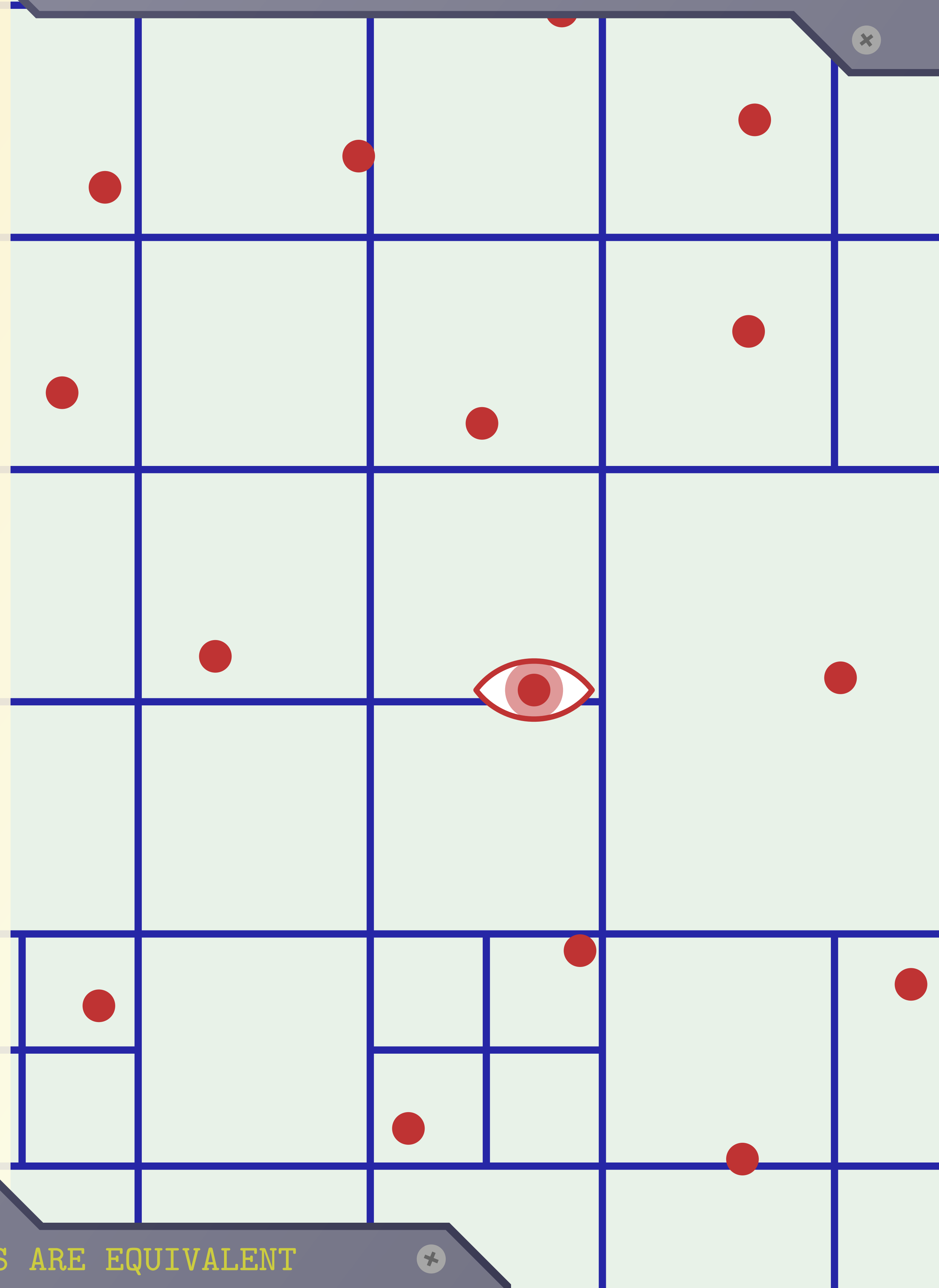

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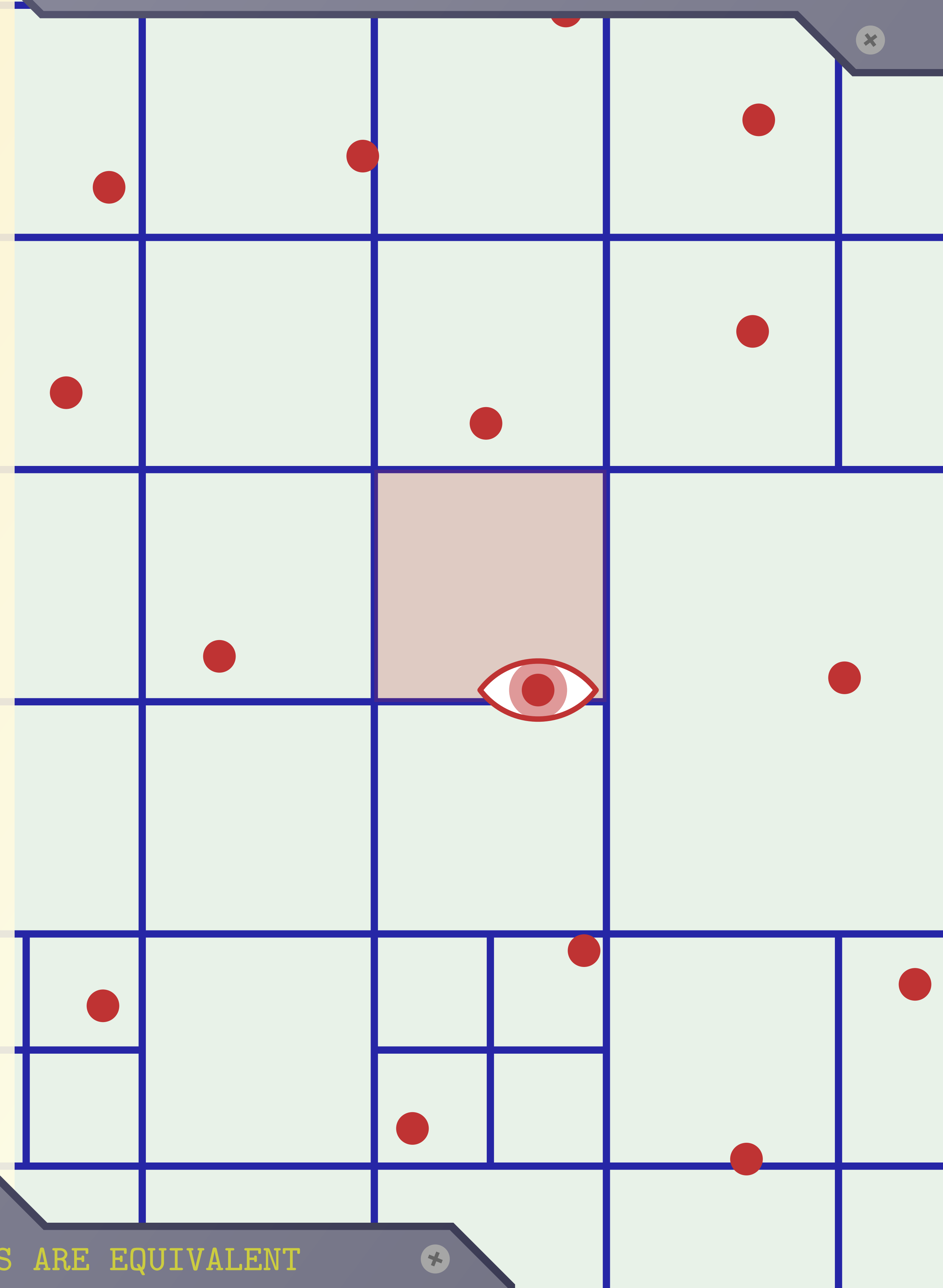

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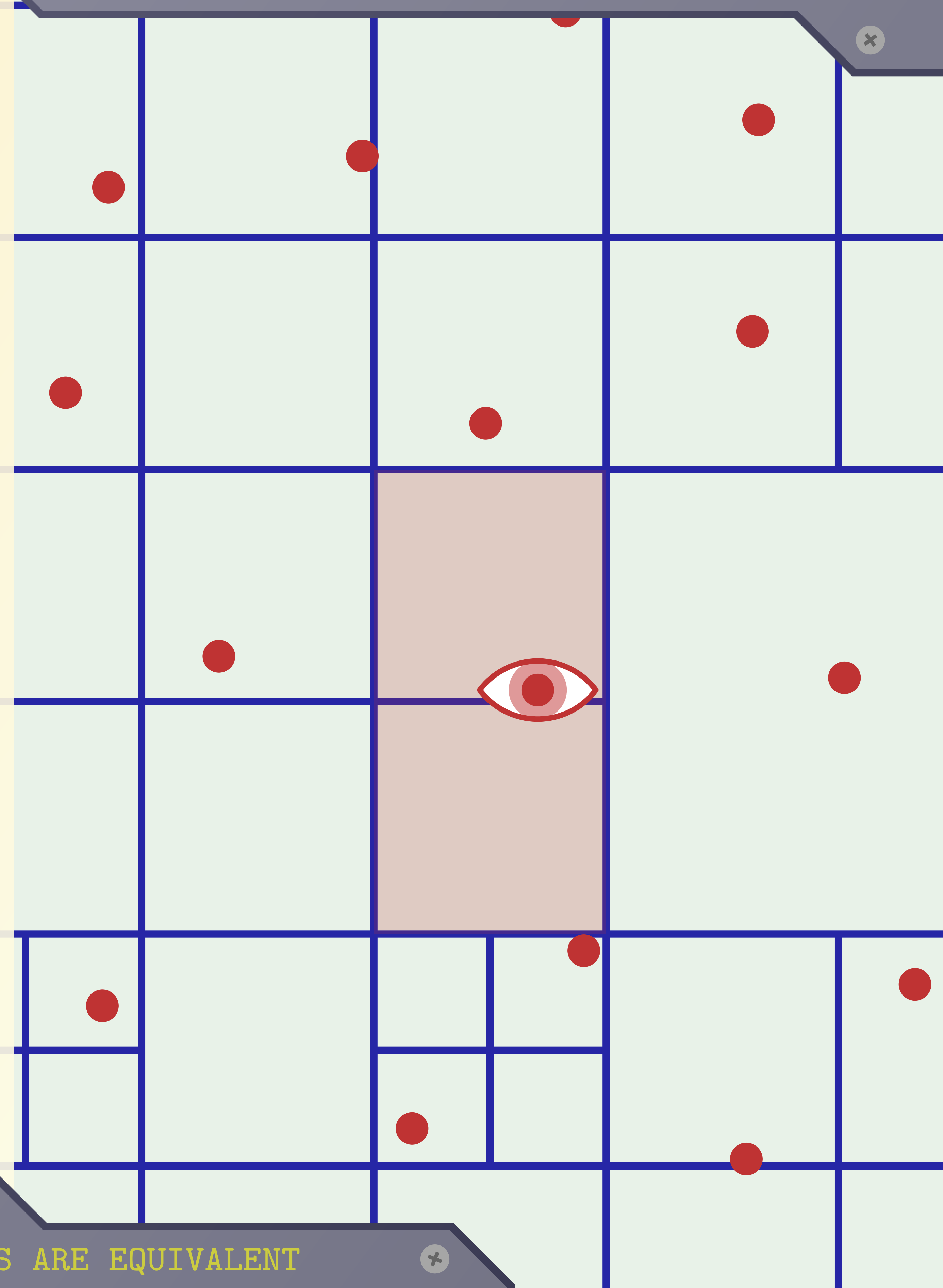

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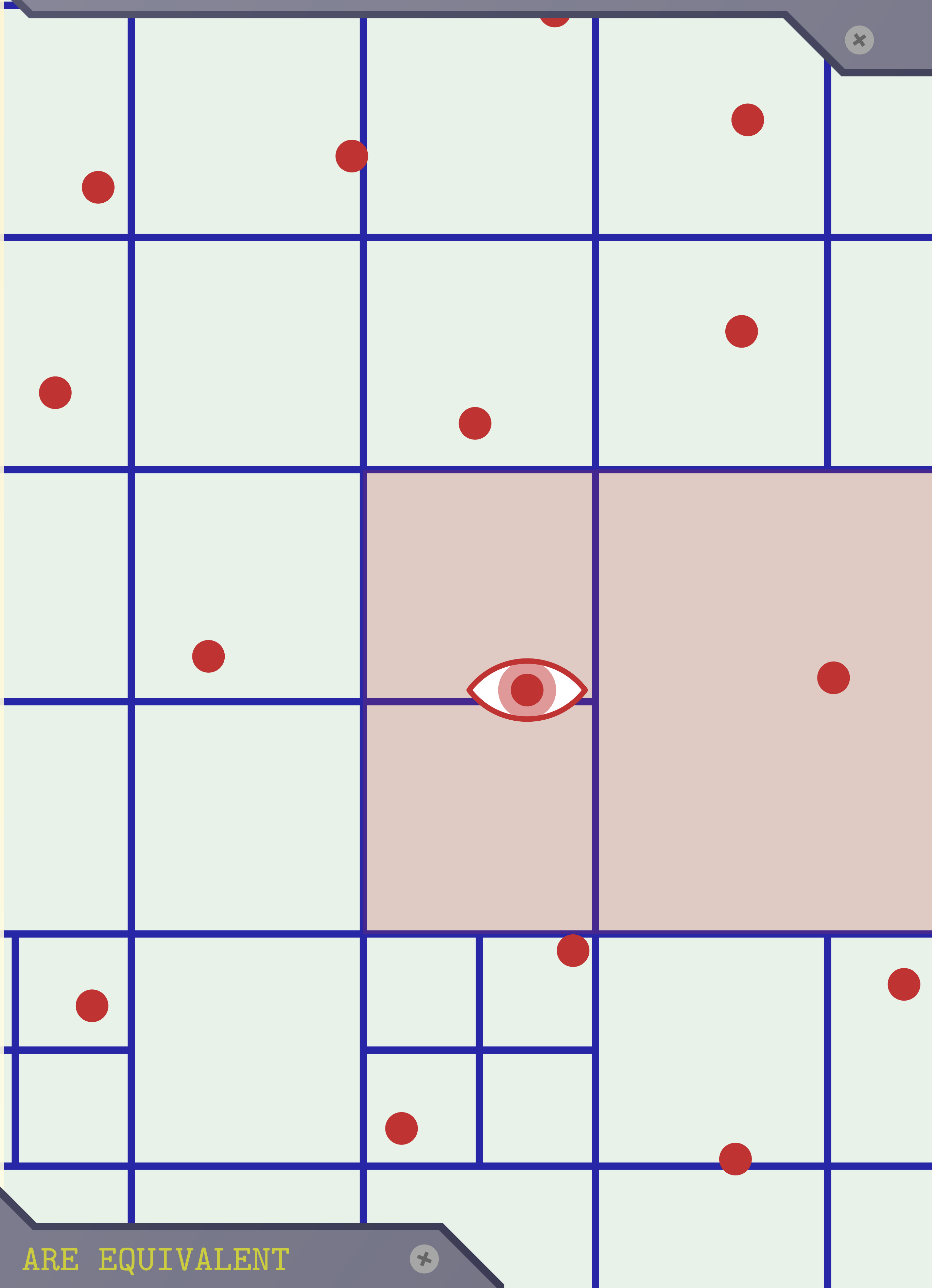

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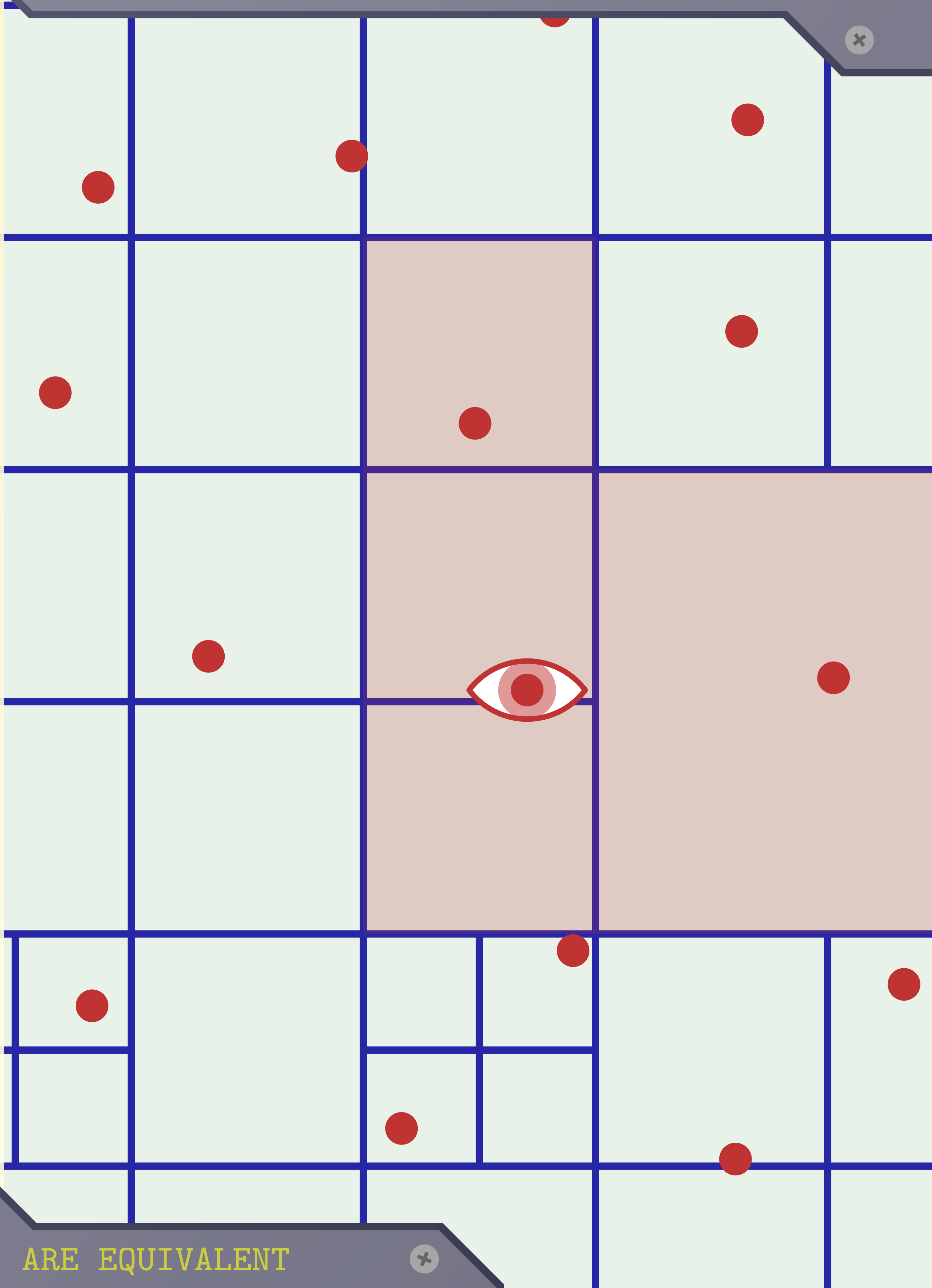

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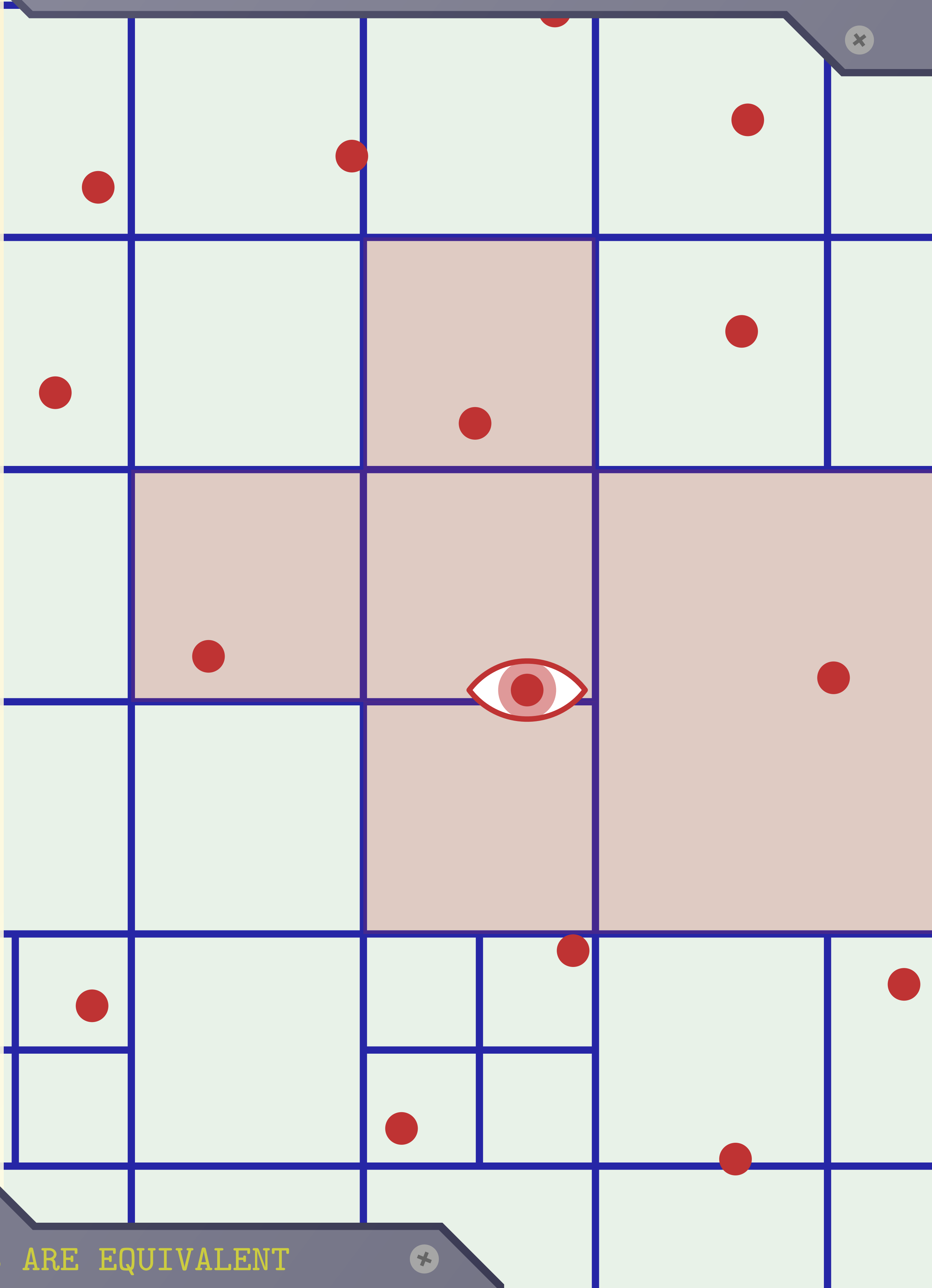

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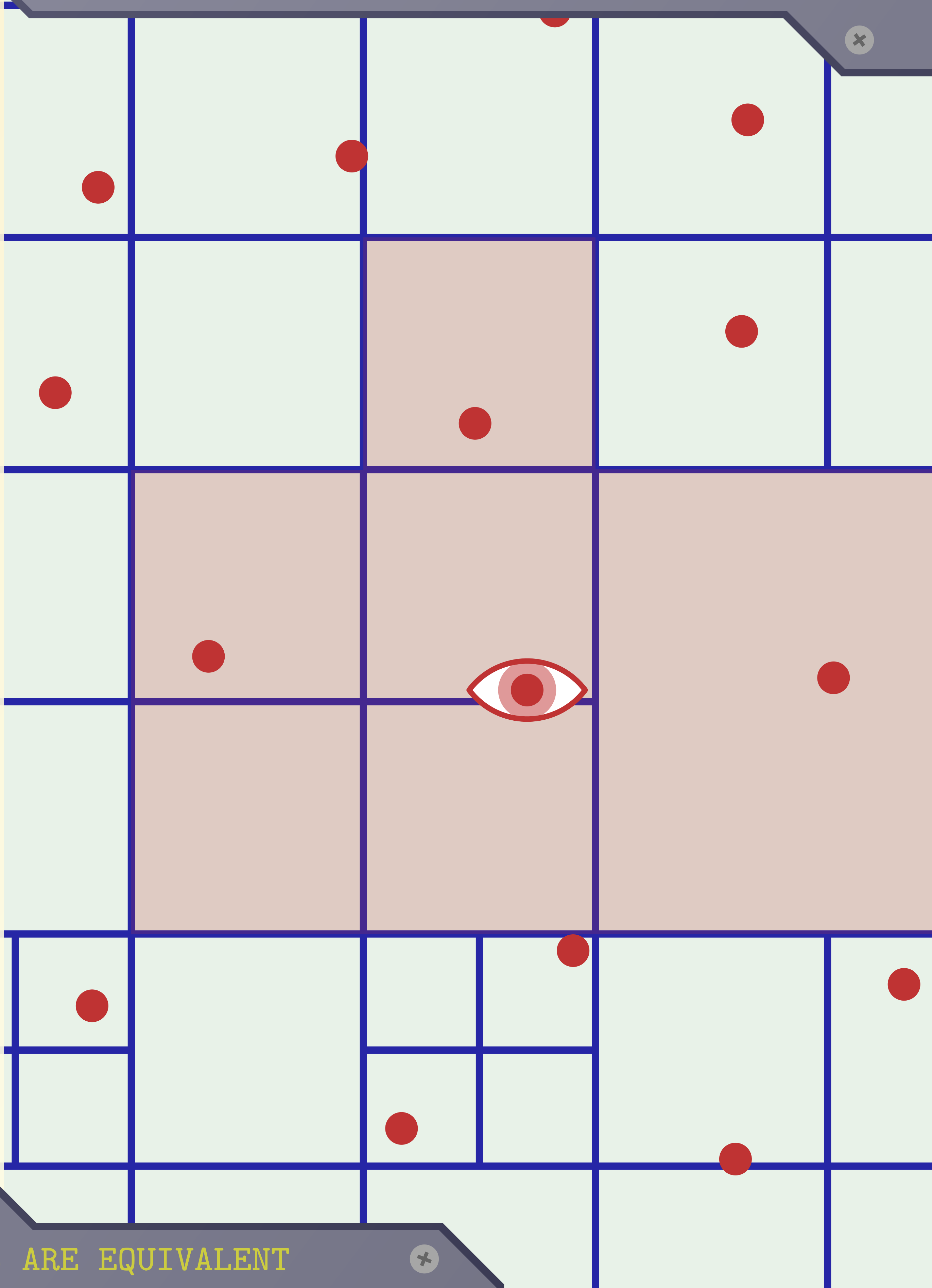

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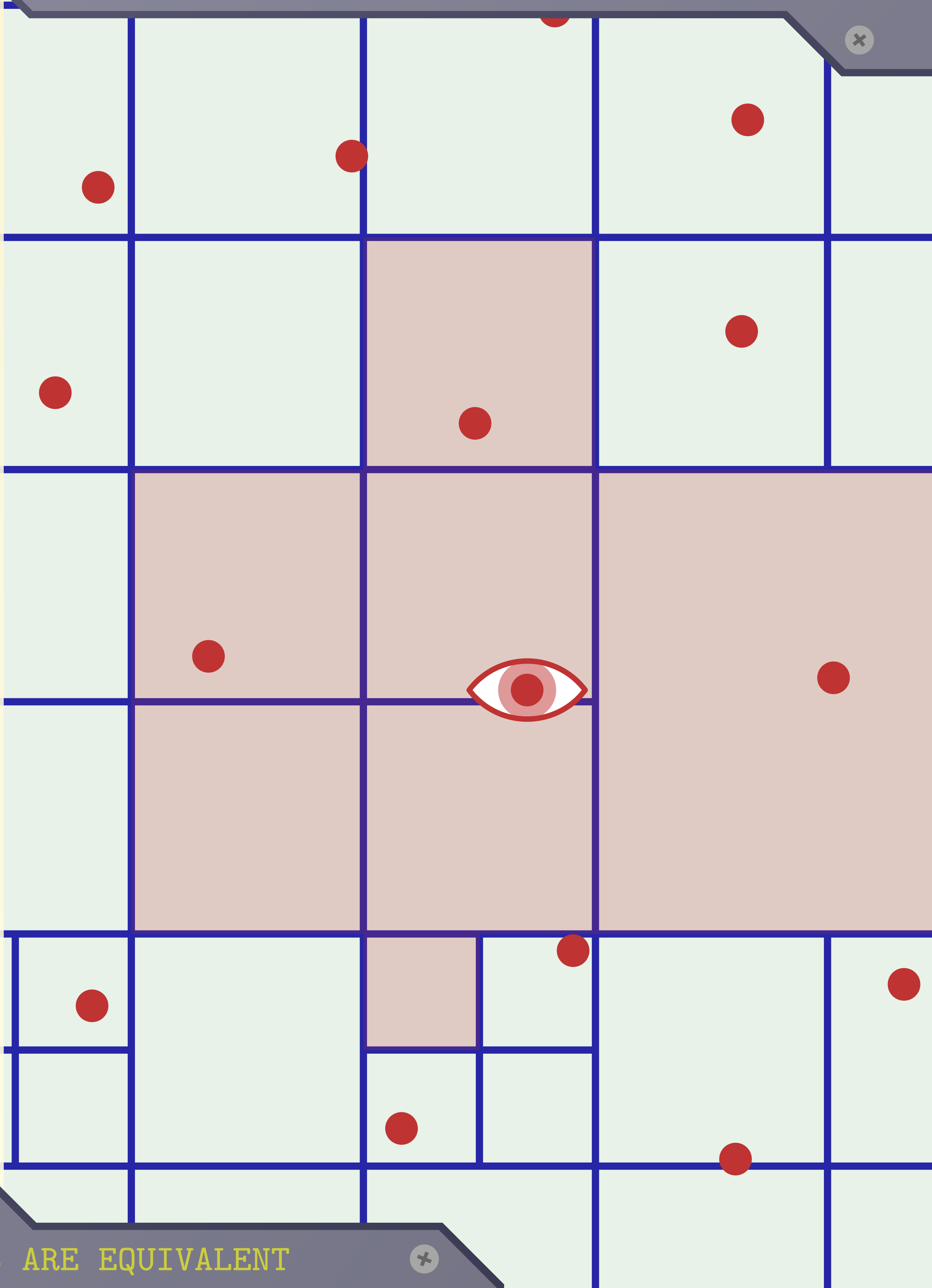

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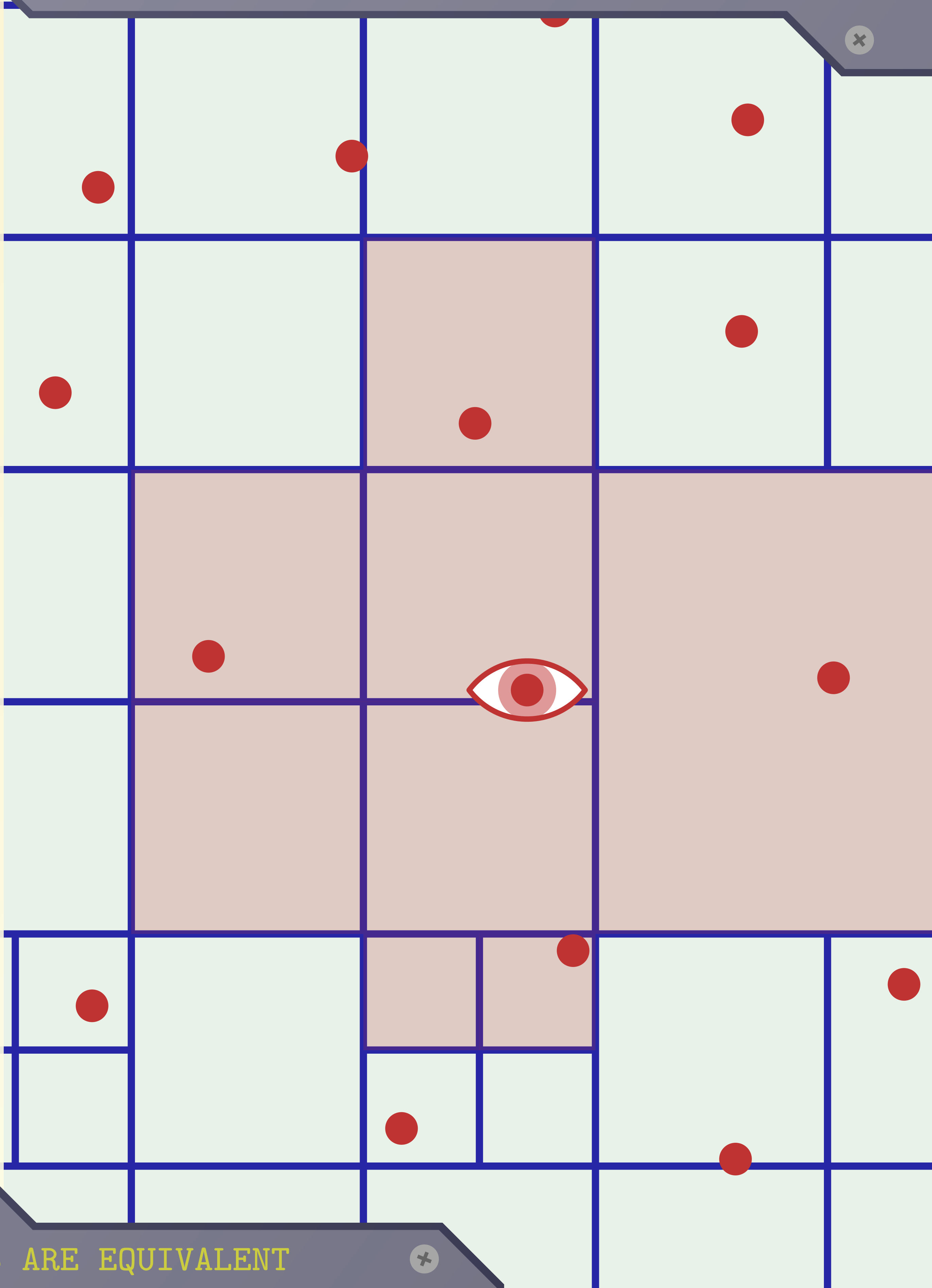

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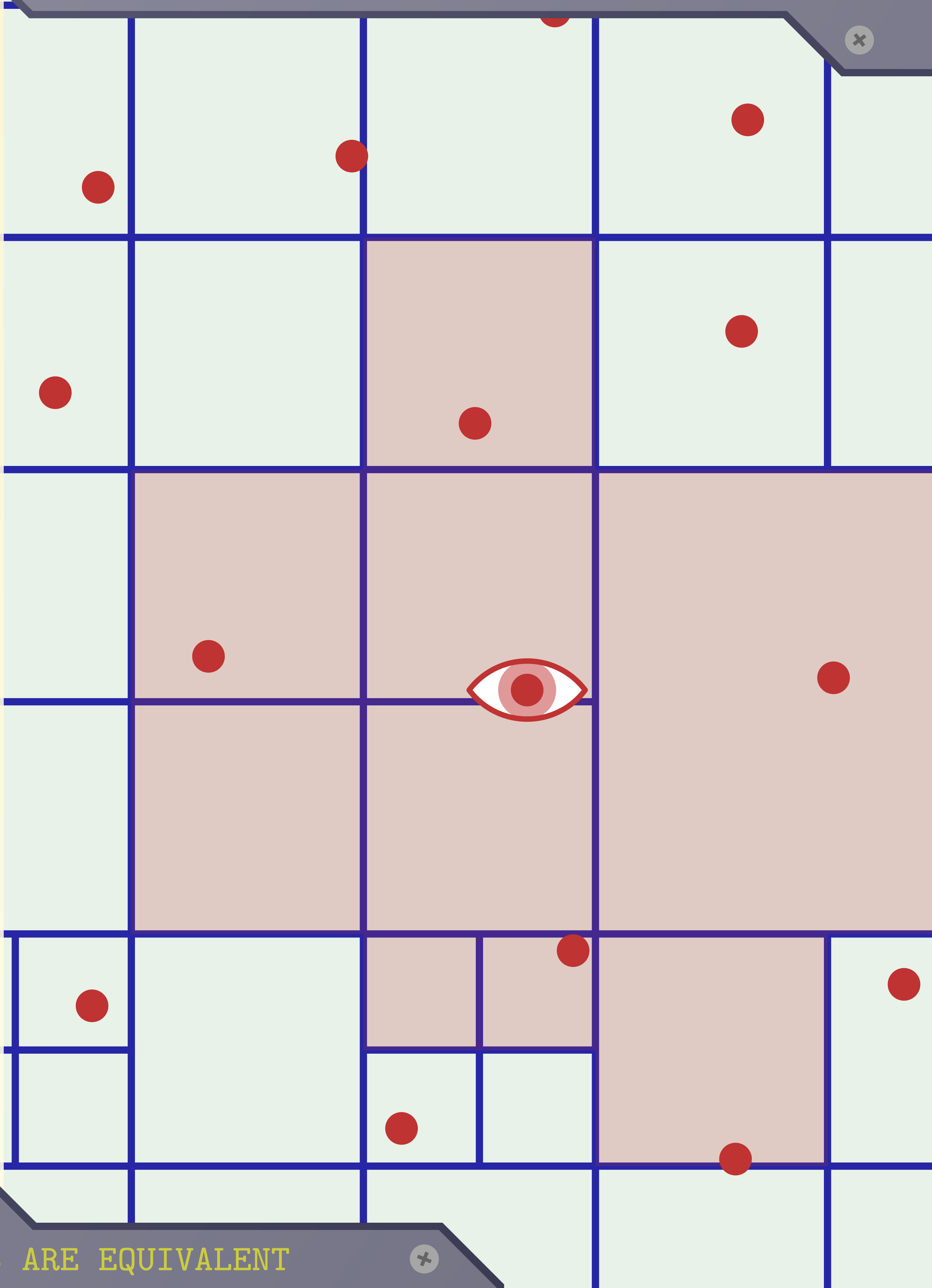

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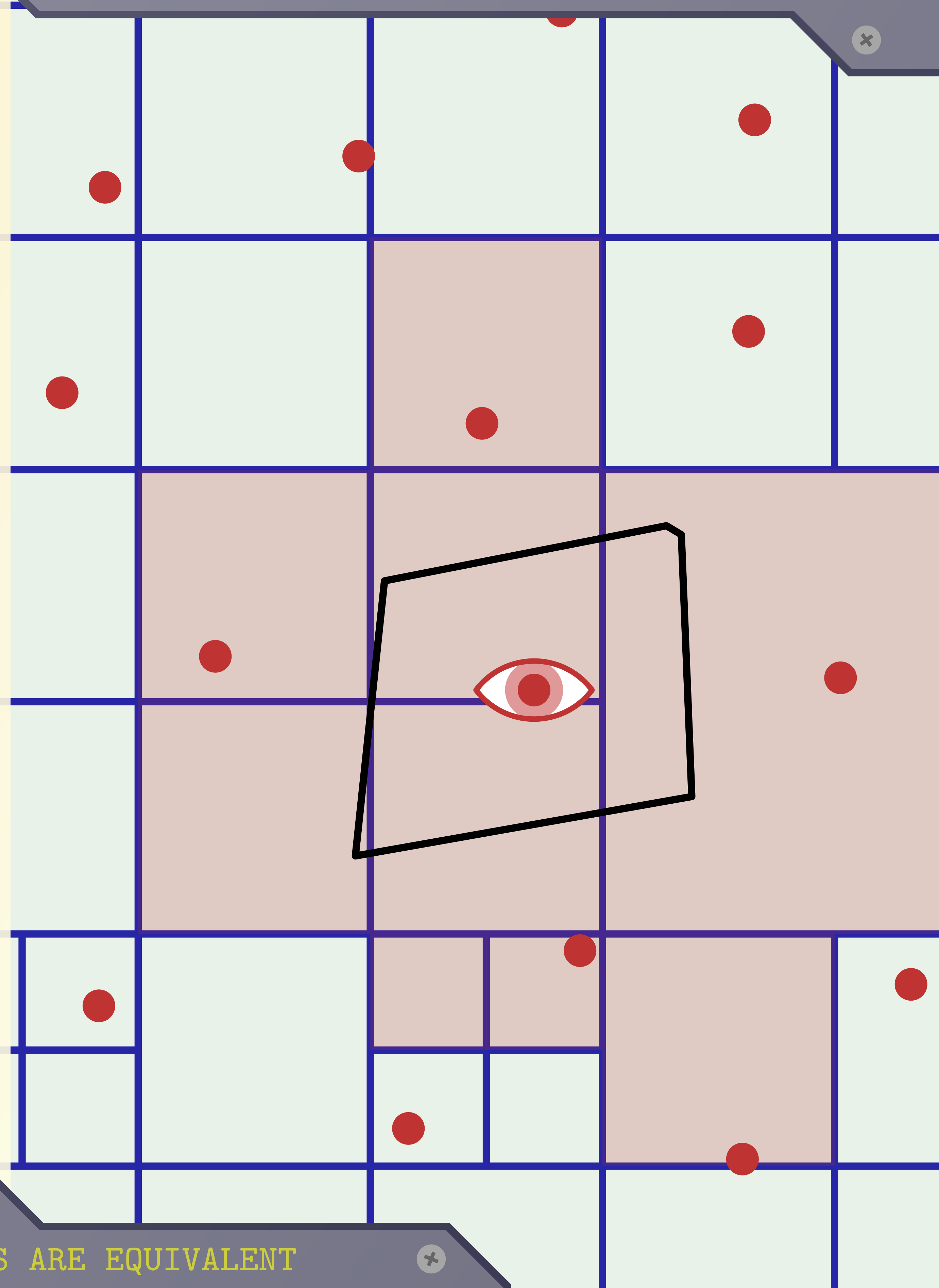

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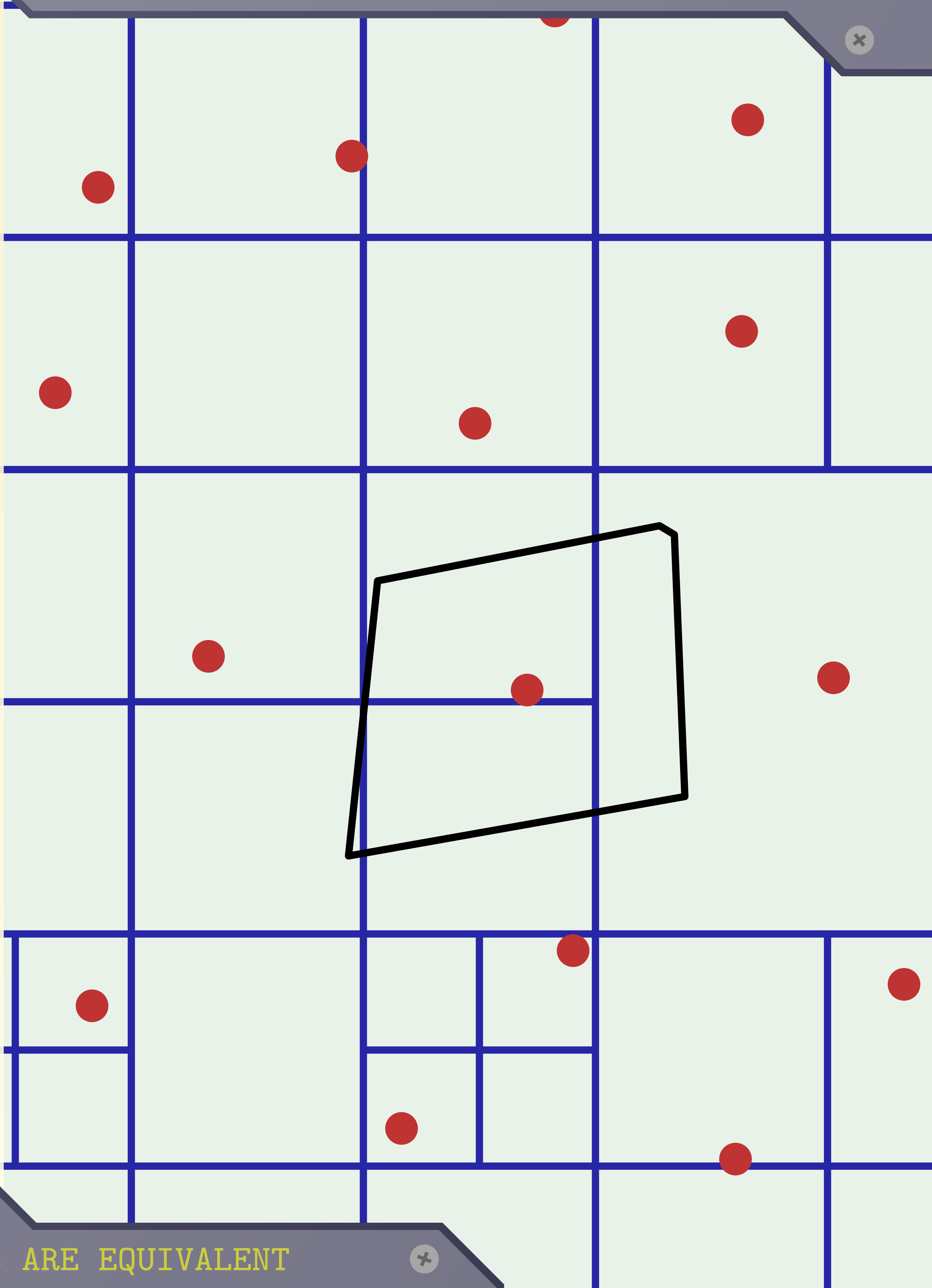

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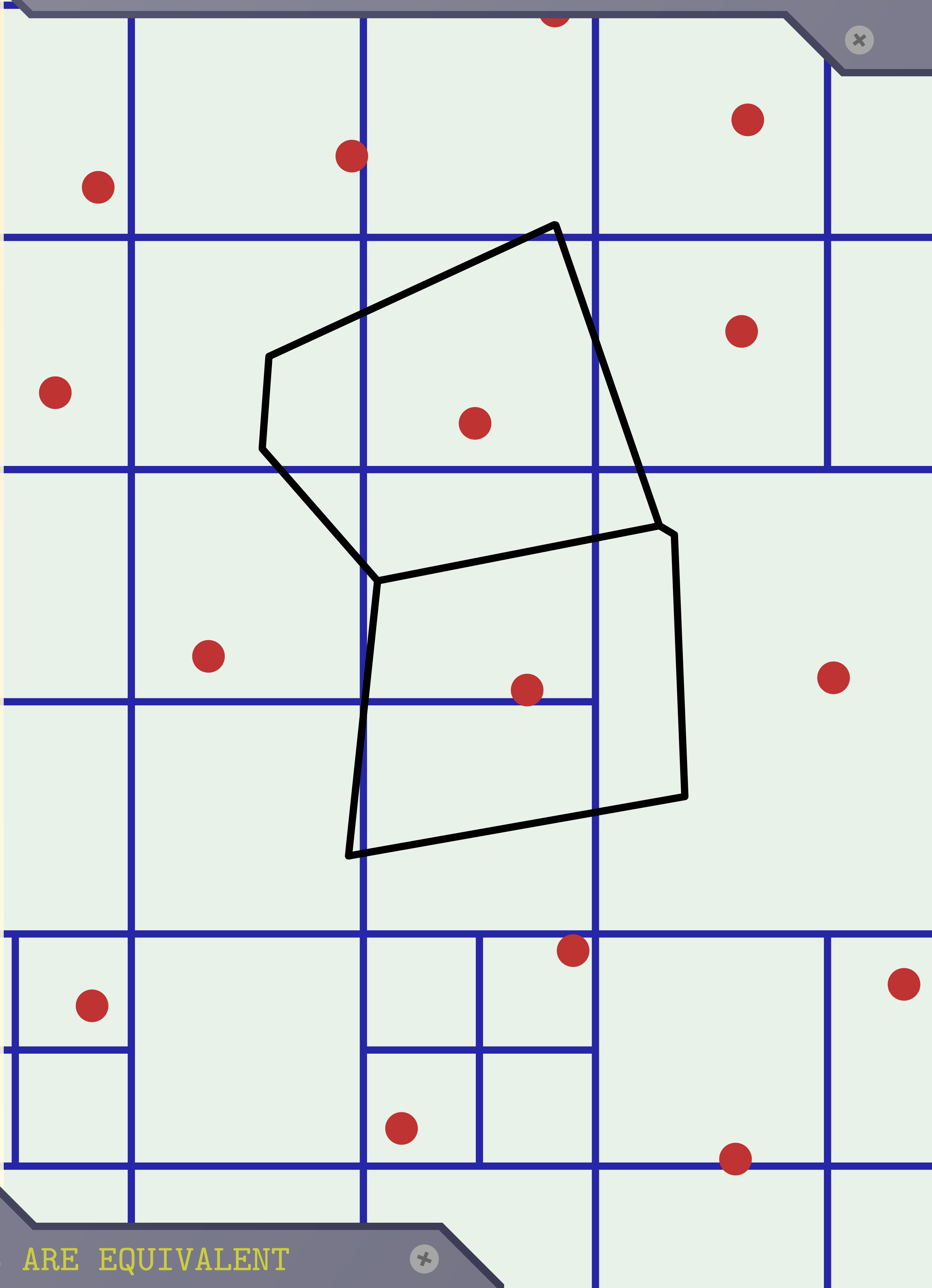

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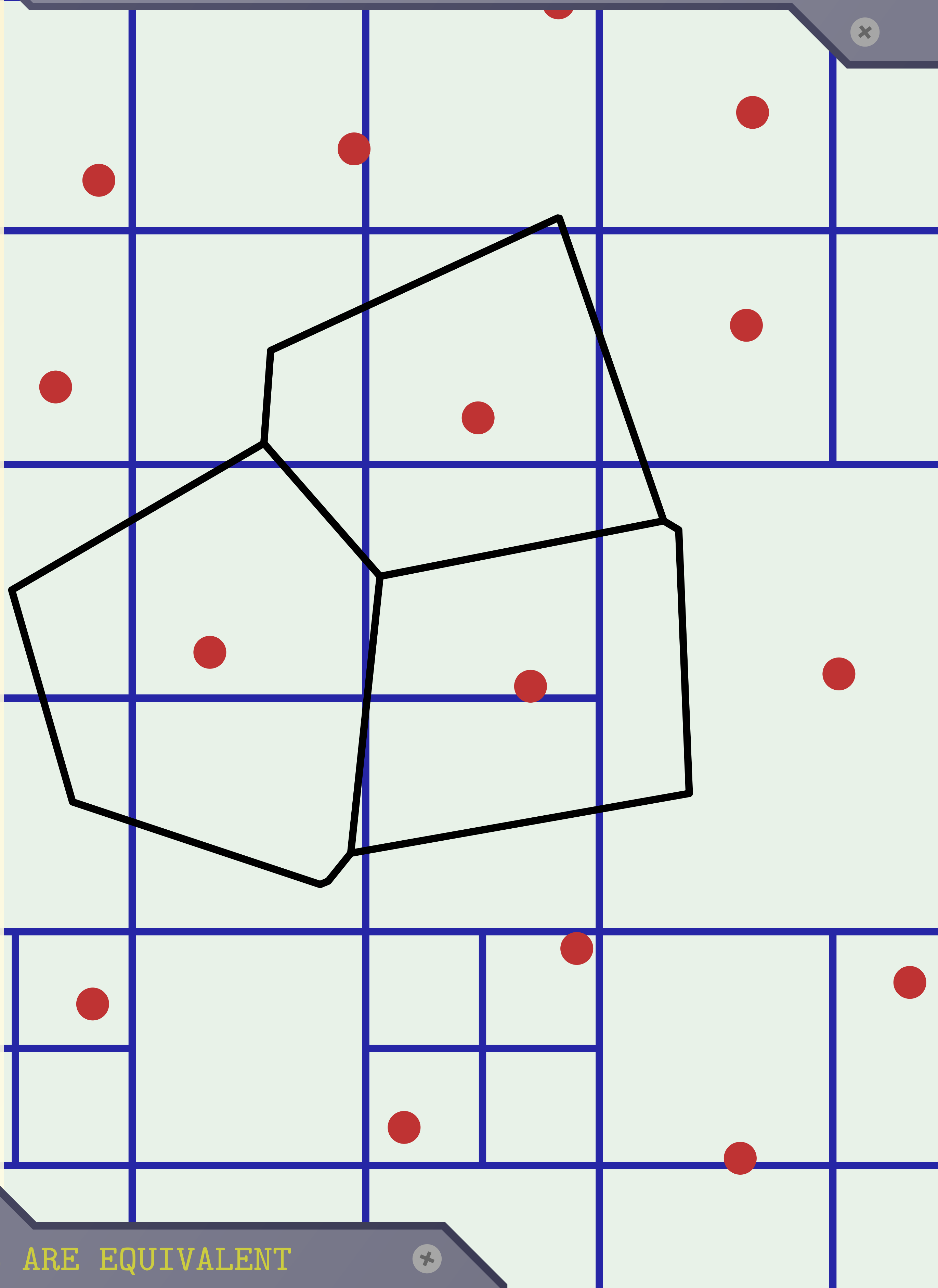

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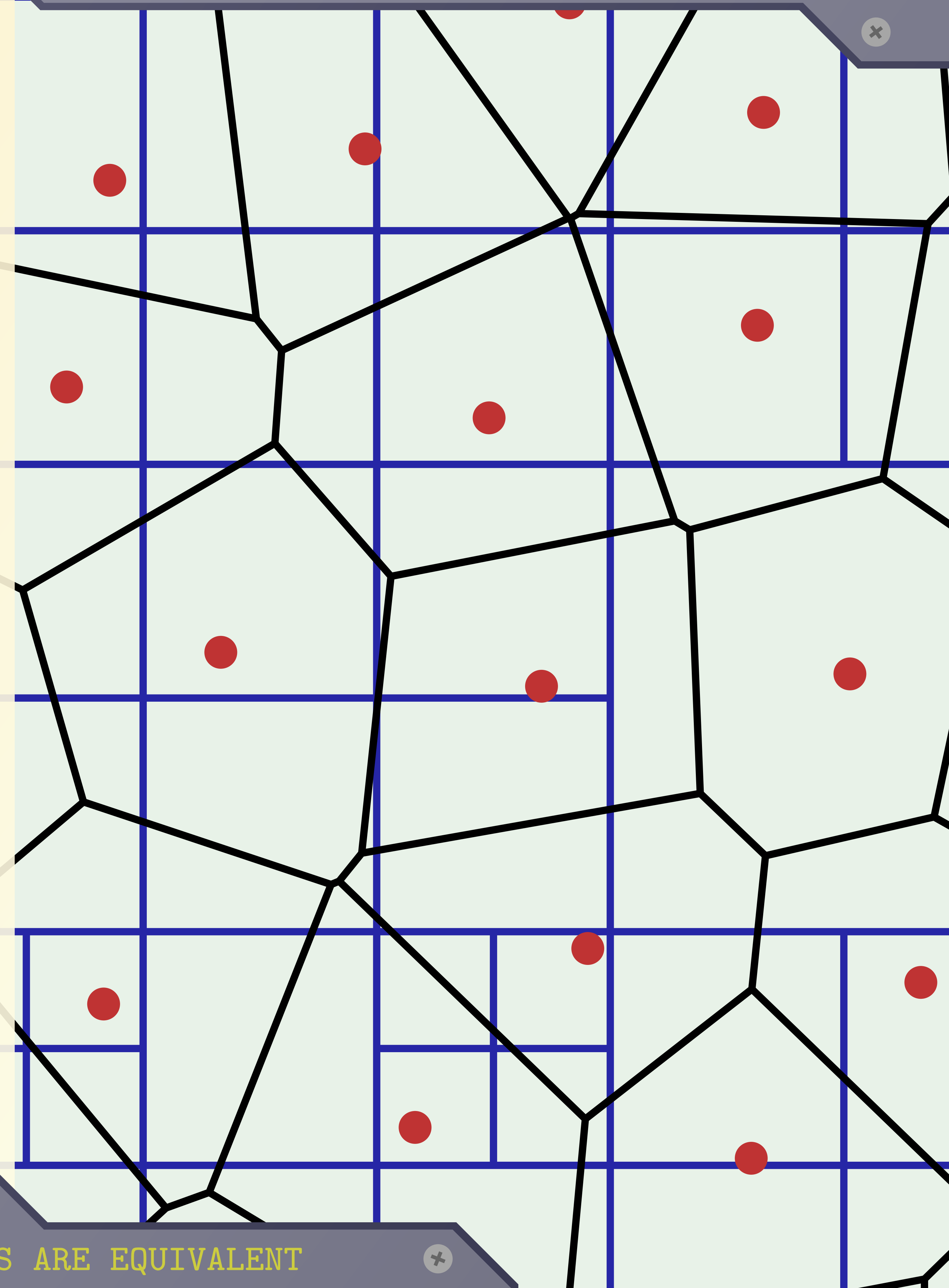

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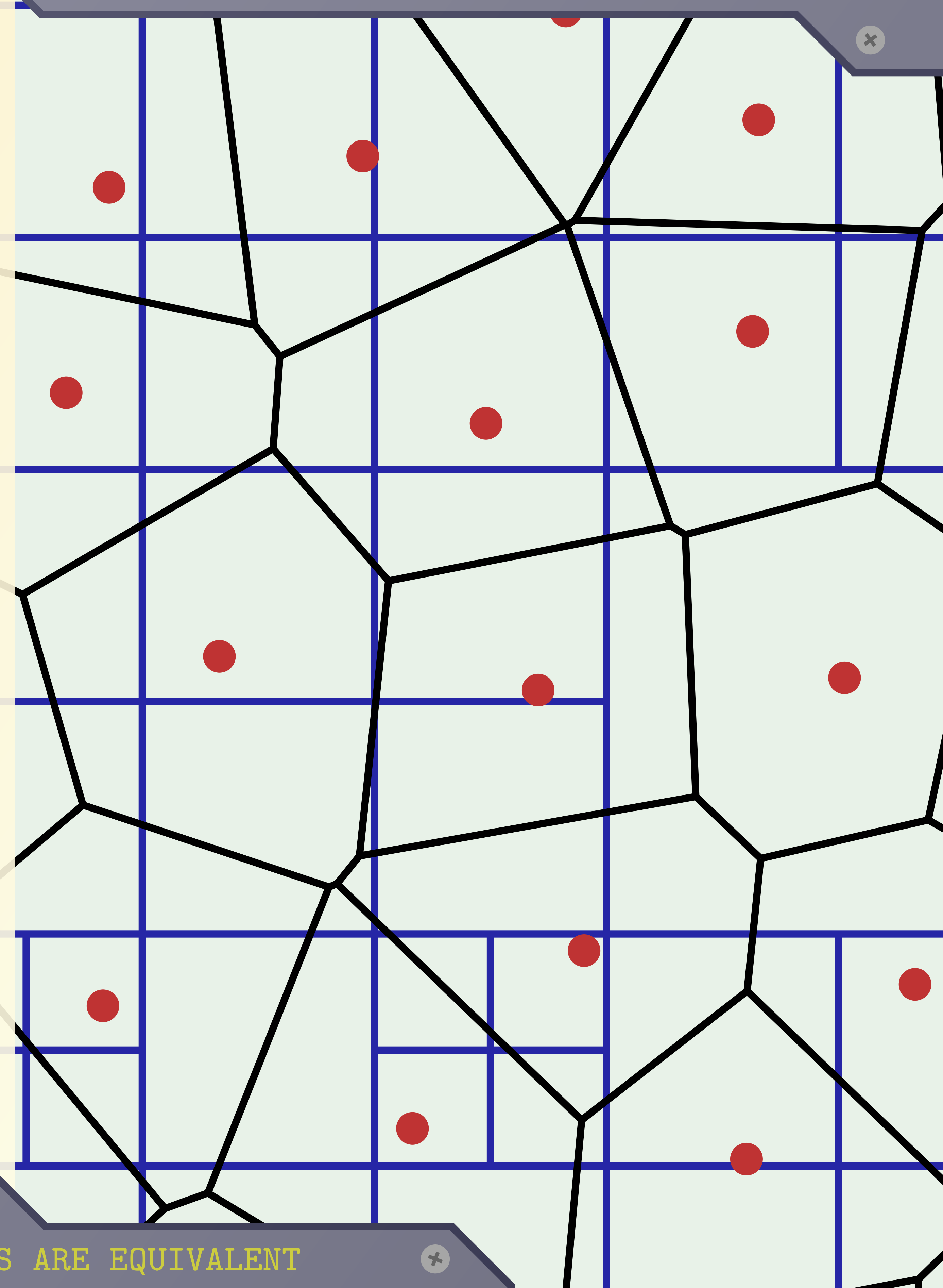
Since information is *local*, such a conversion should take $O(n)$ time.



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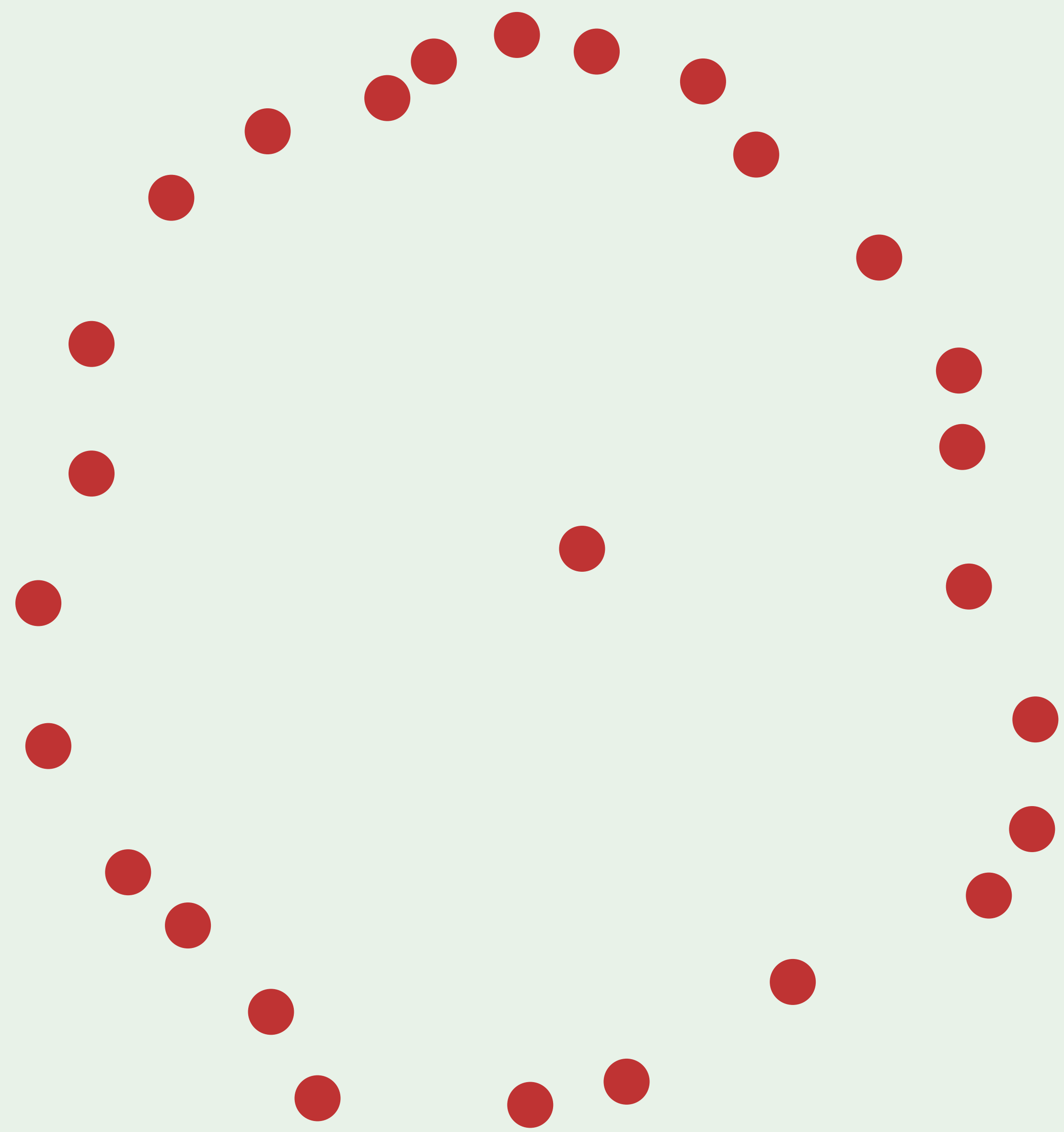

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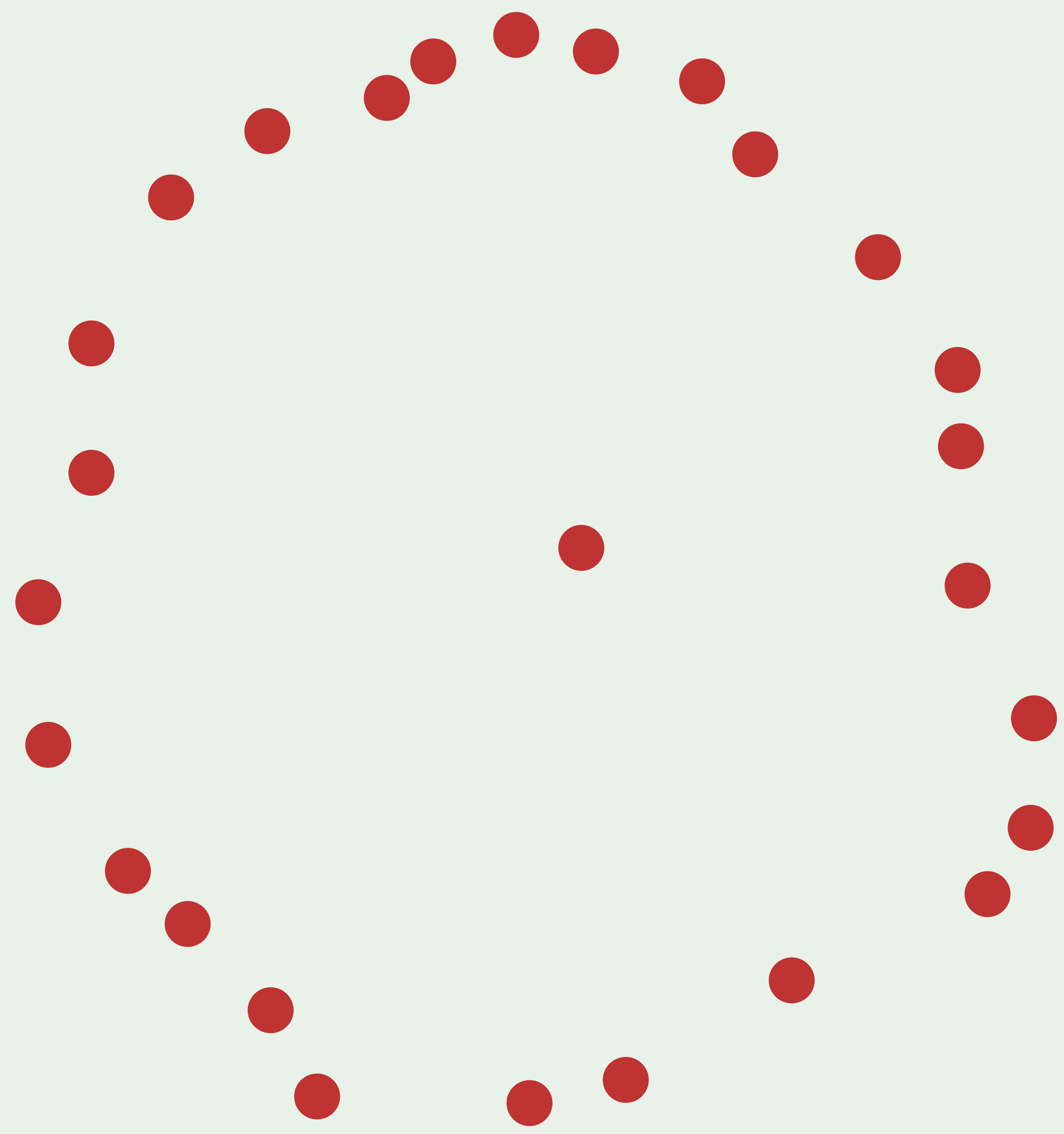
The scale of a point set need not be uniform.



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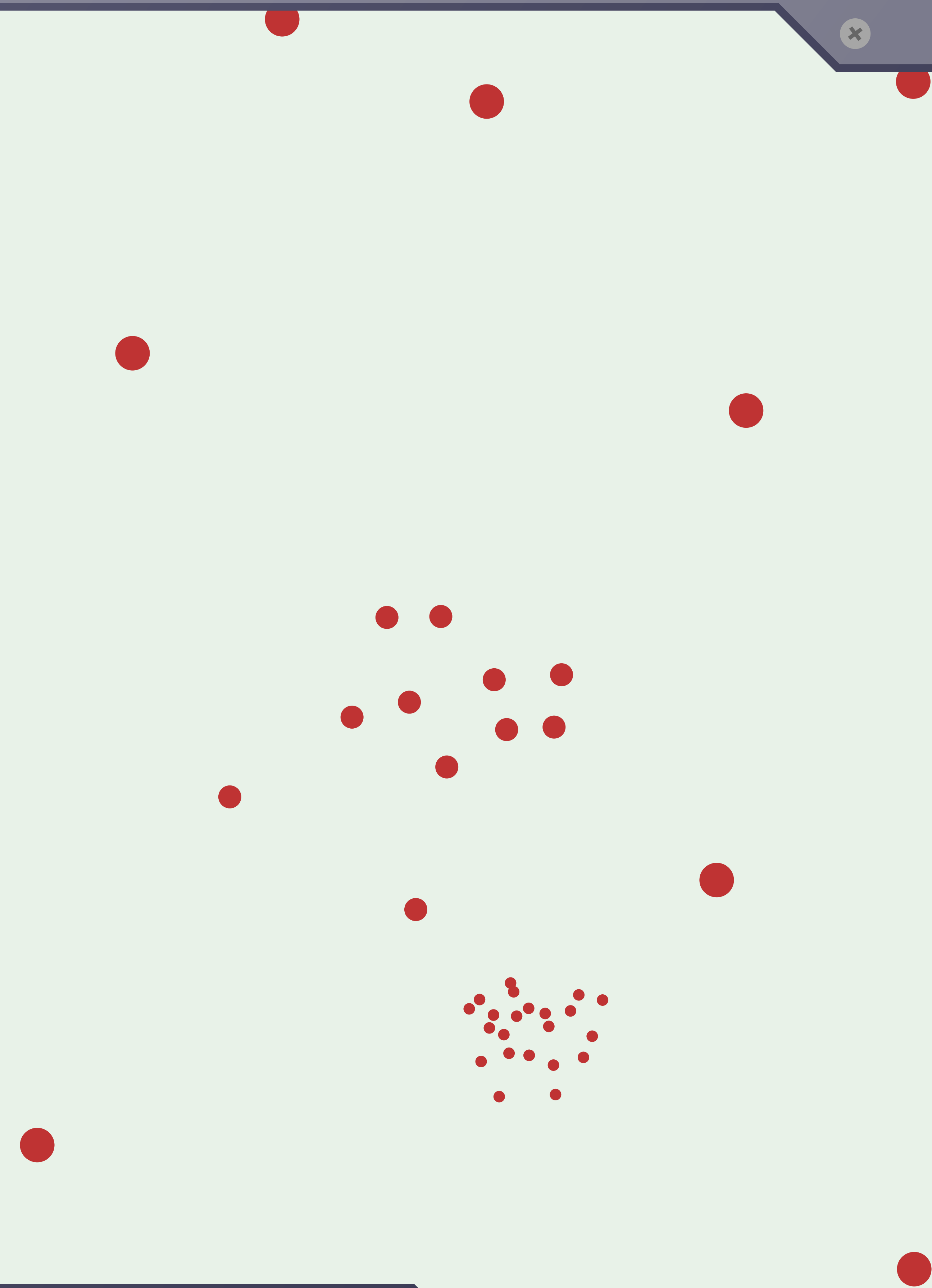

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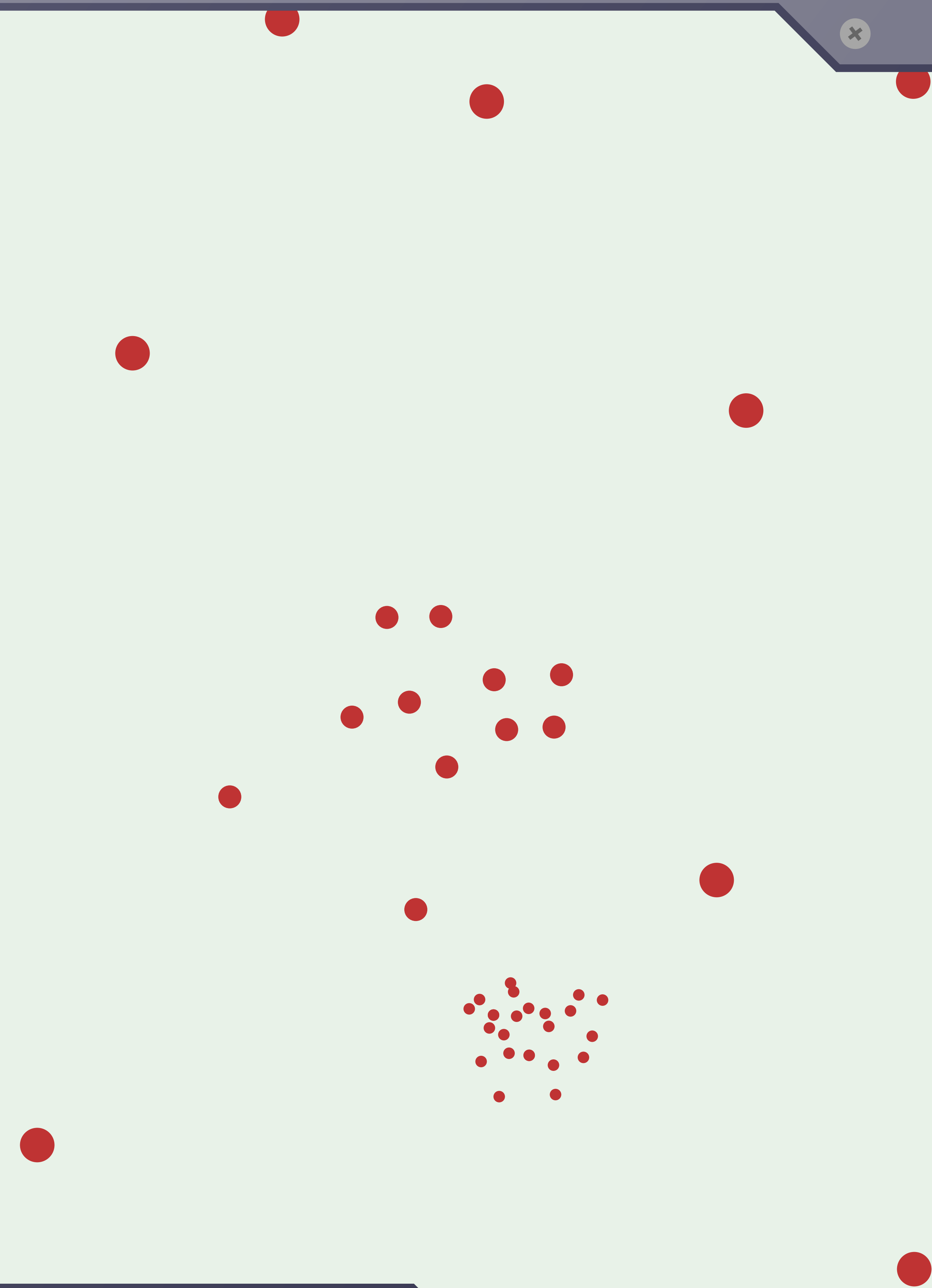
What can we do?



Do we...

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-OR-

Do we...

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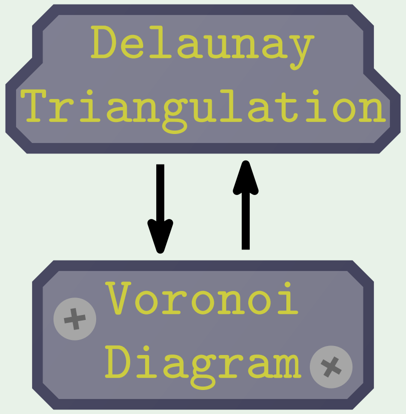
-OR-

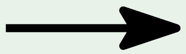
... insist on being general, and design devilishly complicated algorithms that nobody will ever use, but that are able to handle arbitrary point sets?

[Dirichlet, 1850]

Voronoi Diagram

[Dirichlet, 1850]
[Delaunay, 1934]





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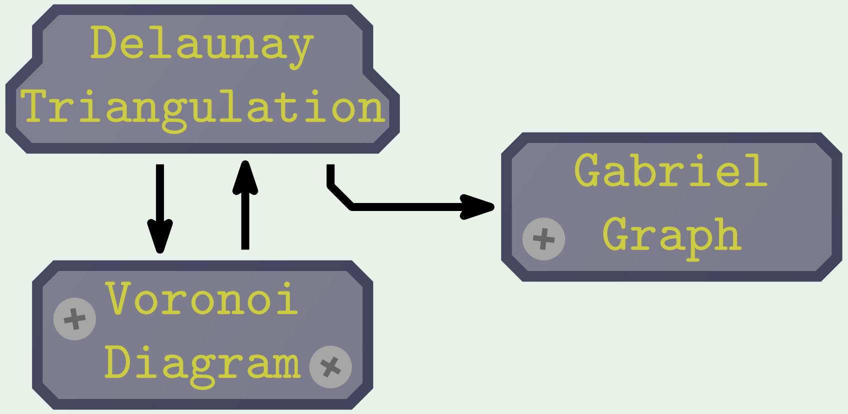

Deterministic linear time

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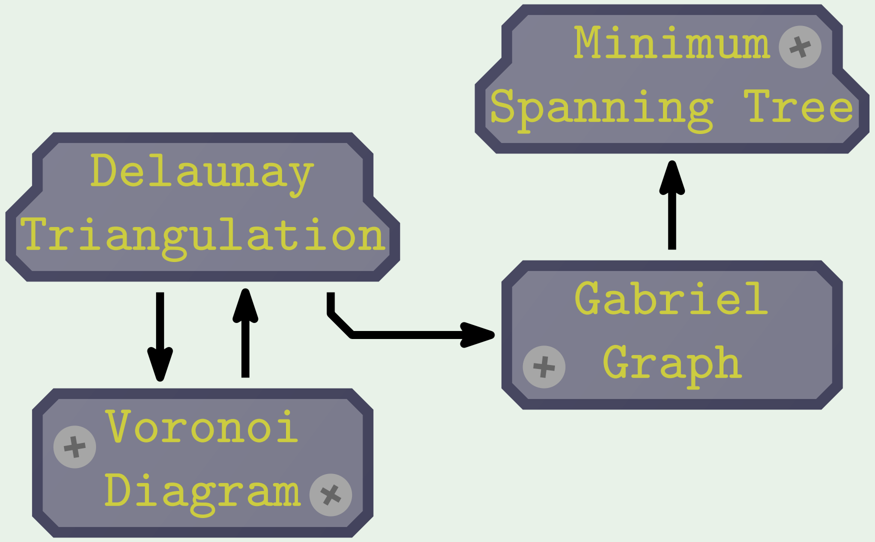

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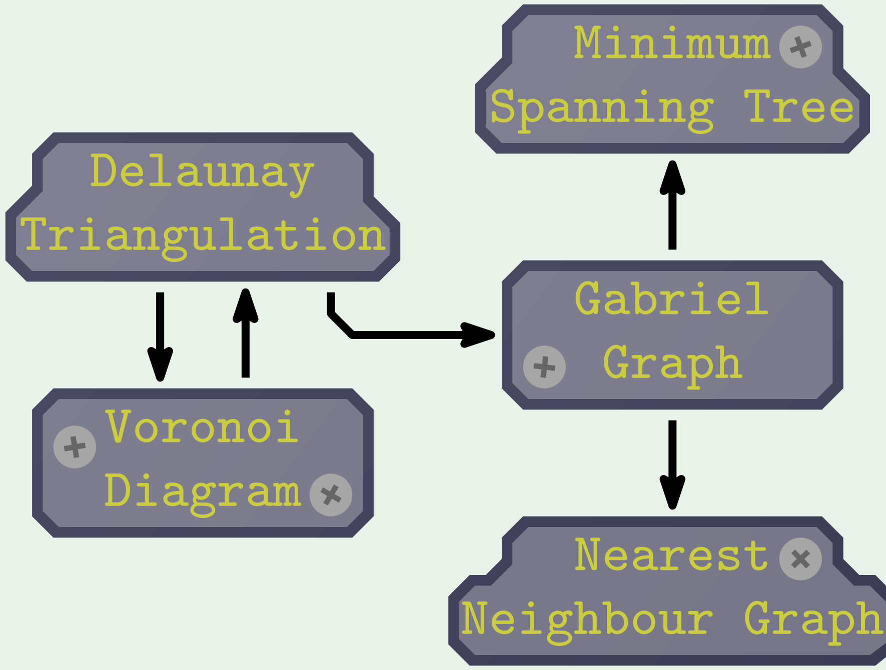


Deterministic linear time

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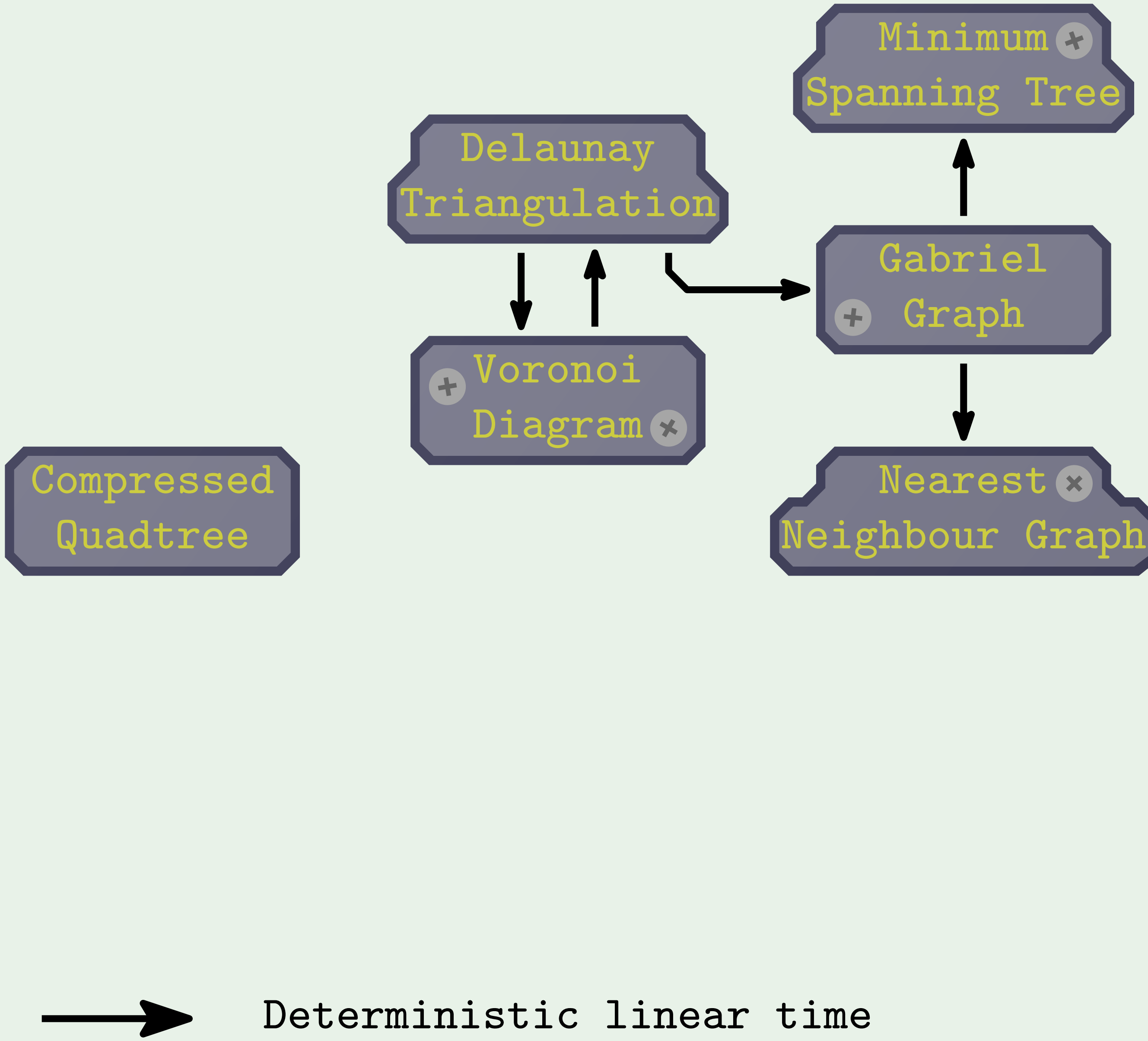

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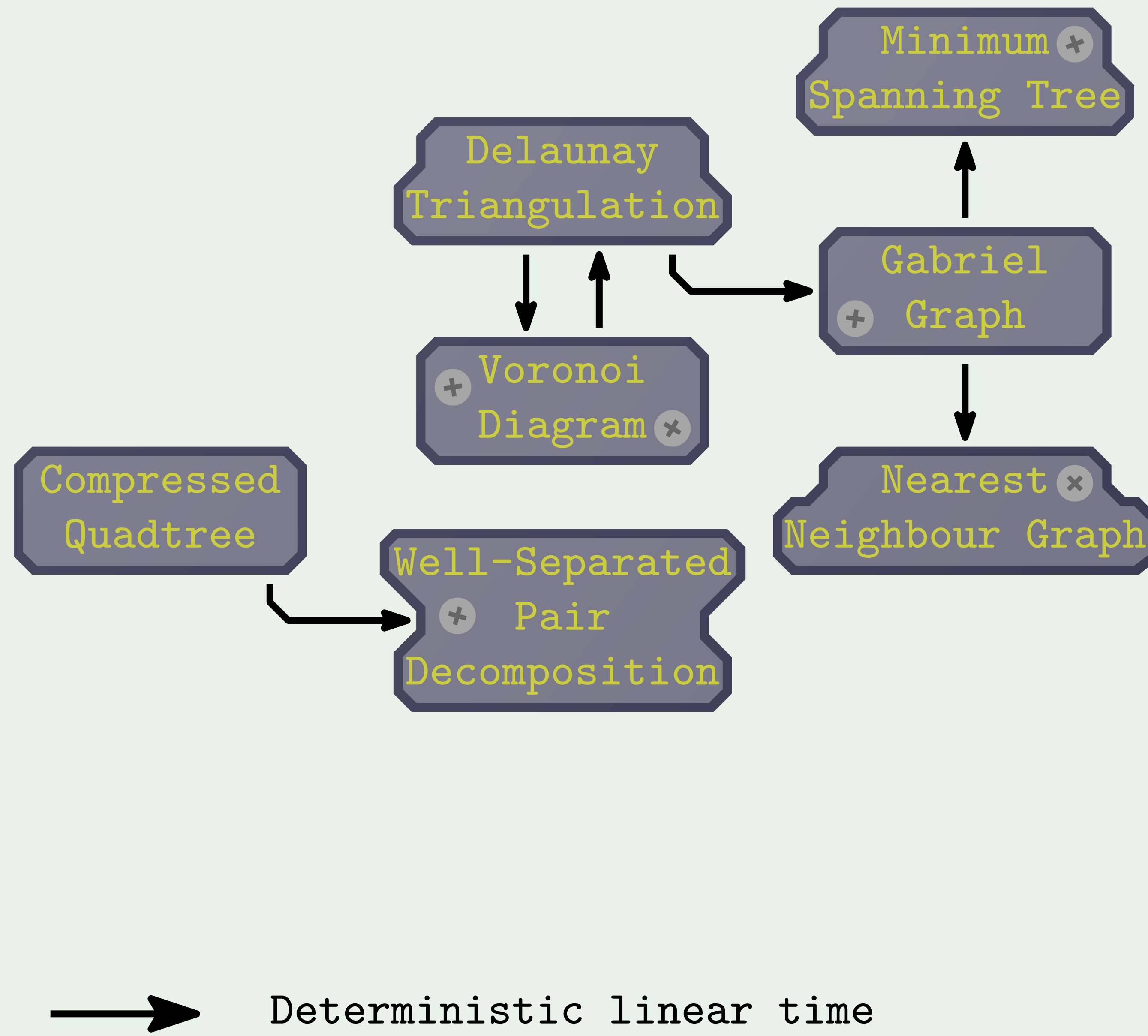

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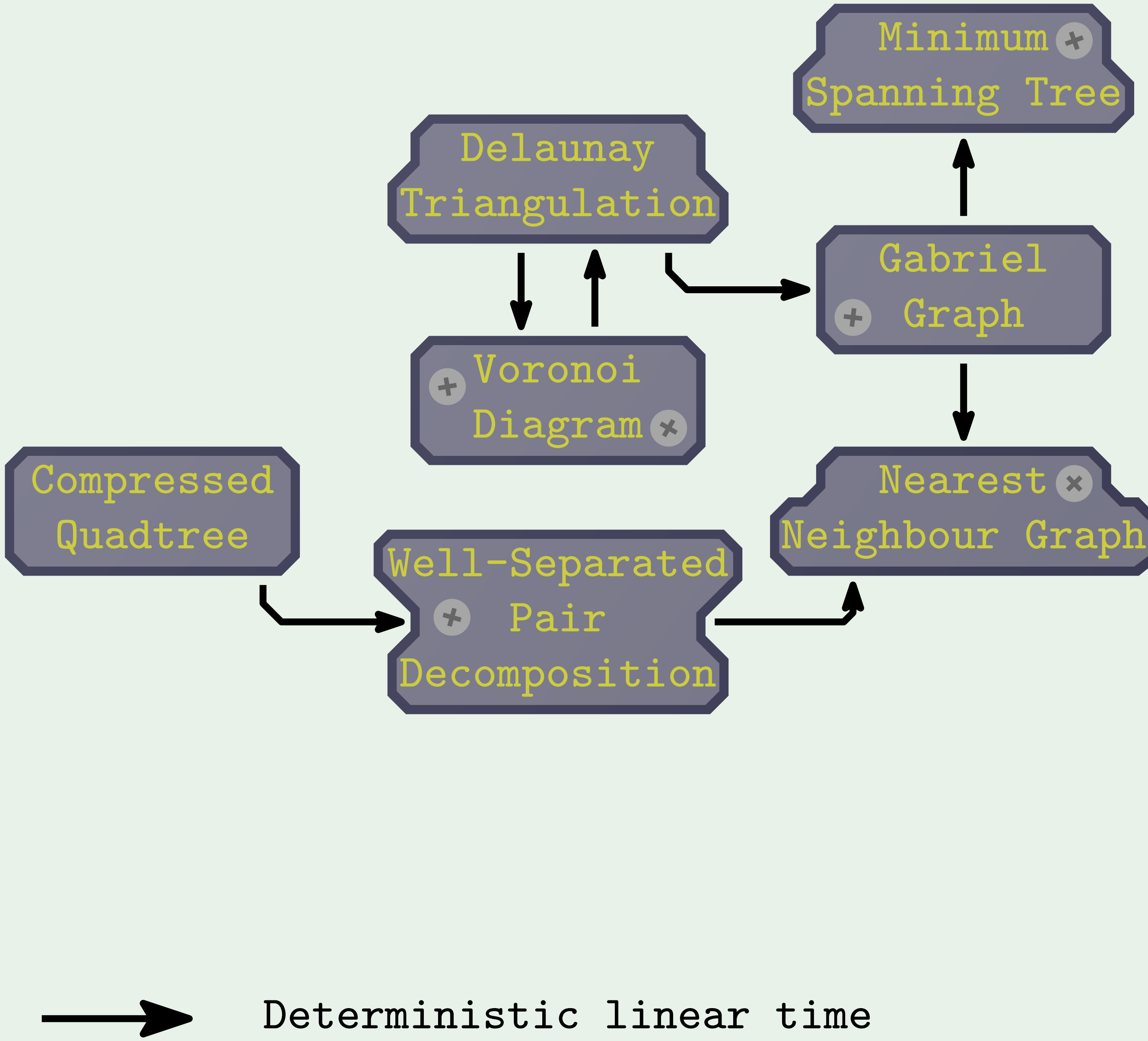

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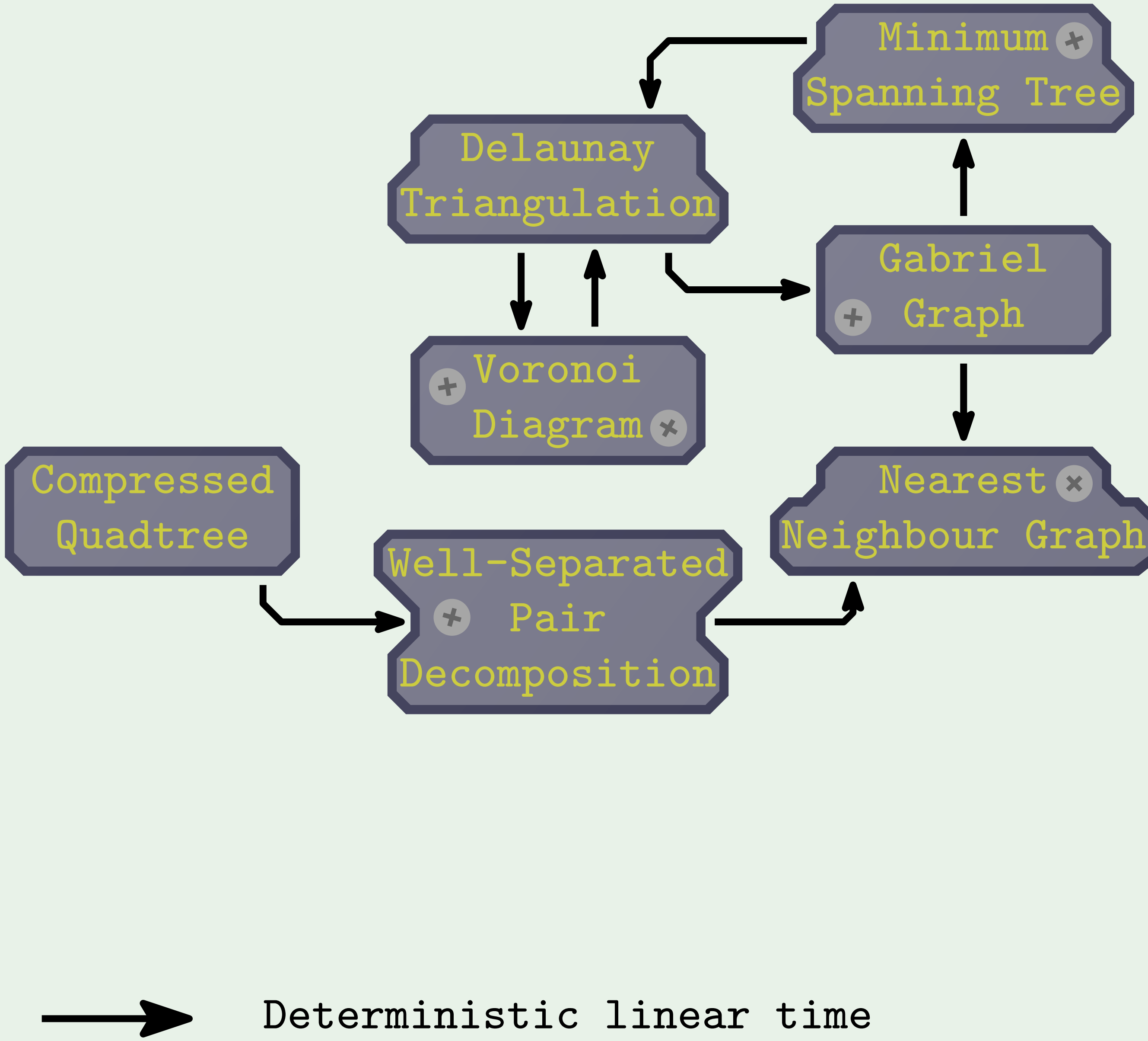

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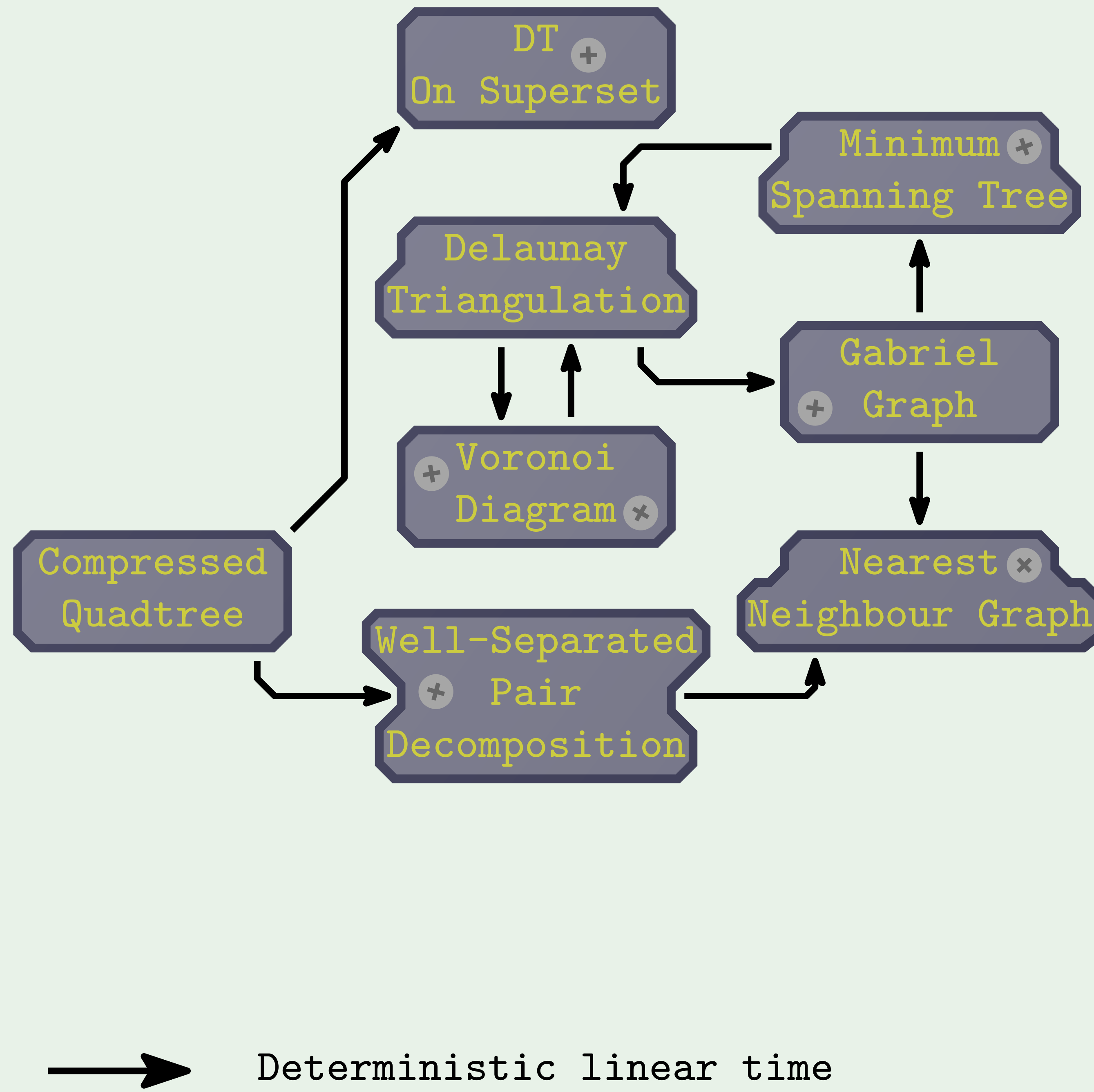

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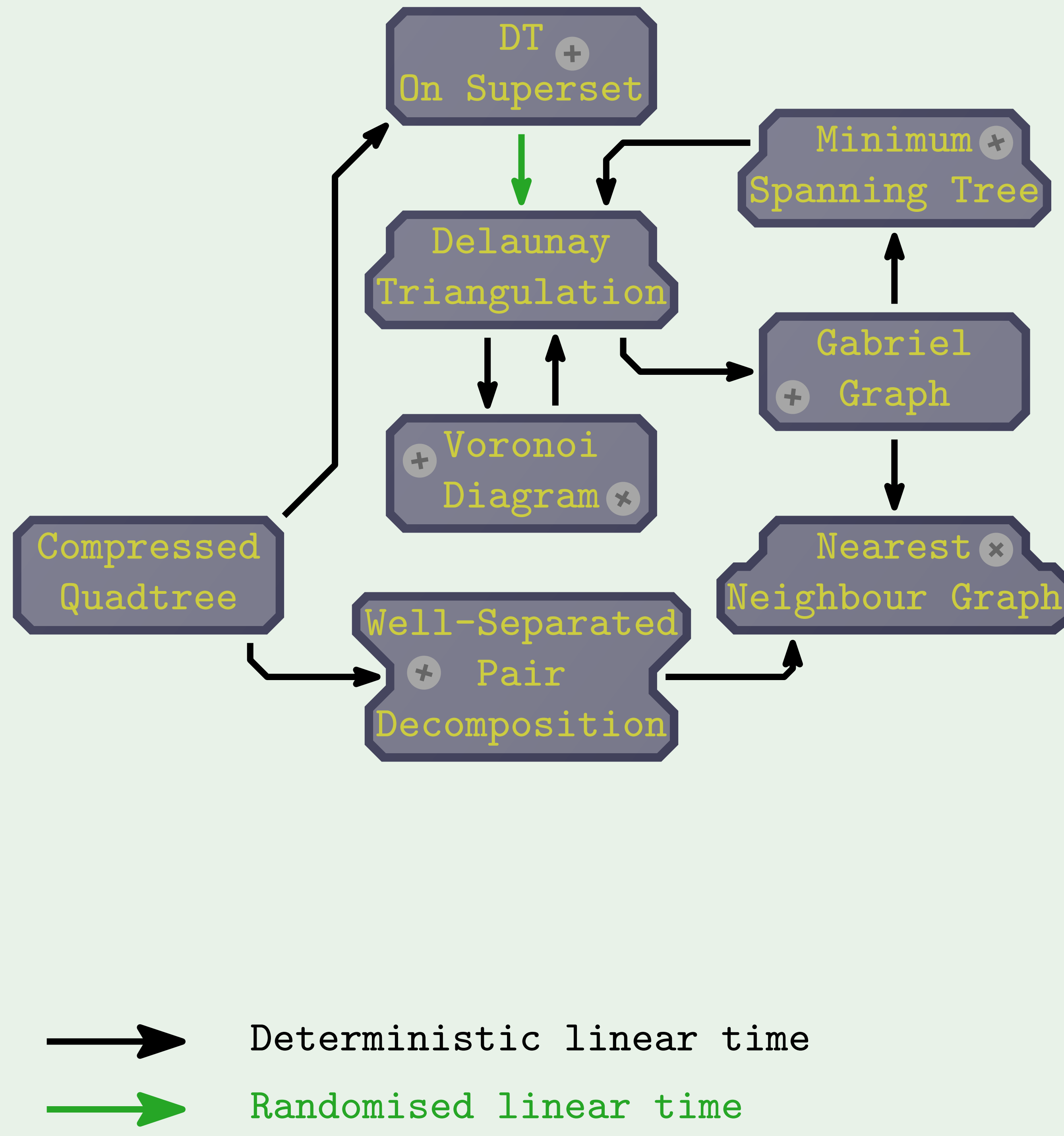

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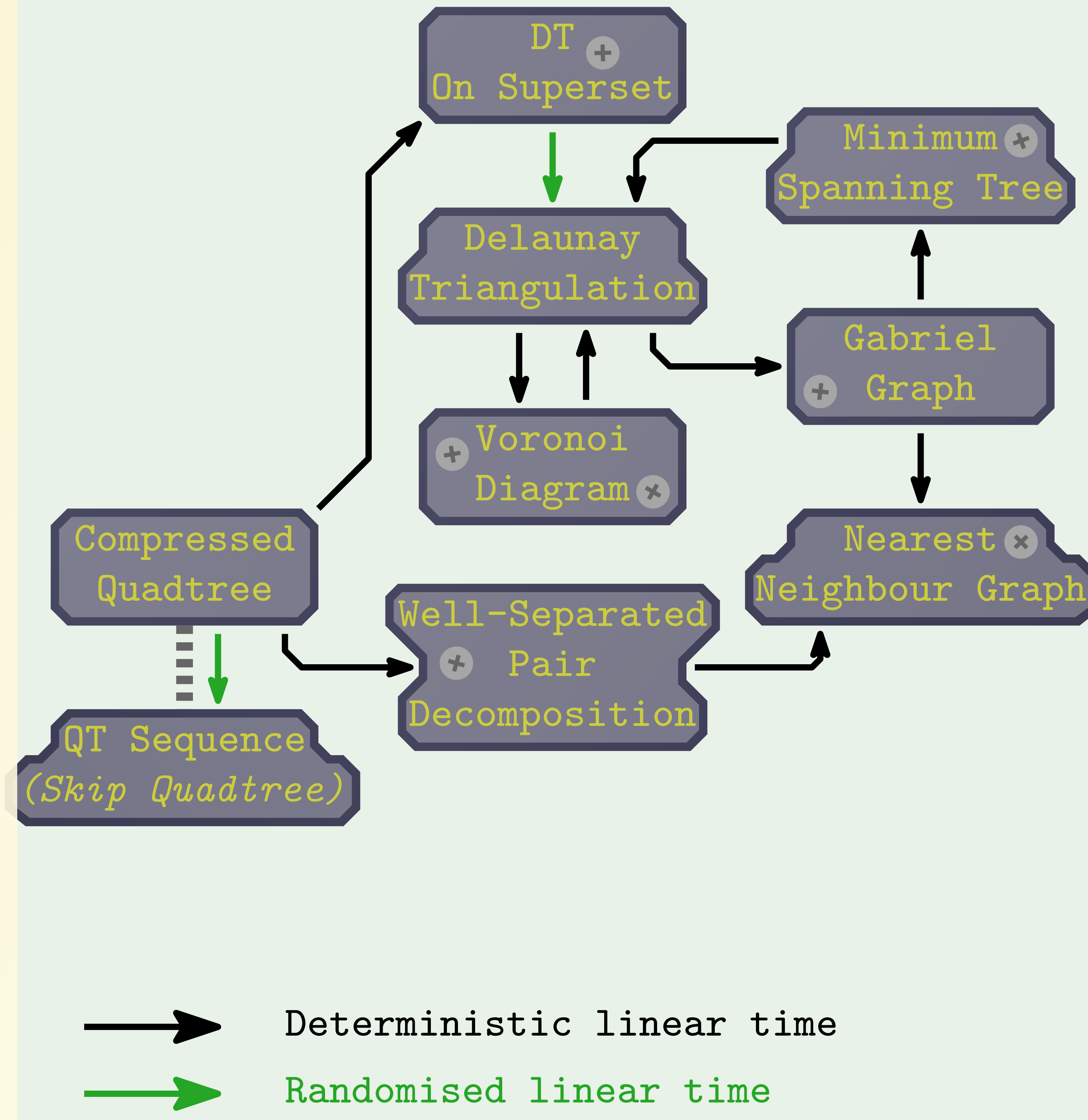

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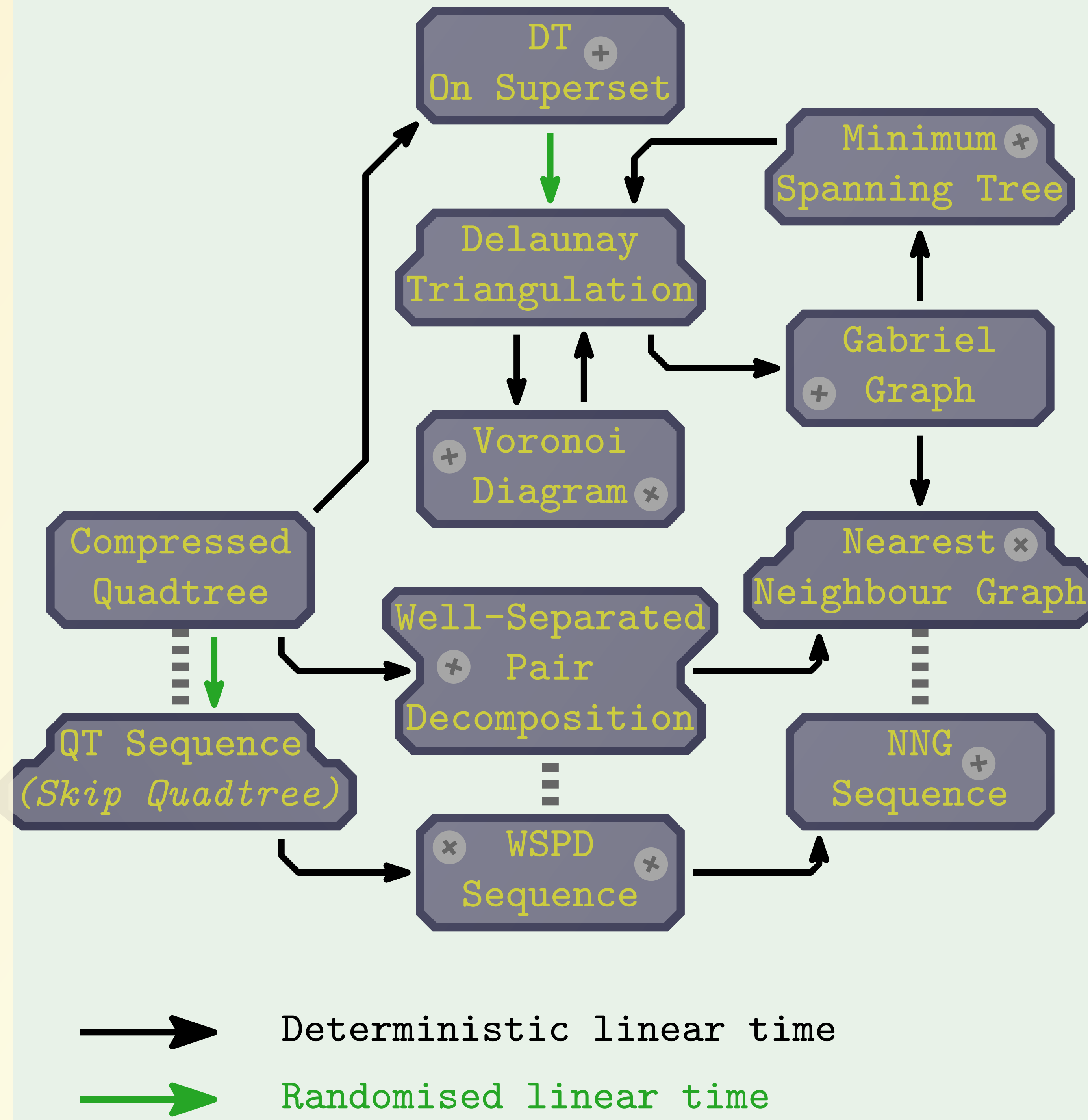

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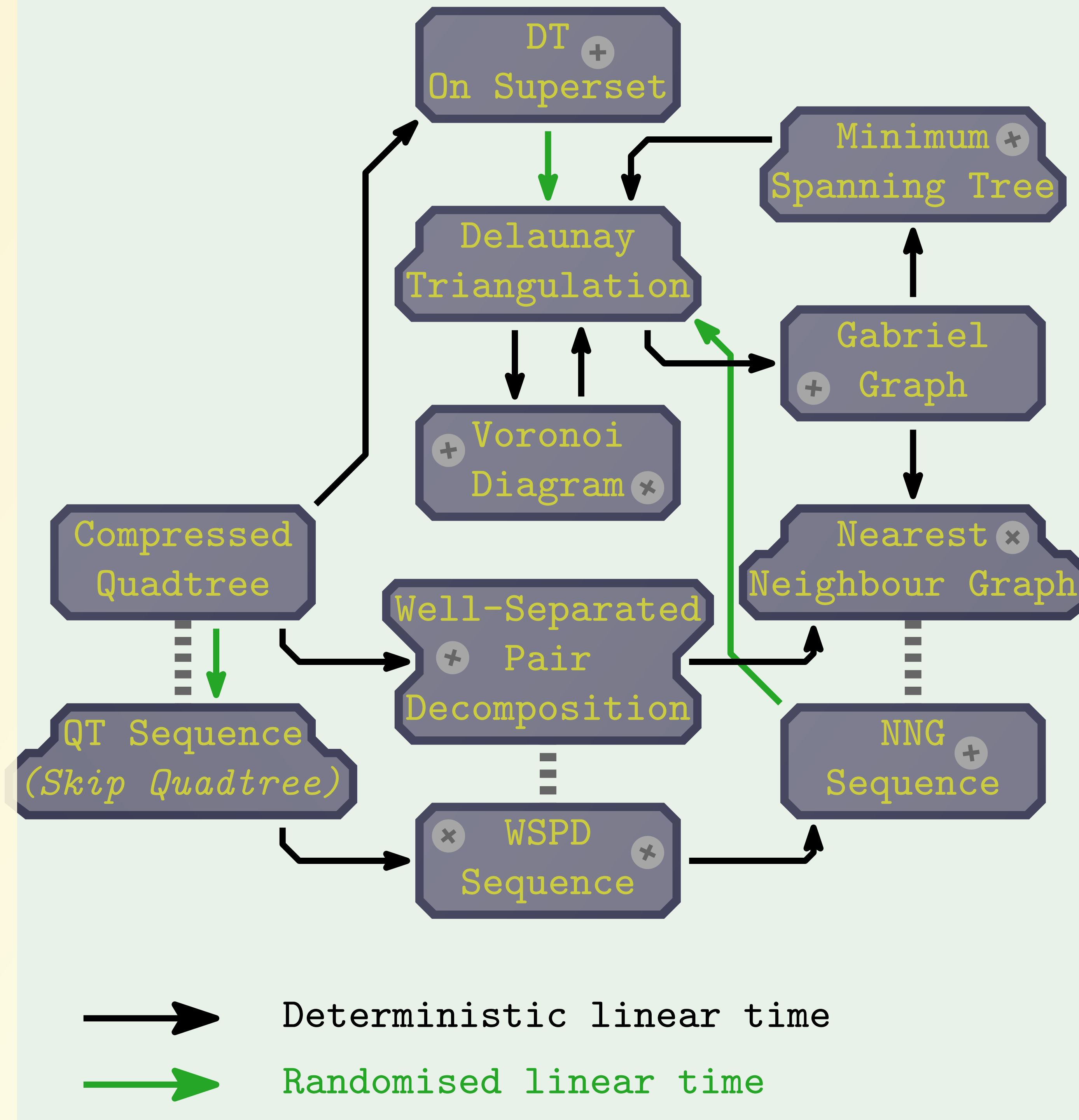

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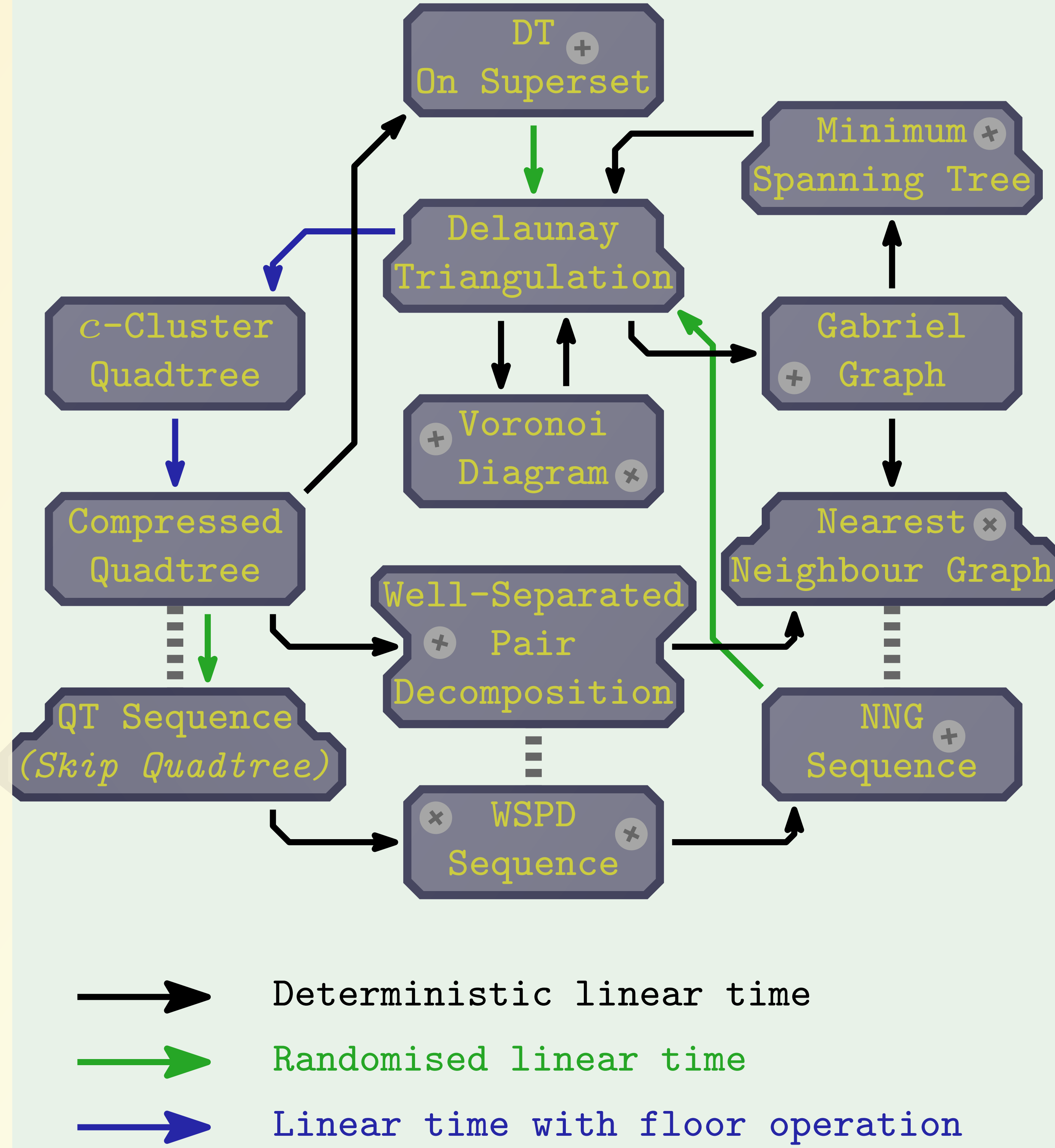

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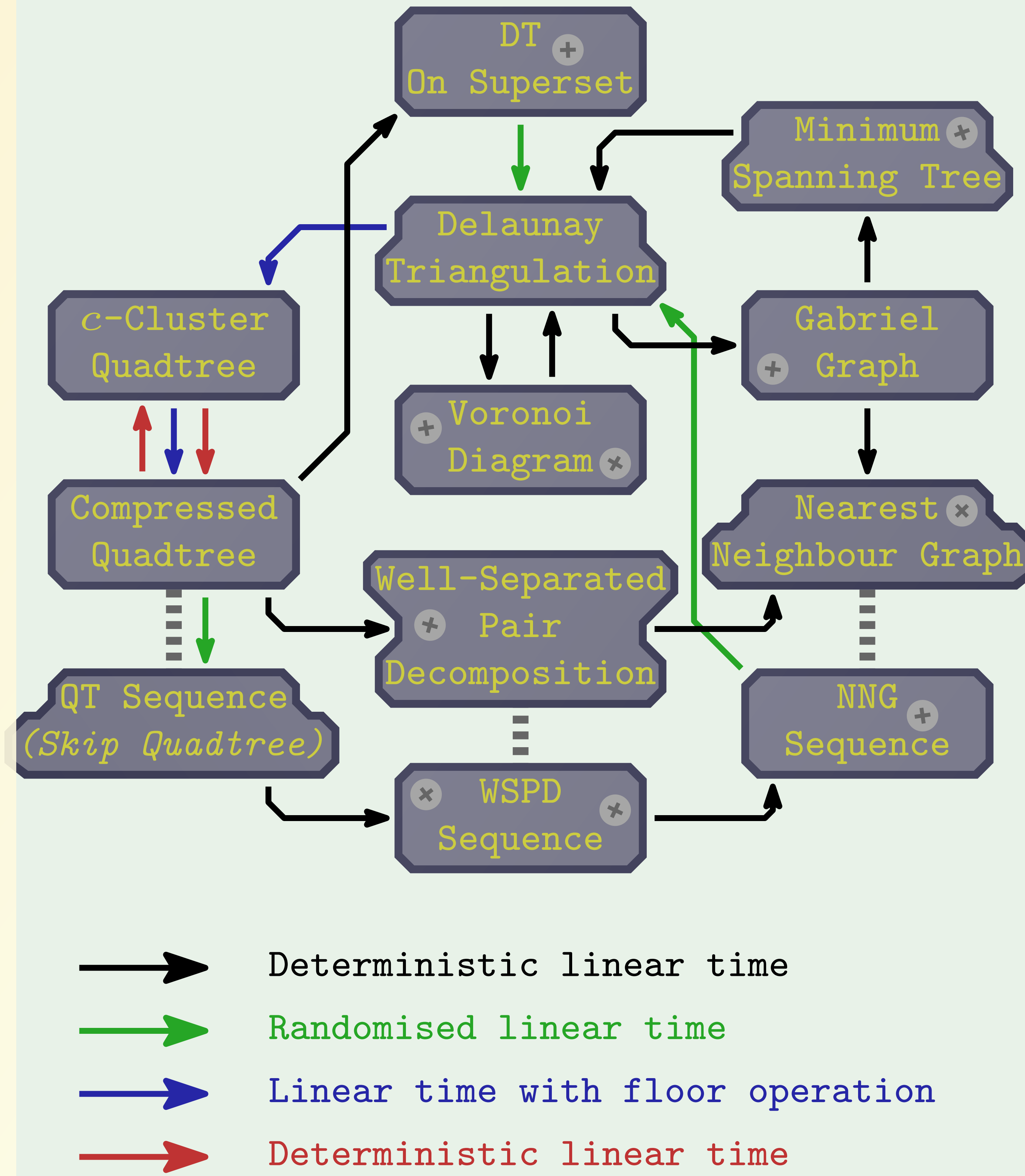

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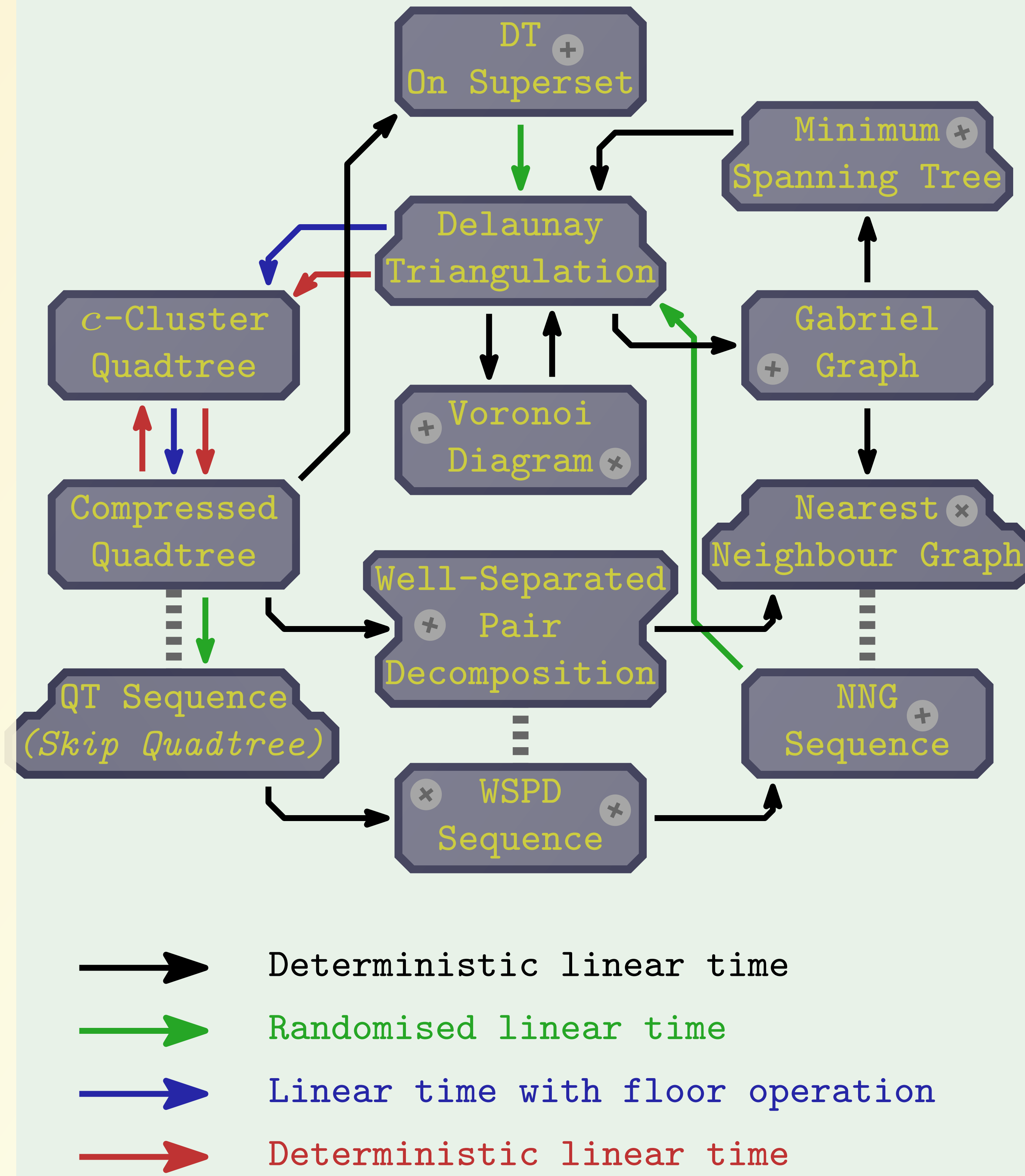

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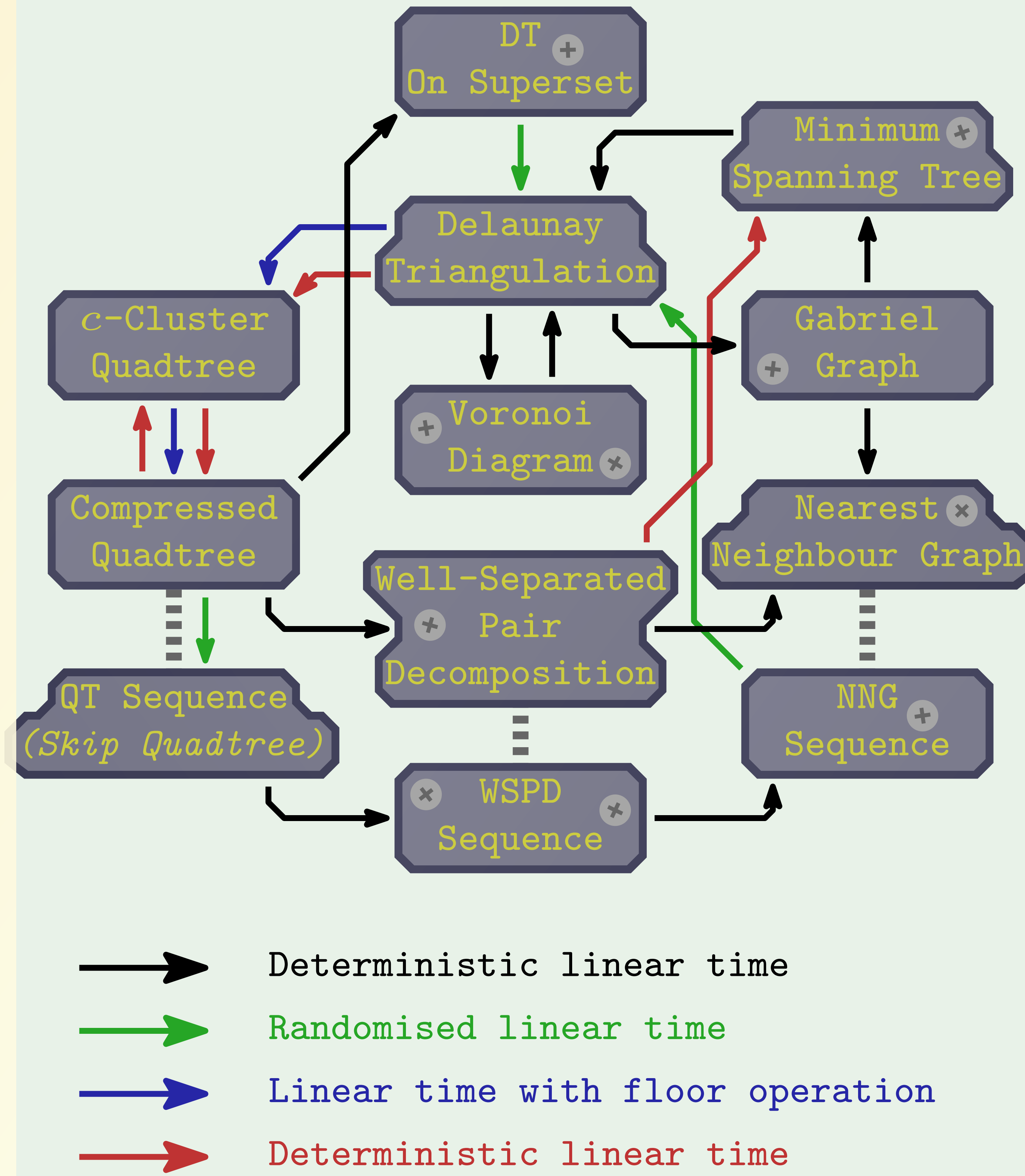

[Dirichlet, 1850]
[Delaunay, 1934]
[Gabriel, 1969]
[Clarkson, 1983]
[Preparata & Shamos, 1985]
[Bern, Eppstein & Gilbert, 1990]
[Chazelle, 1991]
[Matsui, 1995]
[Callahan & Kosaraju, 1995]
[Chin & Wang, 1998]
[Krznaric & Levcopoulos, 1998]
[Chazelle, Devillers, Hurtado, Mora, Sacristán & Teillaud, 2001]
[Eppstein, Goodrich & Sun, 2005]
[Buchin & Mulzer, 2009]
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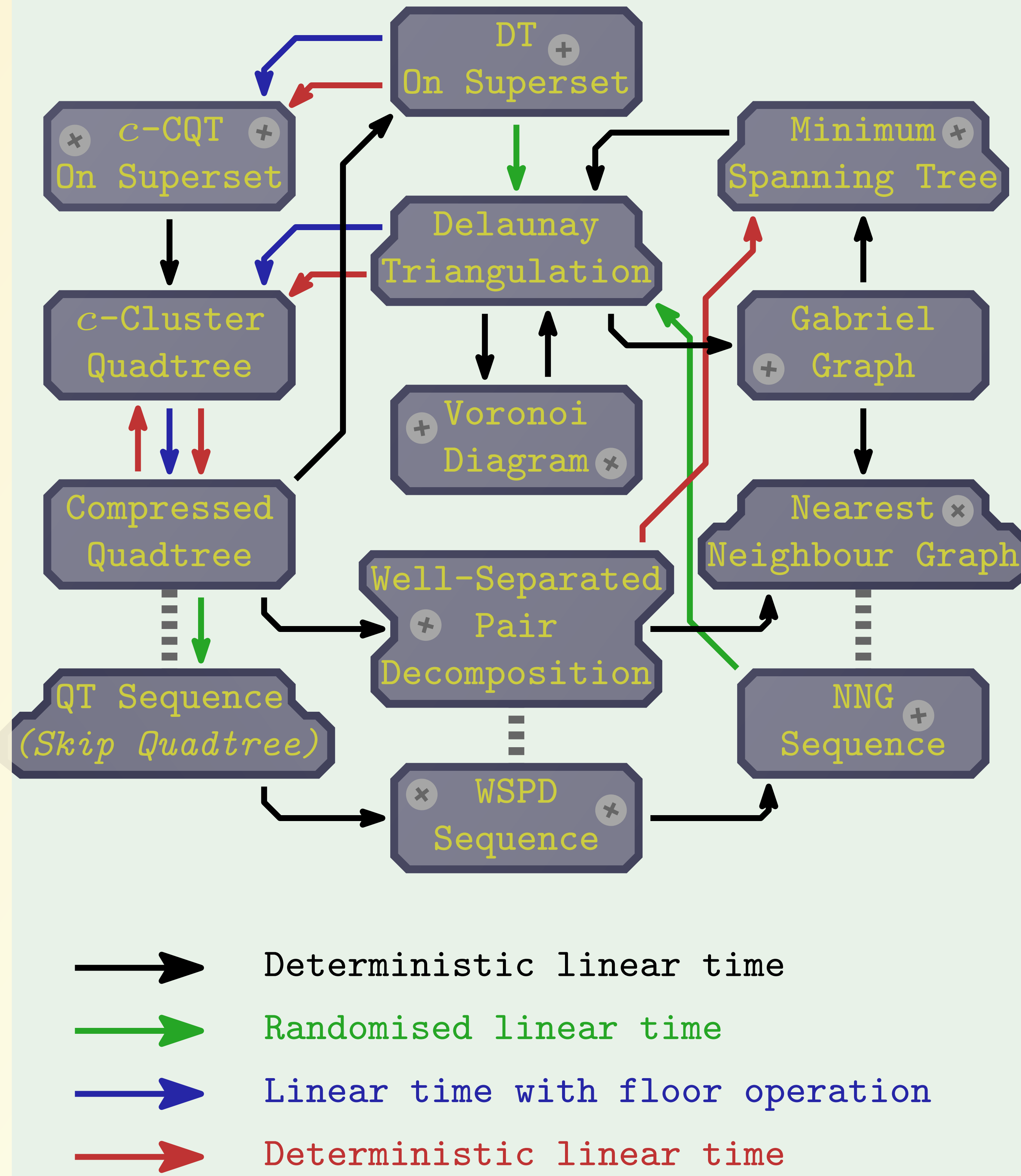

PART II *PRELIMINARIES*

The *Delaunay triangulation (DT)* is a triangulation D of a point set P that has an edge between two points if there is an empty circle through these points.

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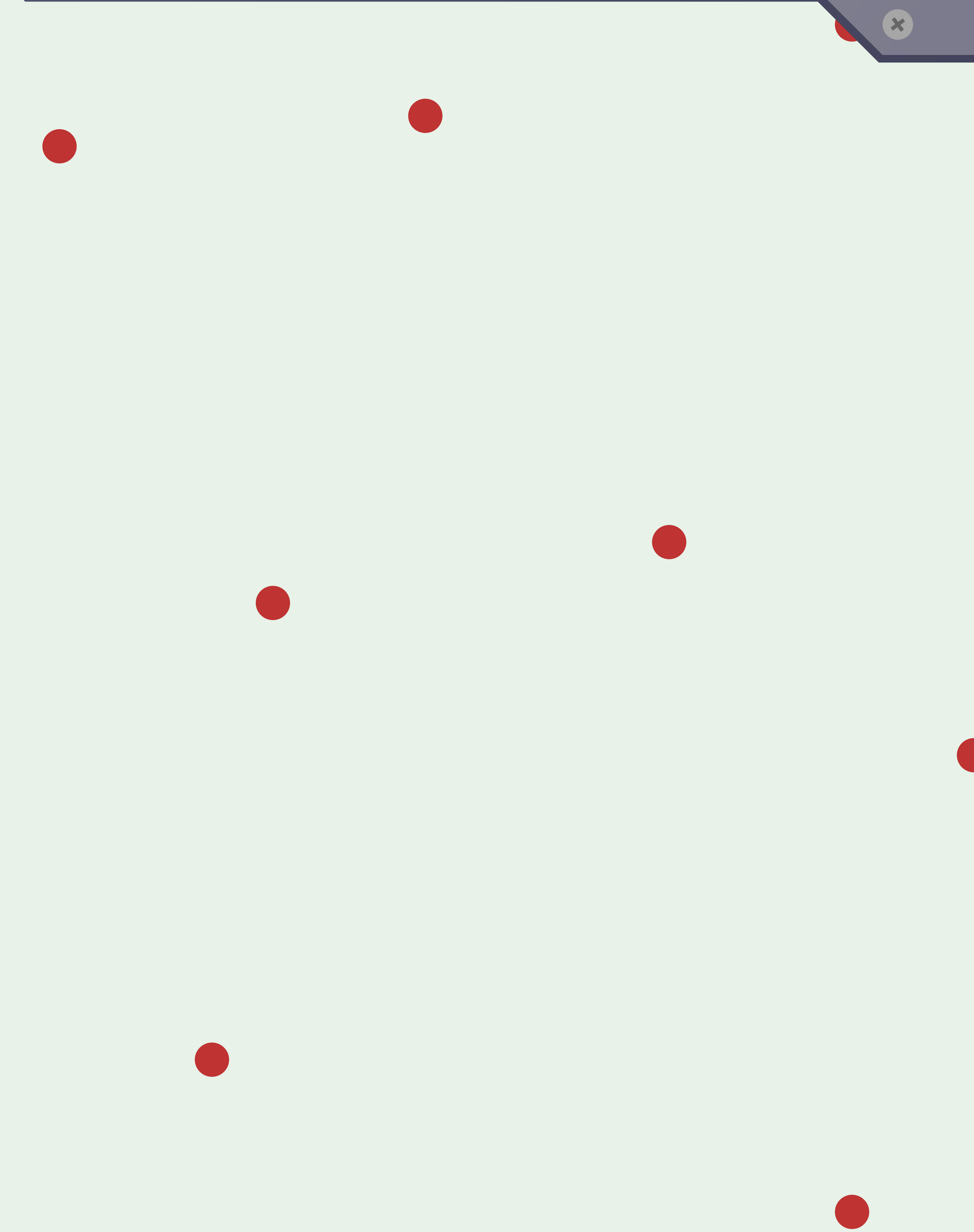

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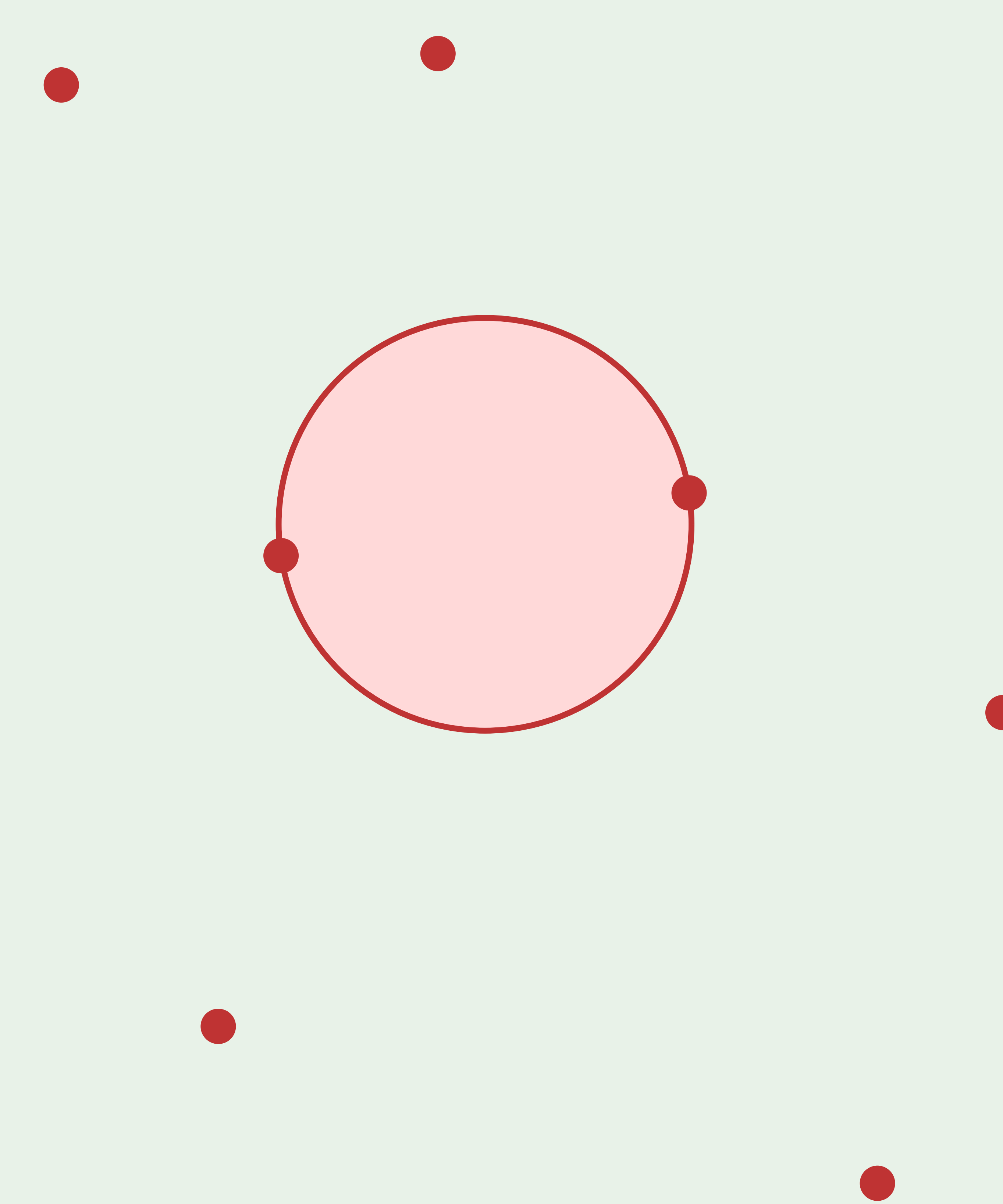

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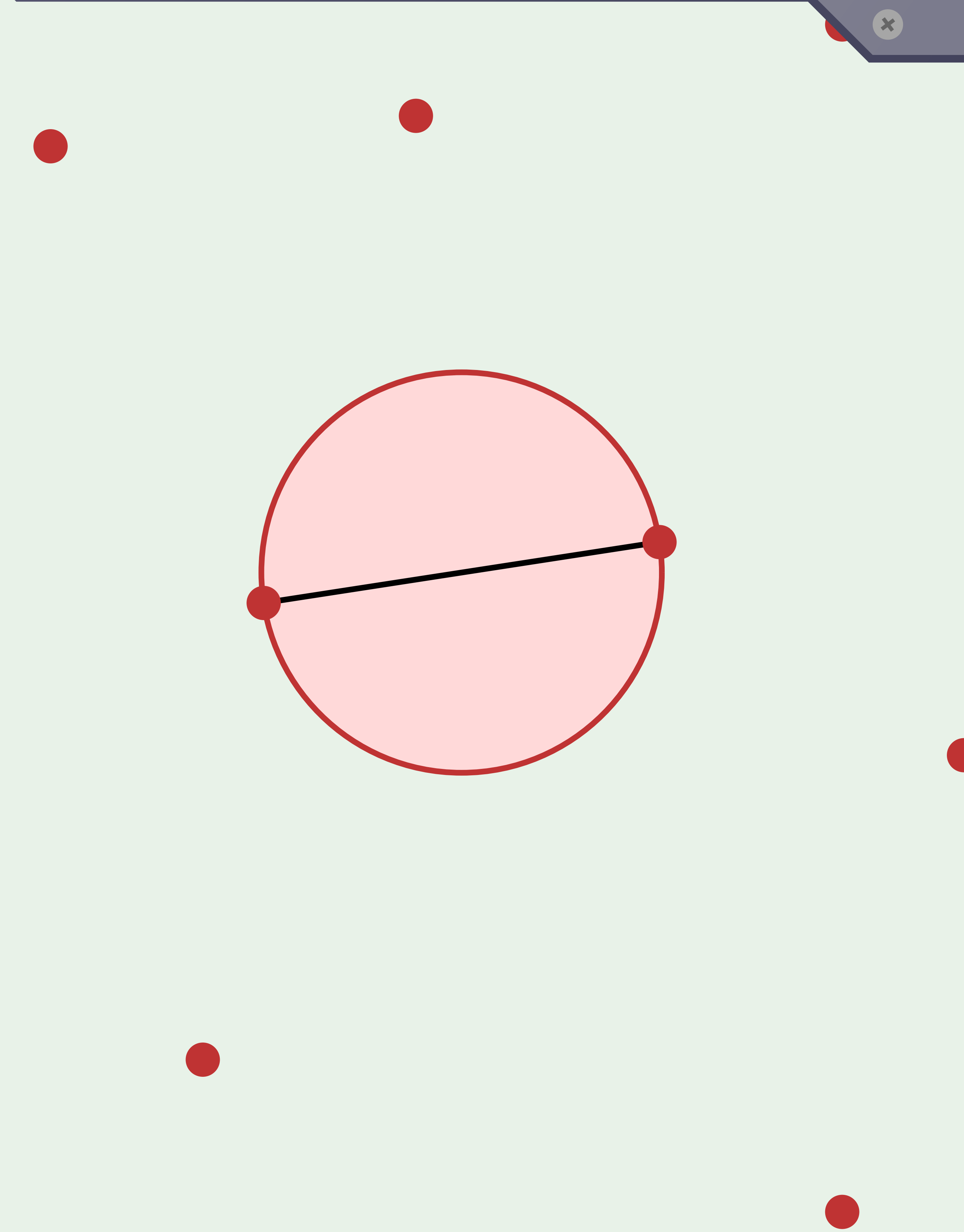

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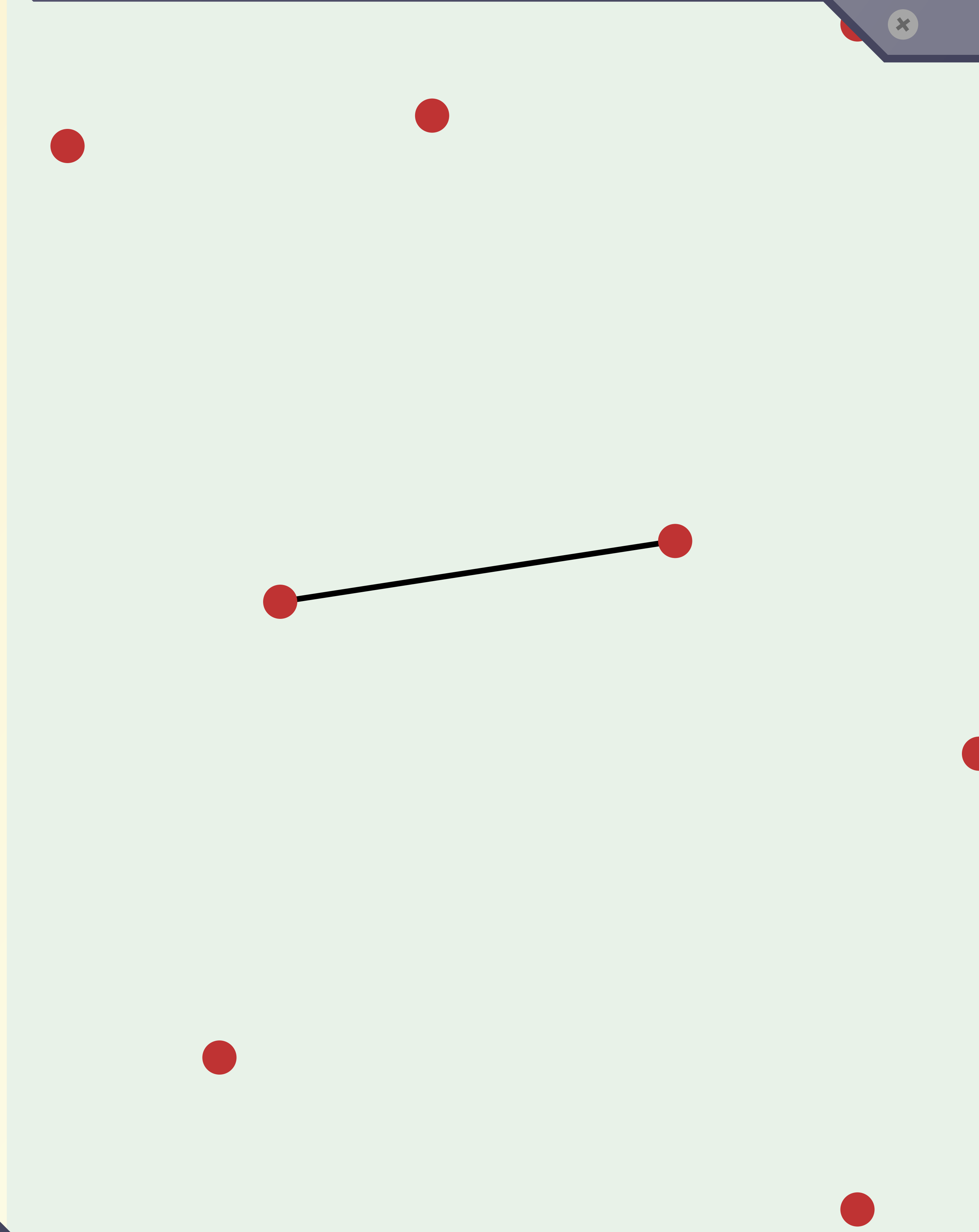

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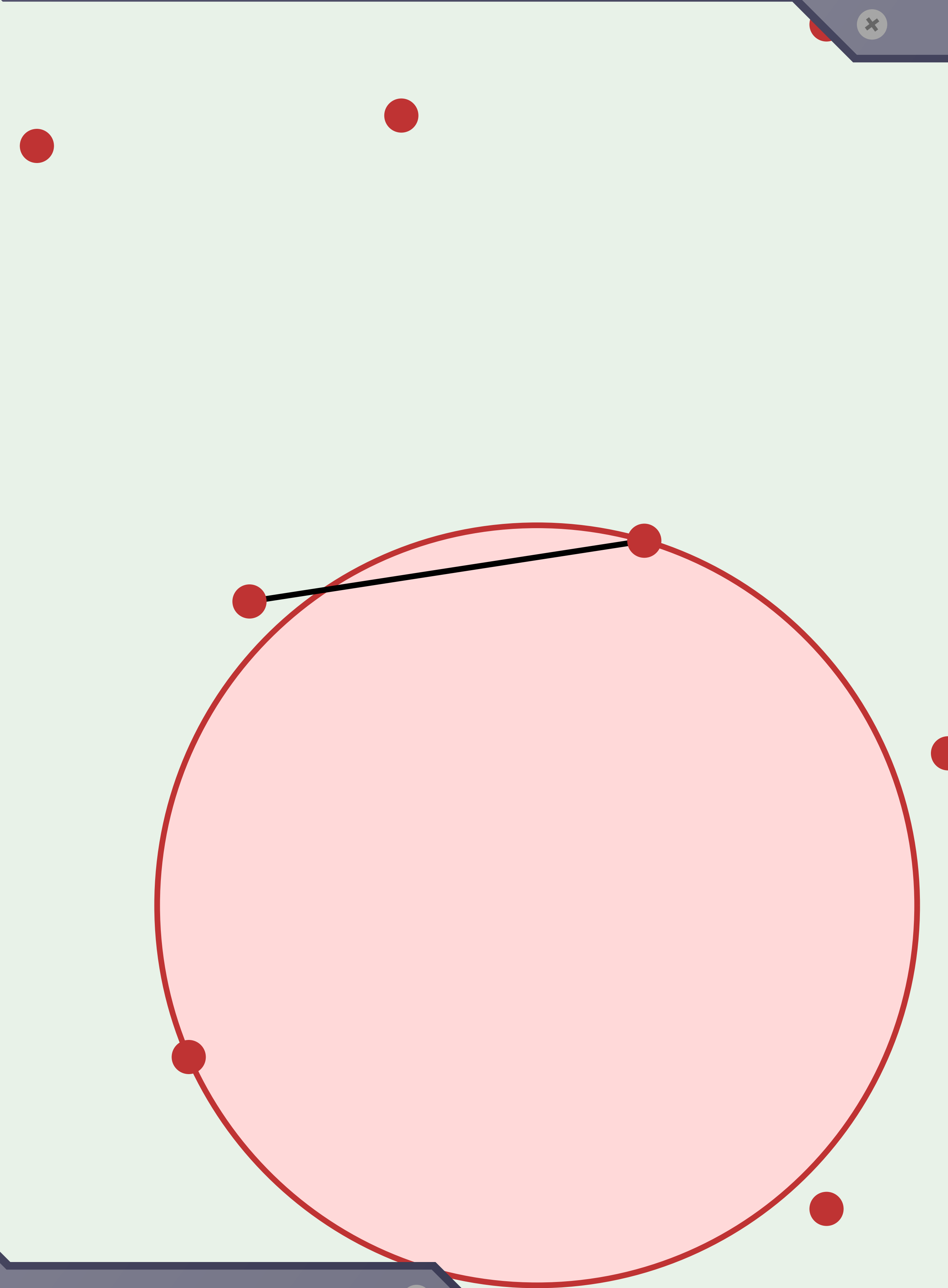

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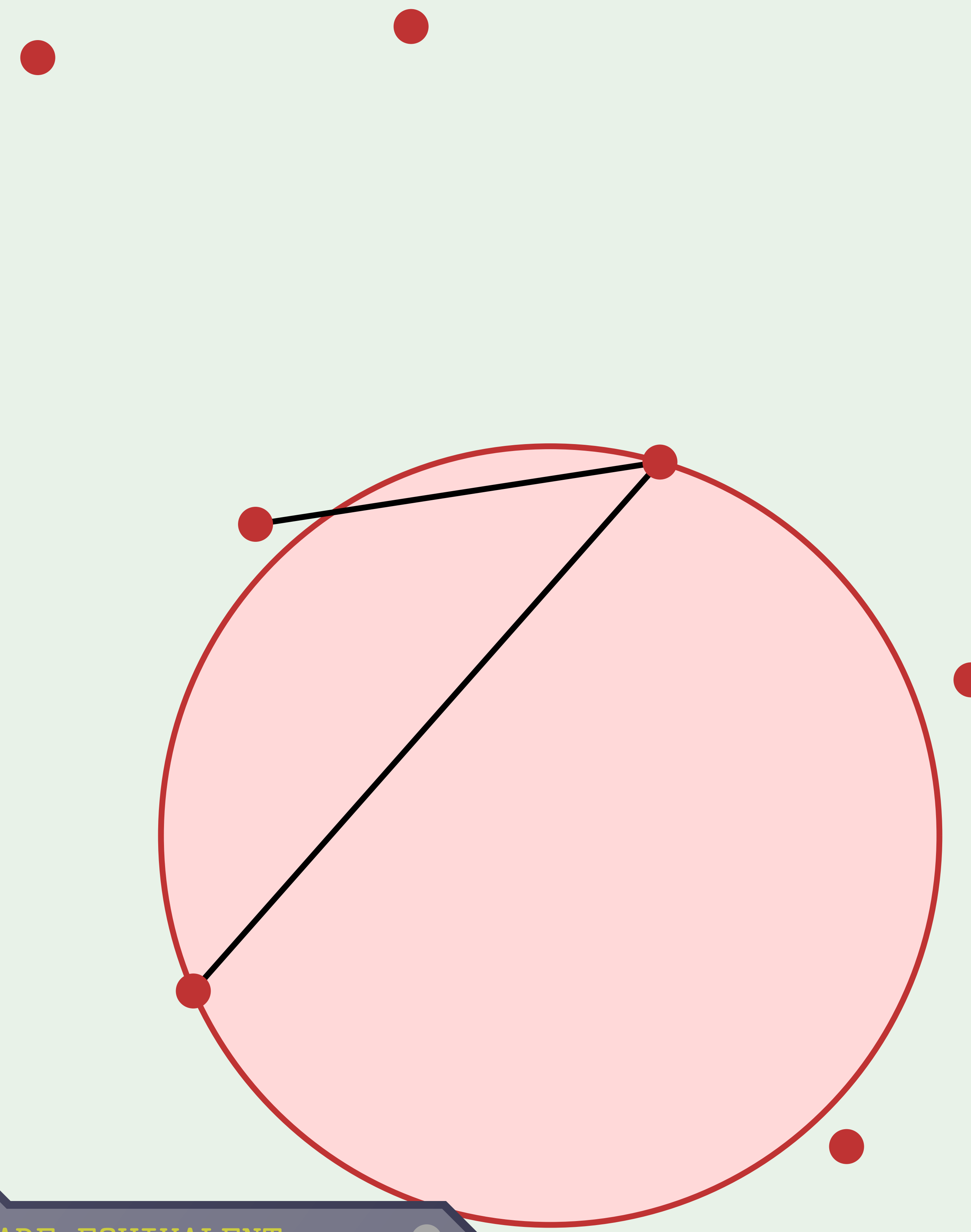

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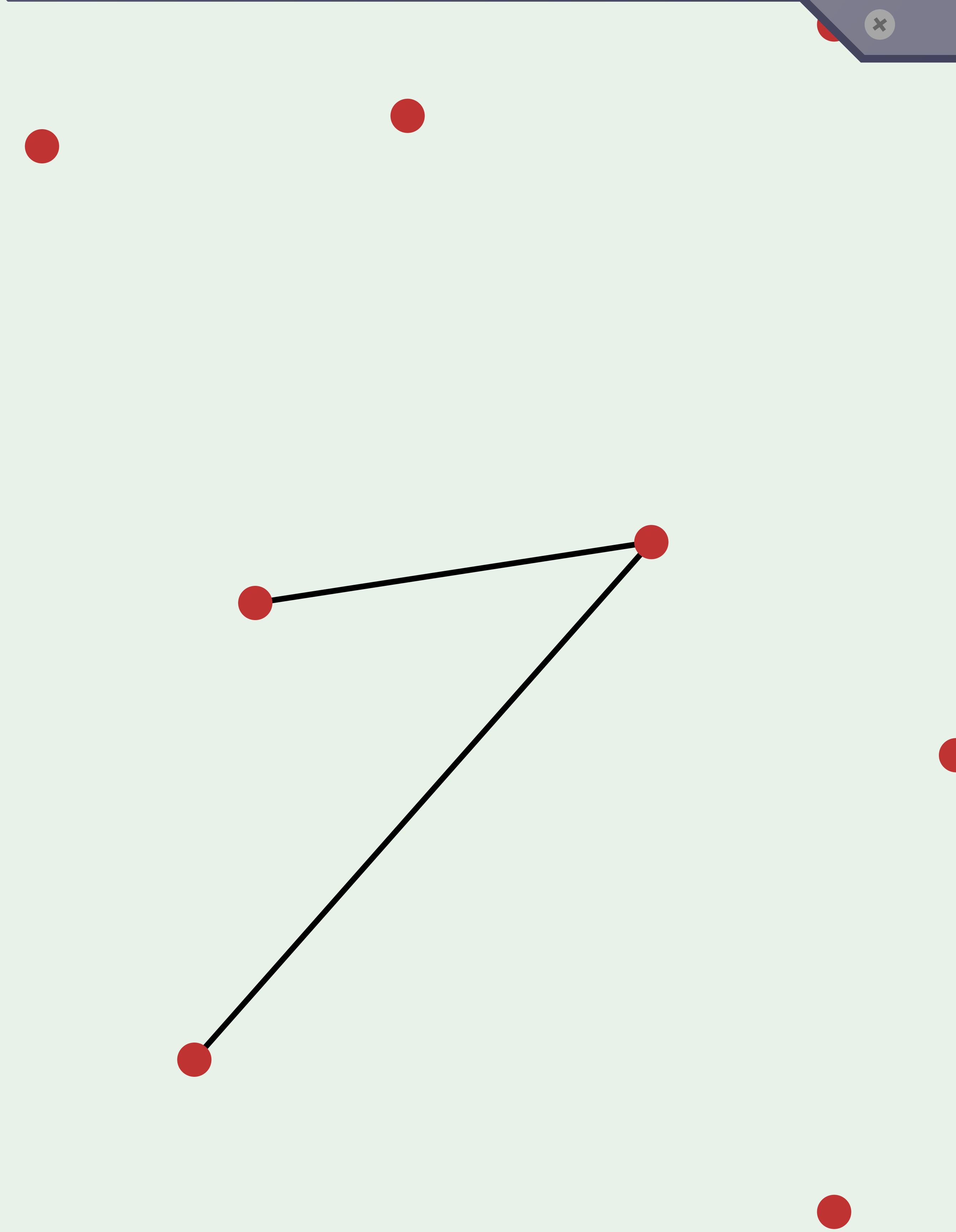

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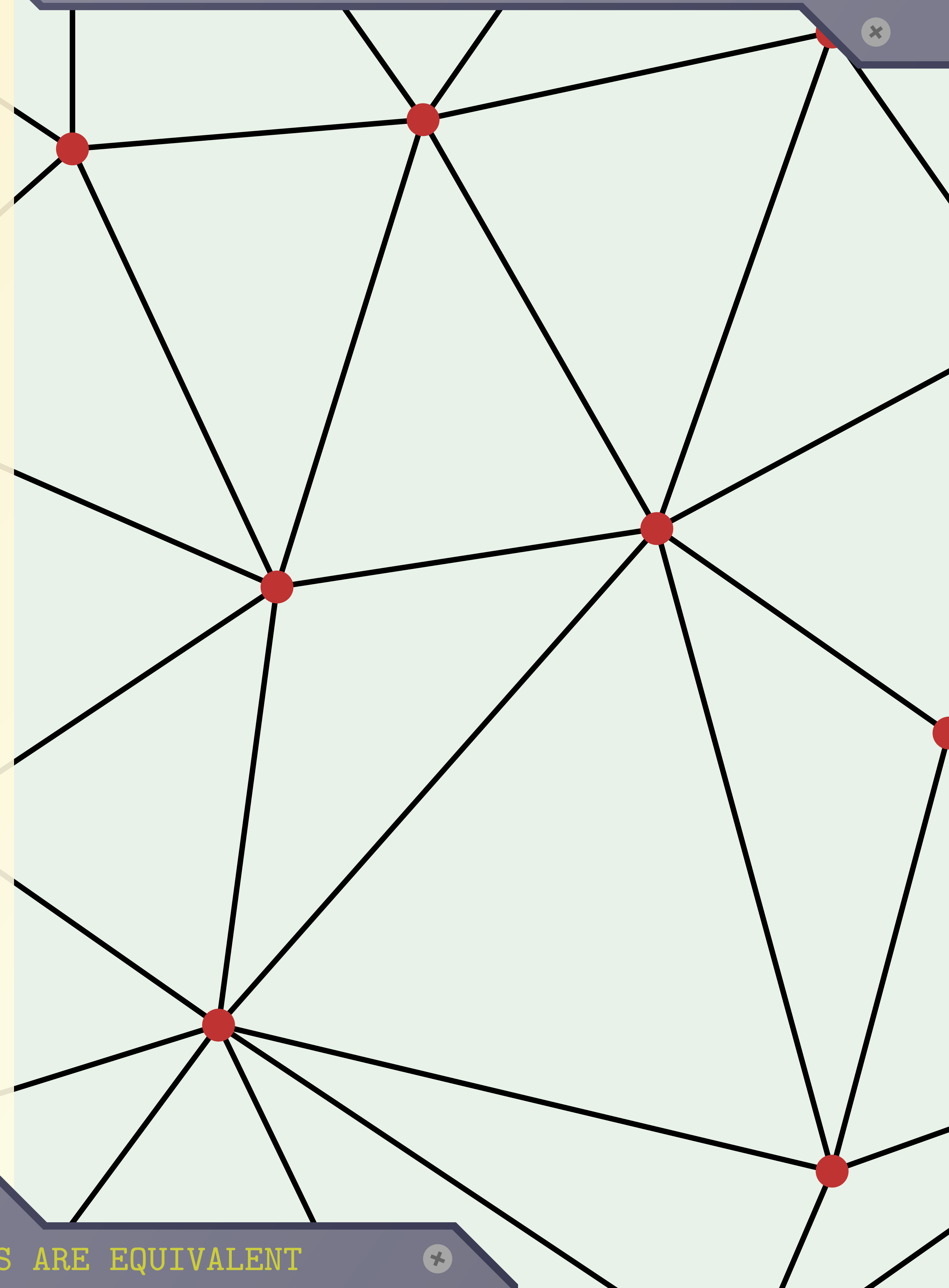

MINIMUM SPANNING TREE

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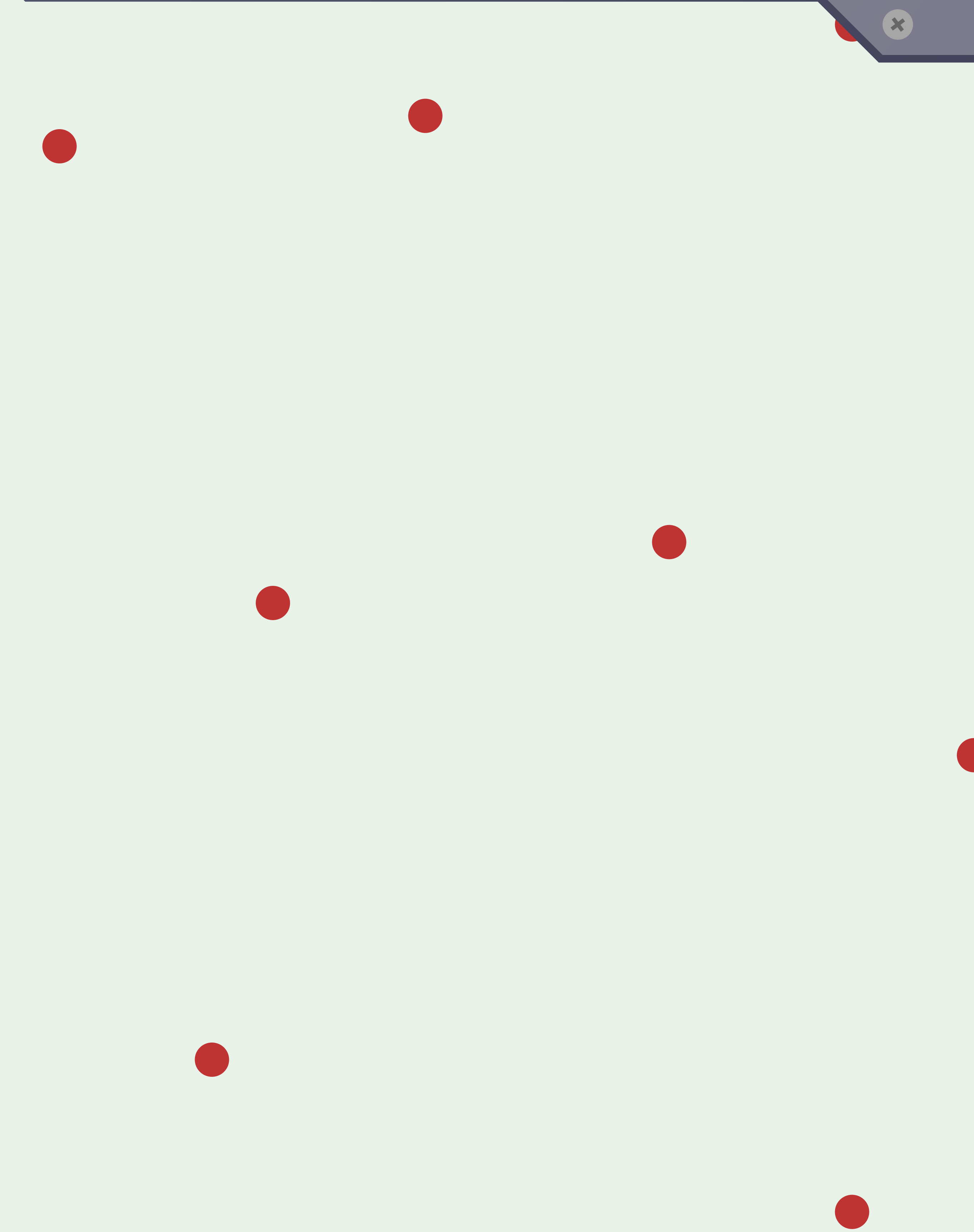

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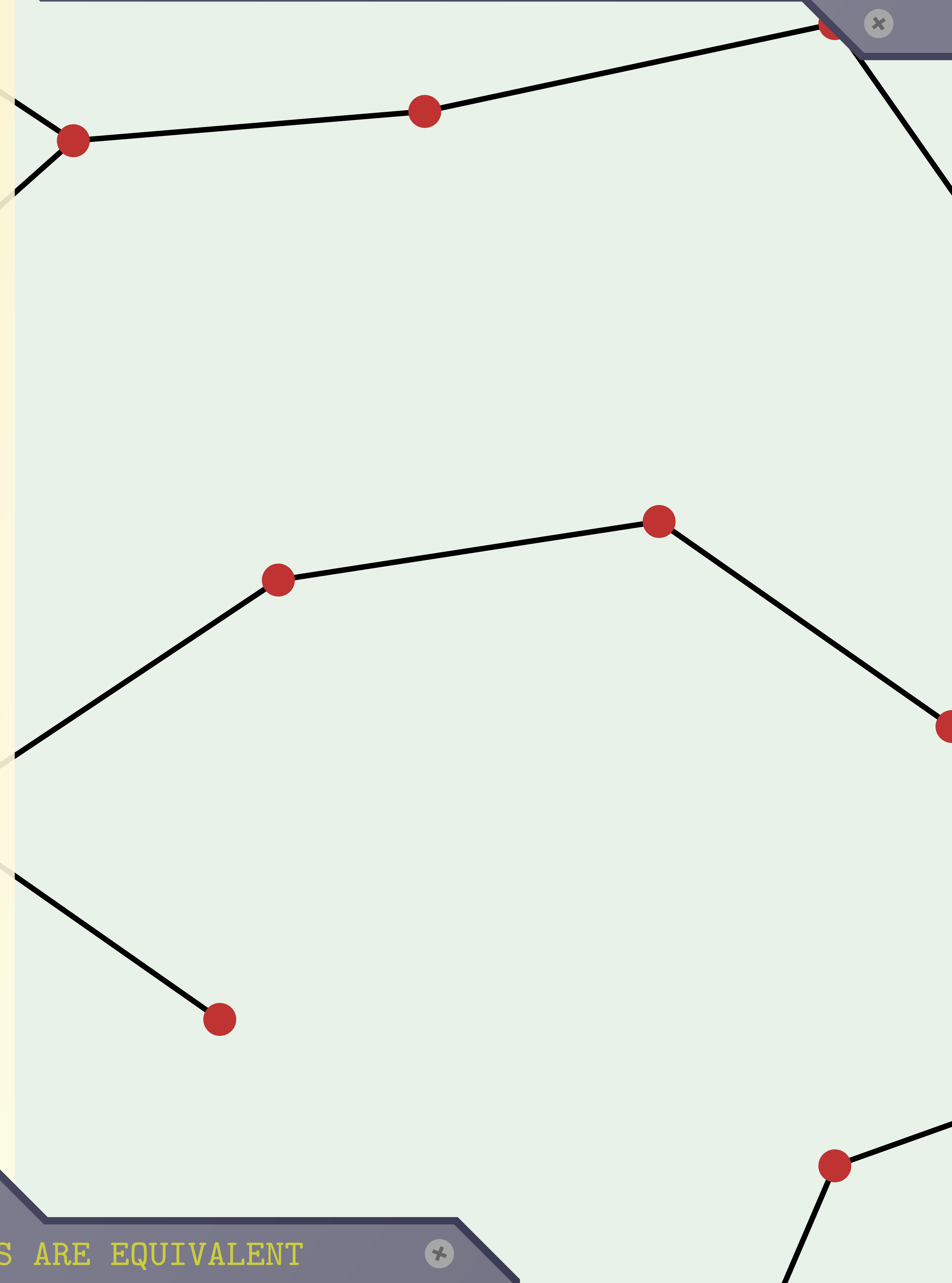

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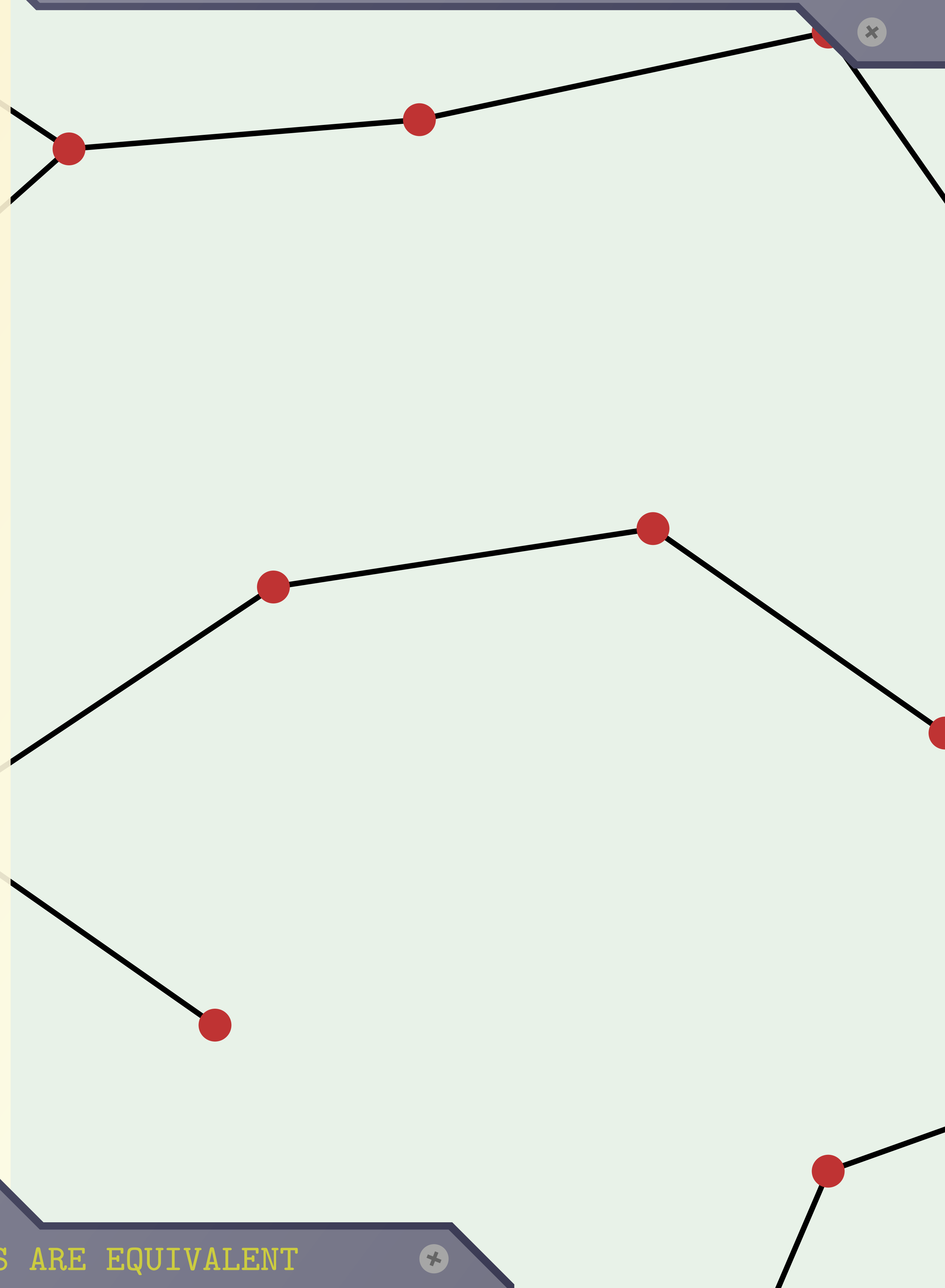
The EMST is a subgraph of the DT.



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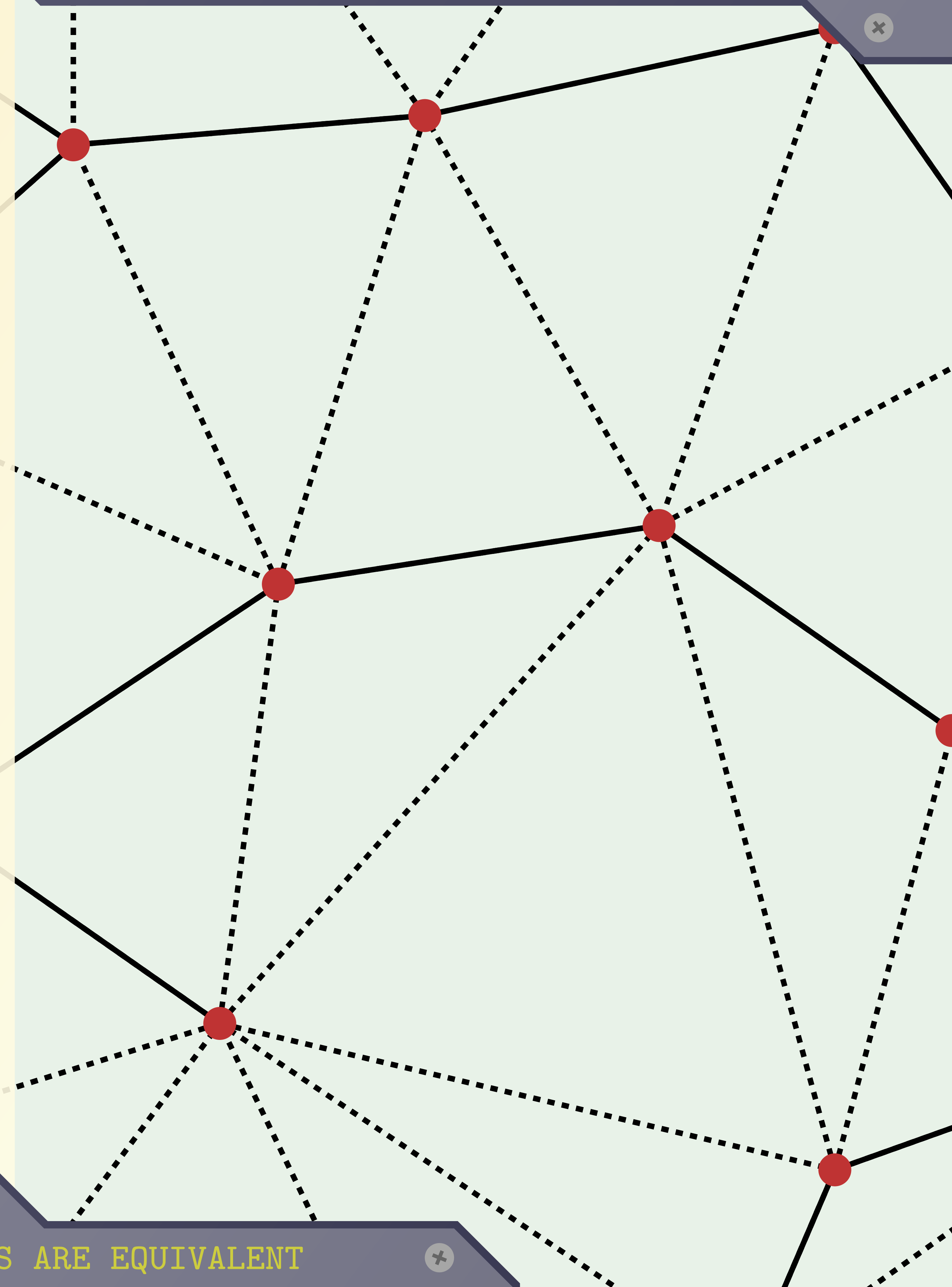

COMPRESSED QUADTREE

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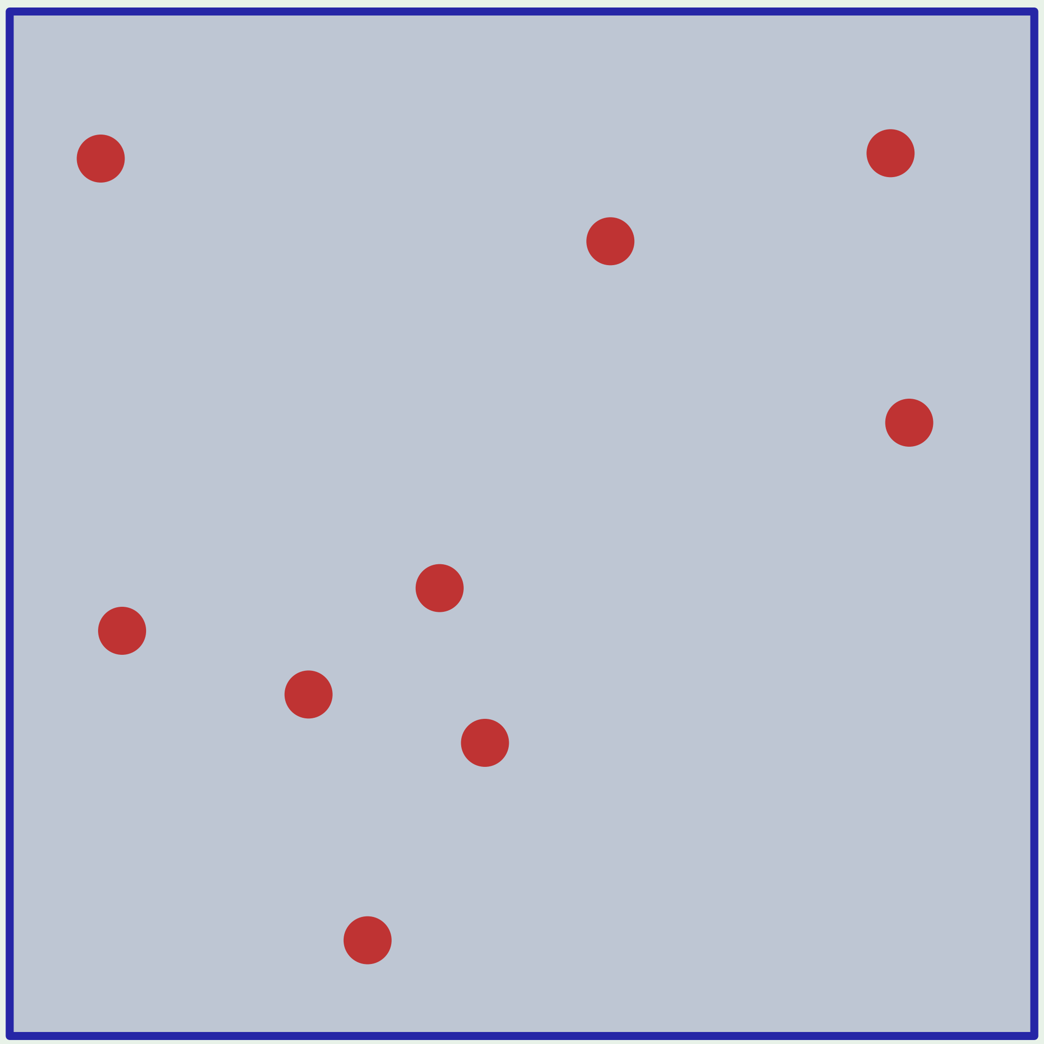

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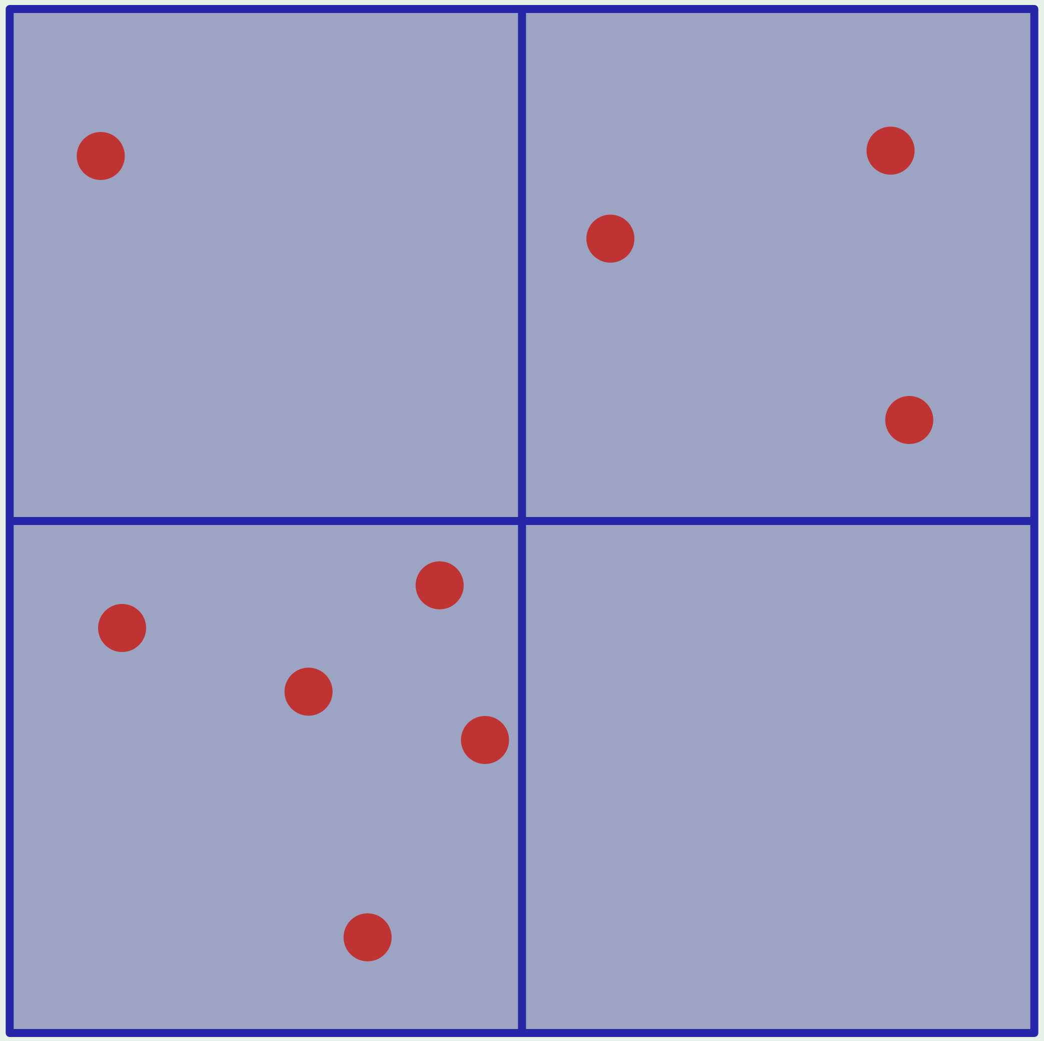

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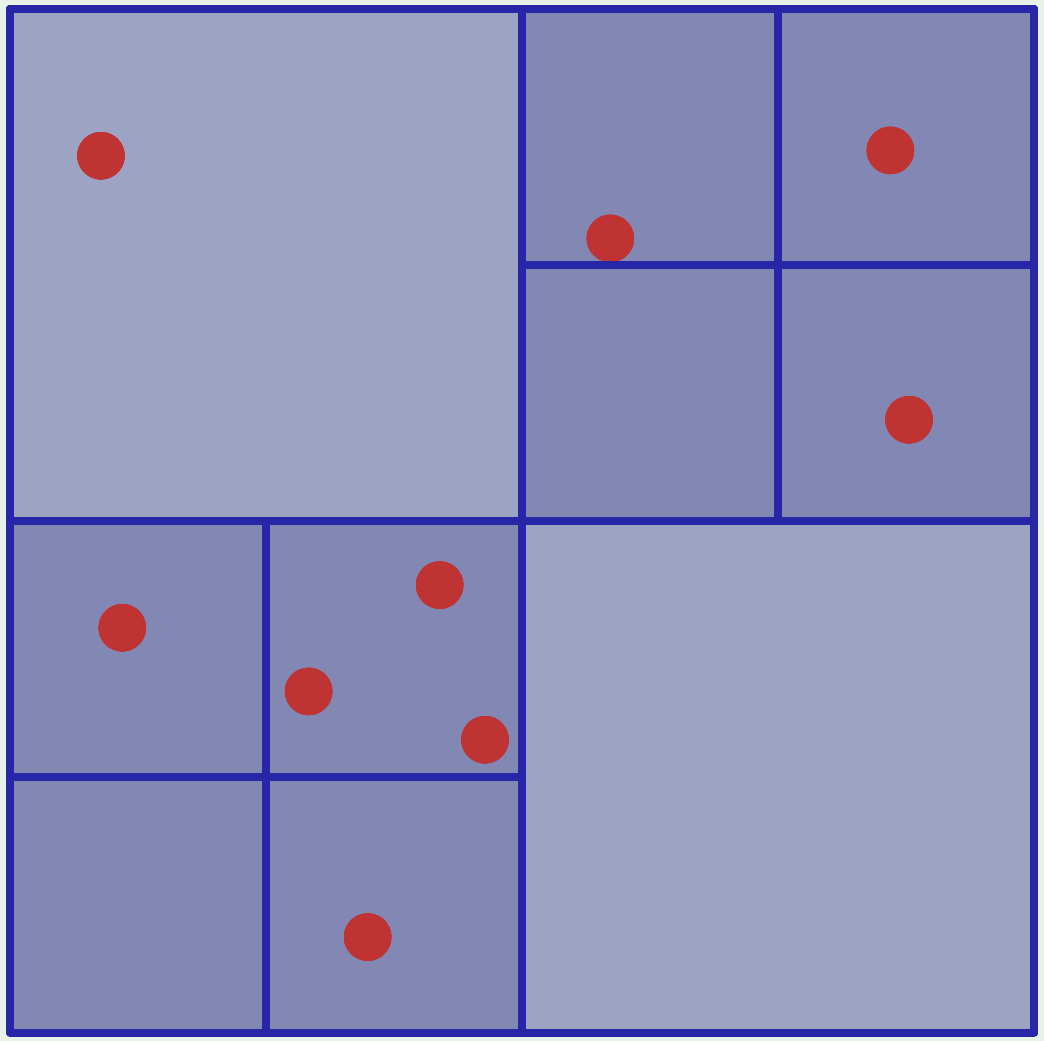

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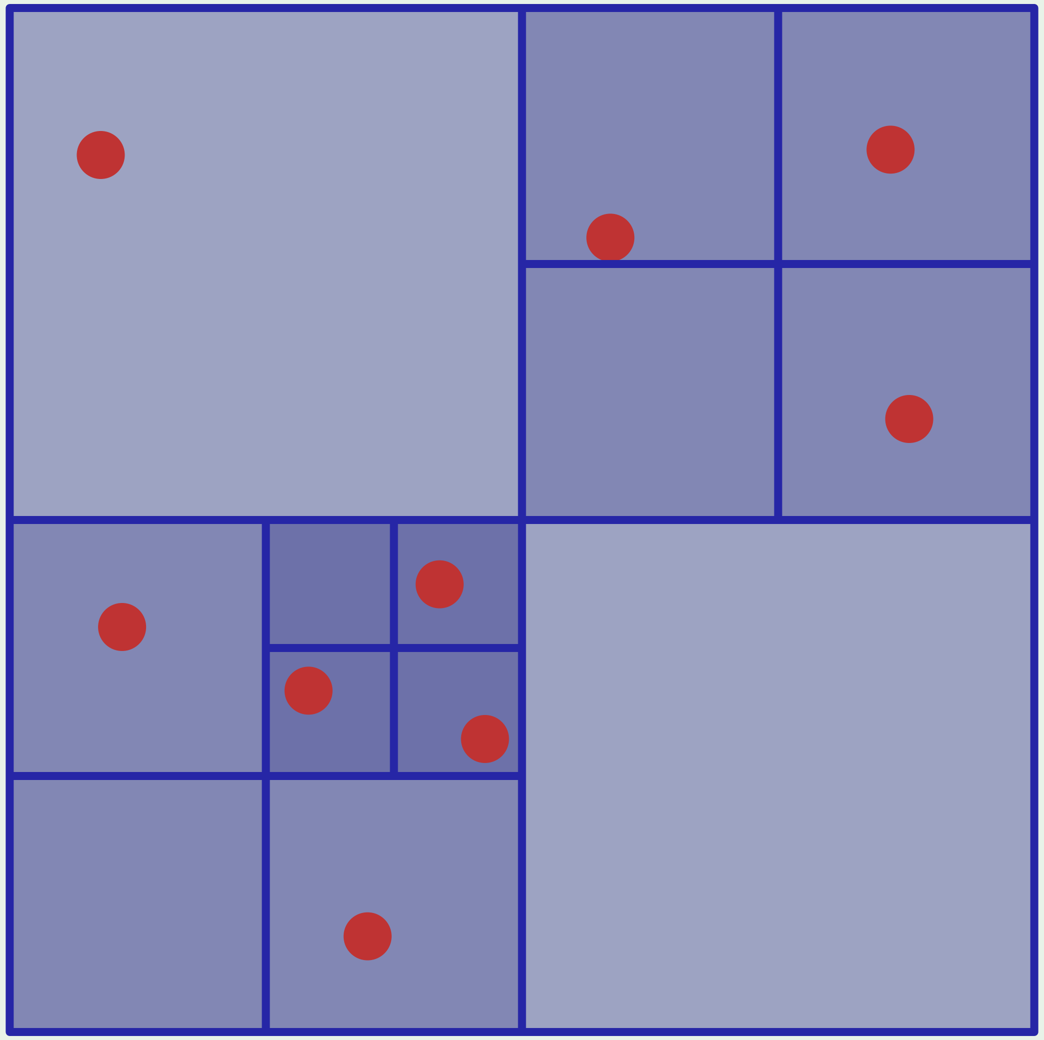

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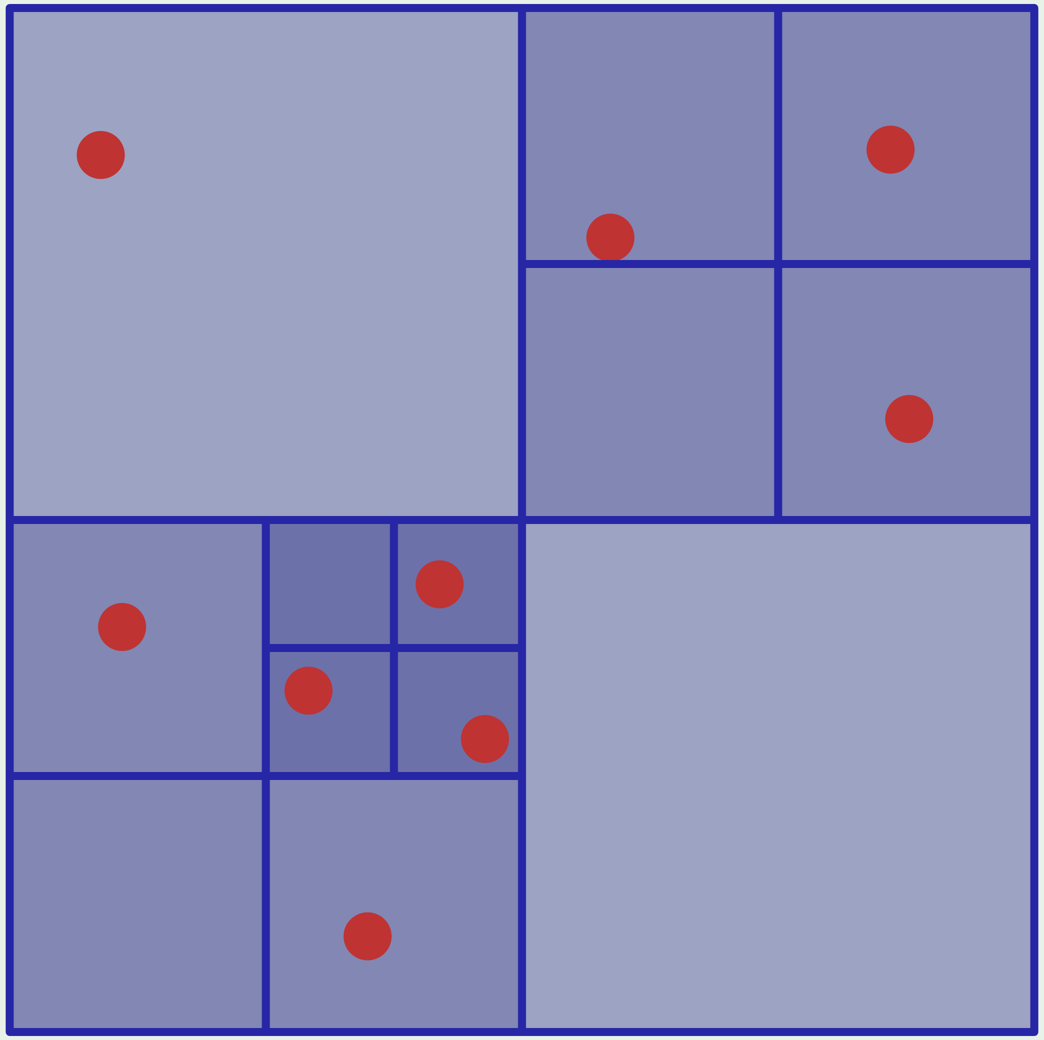
For point sets with large spread, a quadtree can be *compressed*.



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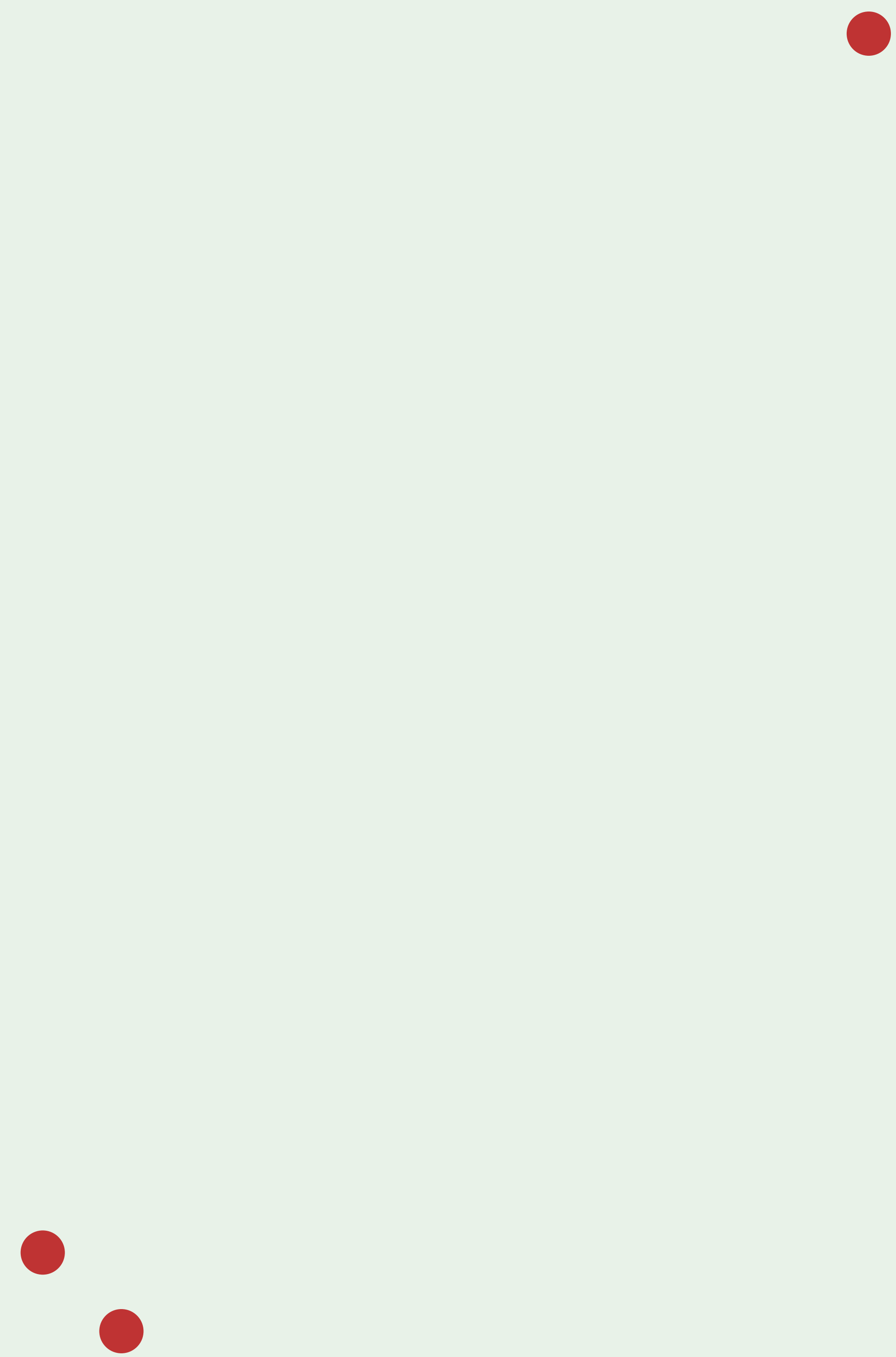

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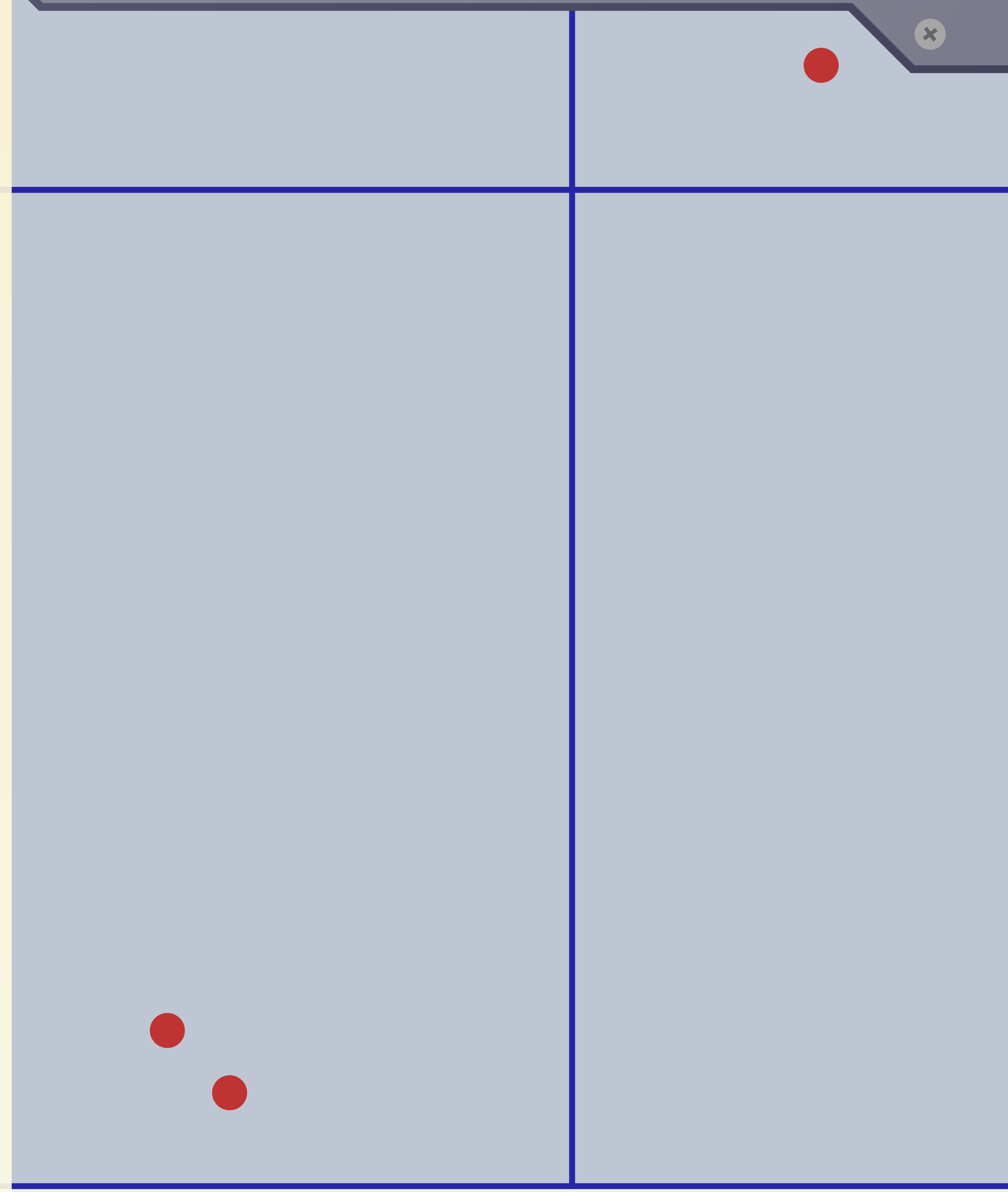

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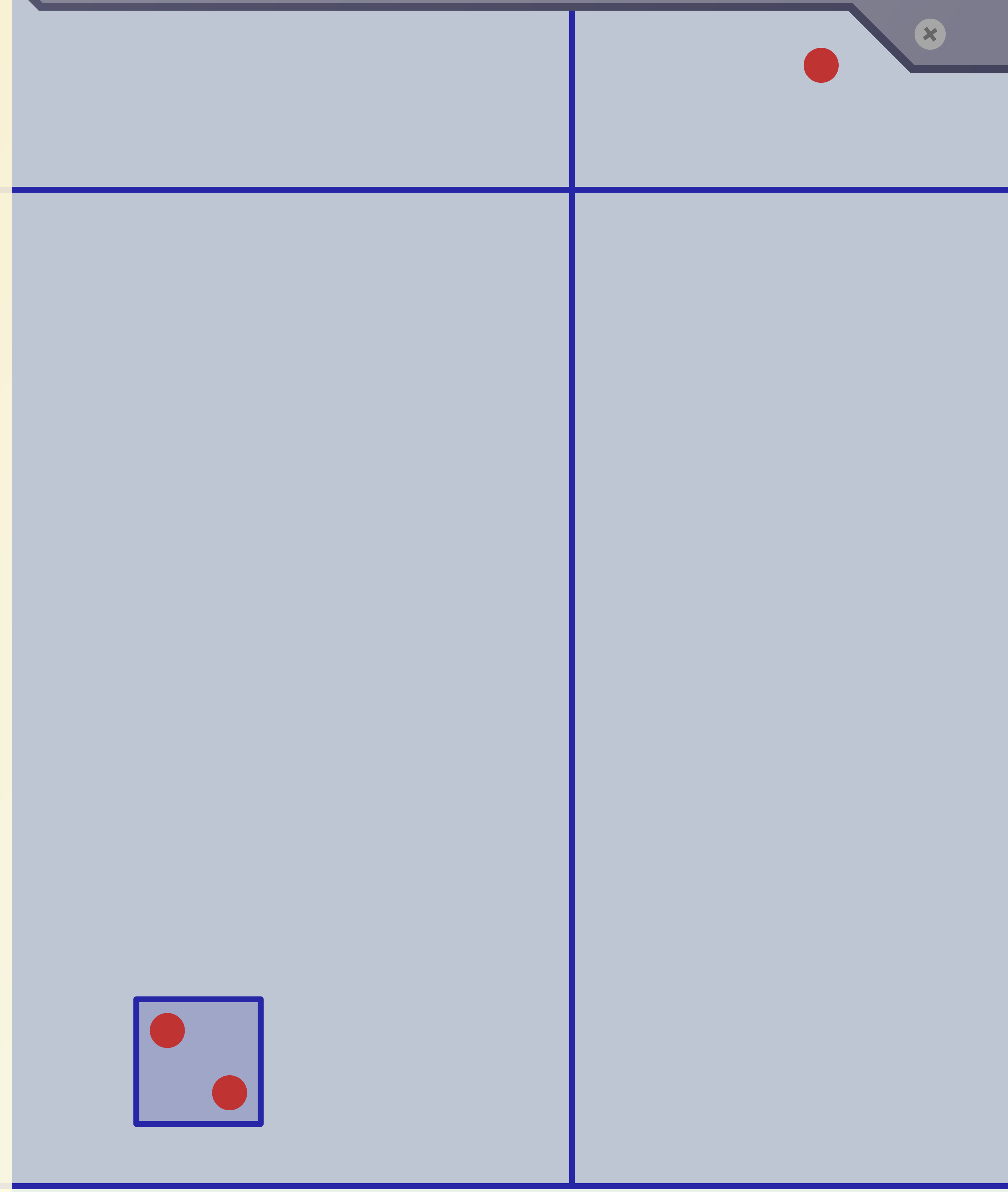

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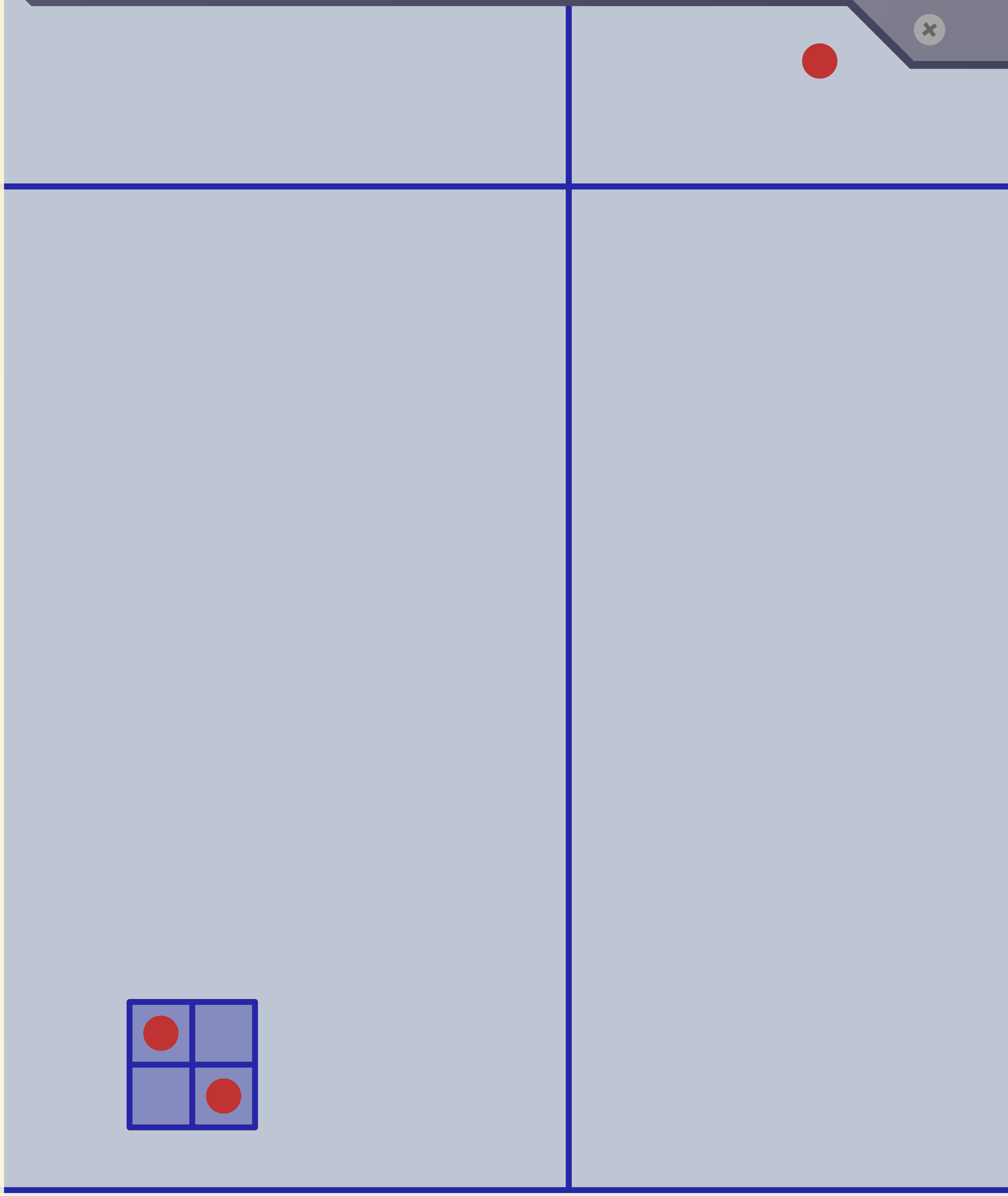

c-CLUSTER QUADTREE

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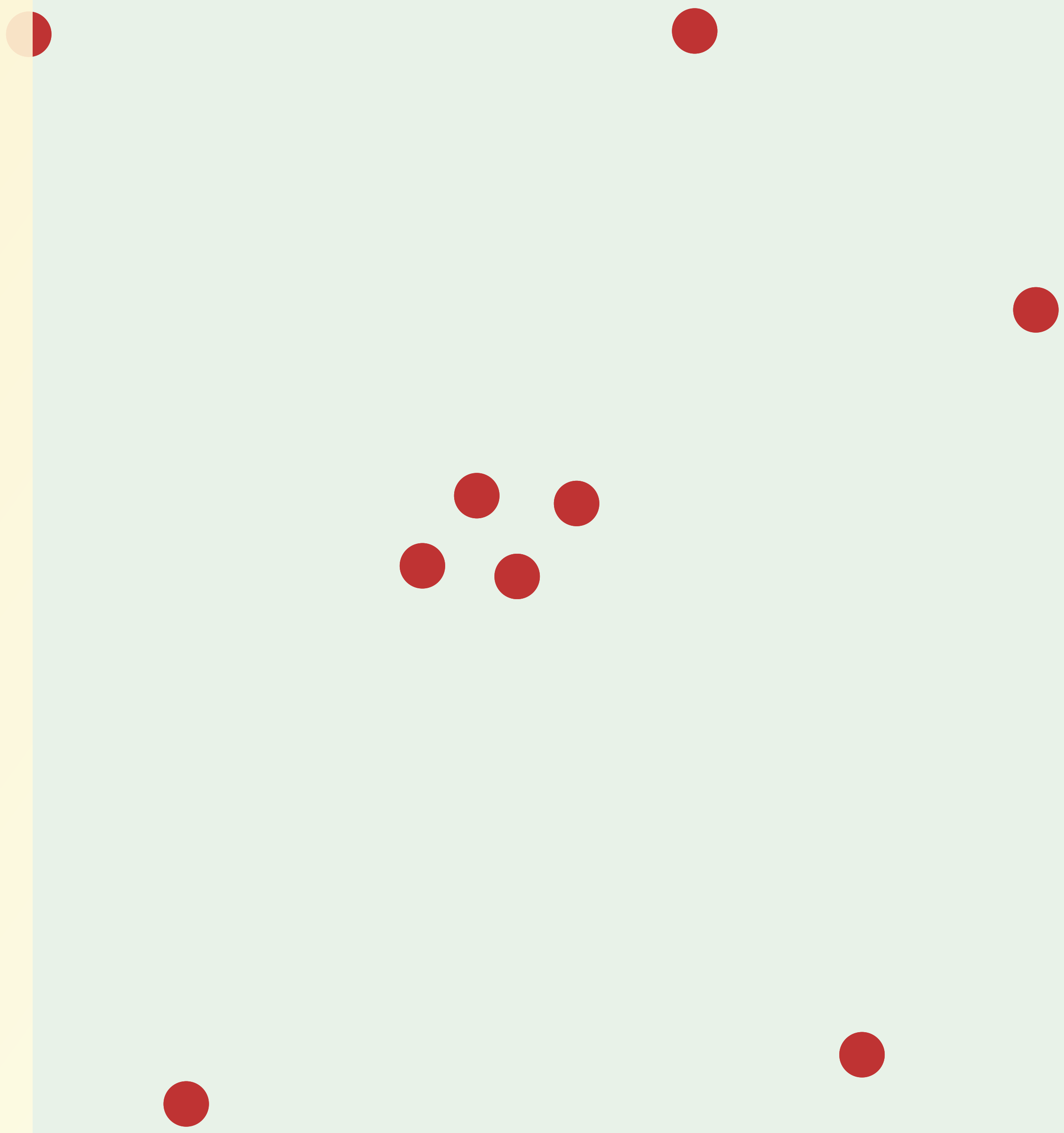

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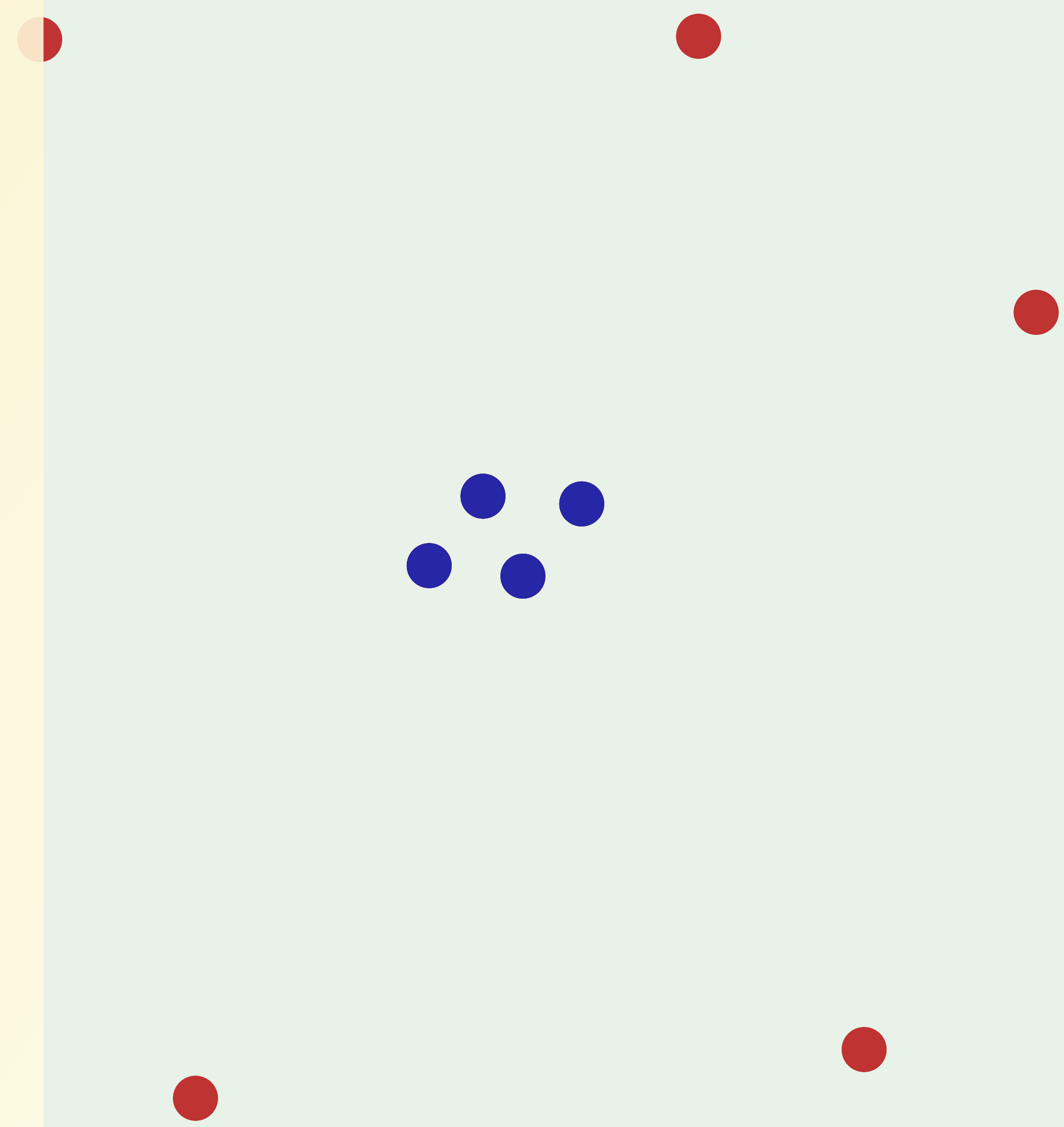

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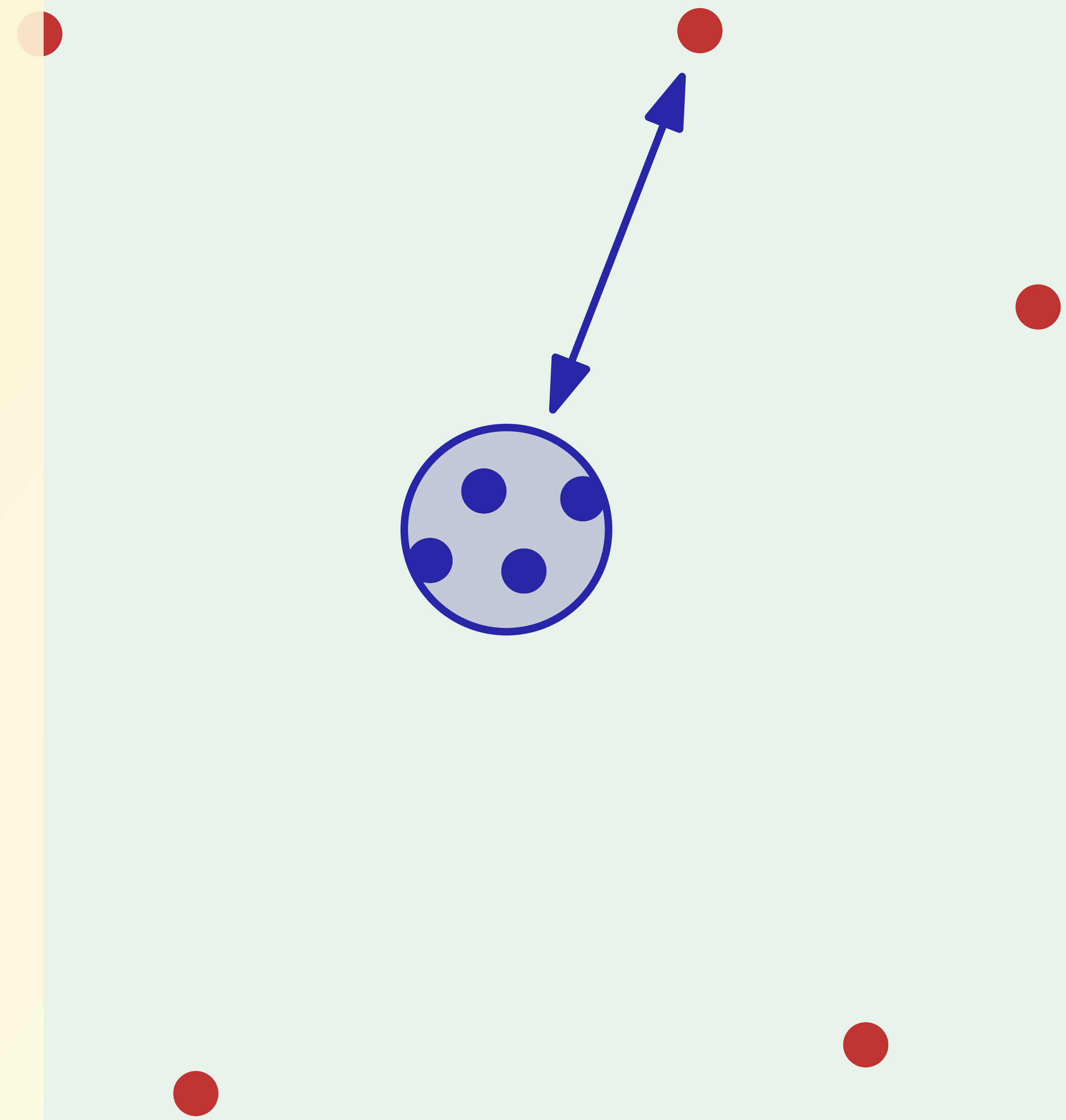

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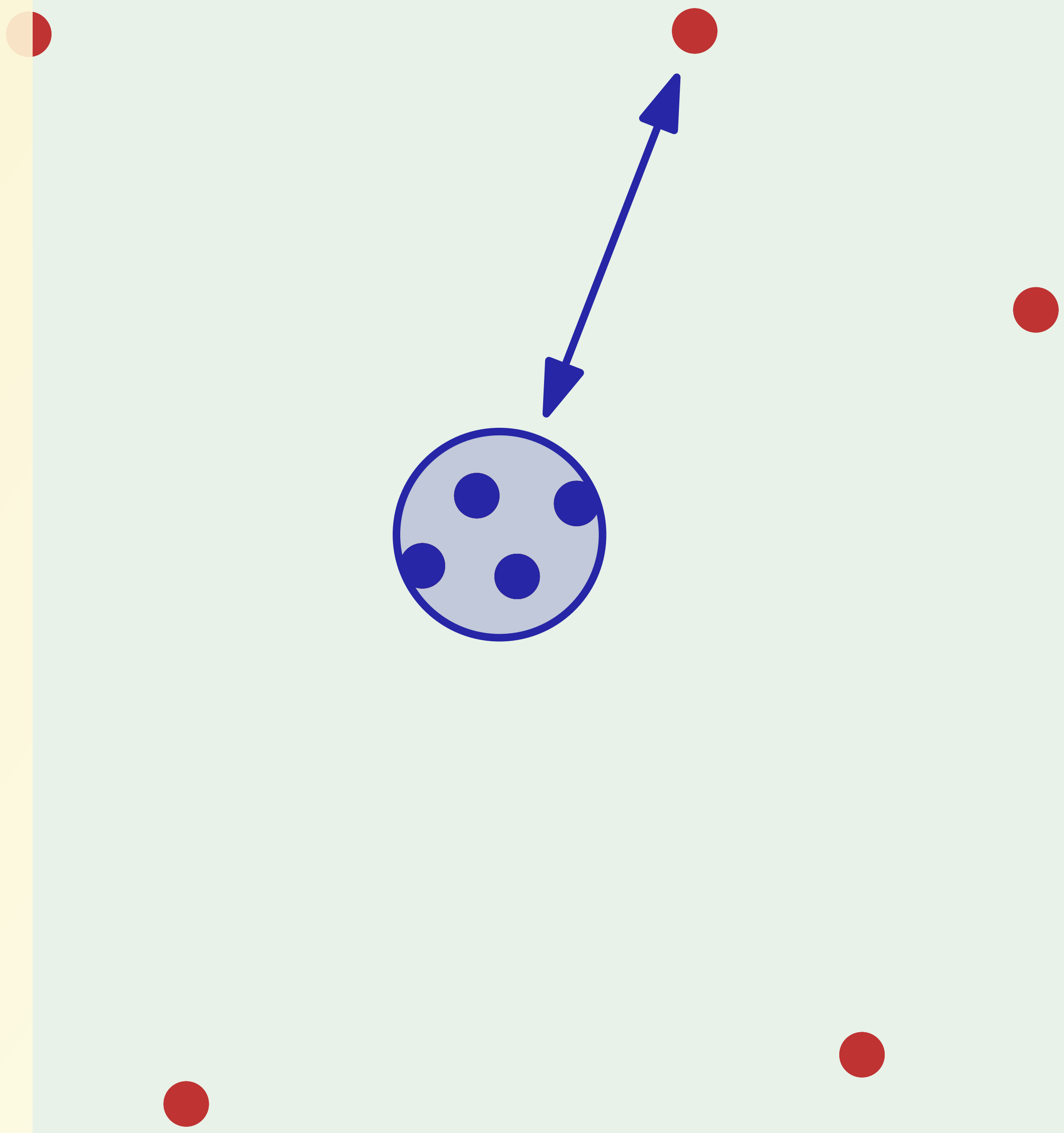
The c-clusters on P form a hierarchy.



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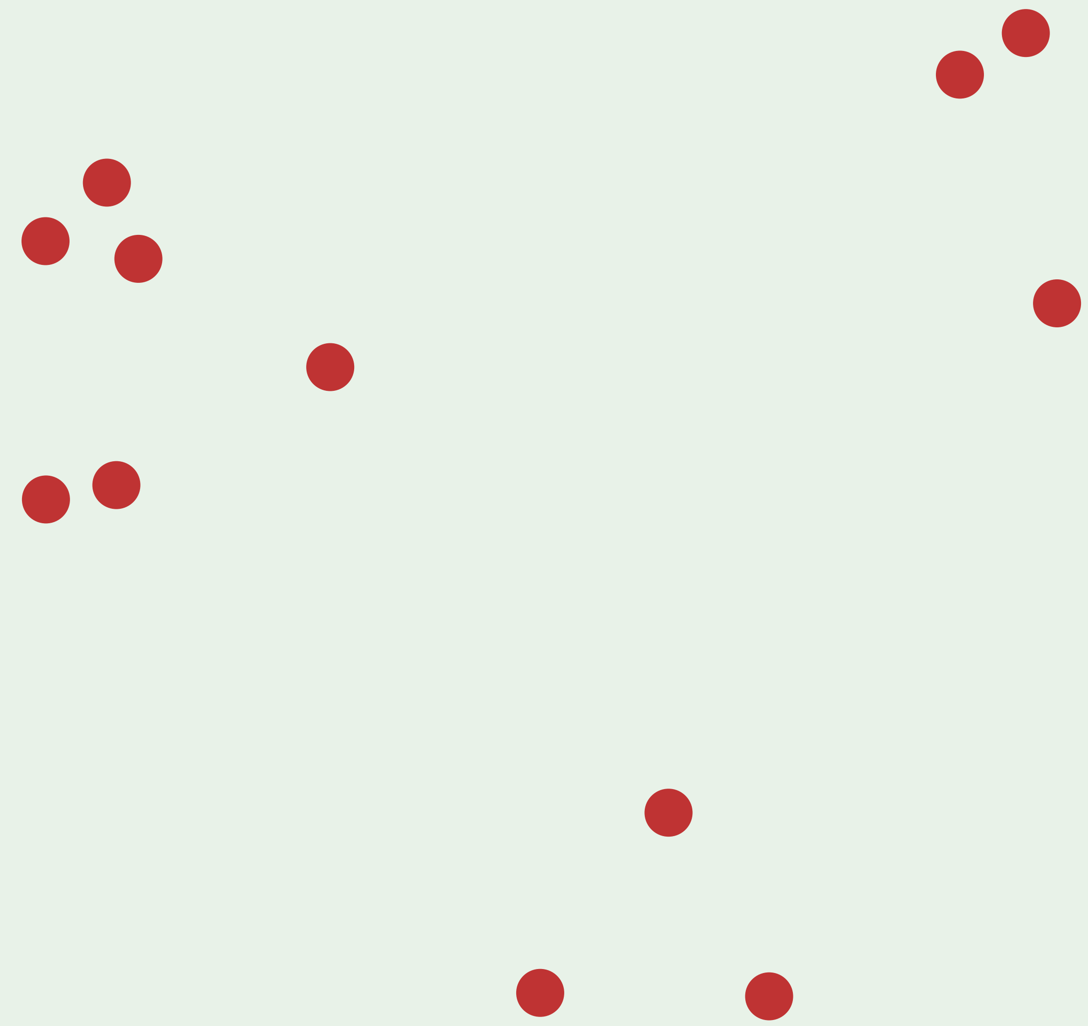

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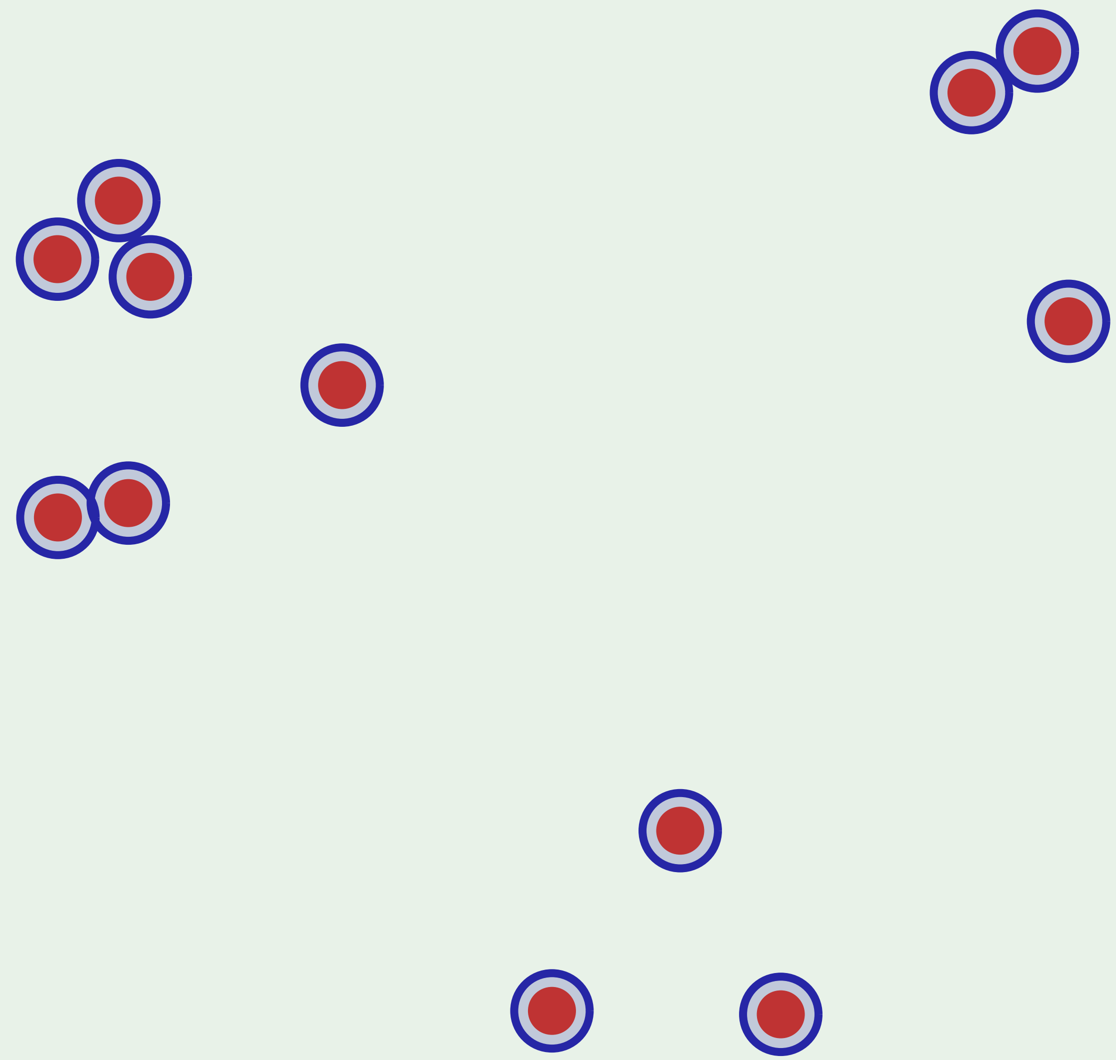

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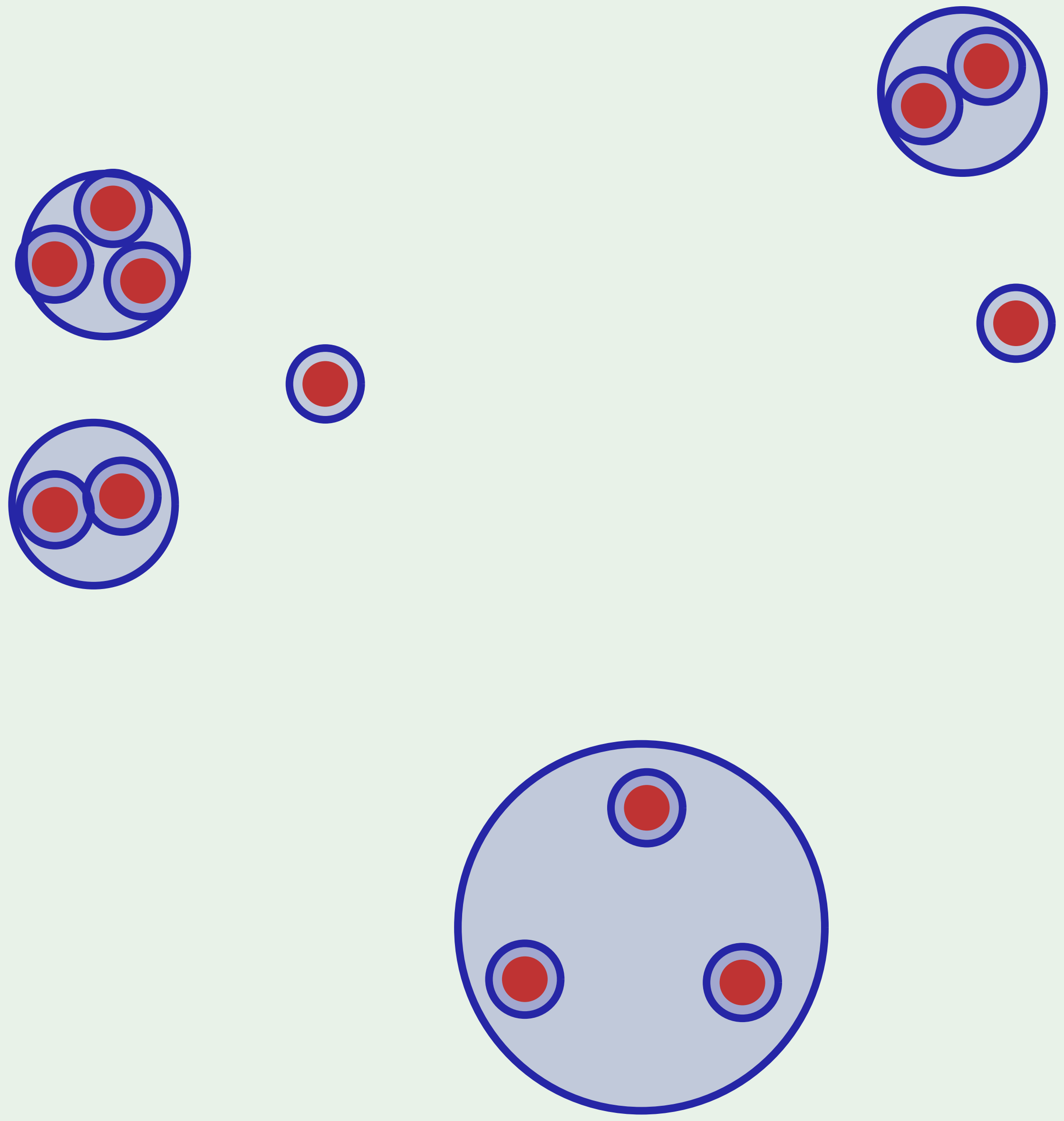

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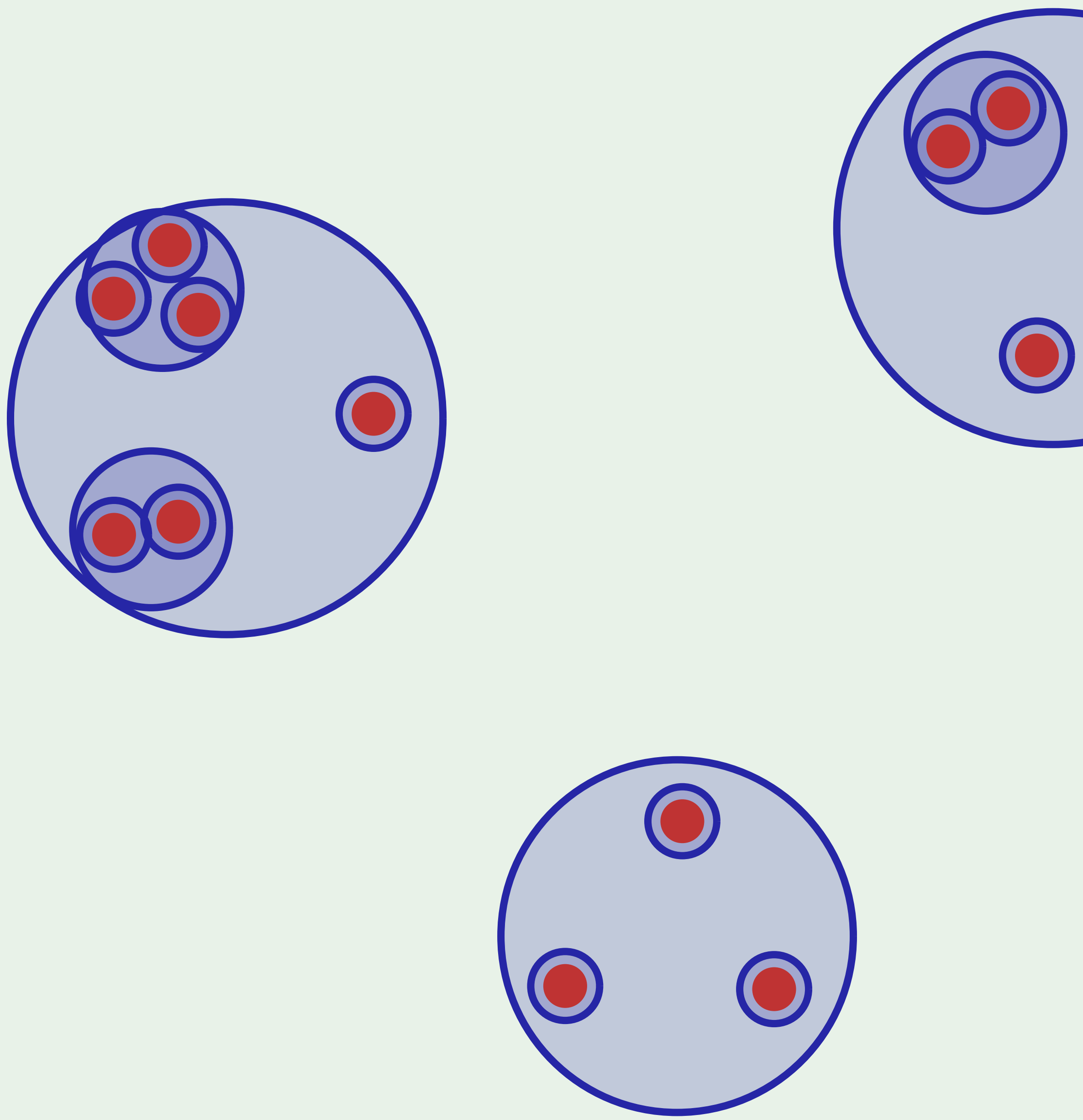

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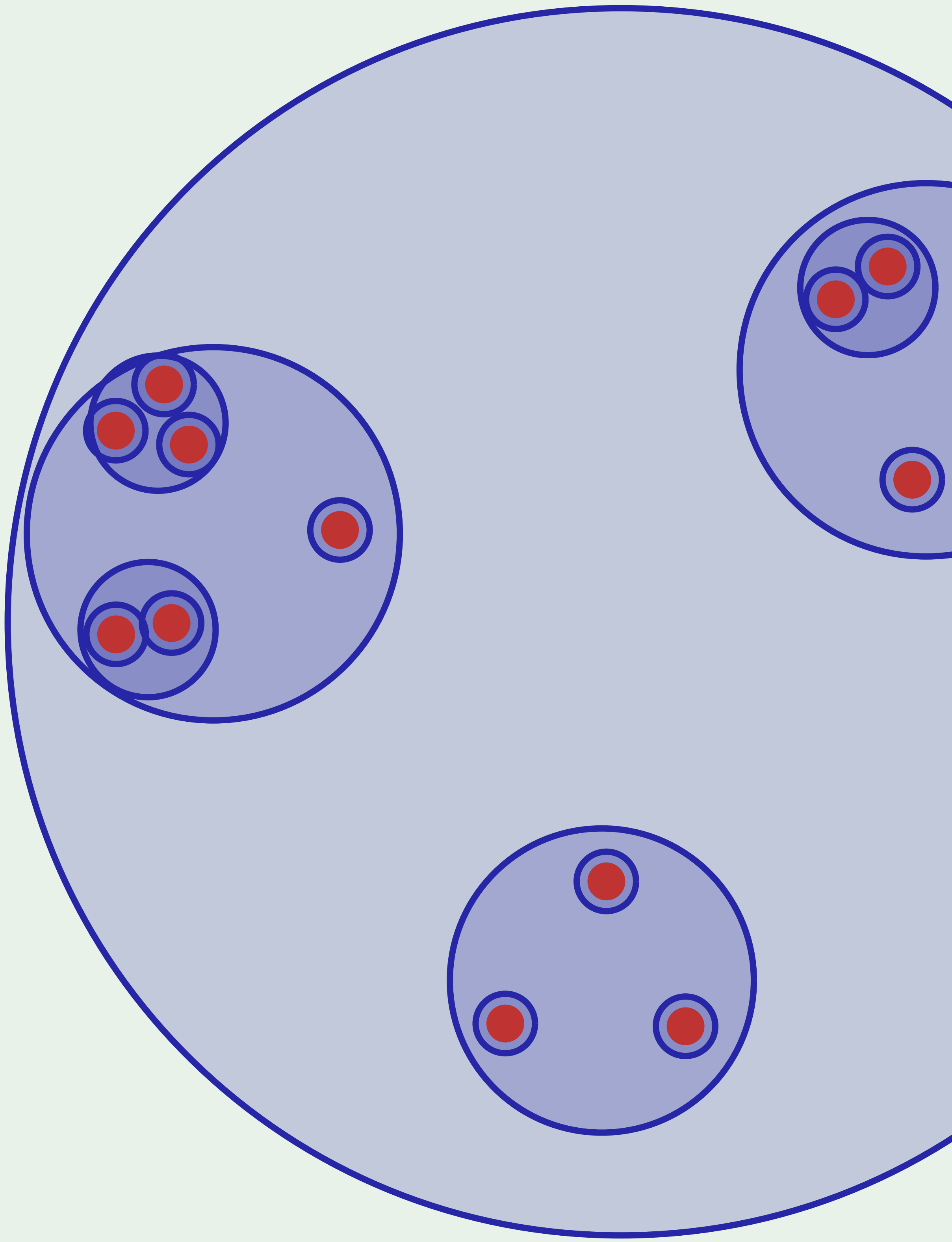

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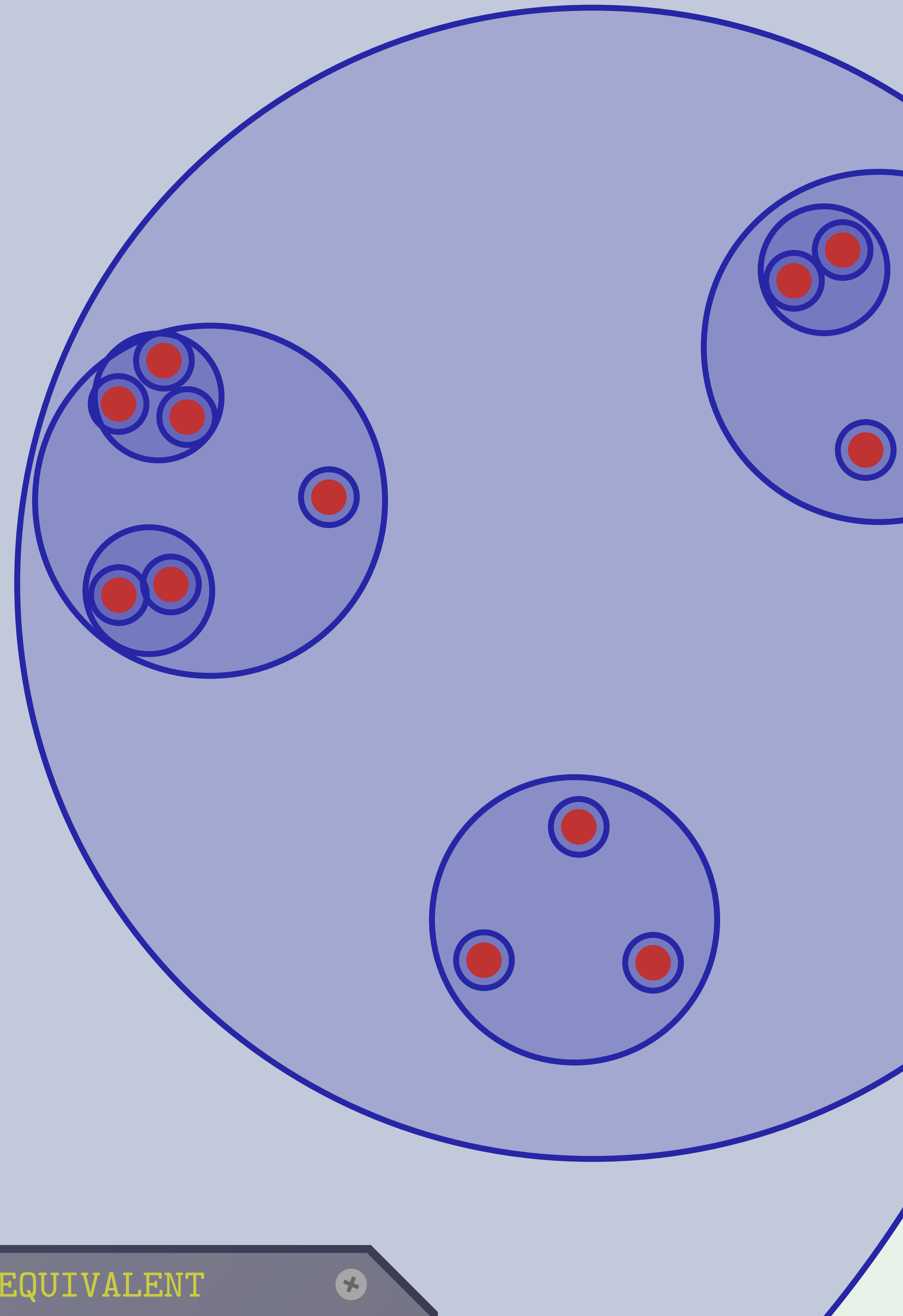

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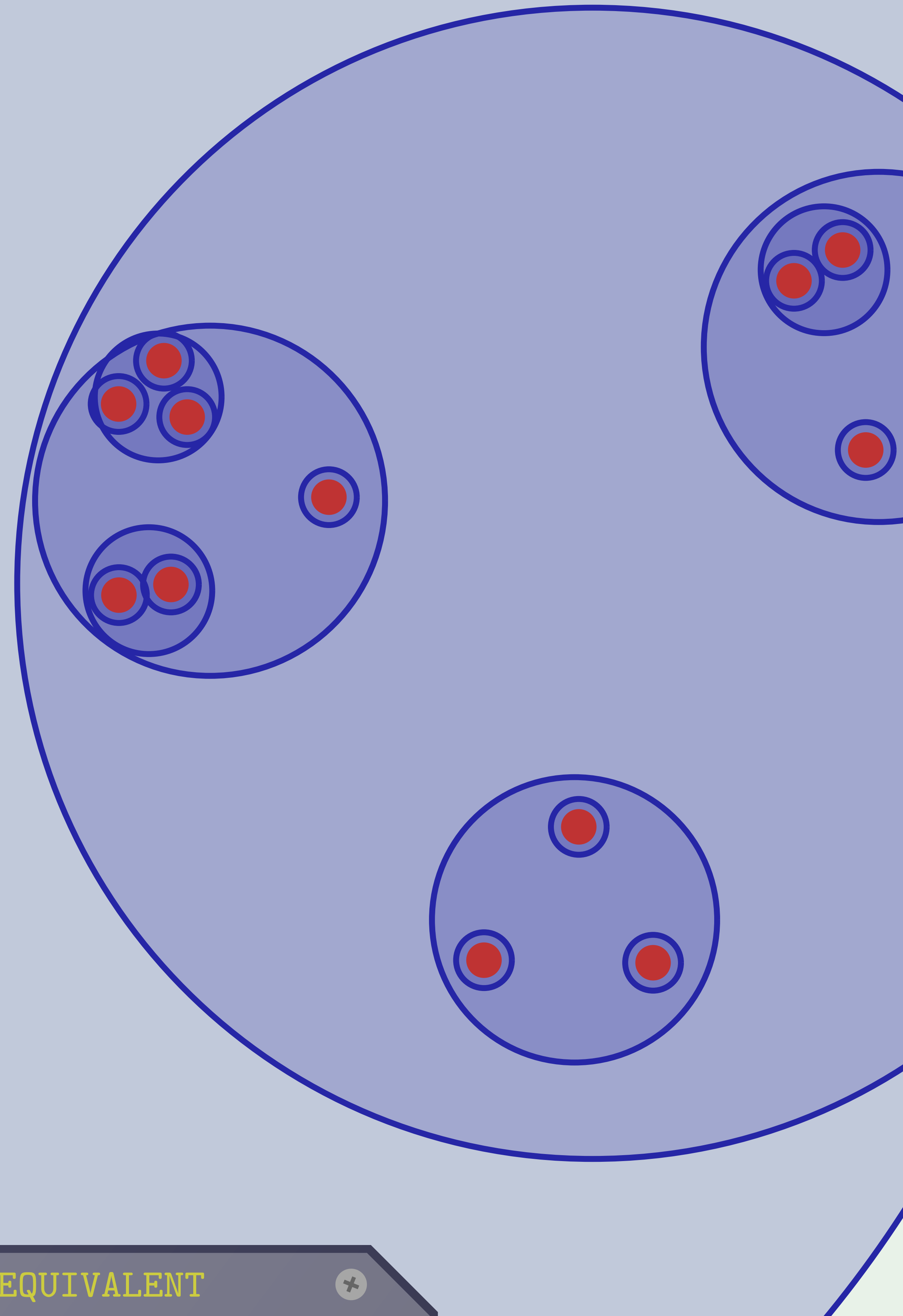
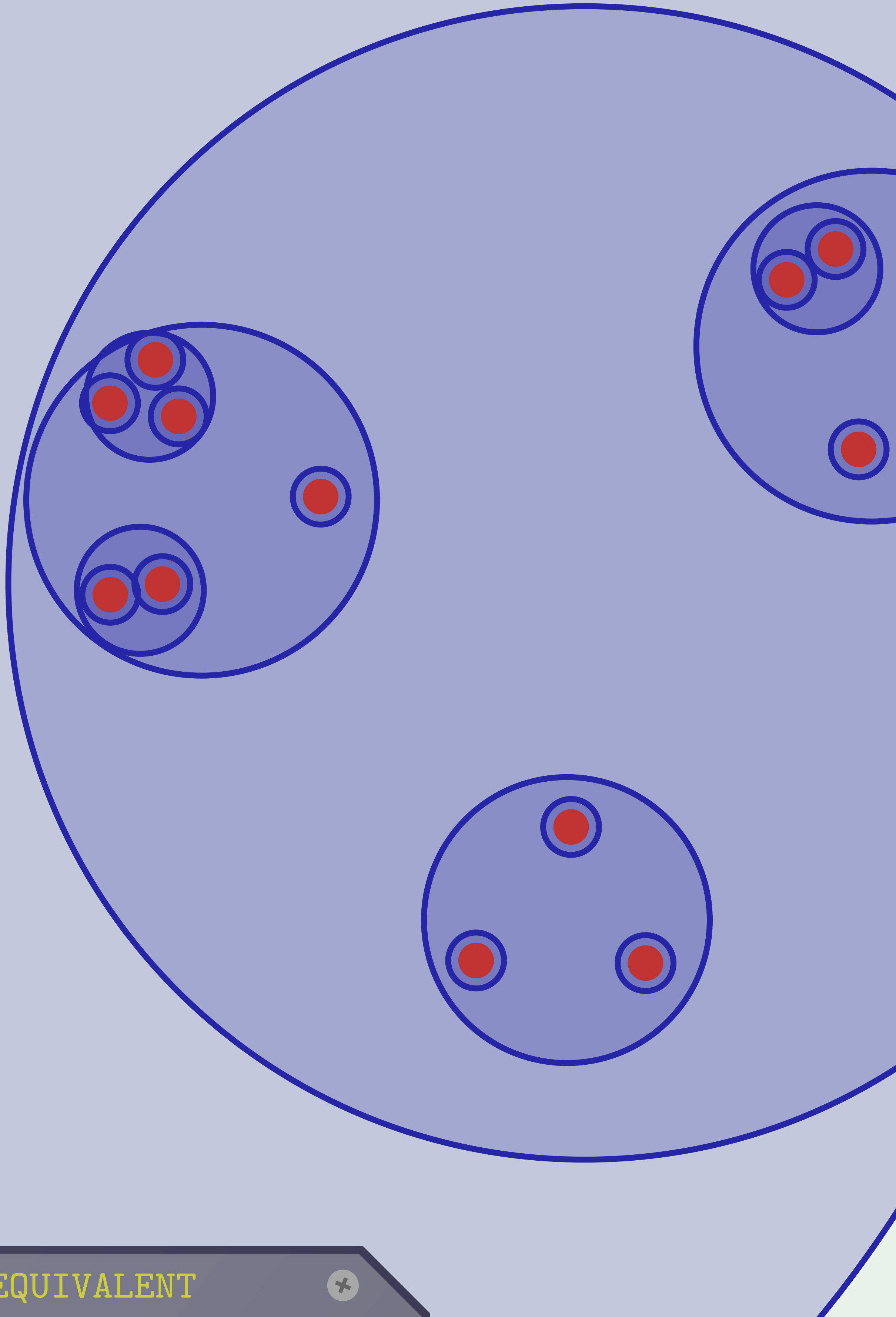
This c-cluster tree can have linear degree.



A *c-cluster quadtree* is a c-cluster tree augmented with a quadtree on its high-degree nodes.



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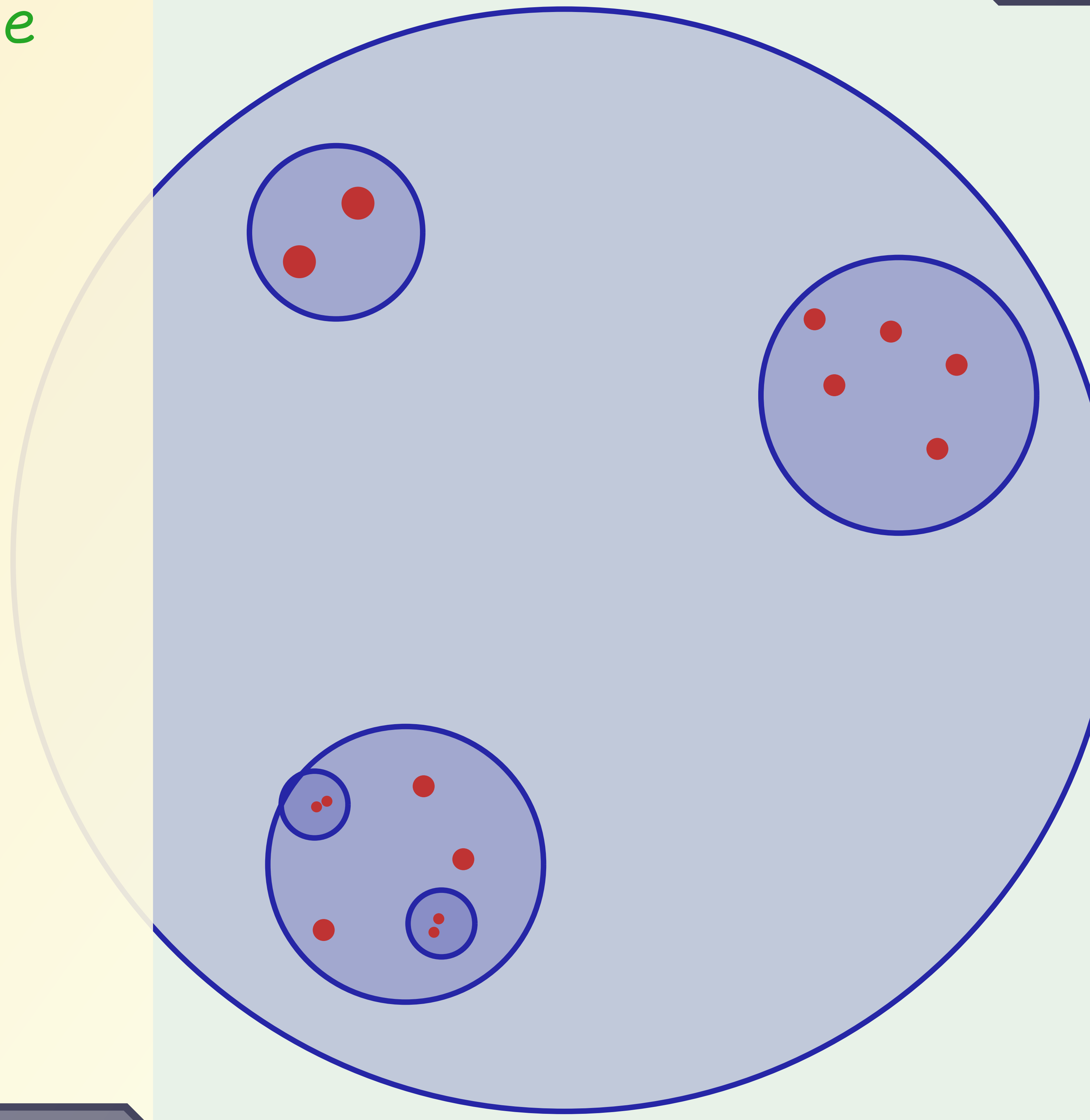

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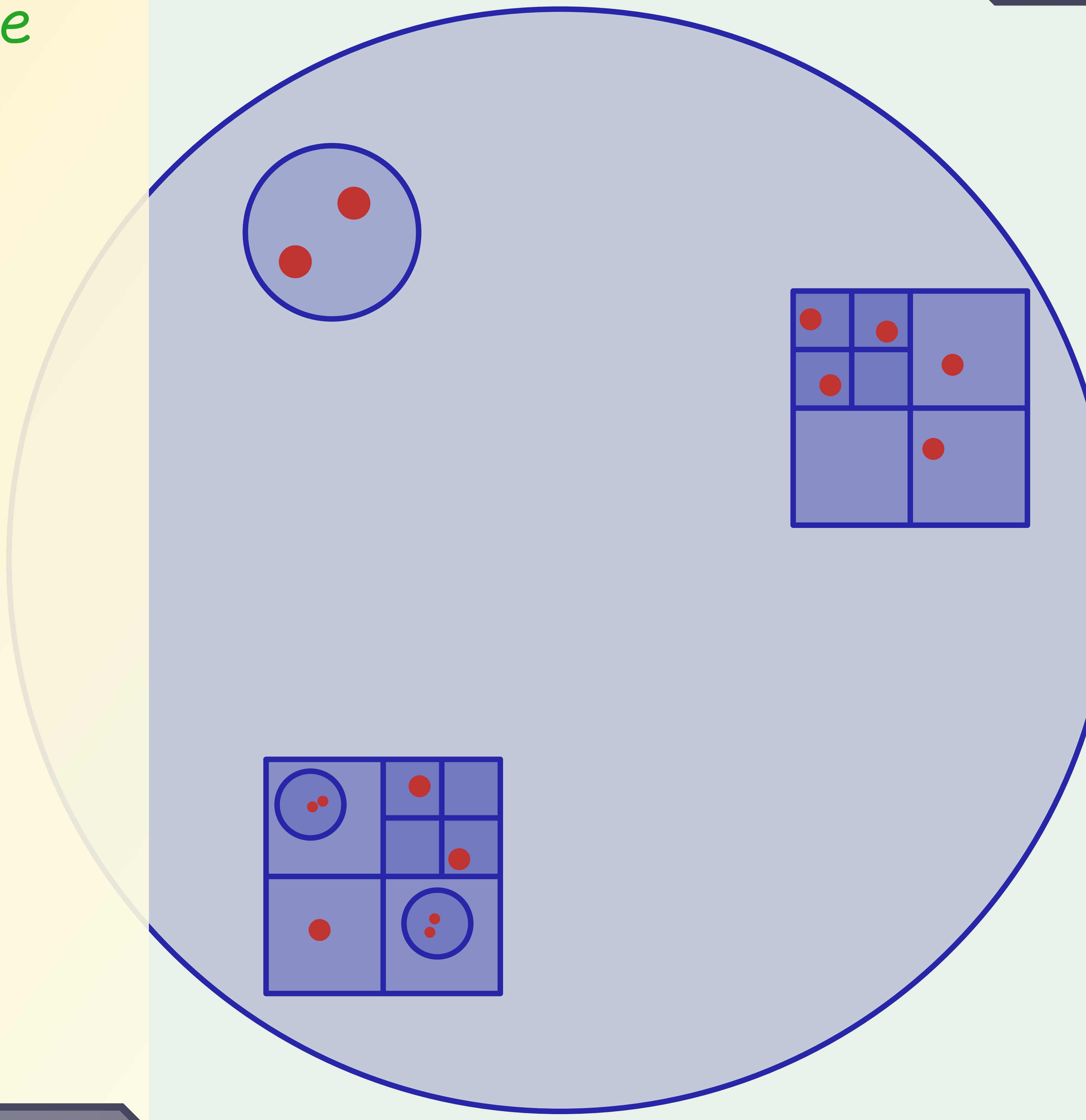

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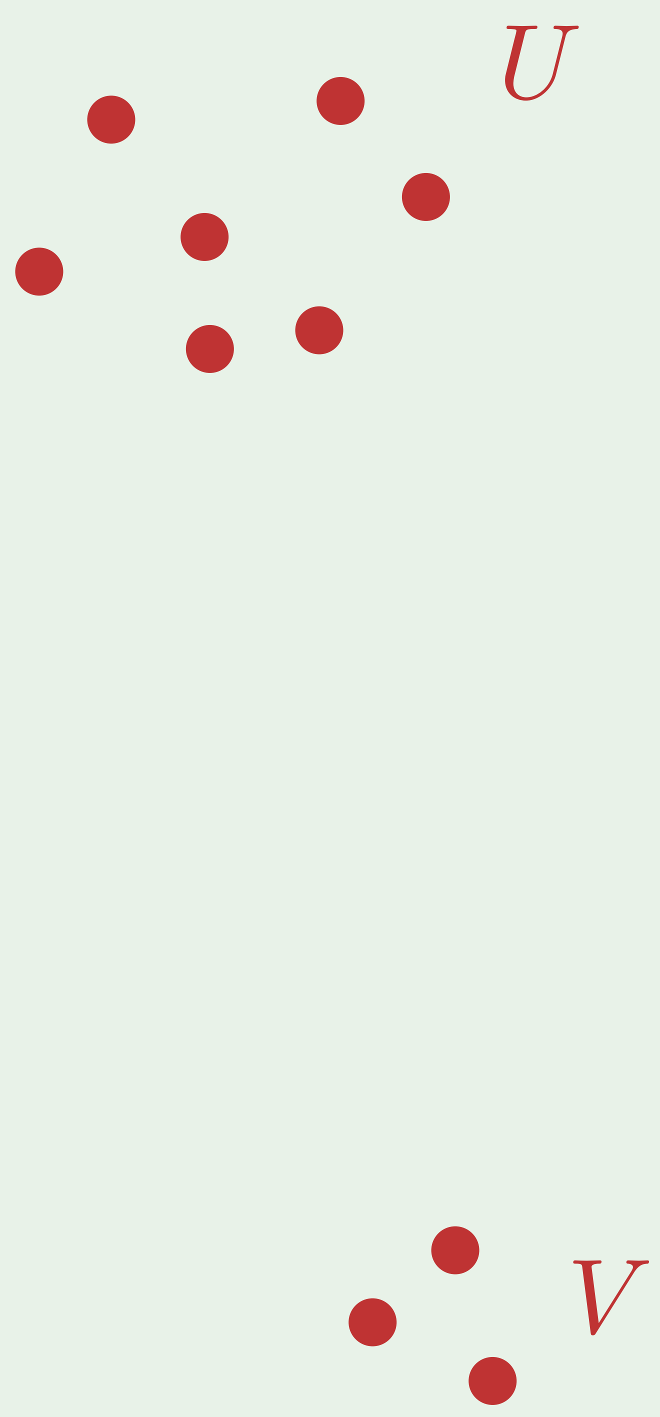

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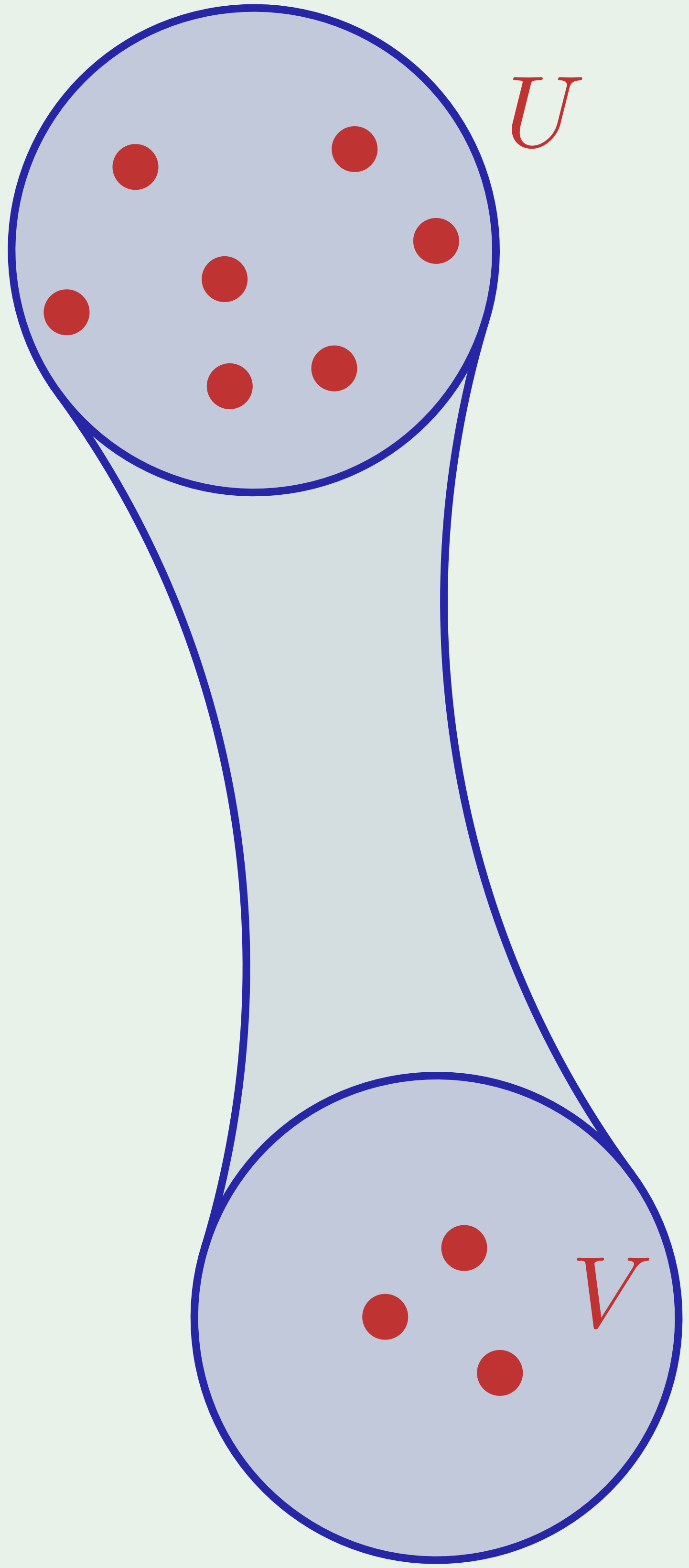
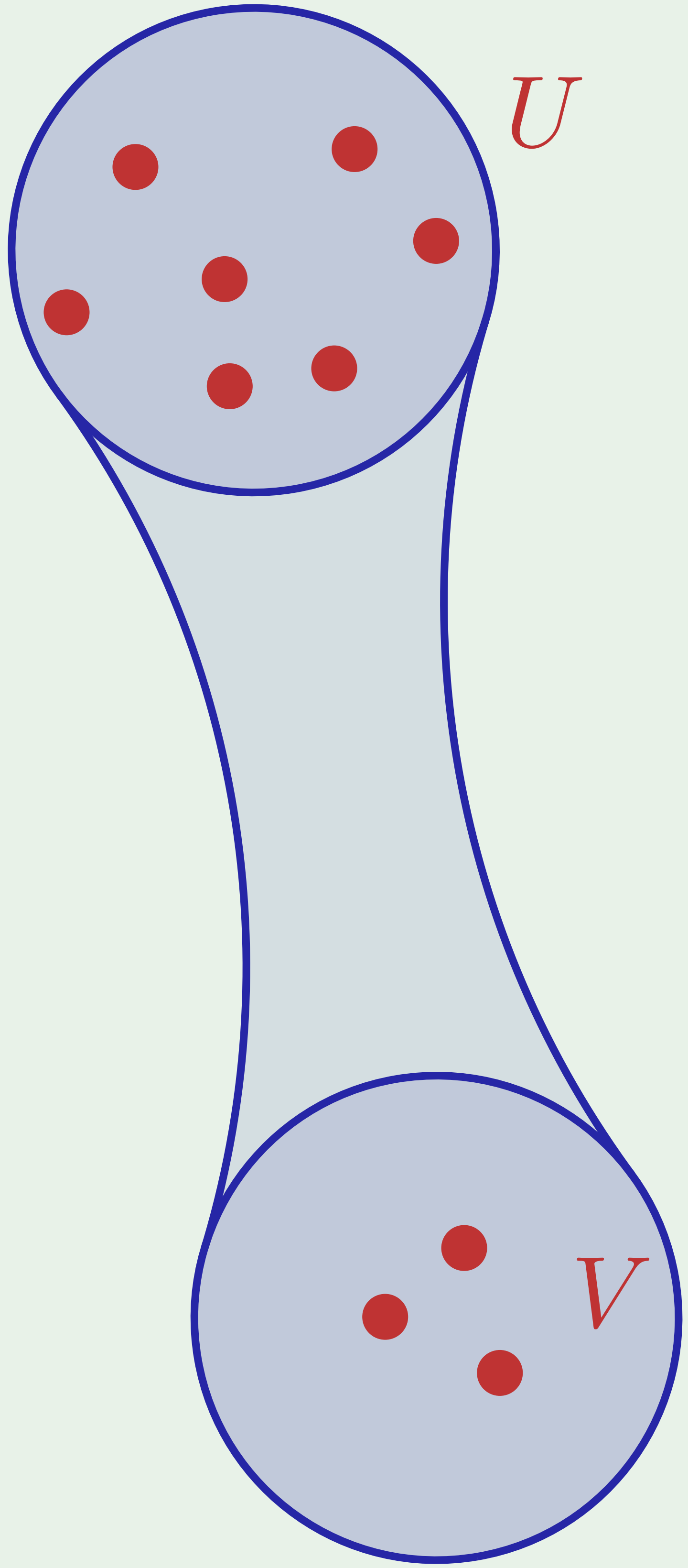


A *c-well-separated pair decomposition (WSPD)* of a set of points P is a set of well-separated pairs of subsets of P such that every pair of points $p, q \in P$ appear in exactly one pair of subsets.



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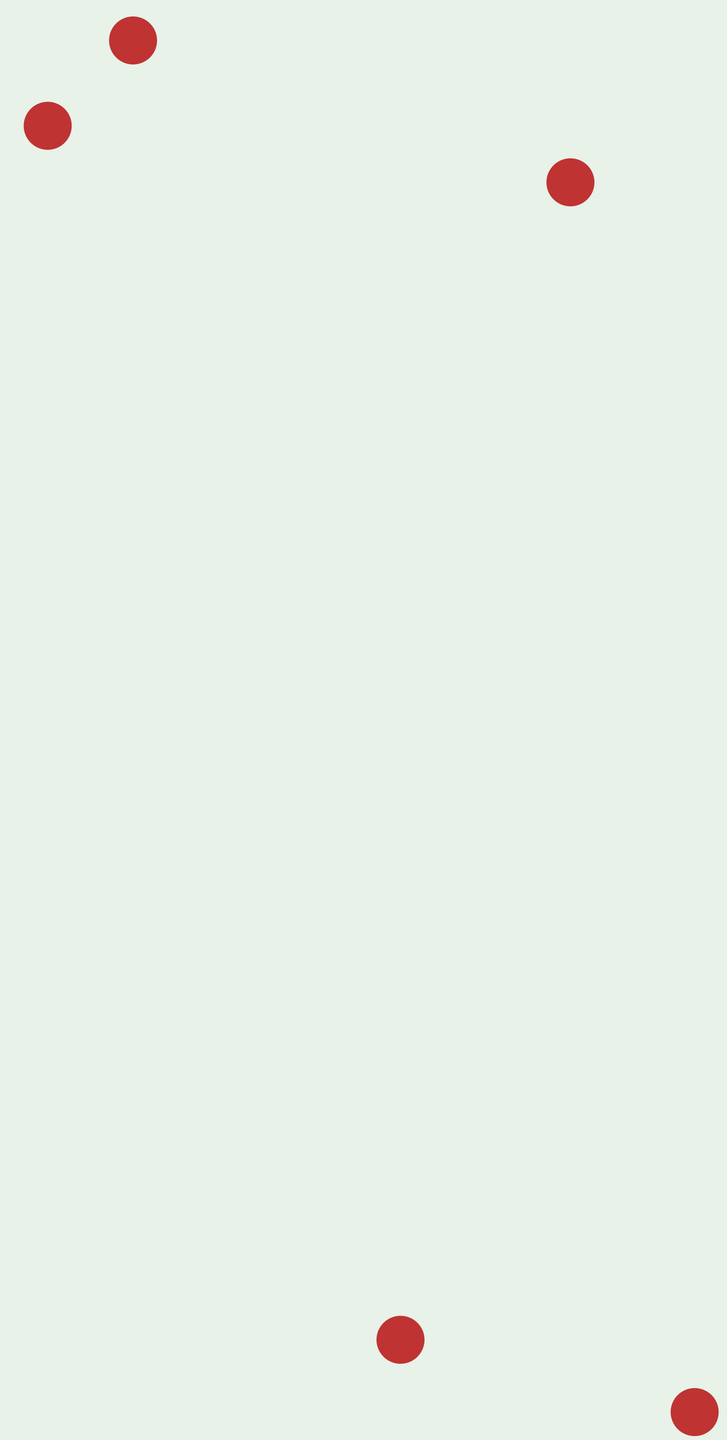

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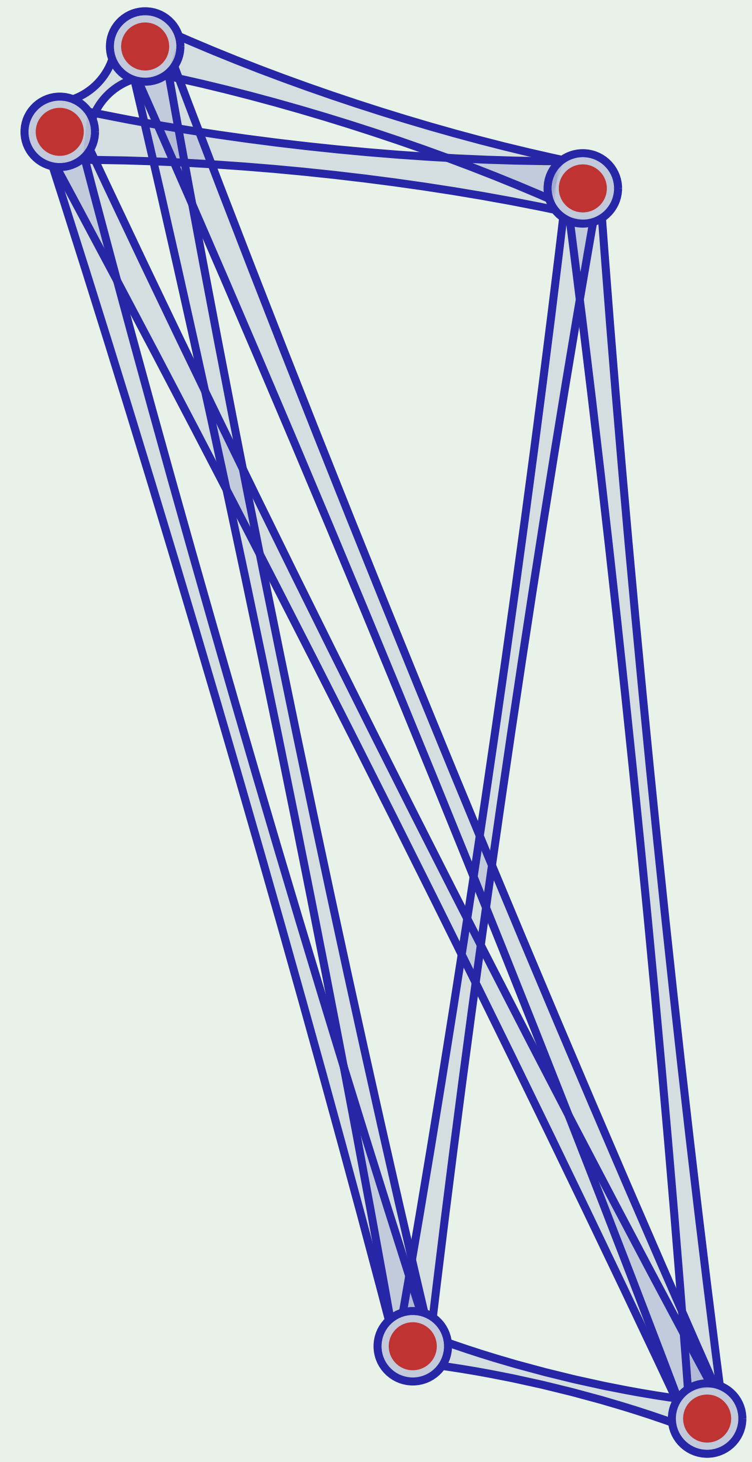

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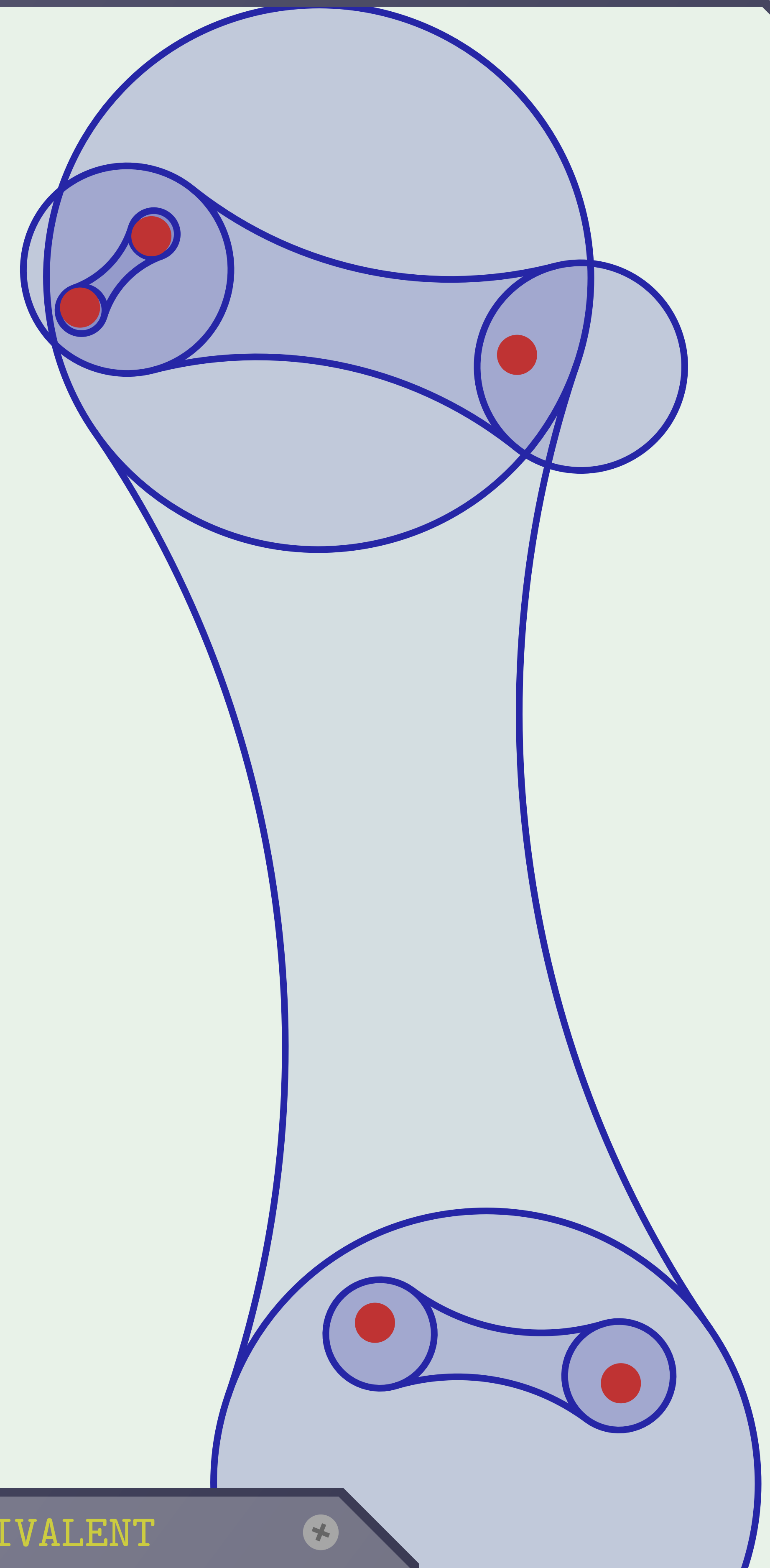
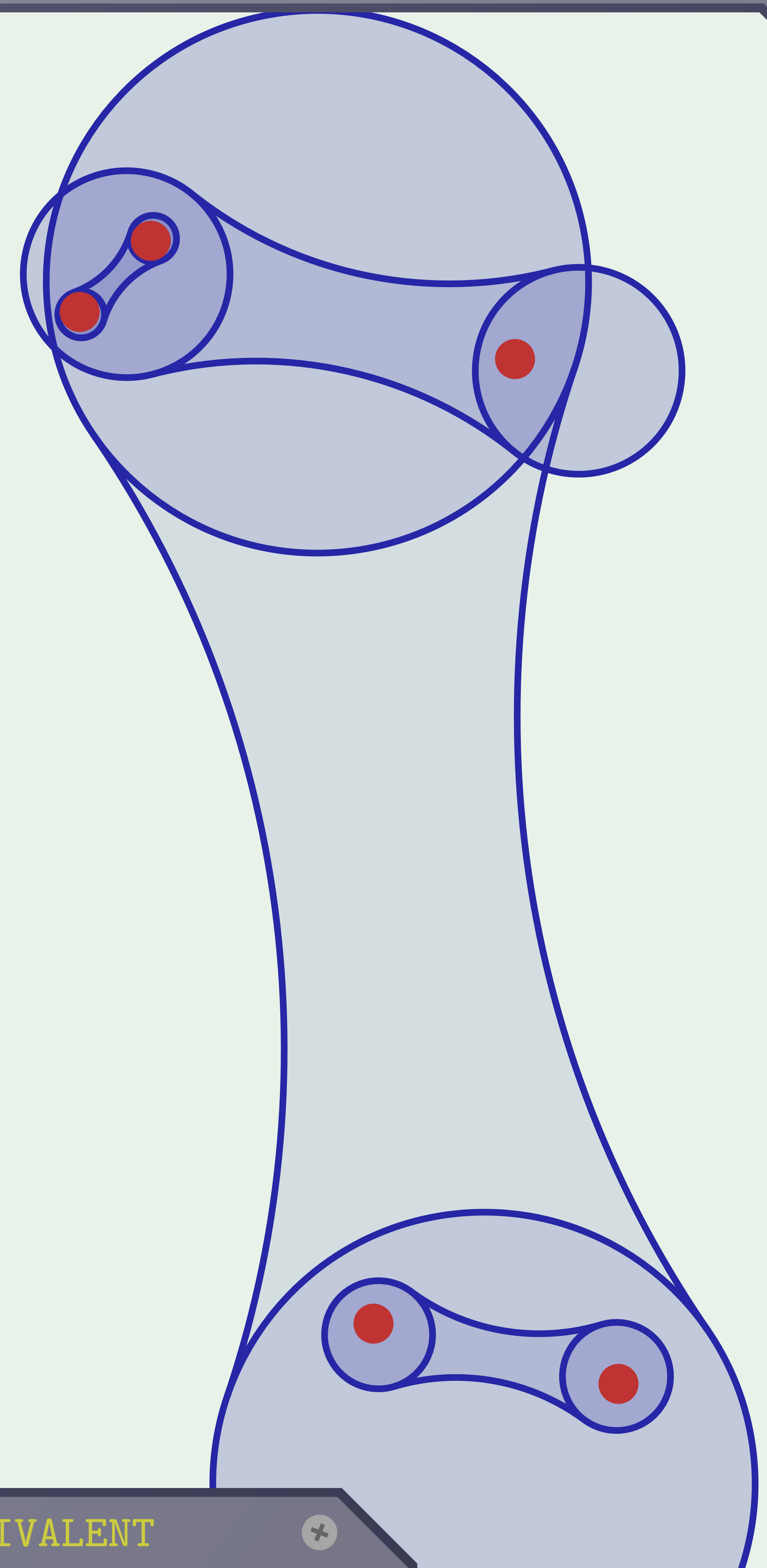
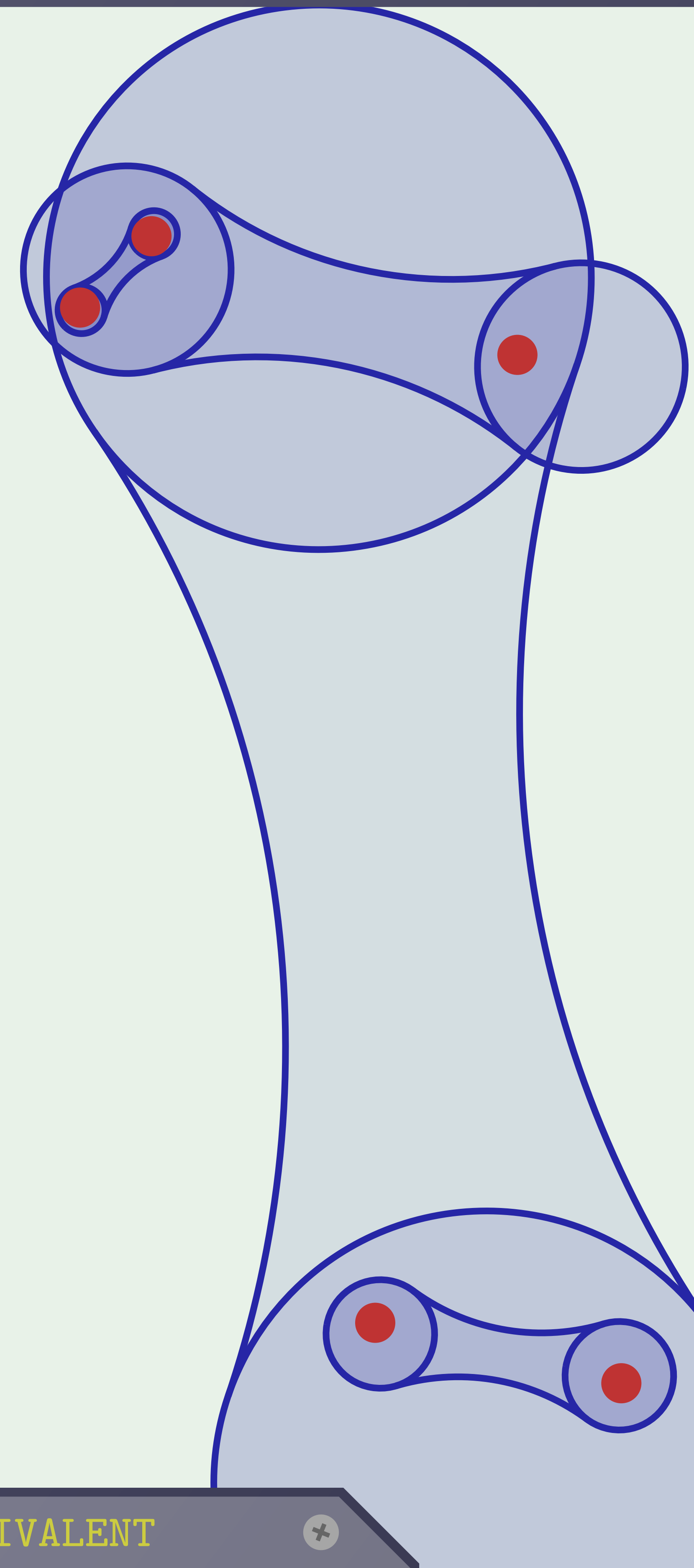


A WSPD with $O(|P|)$ pairs always exists.



A WSPD with $O(|P|)$ pairs always exists.

One way to construct one is from a hierarchical subdivision (such as a c-cluster quadtree).



PART III

THE PART YOU'VE BEEN WAITING FOR

Suppose we have a WSPD $\mathcal{P}$ on P.

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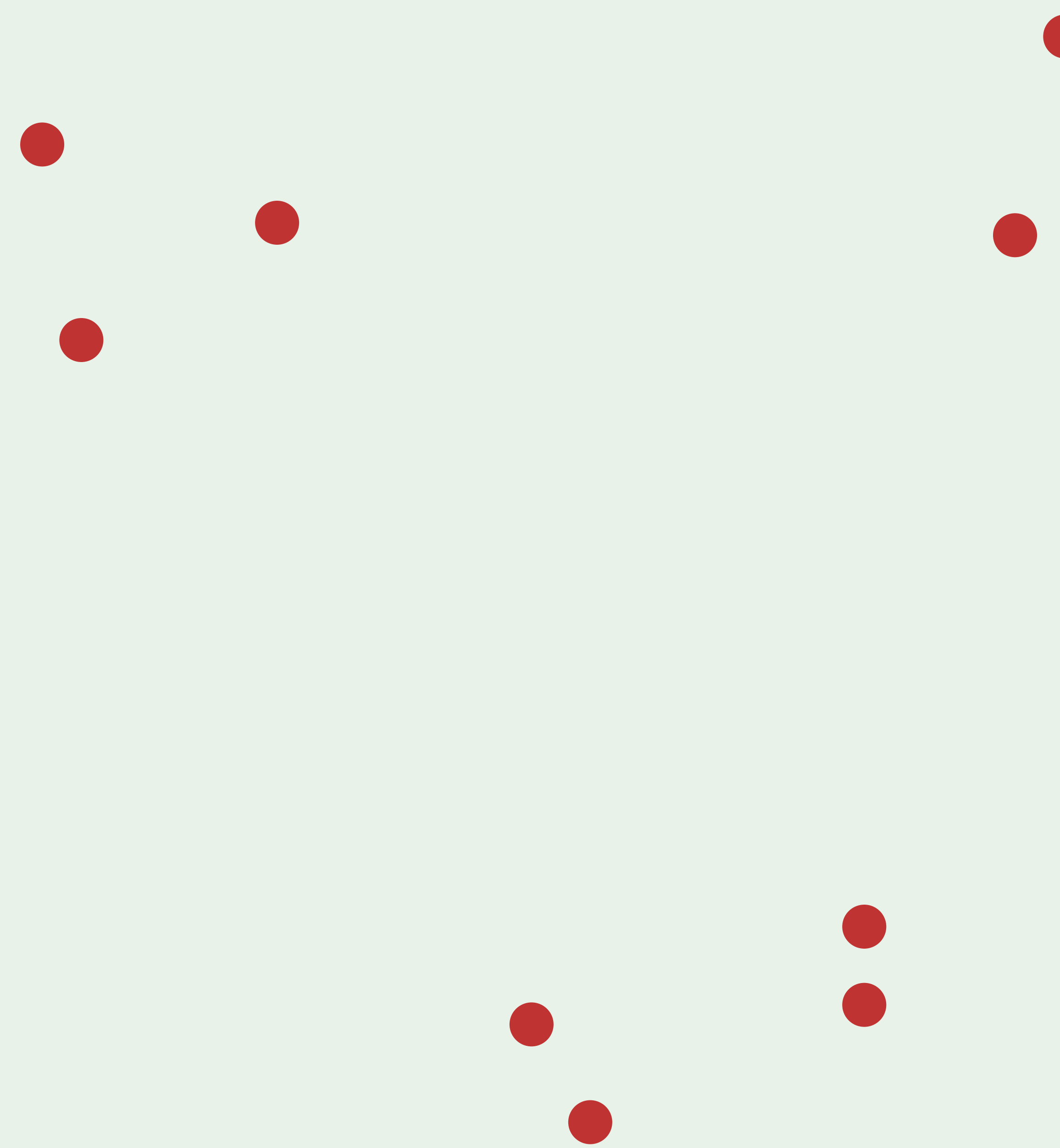

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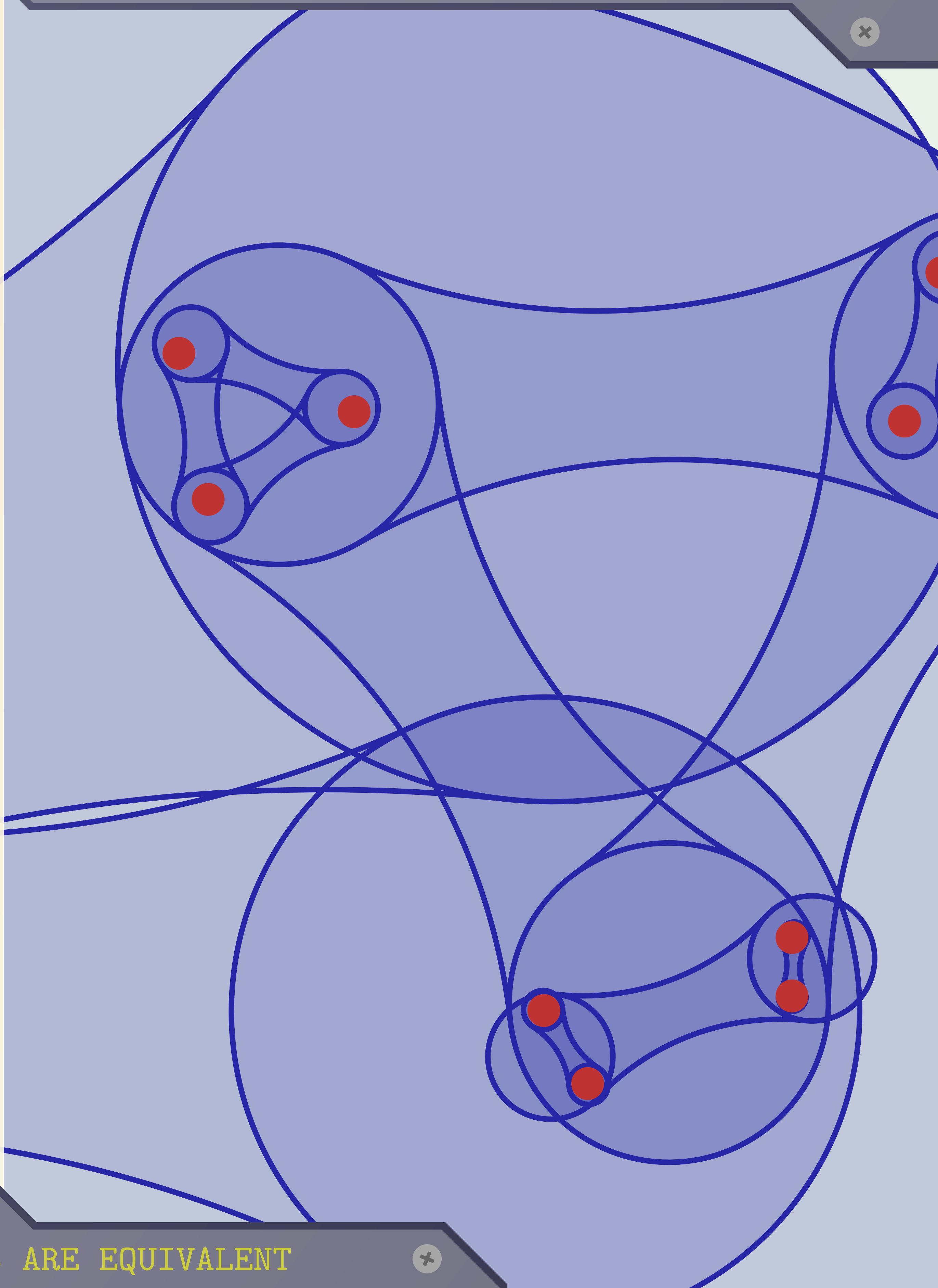
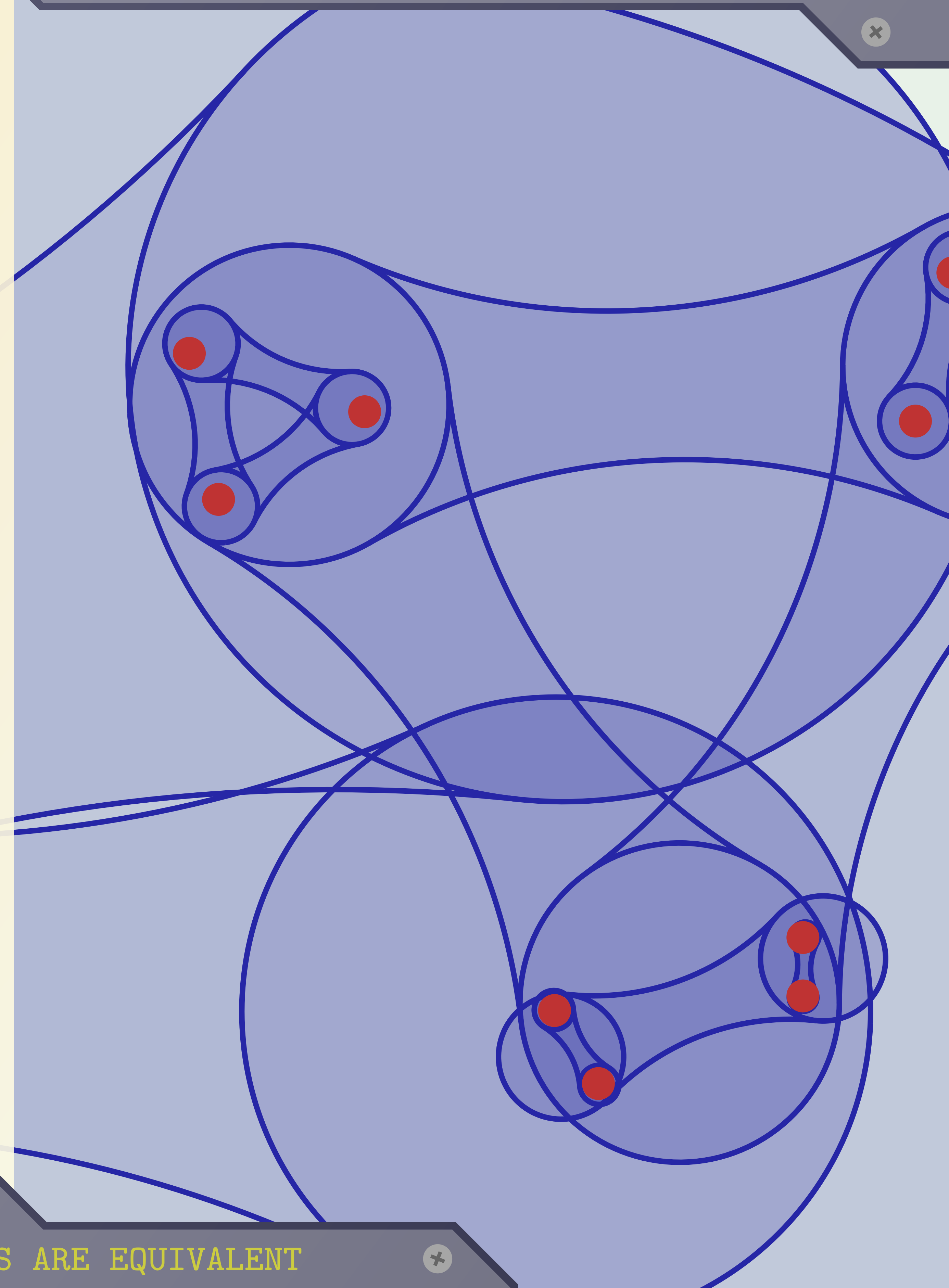


Suppose we have a WSPD $\mathcal{P}$ on P.

Let G be the graph that contains the shortest edge between two points in any pair of $\mathcal{P}$.



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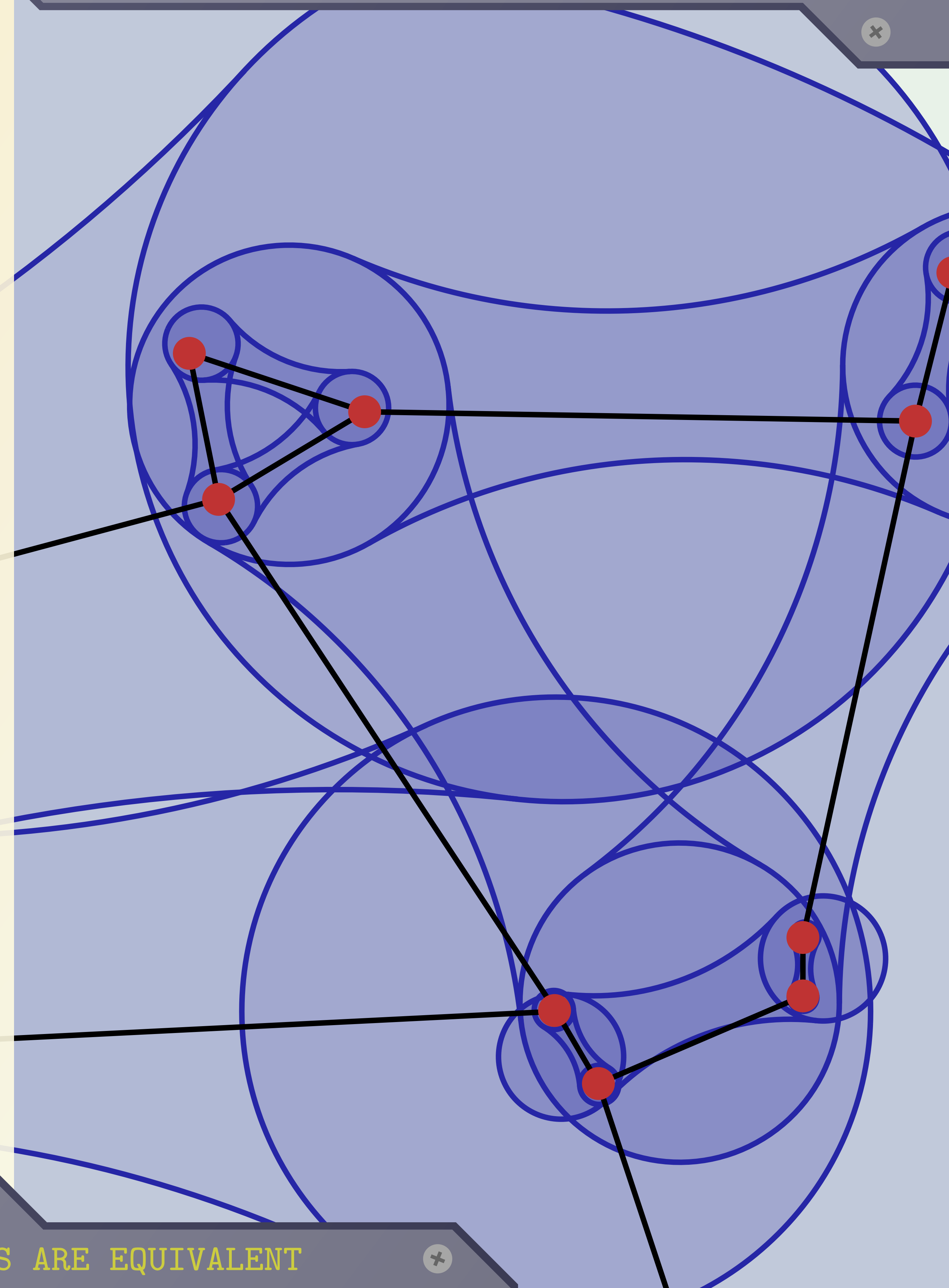
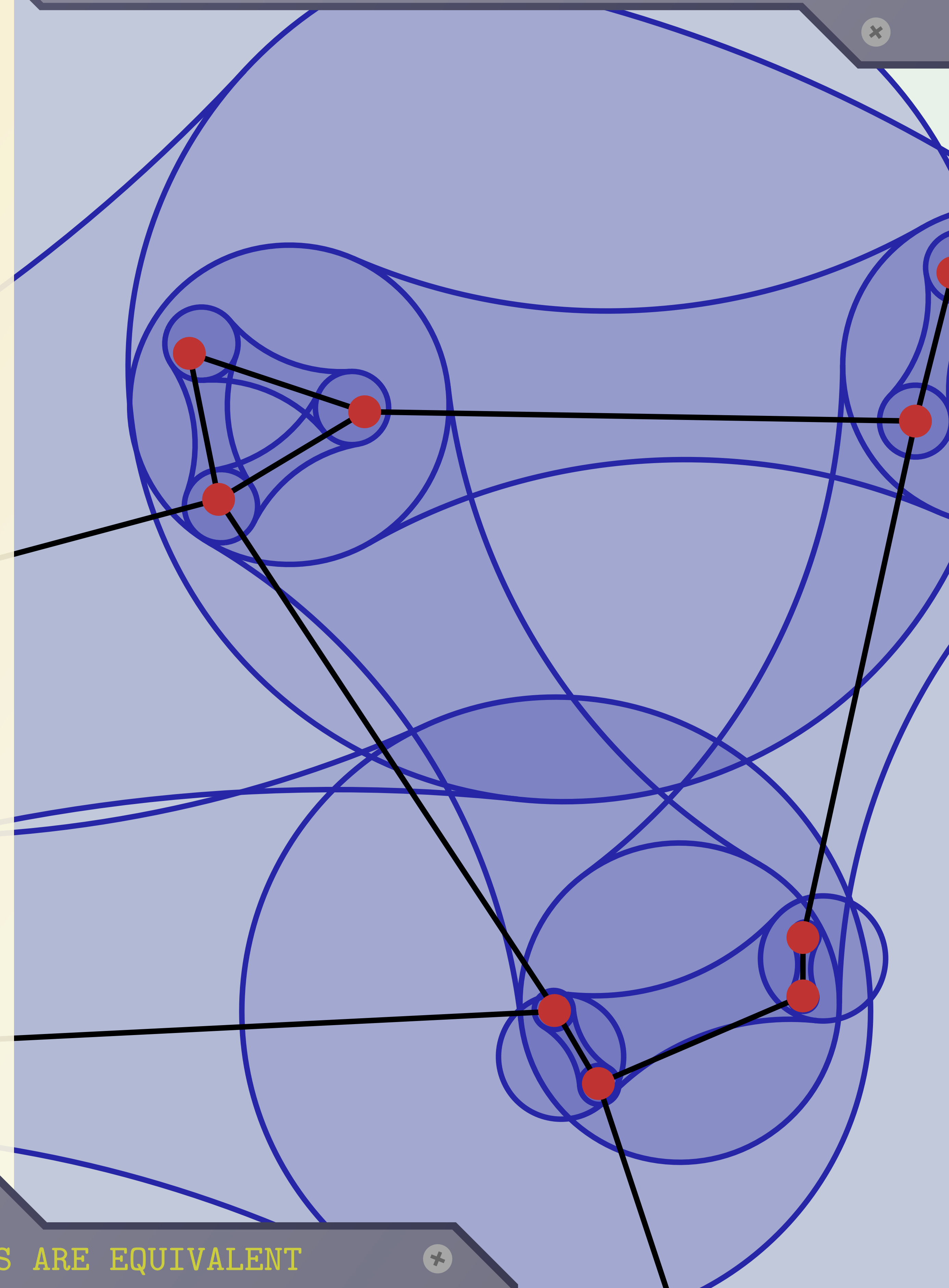


Suppose we have a WSPD $\mathcal{P}$ on P.

Let G be the graph that contains the shortest edge between two points in any pair of $\mathcal{P}$.

Claim: G has linear size and contains the MST of P.



REDUCING THE WEIGHT

Problem: $\mathcal{P}$ has quadratic weight.

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Idea: most heavy pairs are far away and don't contribute to the MST anyway.

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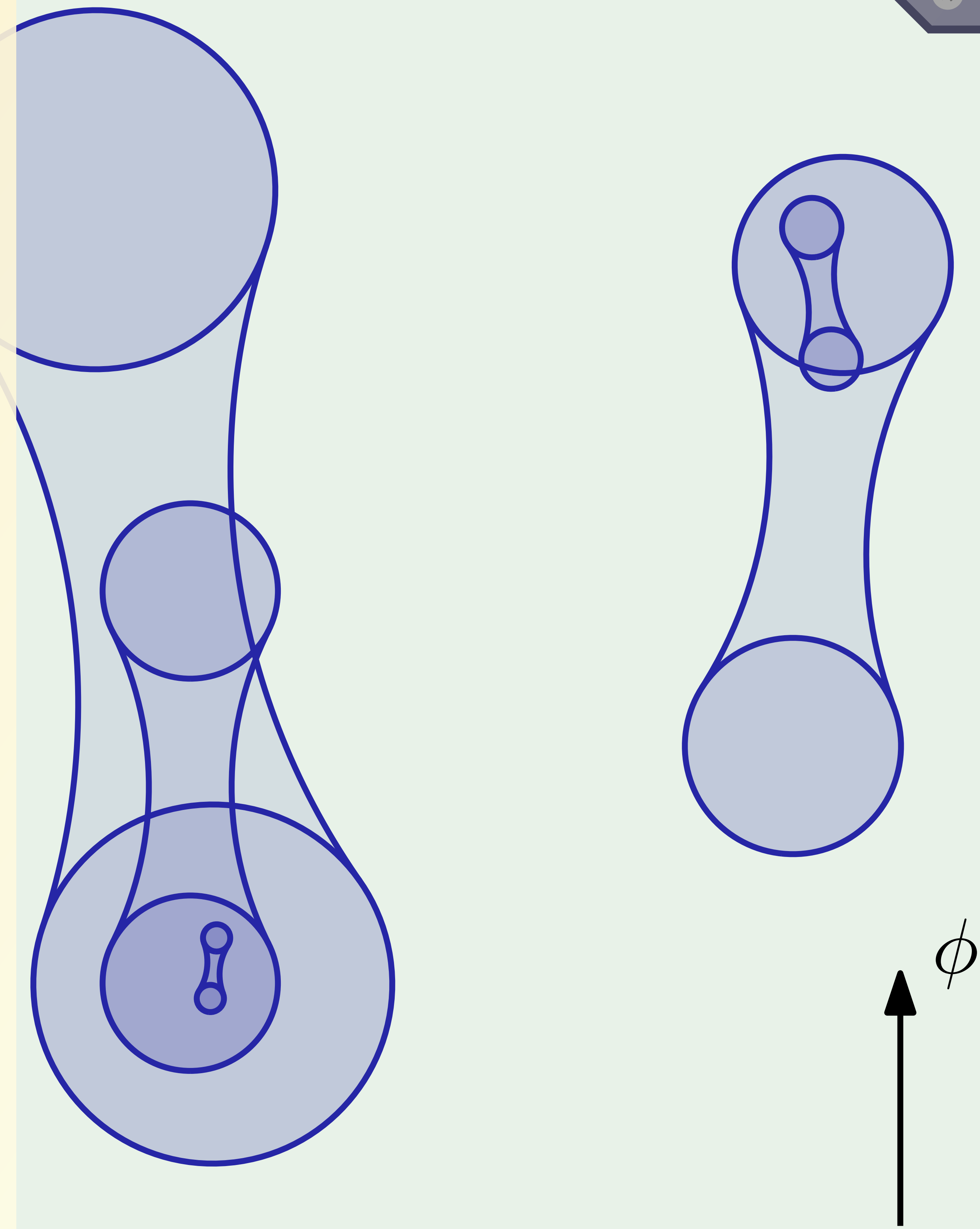

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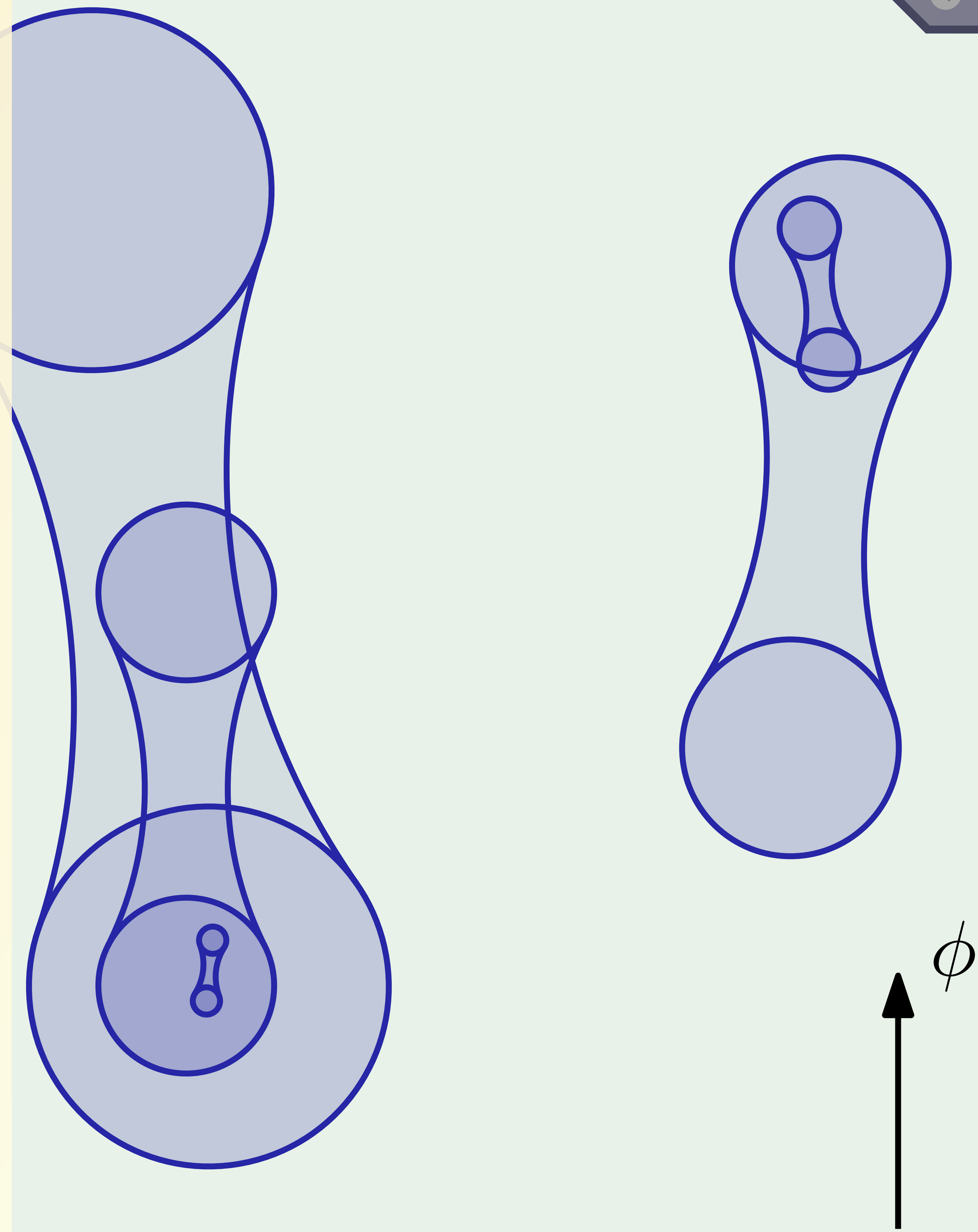
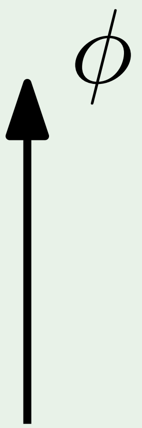
Let $\mathcal{P}_\phi$ be the pairs in $\mathcal{P}$ with general direction ϕ.

We construct a G_ϕ separately from $\mathcal{P}_\phi$.



For each $p \in P$, find the k closest pairs in $\mathcal{P}_\phi$, and remove p from all other pairs.



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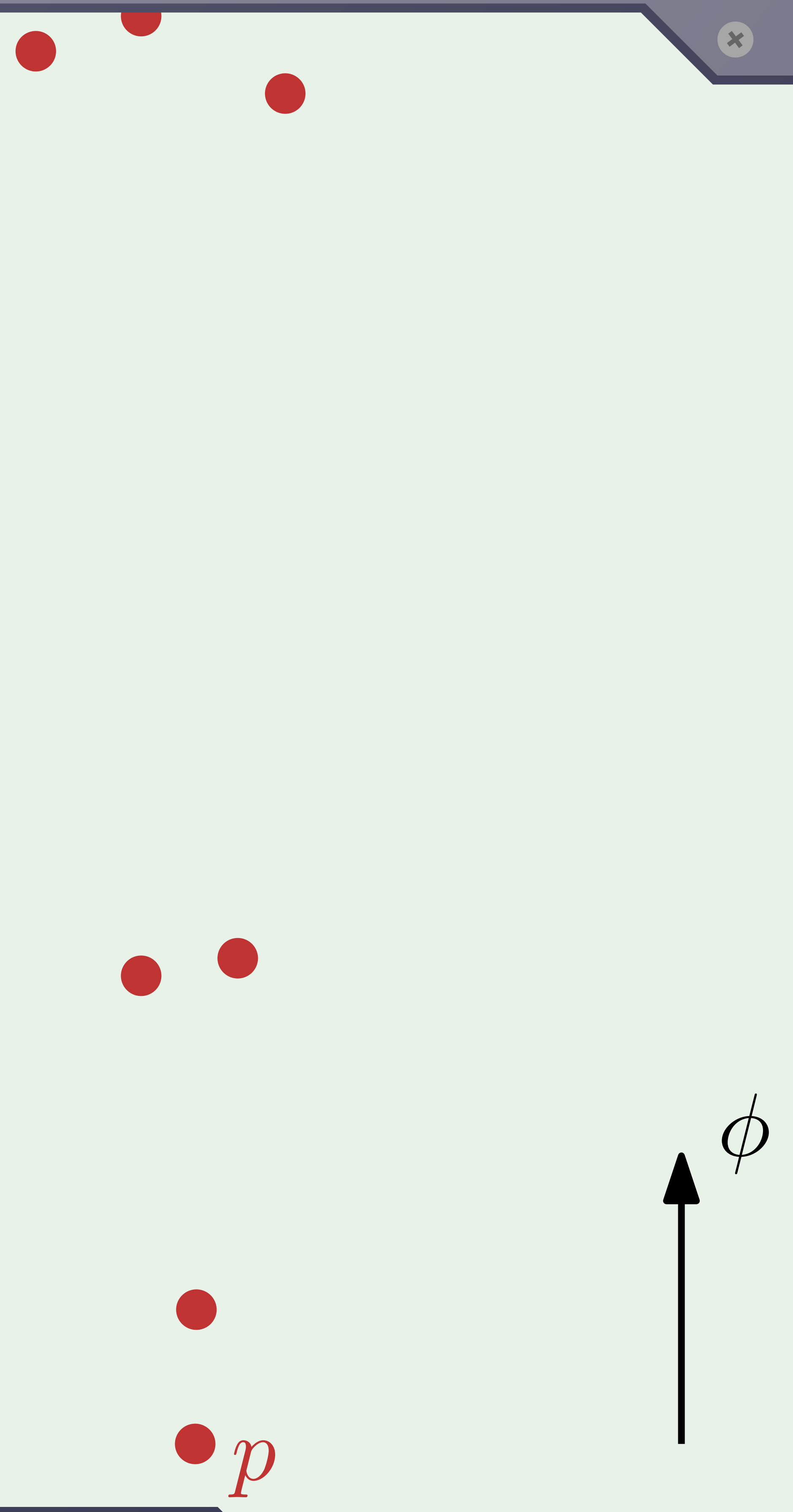

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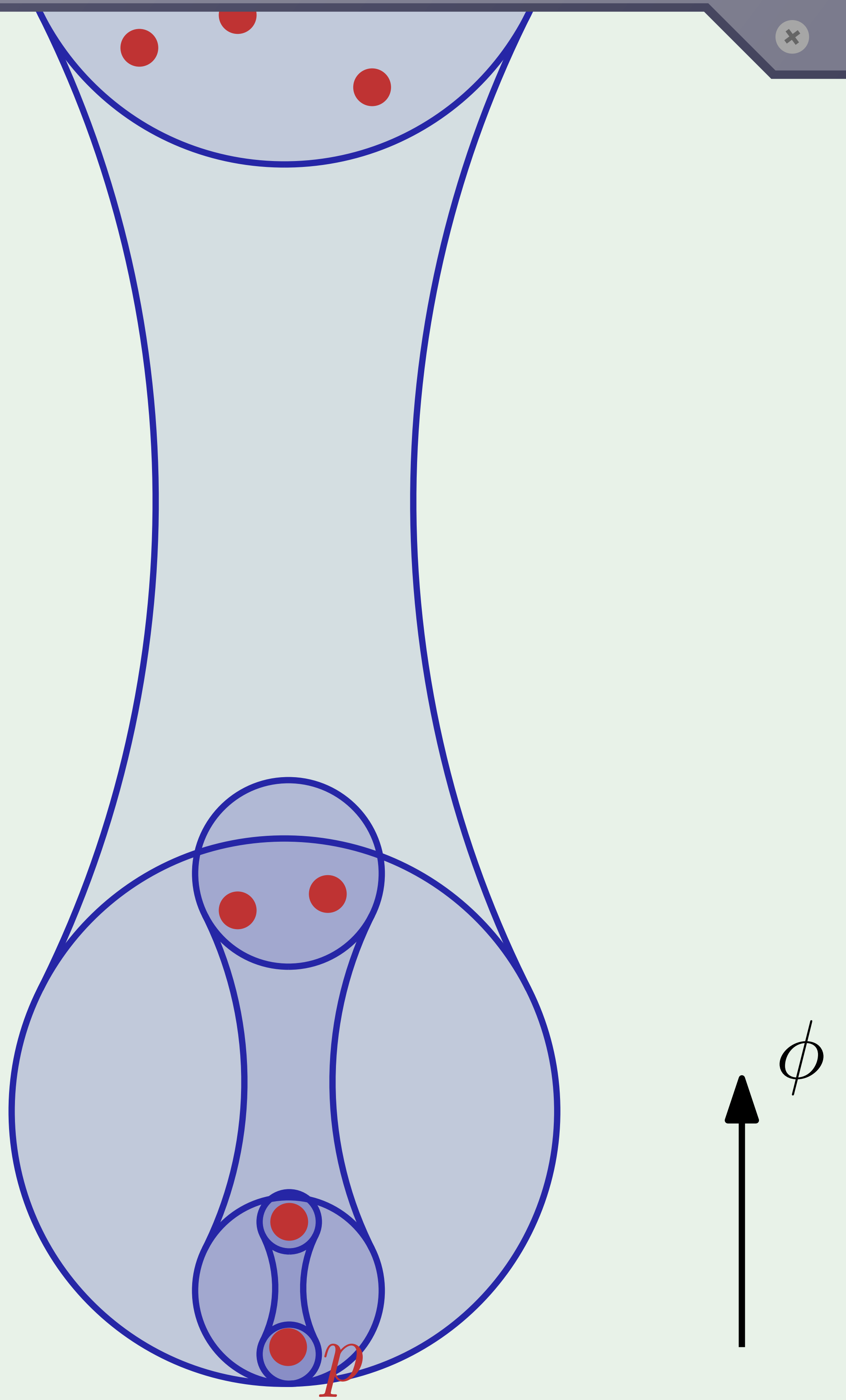

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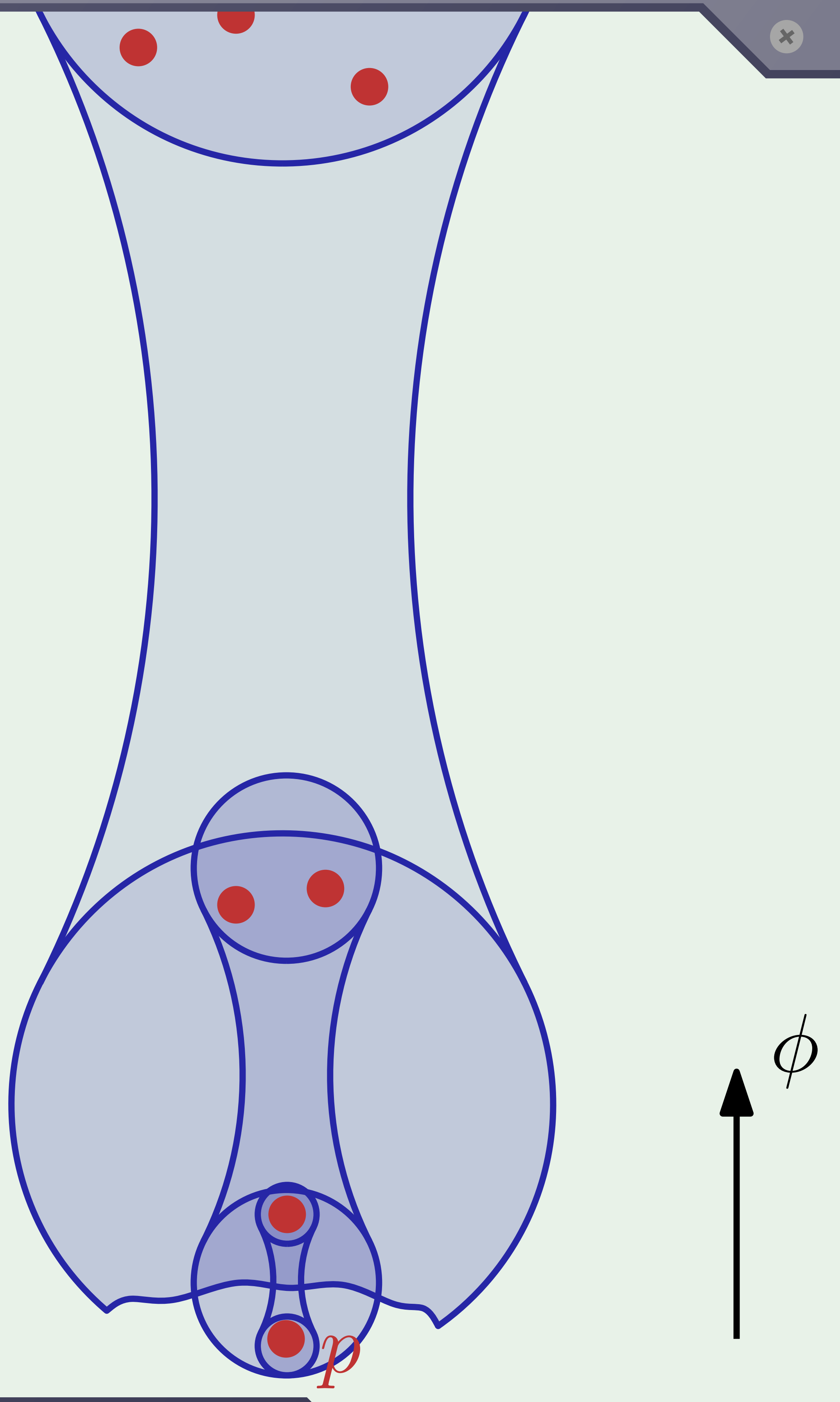

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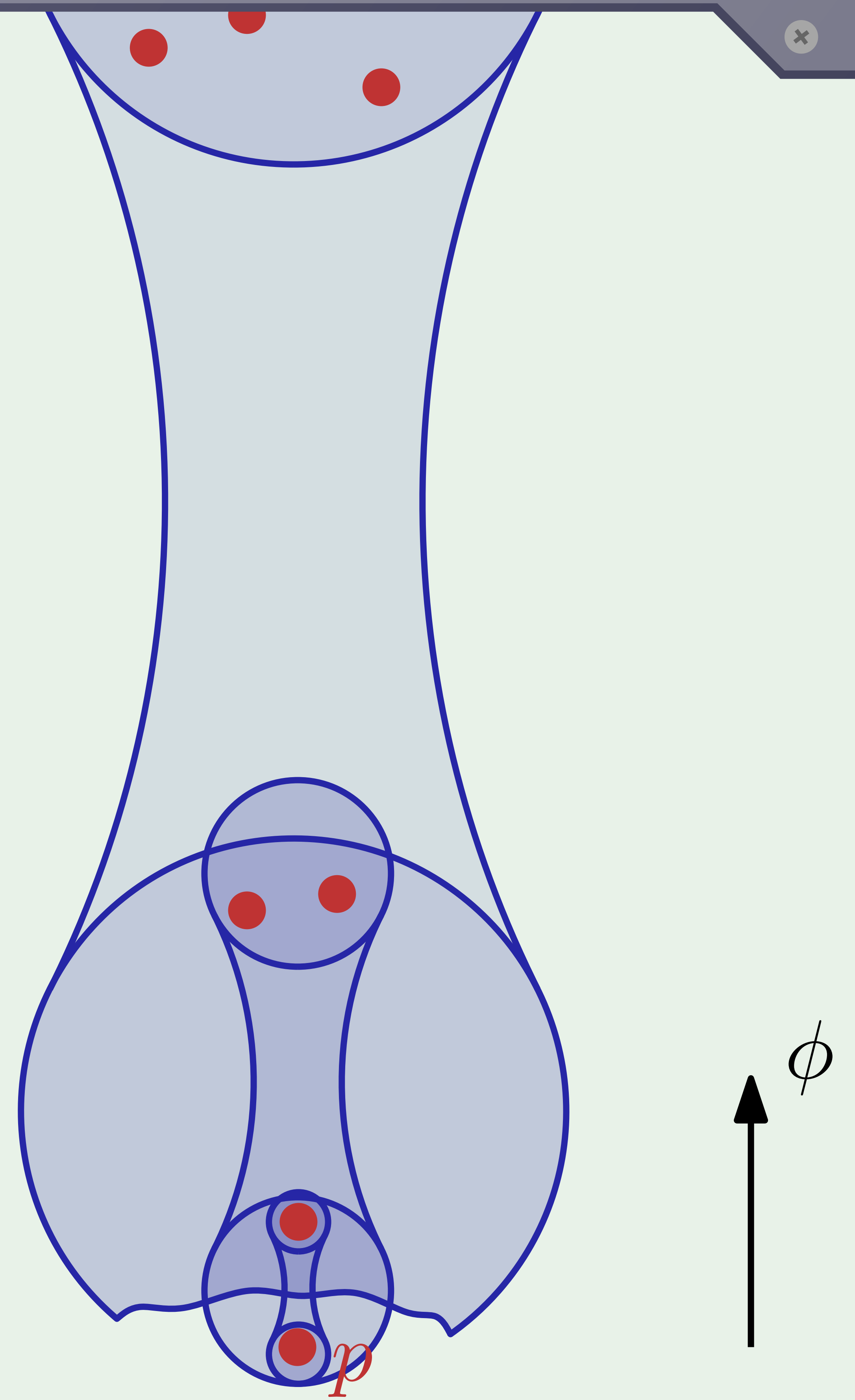
Let $\mathcal{P}'_\phi$ be the resulting pair set.



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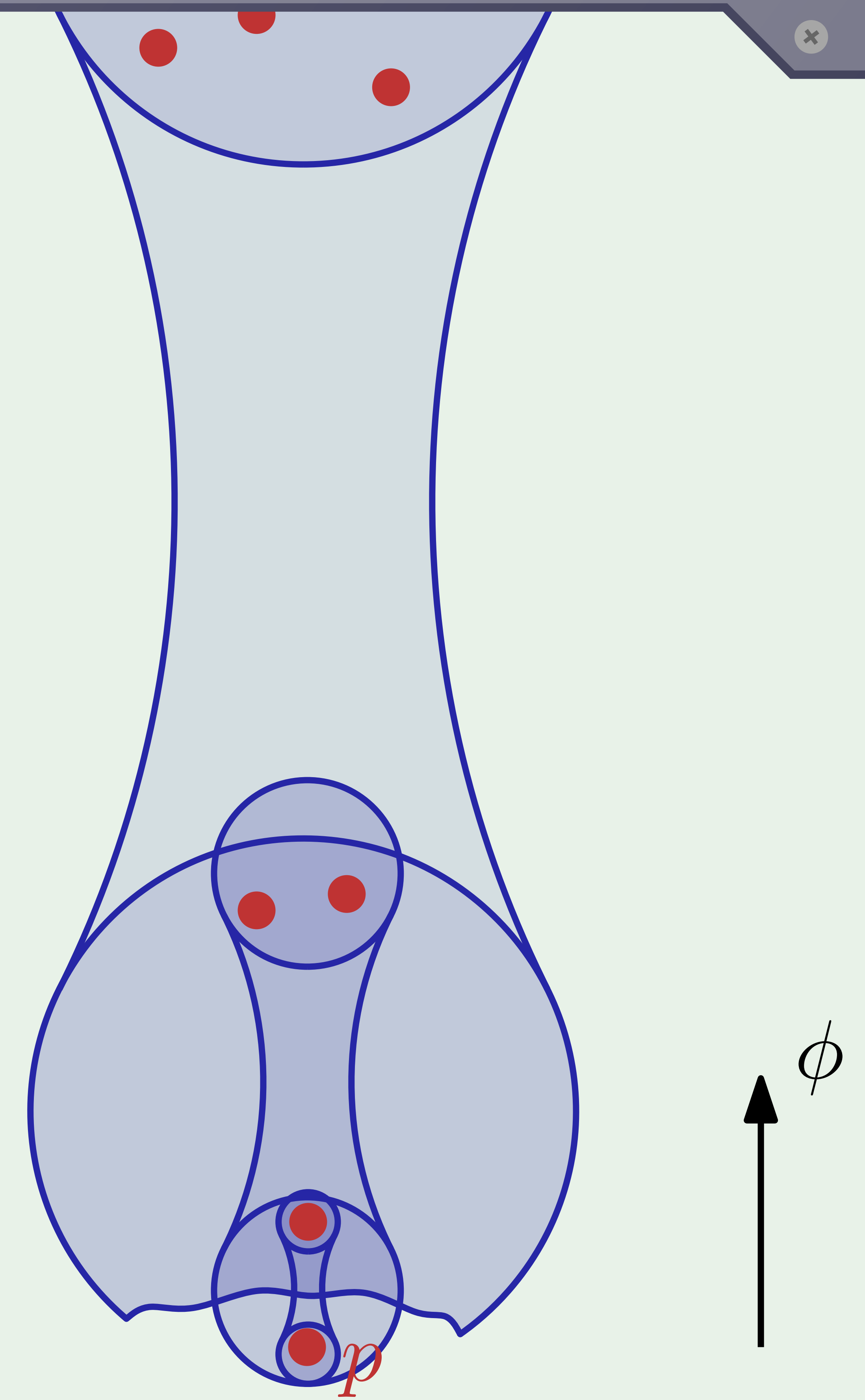
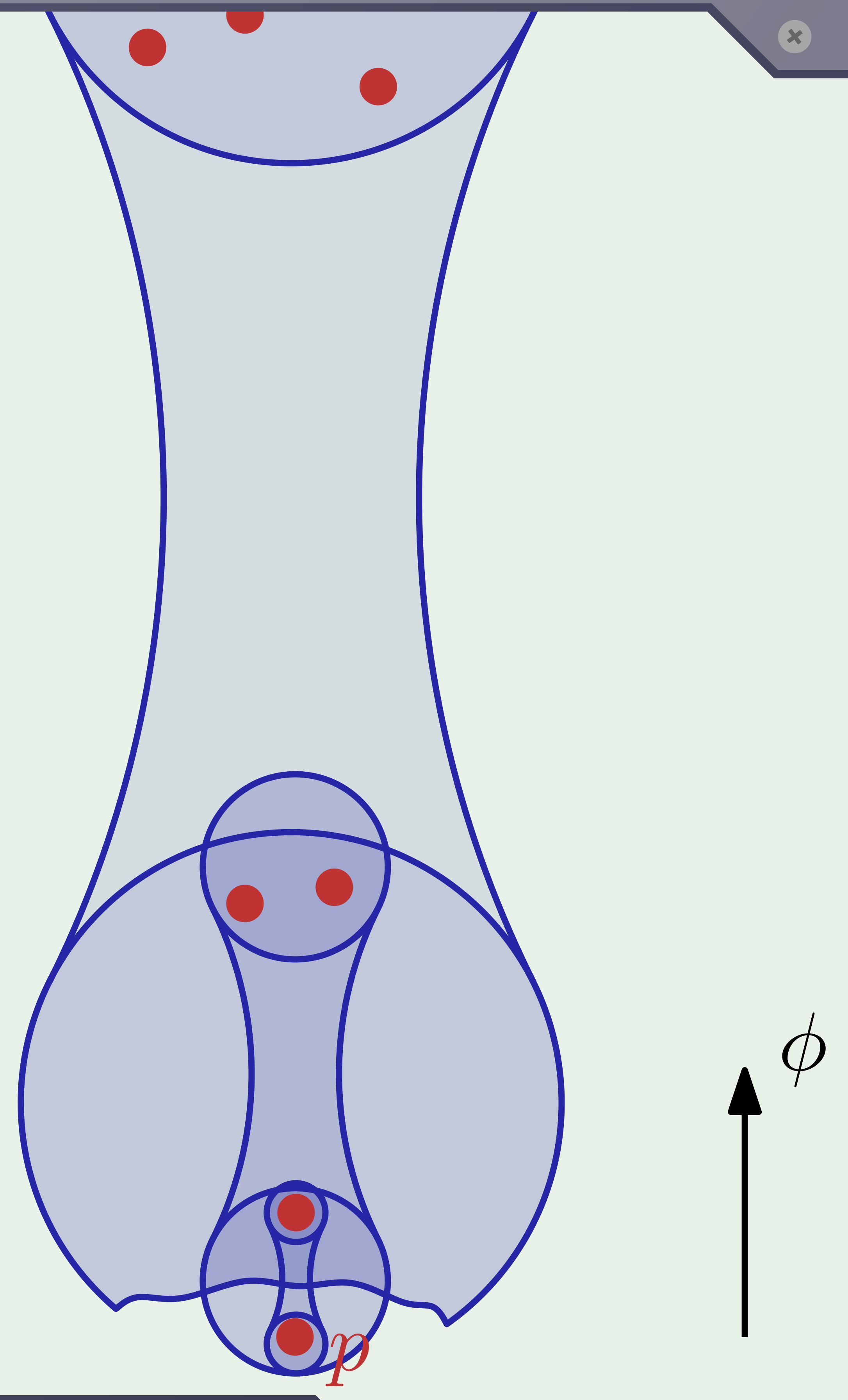
Let $\mathcal{P}'_\phi$ be the resulting pair set.

$\mathcal{P}'$, the union of $\mathcal{P}'_\phi$ over all ϕ, is no longer a pair decomposition of P.

But...



Claim: $\mathcal{P}'$ has linear weight.



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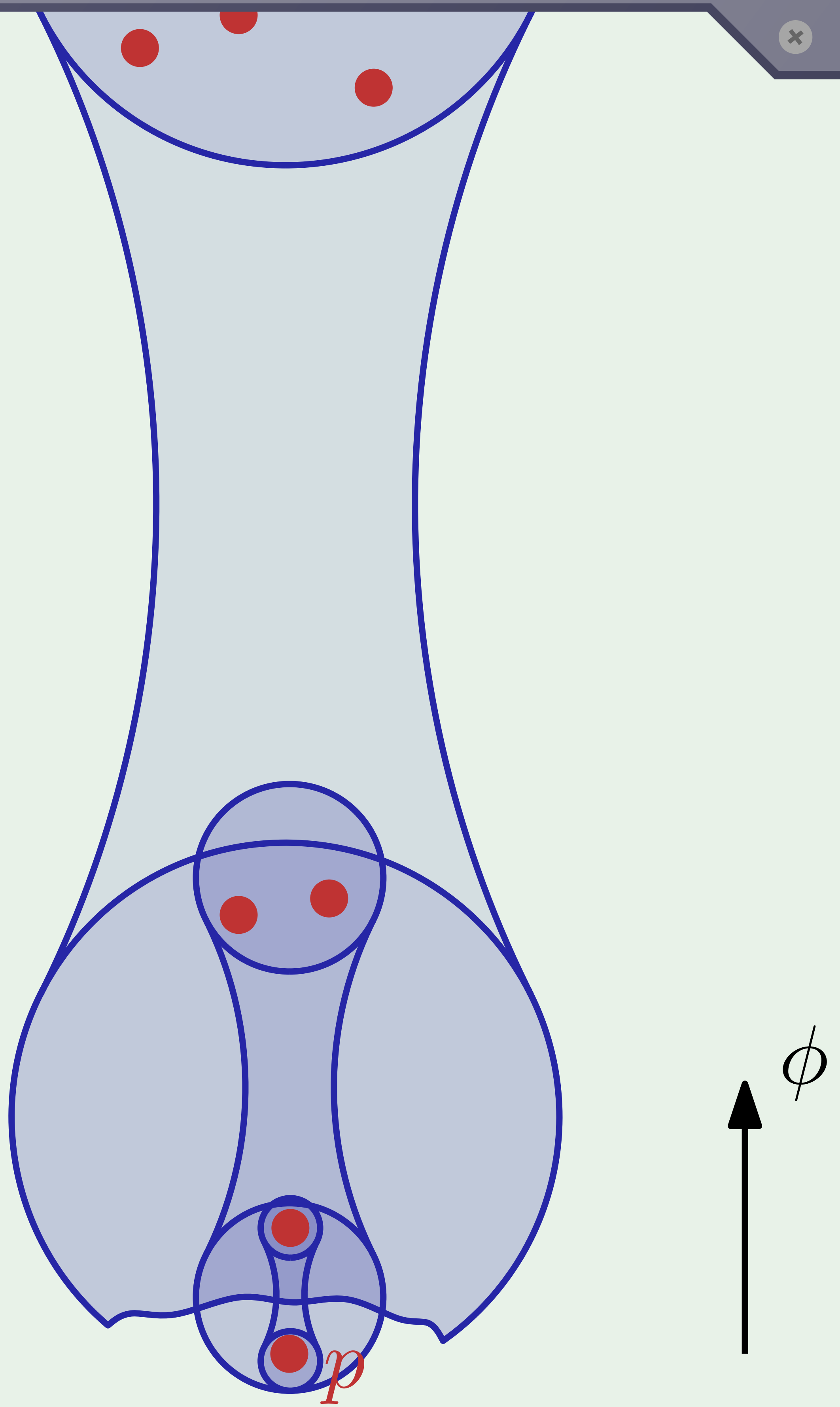
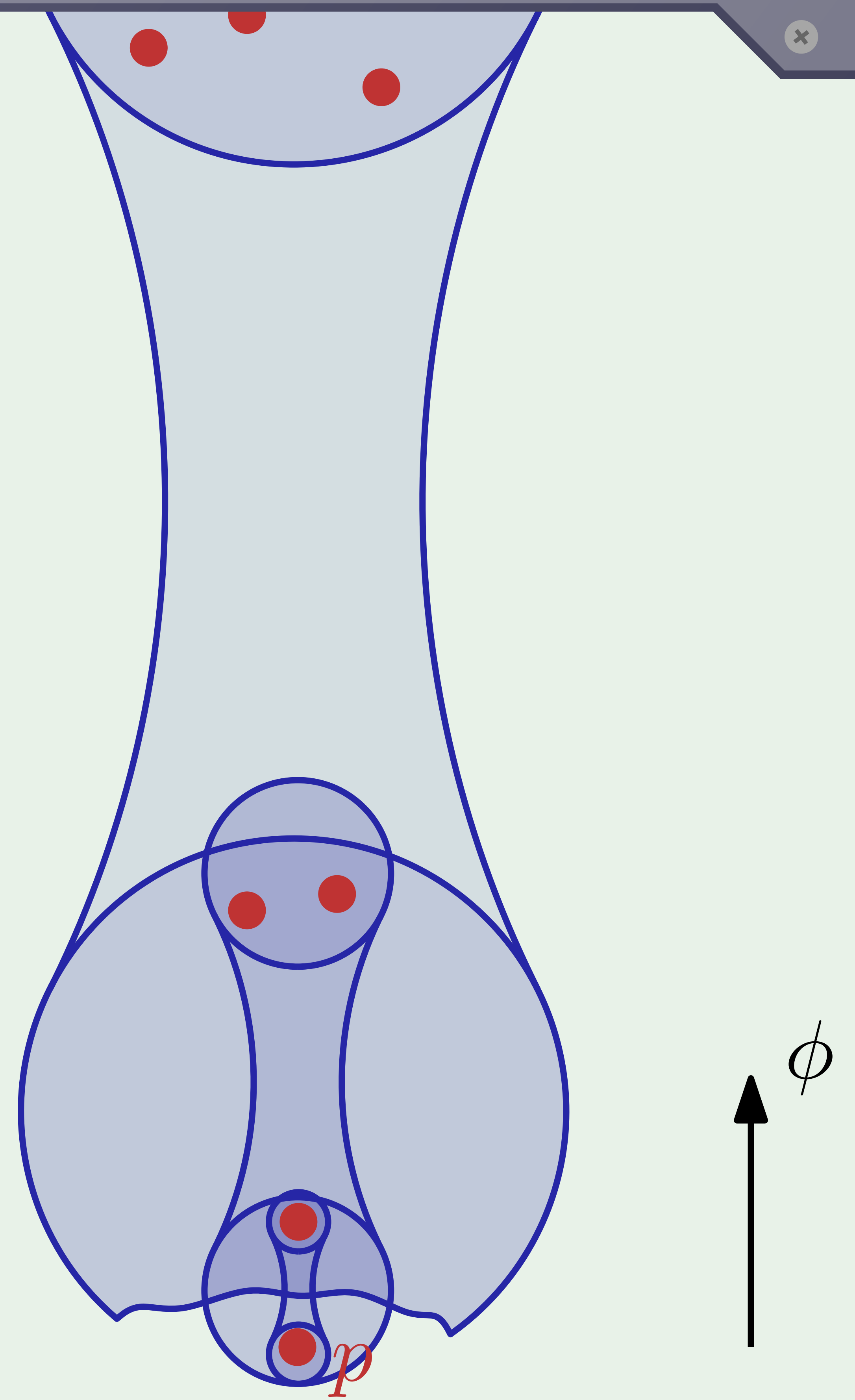
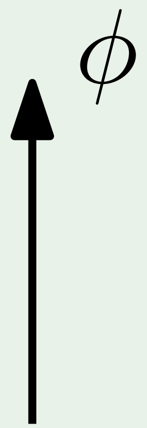


Claim: $\mathcal{P}'$ has linear weight.

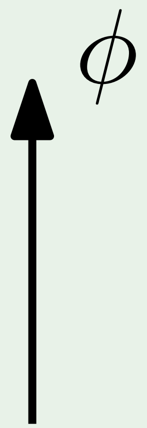
Claim: $\mathcal{P}'$ still contains all edges of the MST of P.

Claim: and as if that wasn't cool enough, $\mathcal{P}'$ can even be computed in linear time.




ϕ

Now, for each pair $(U, V) \in \mathcal{P}'_\phi$, we need to find the closest pair of points.



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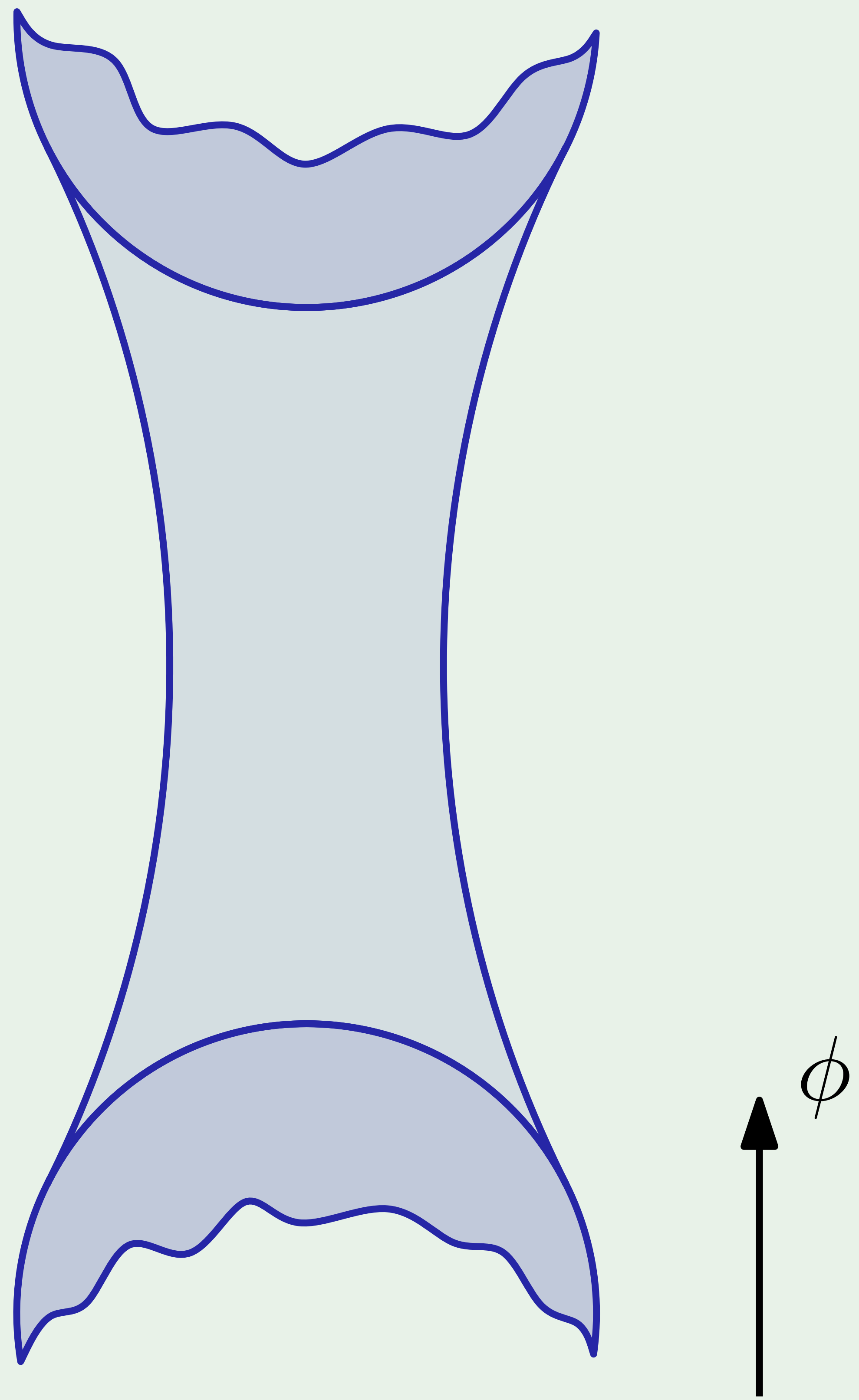

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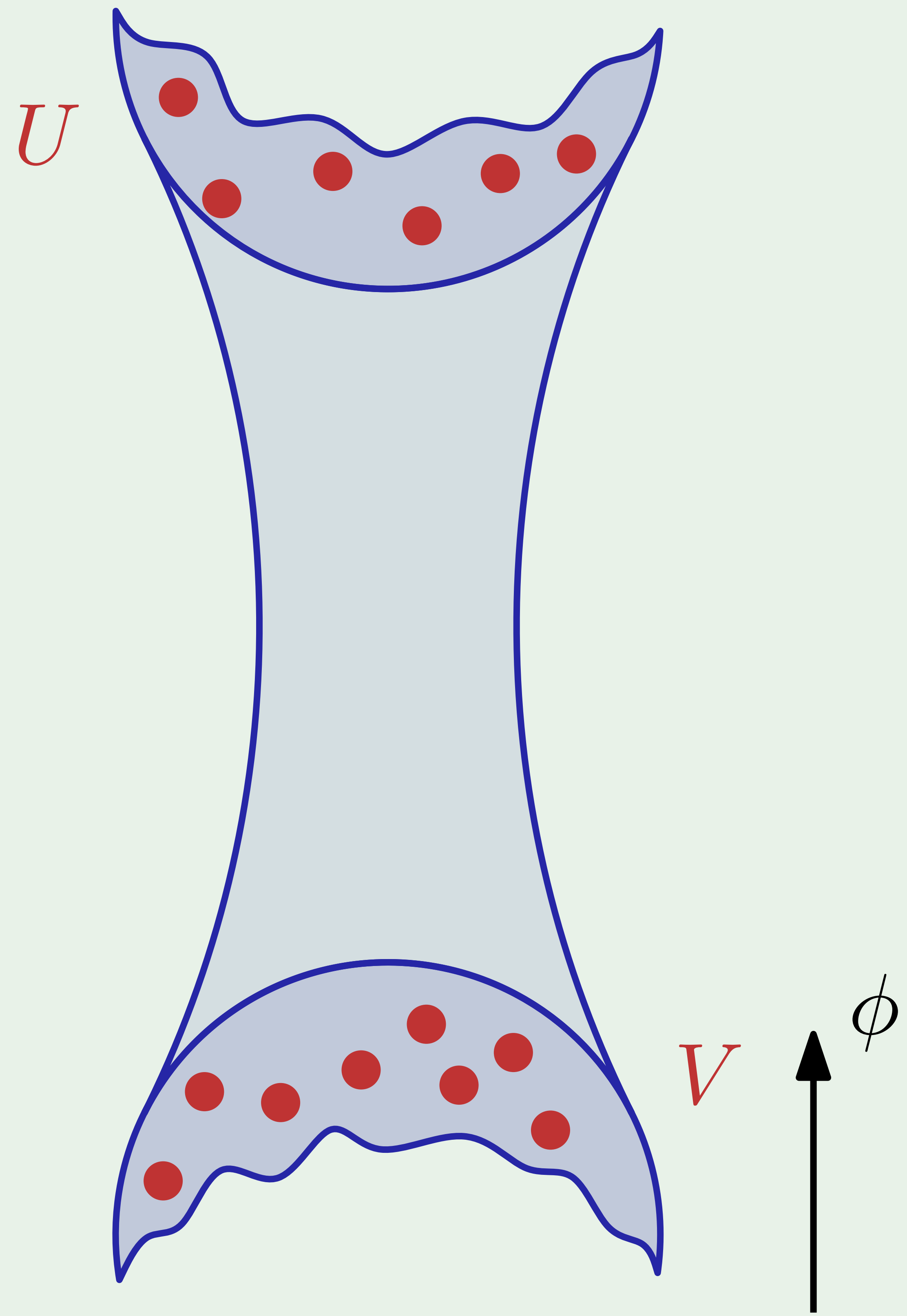
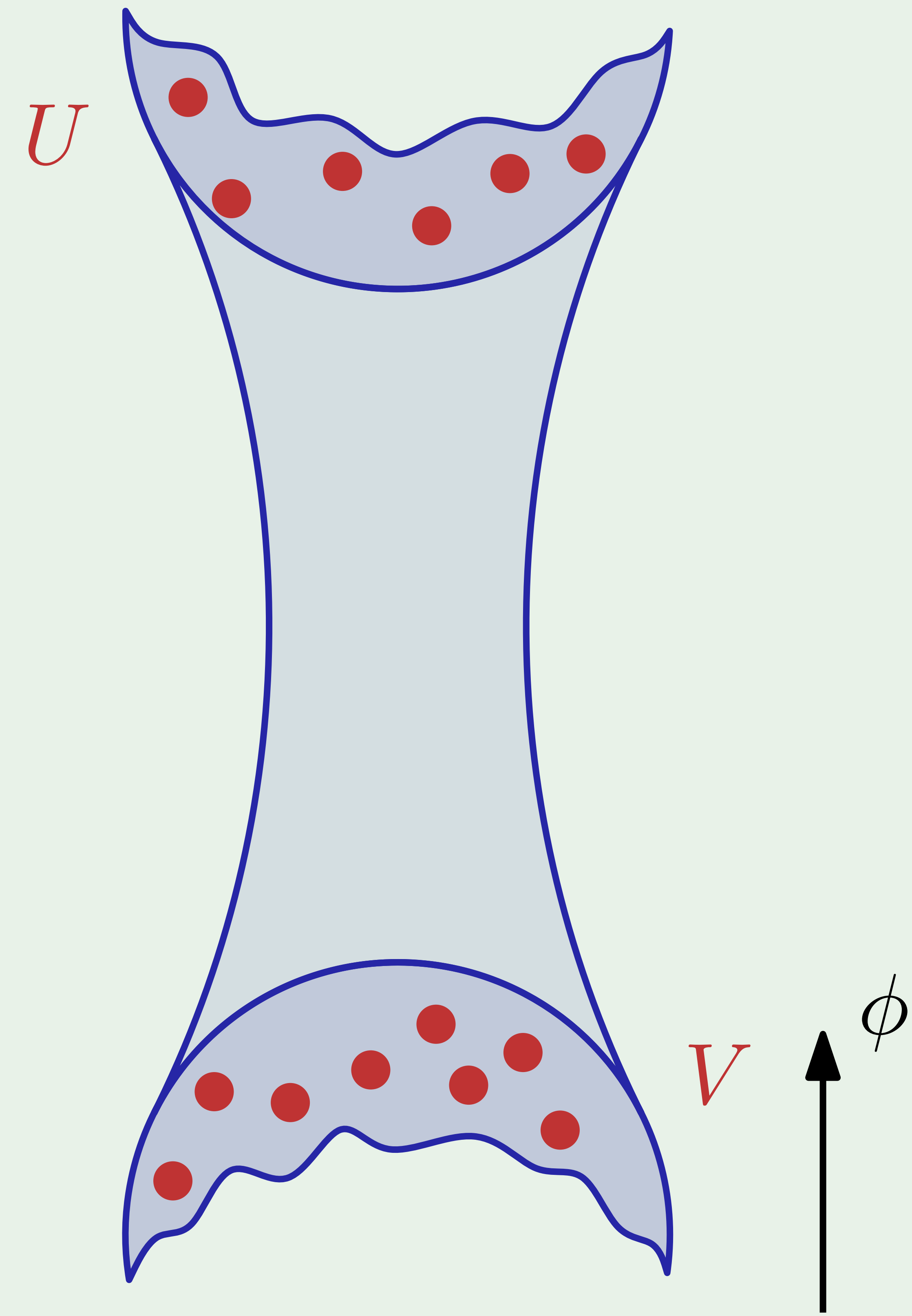


Now, for each pair $(U, V) \in \mathcal{P}'_\phi$, we need to find the closest pair of points.

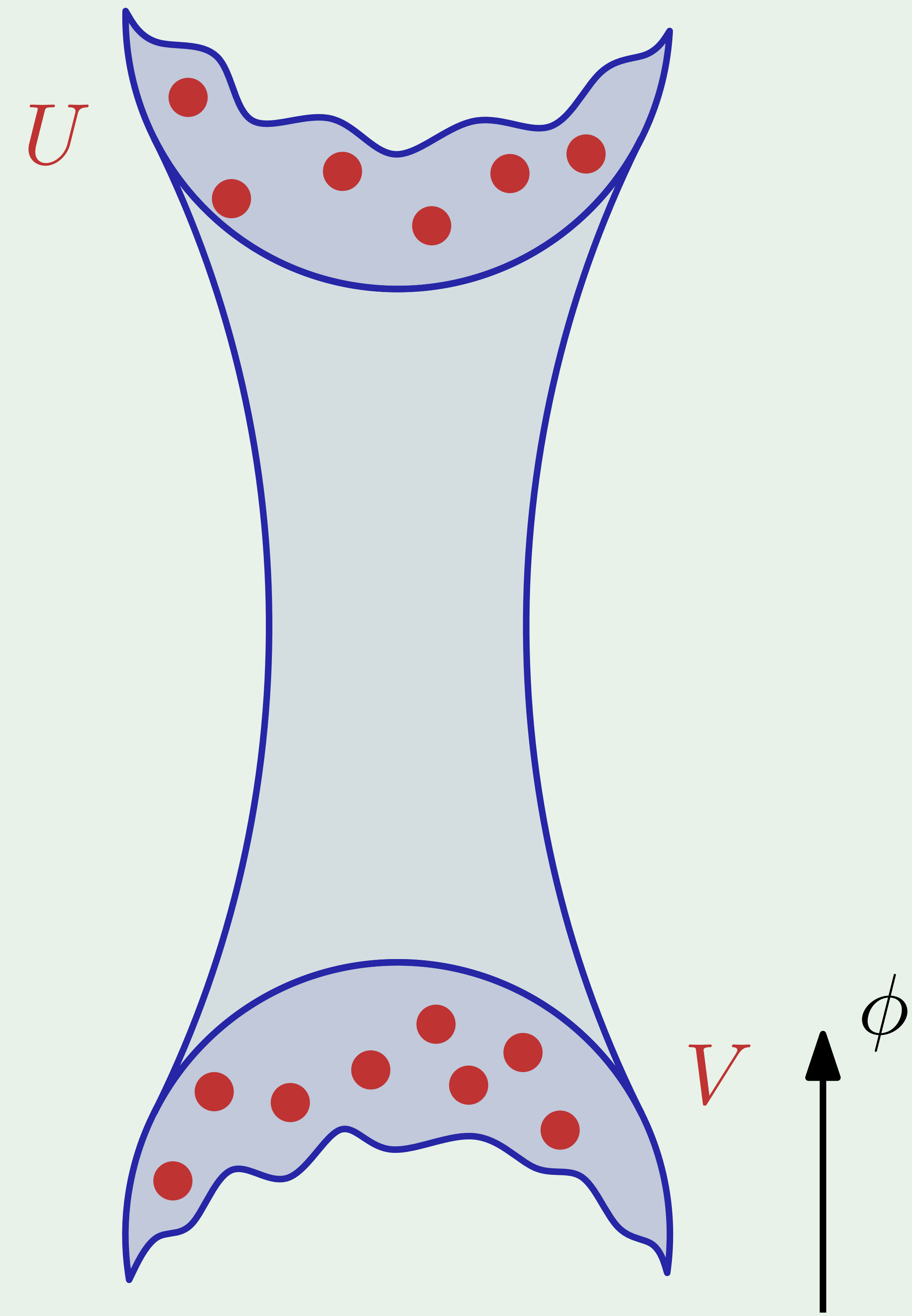
Claim: if the points would be sorted on x-coordinate, we could find it in linear time.



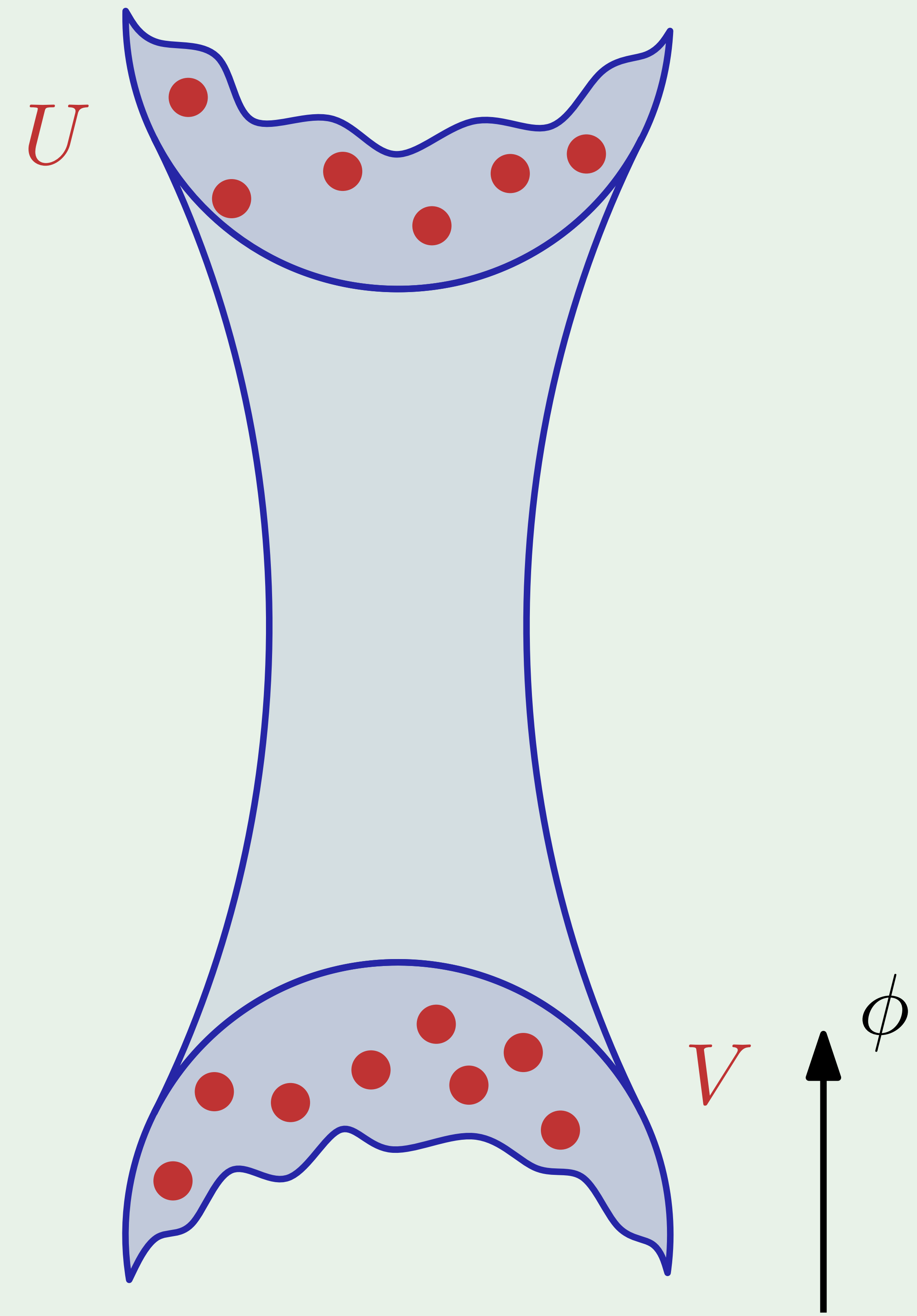
Now, for each pair $(U,V) \in \mathcal{P}'_\phi$, we need to find the closest pair of points.

Claim: if the points would be sorted on x-coordinate, we could find it in linear time.

Problem: sorting takes $\Theta(n \log n)$ time.



Claim: we only need to sort the points whose 'upward cone' is empty.



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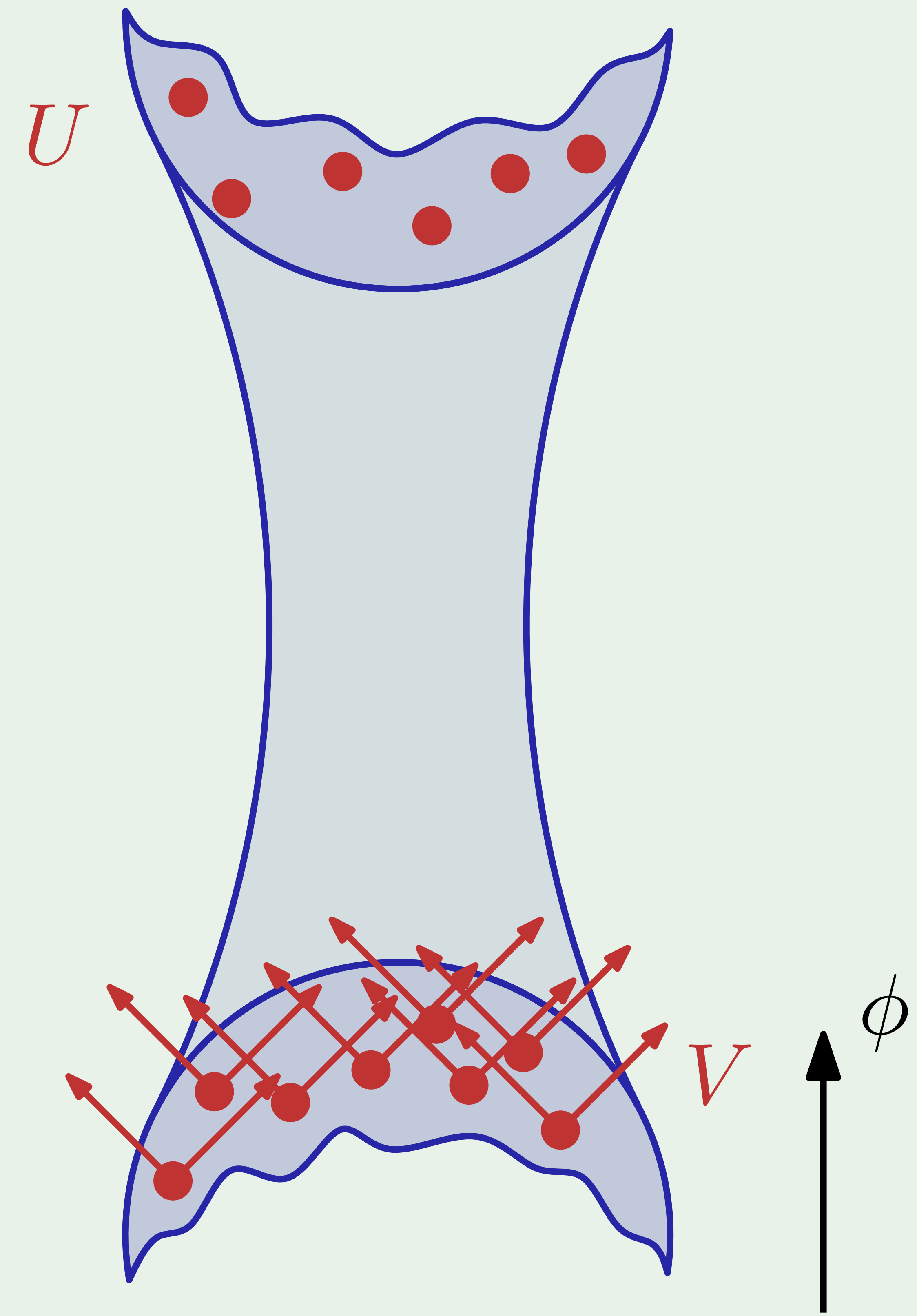

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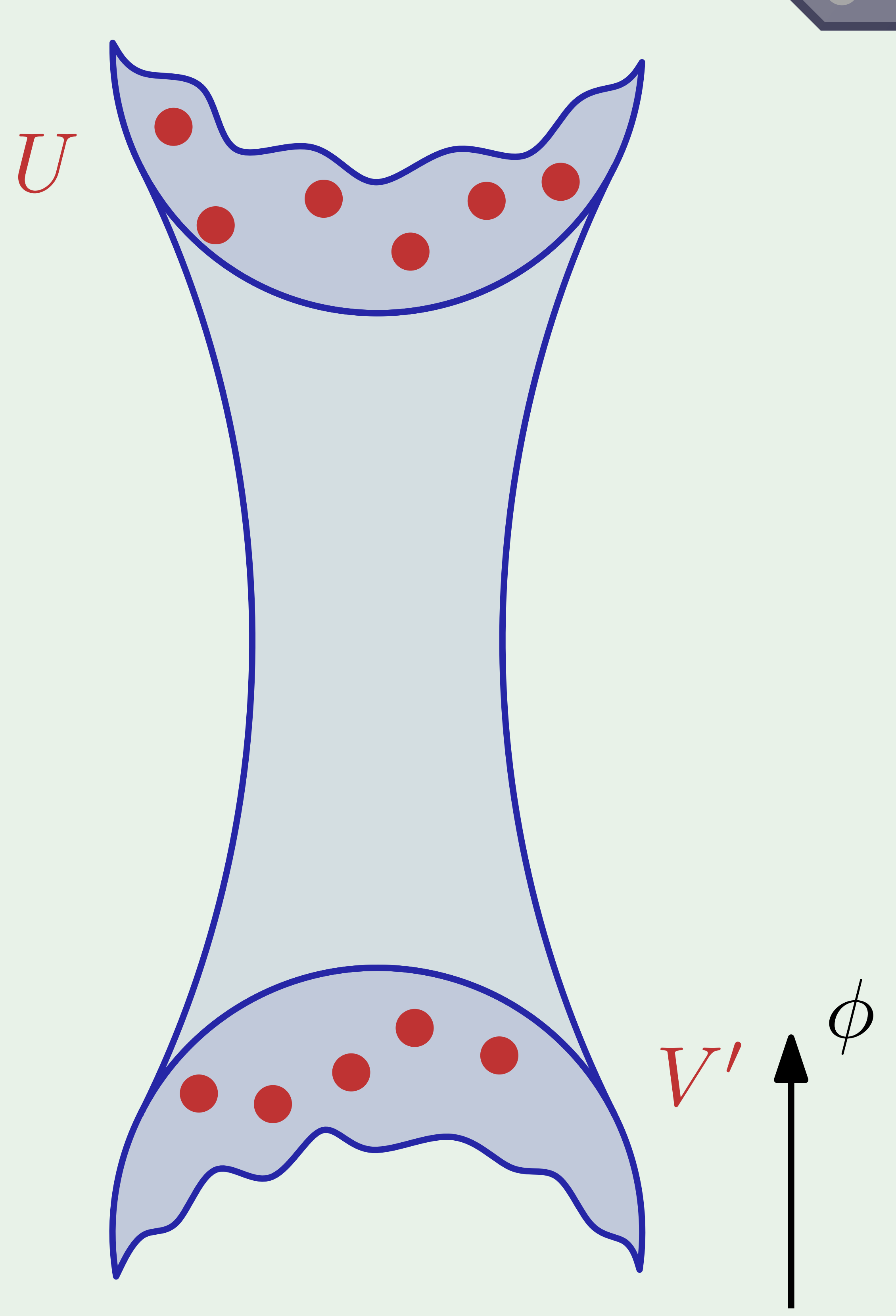

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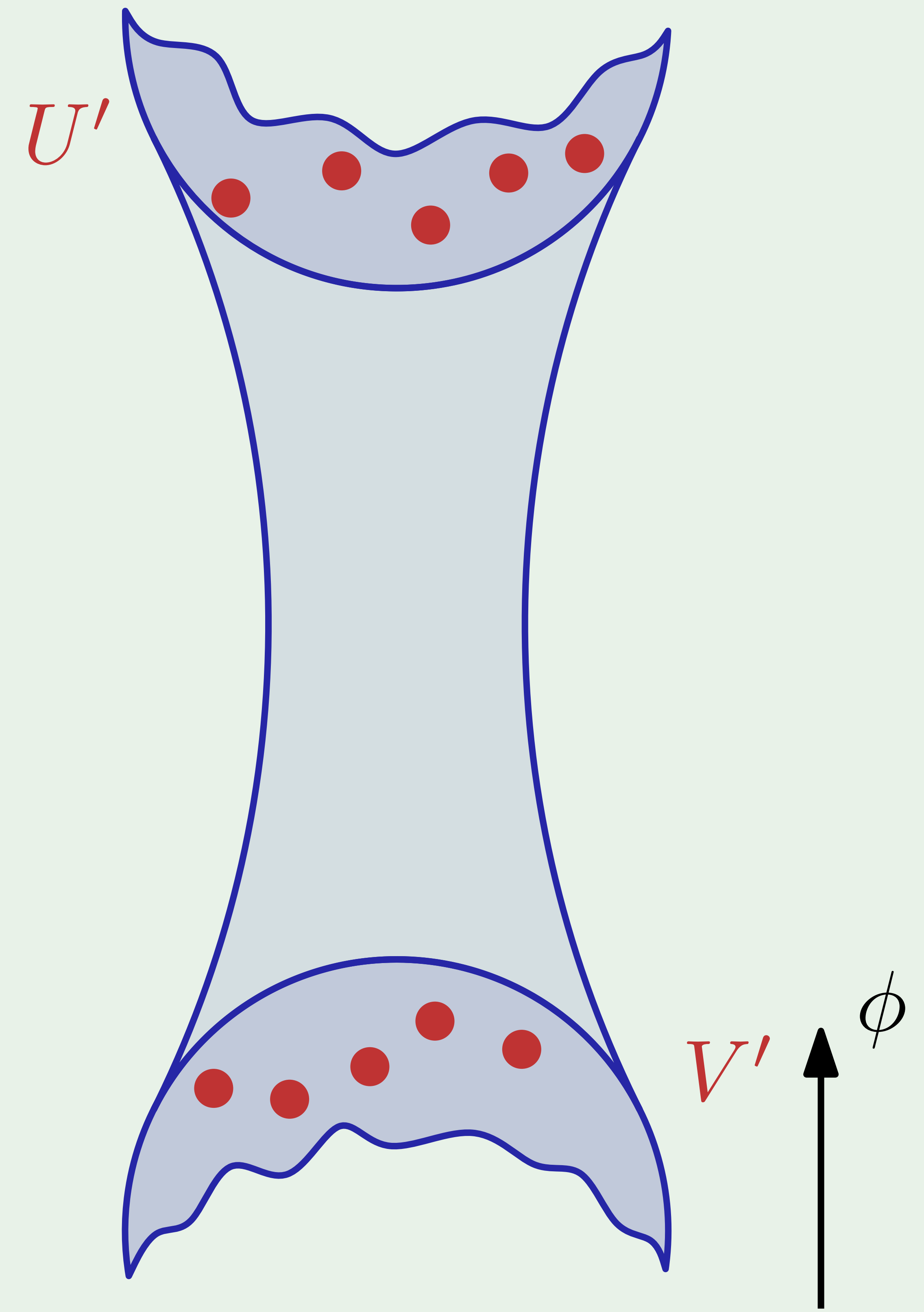

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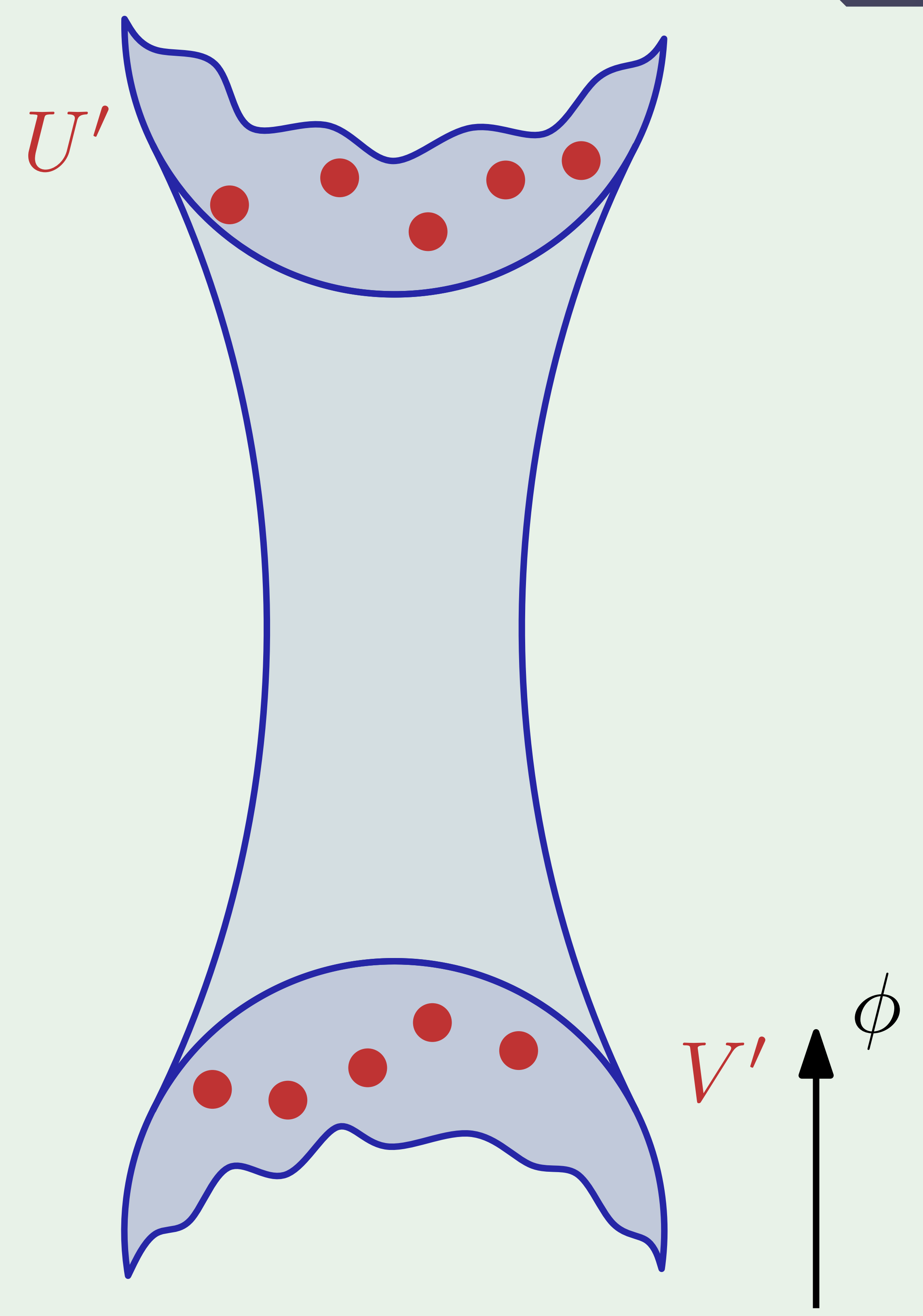

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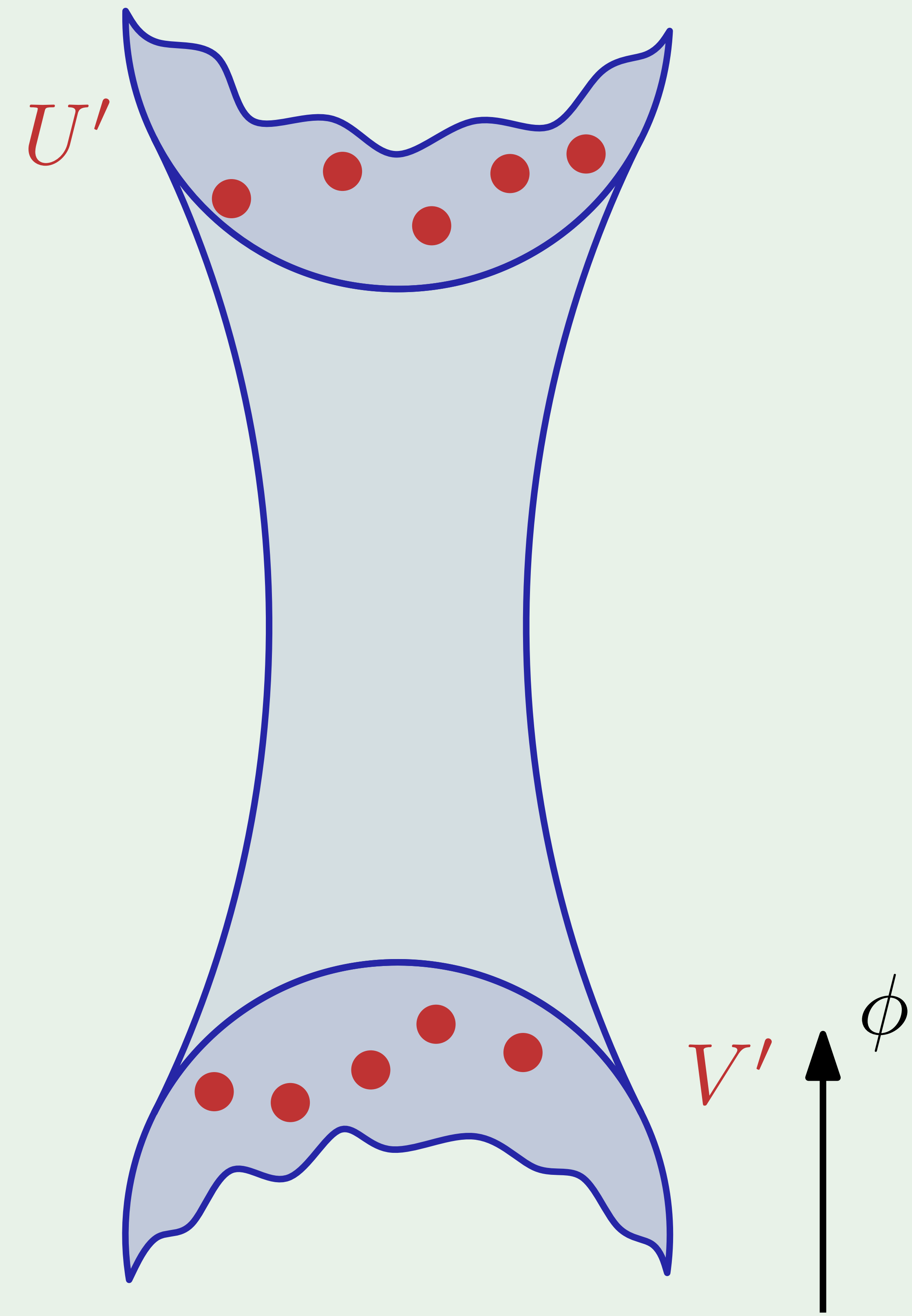
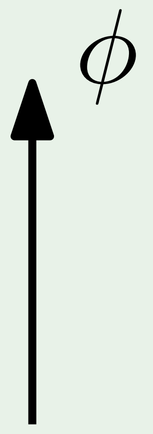
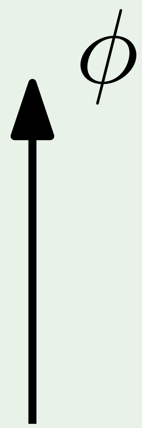
But we have that information: we started with a WSPD of P!




ϕ

Consider $\mathcal{P}_{\phi+\frac{1}{2}\pi}$, the pairs *perpendicular* to those in $\mathcal{P}_\phi$.



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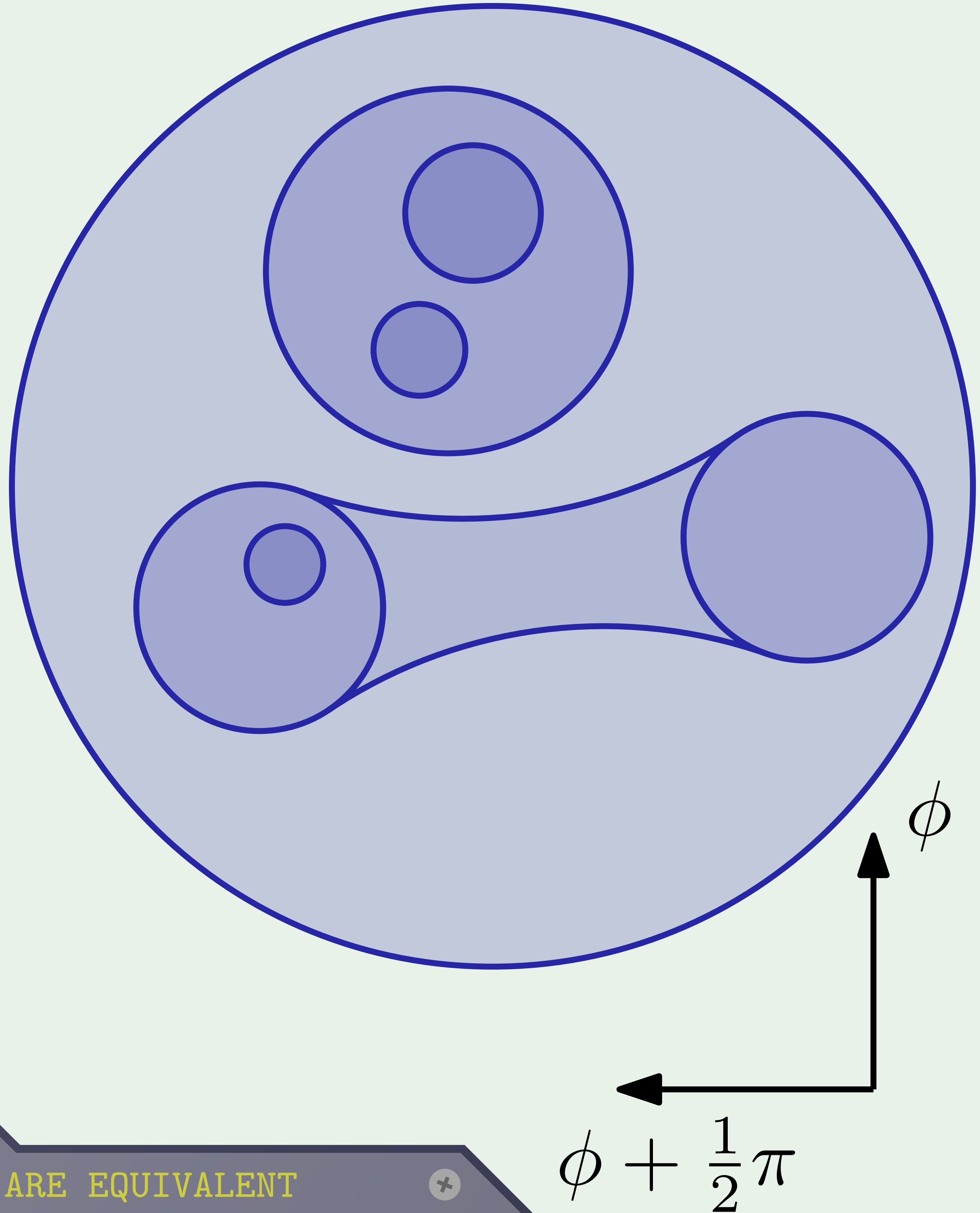

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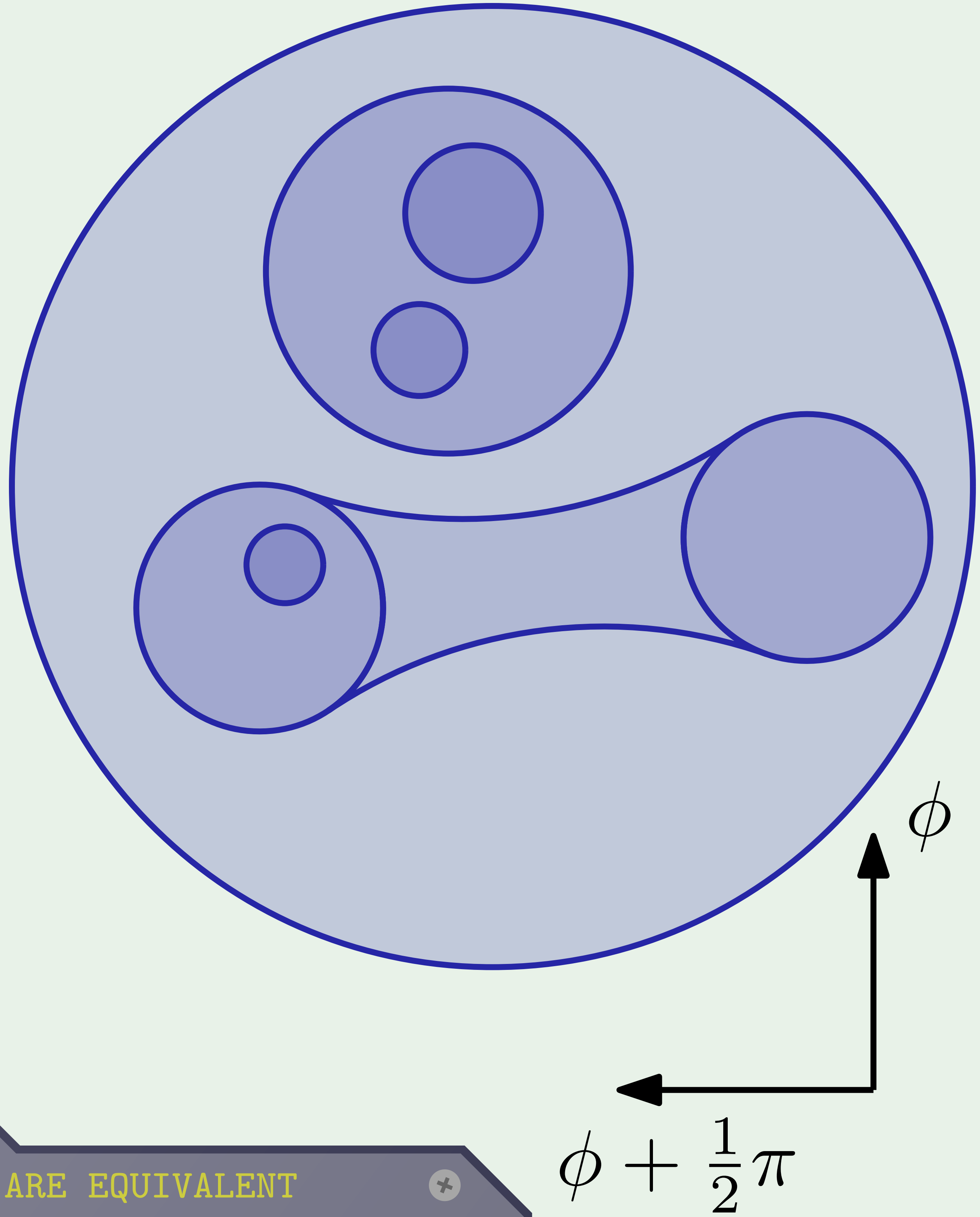
Let V be a set with two copies of each set in $\mathcal{P}_{\phi+\frac{1}{2}\pi}$.



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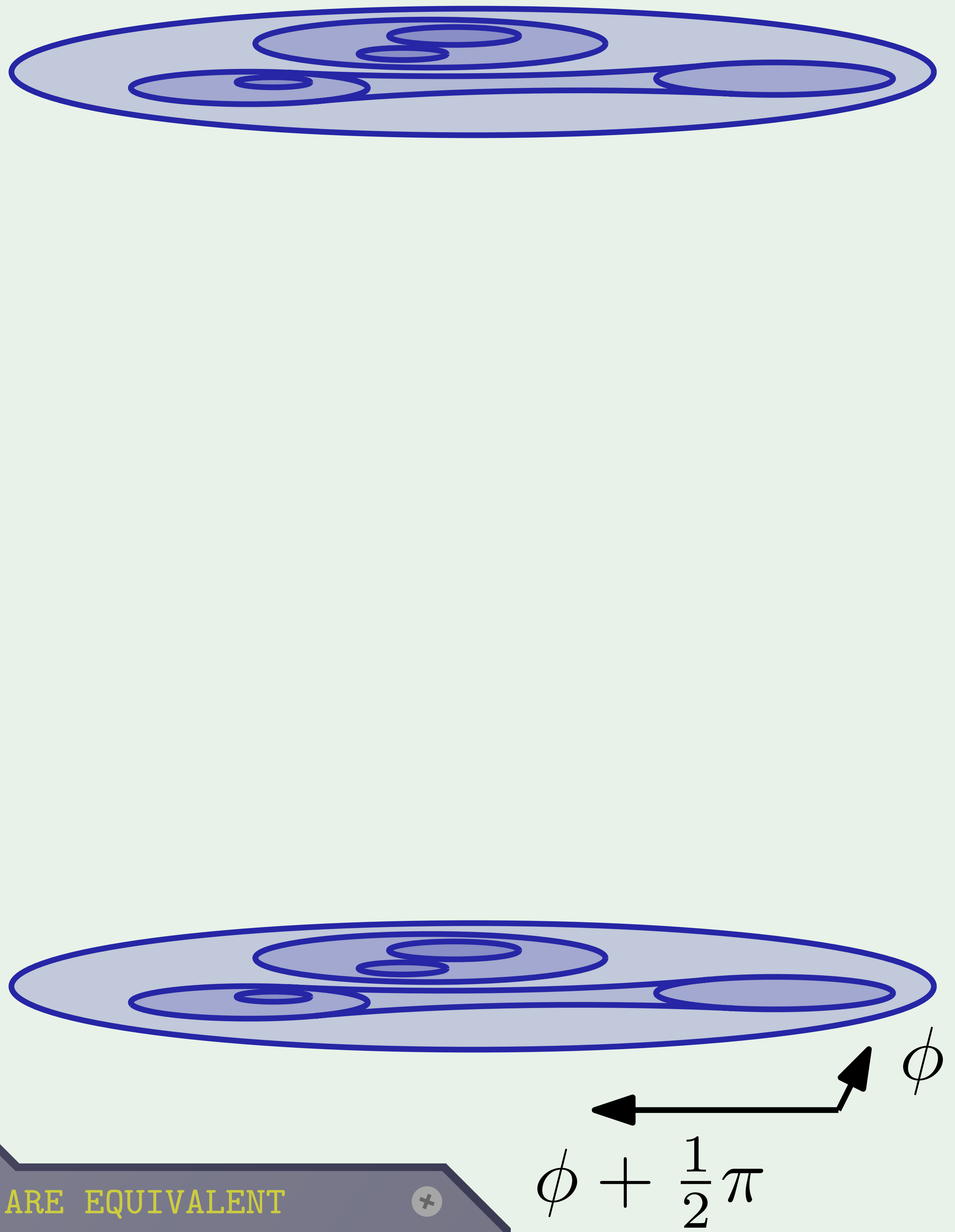

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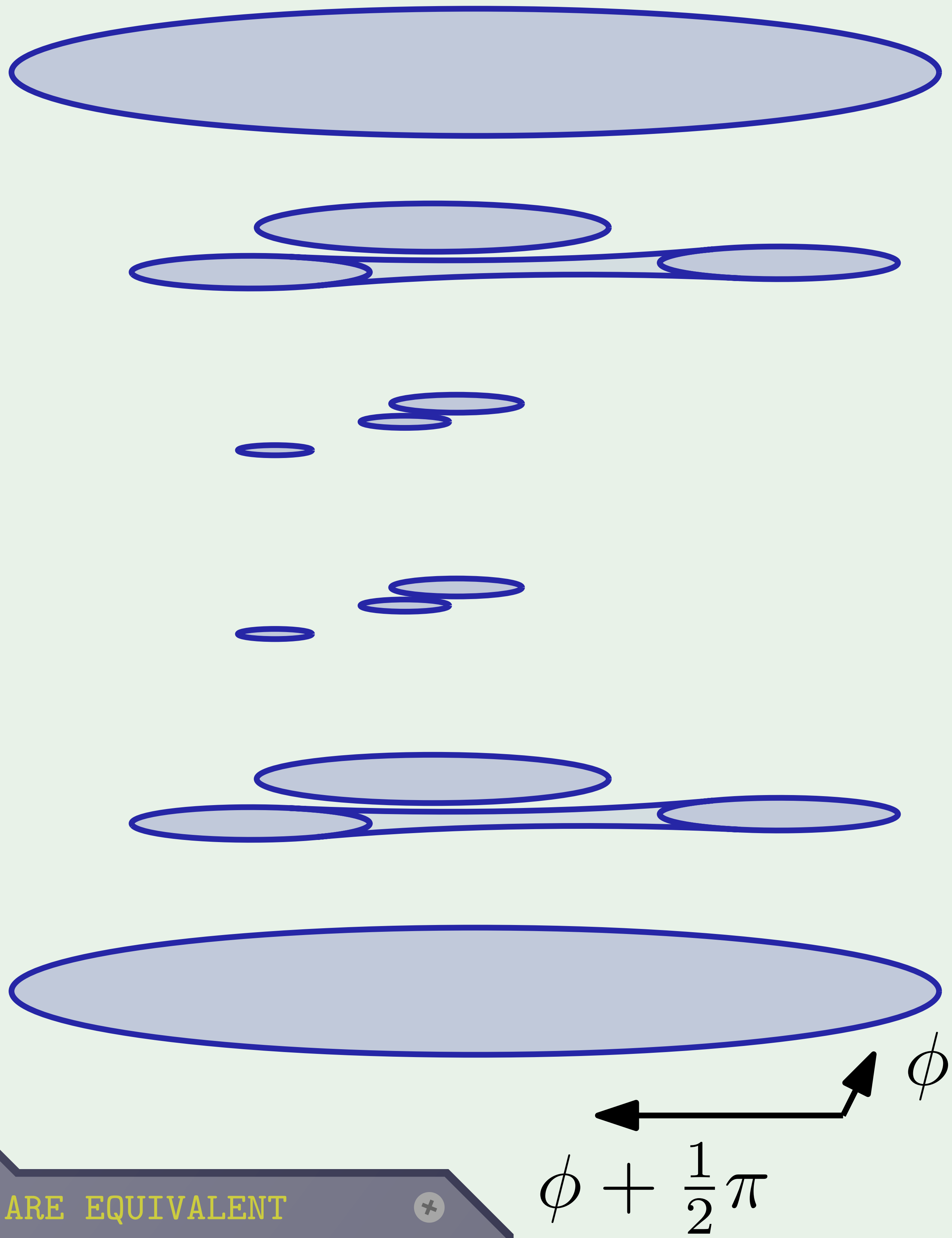

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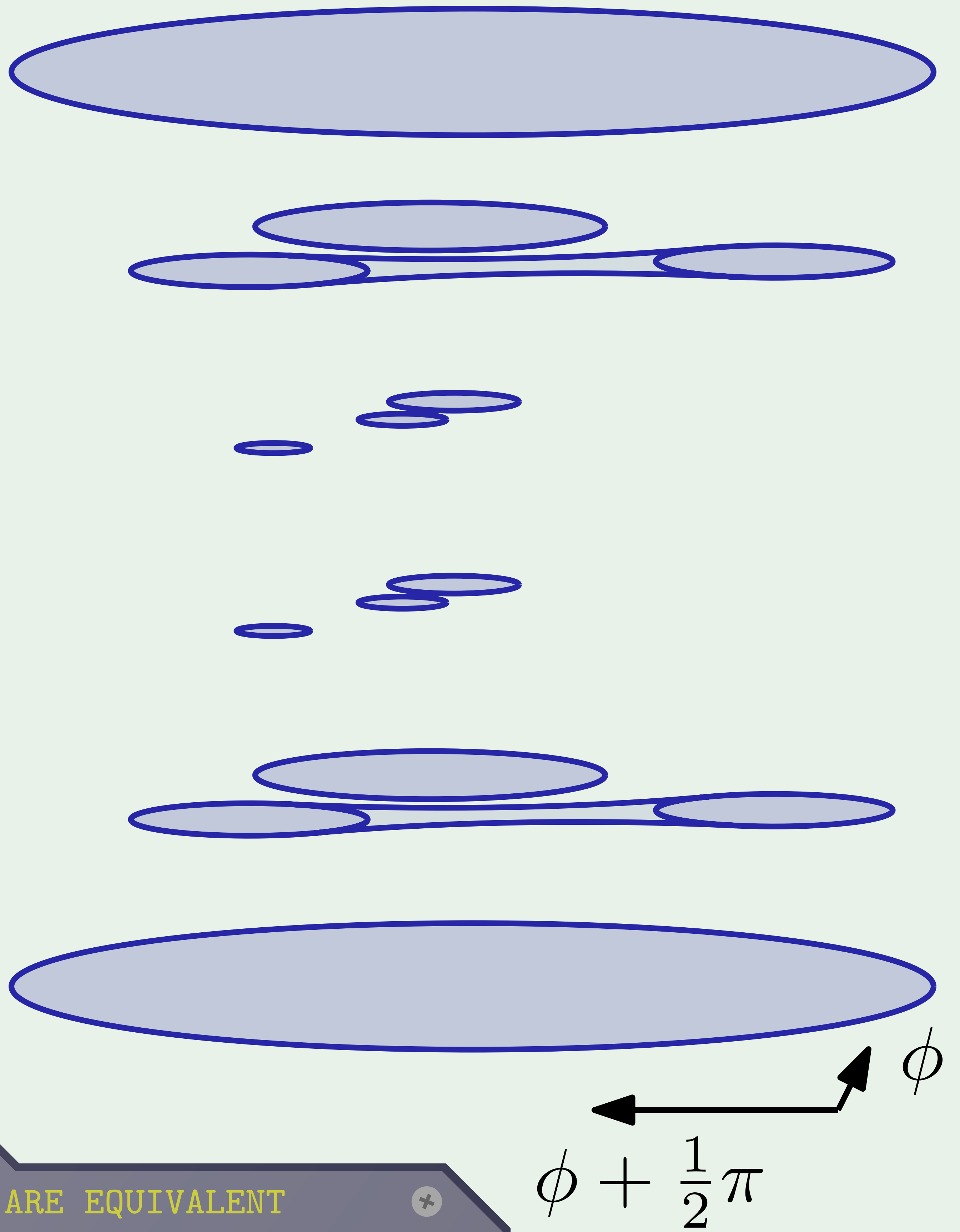
Let Γ be a graph on V that follows the hierarchical structure and WSPD pairs.



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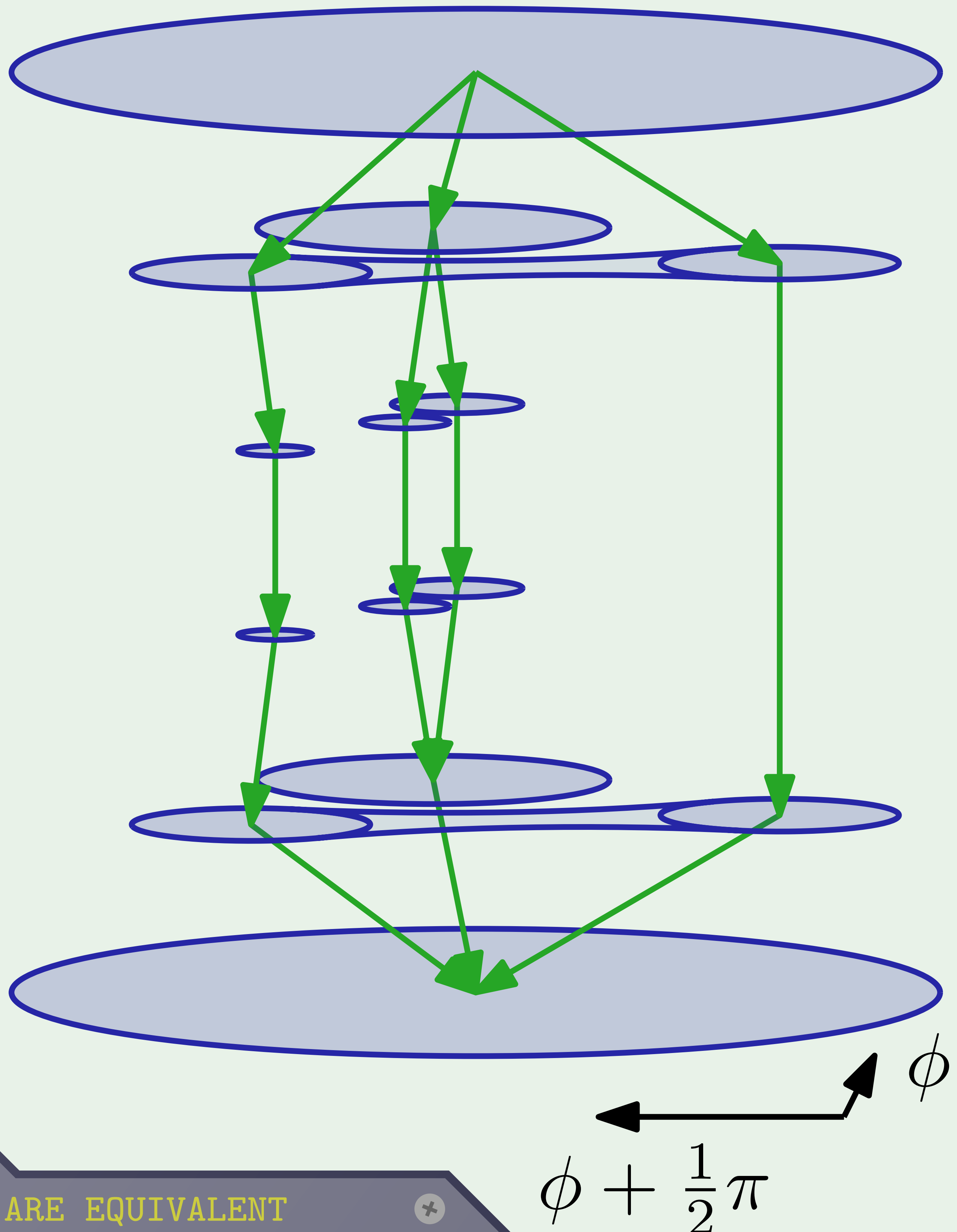

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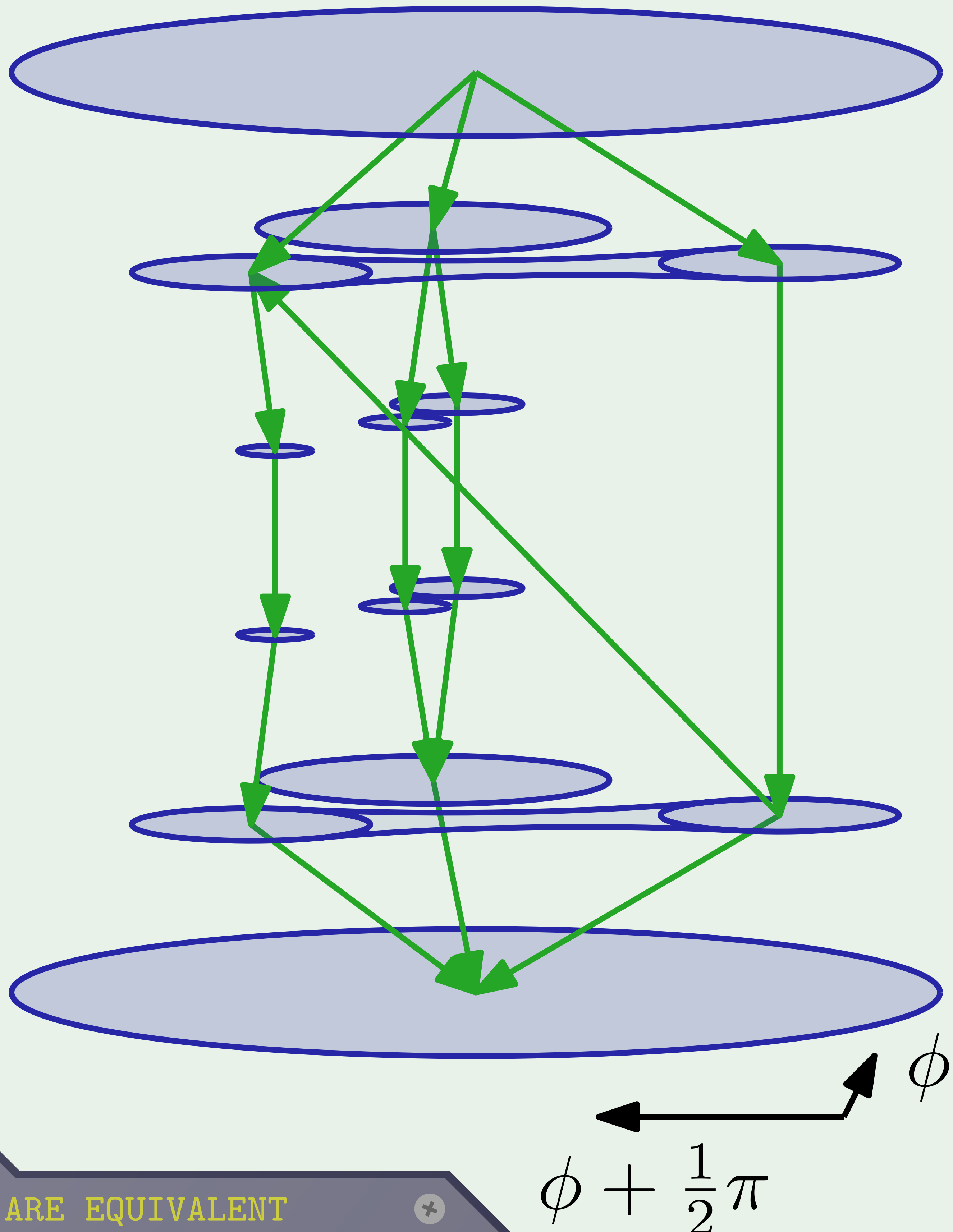
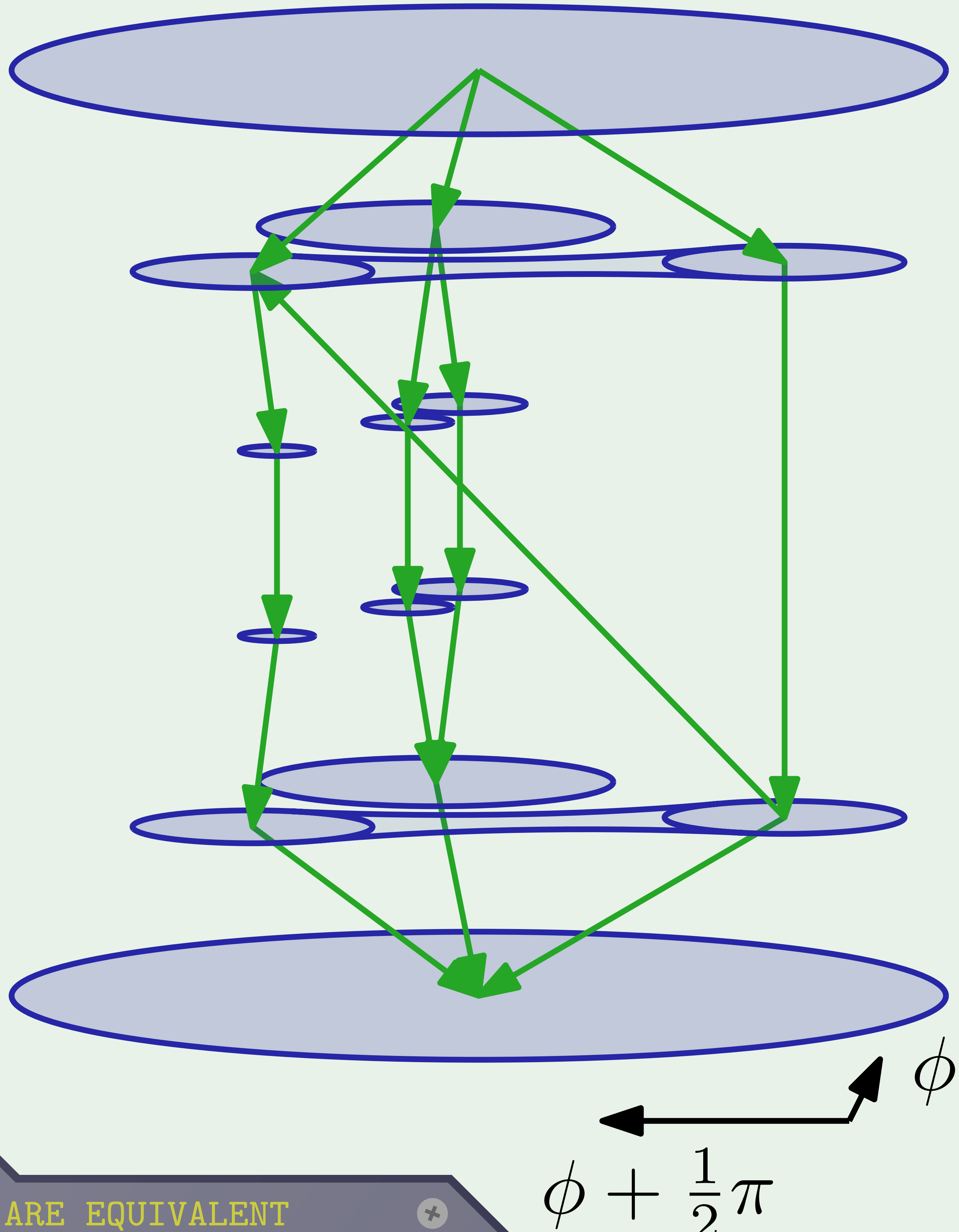


Claim: Γ is acyclic.



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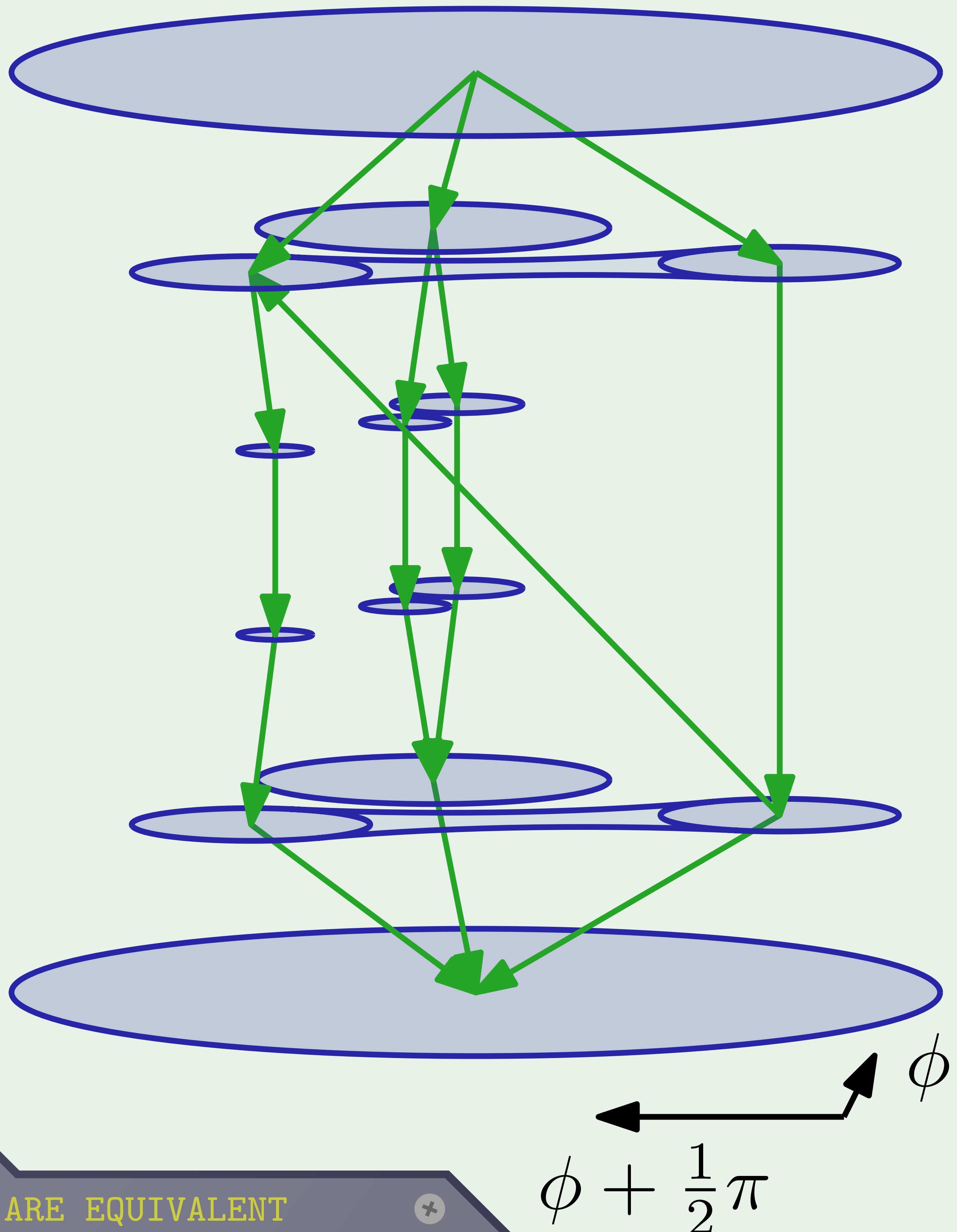

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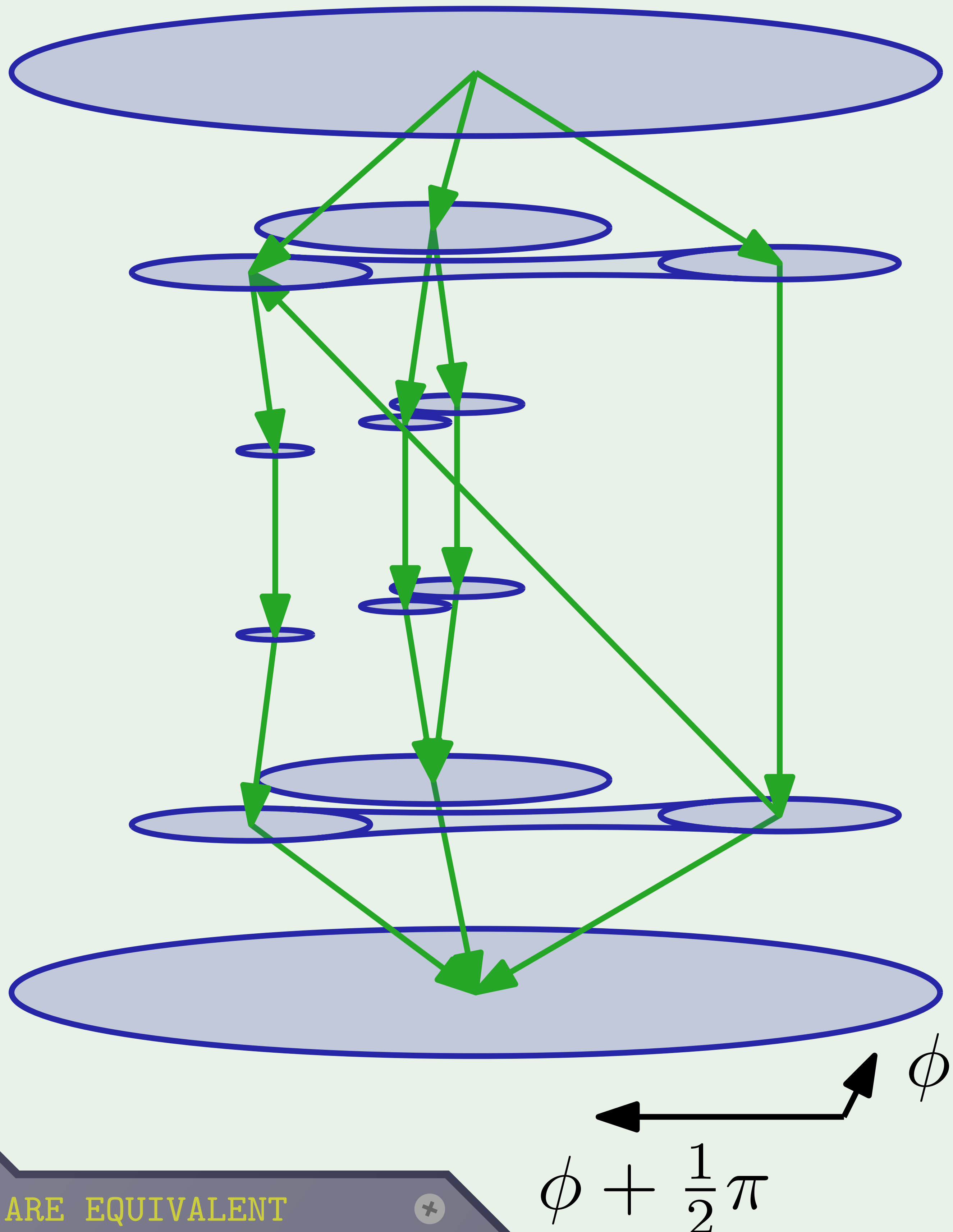
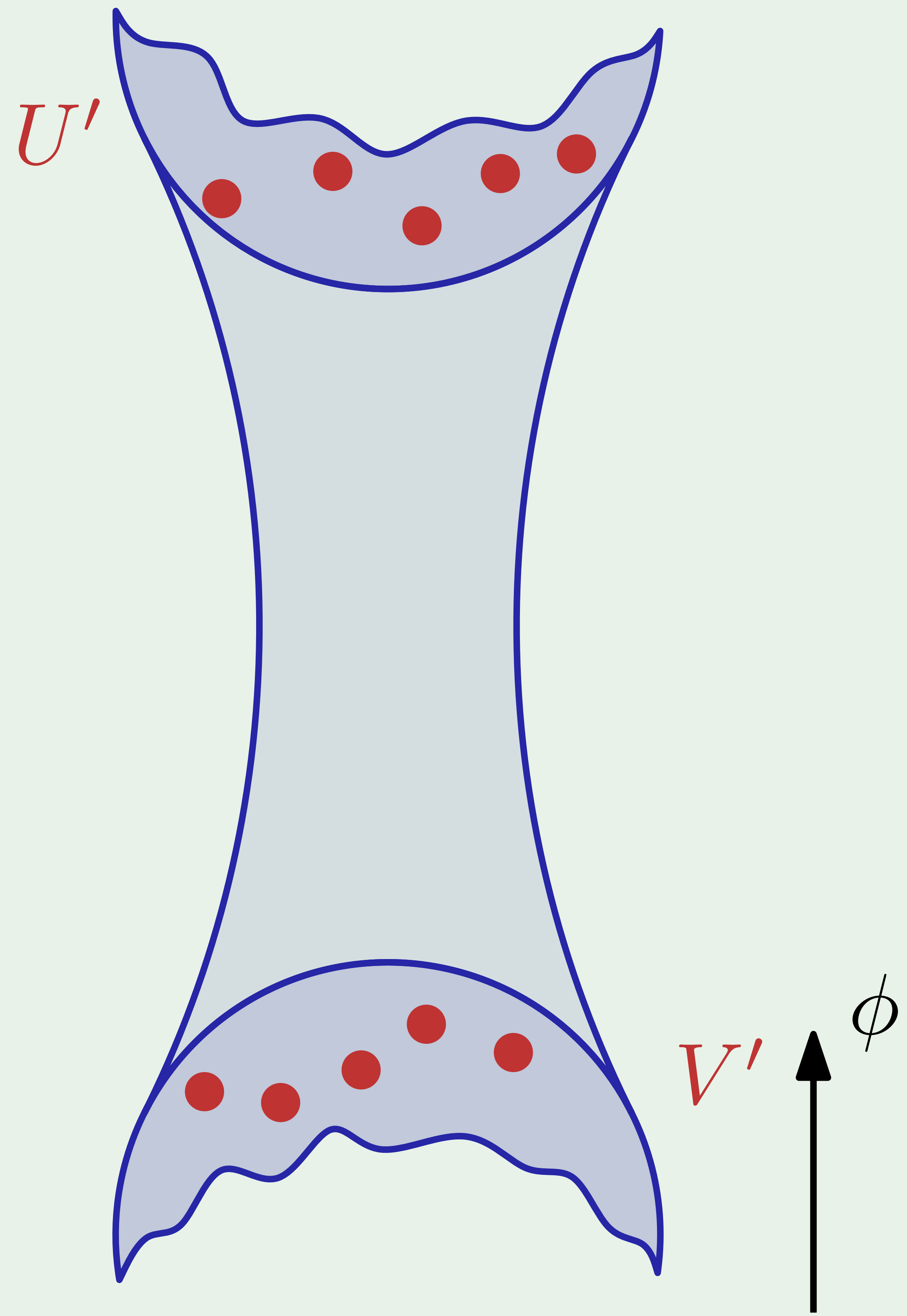
Claim: We can sort P conforming Γ in linear time.



Now, we can find the closest edge e between U' and V' in $O(|U| + |V|)$ time.



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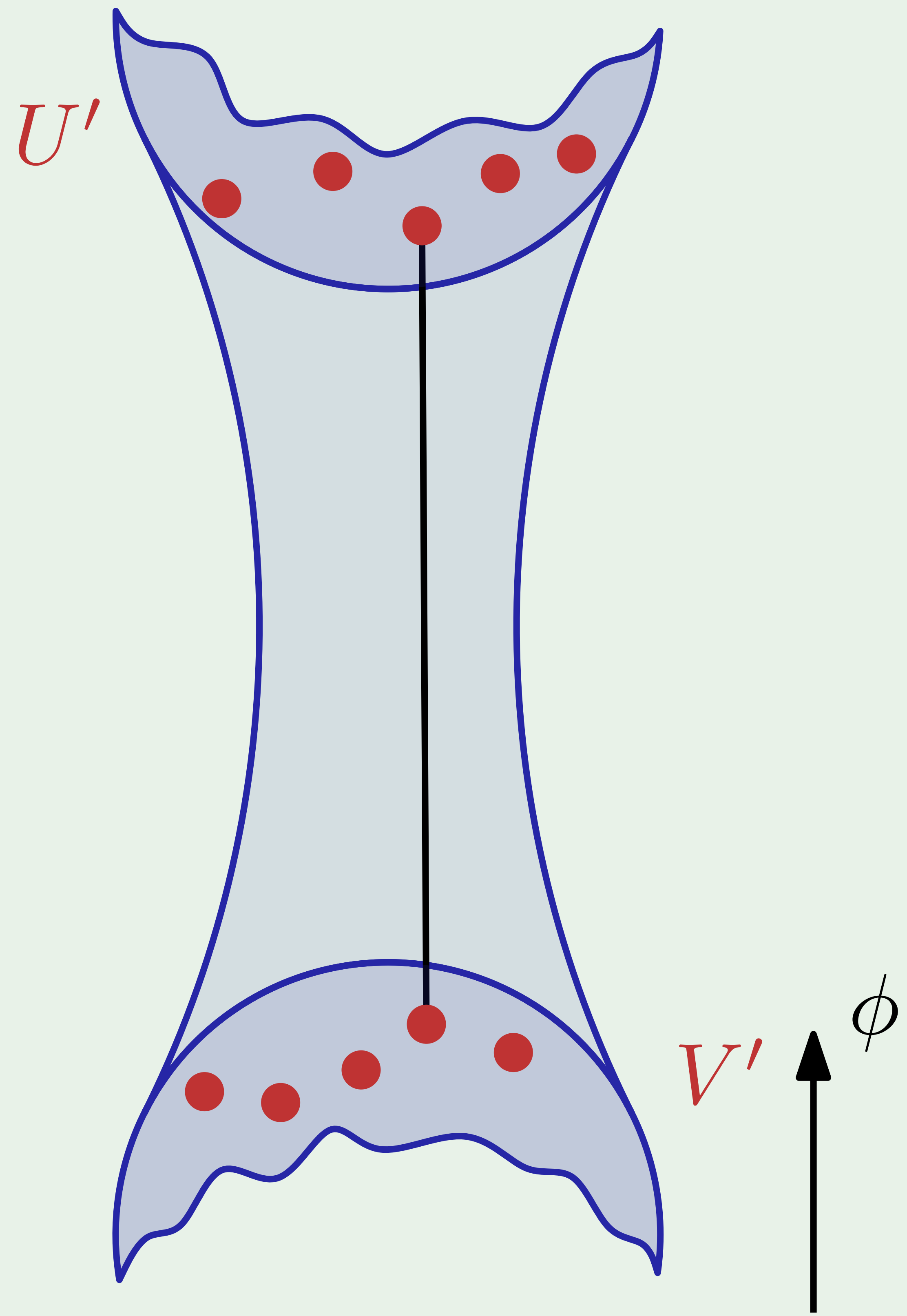

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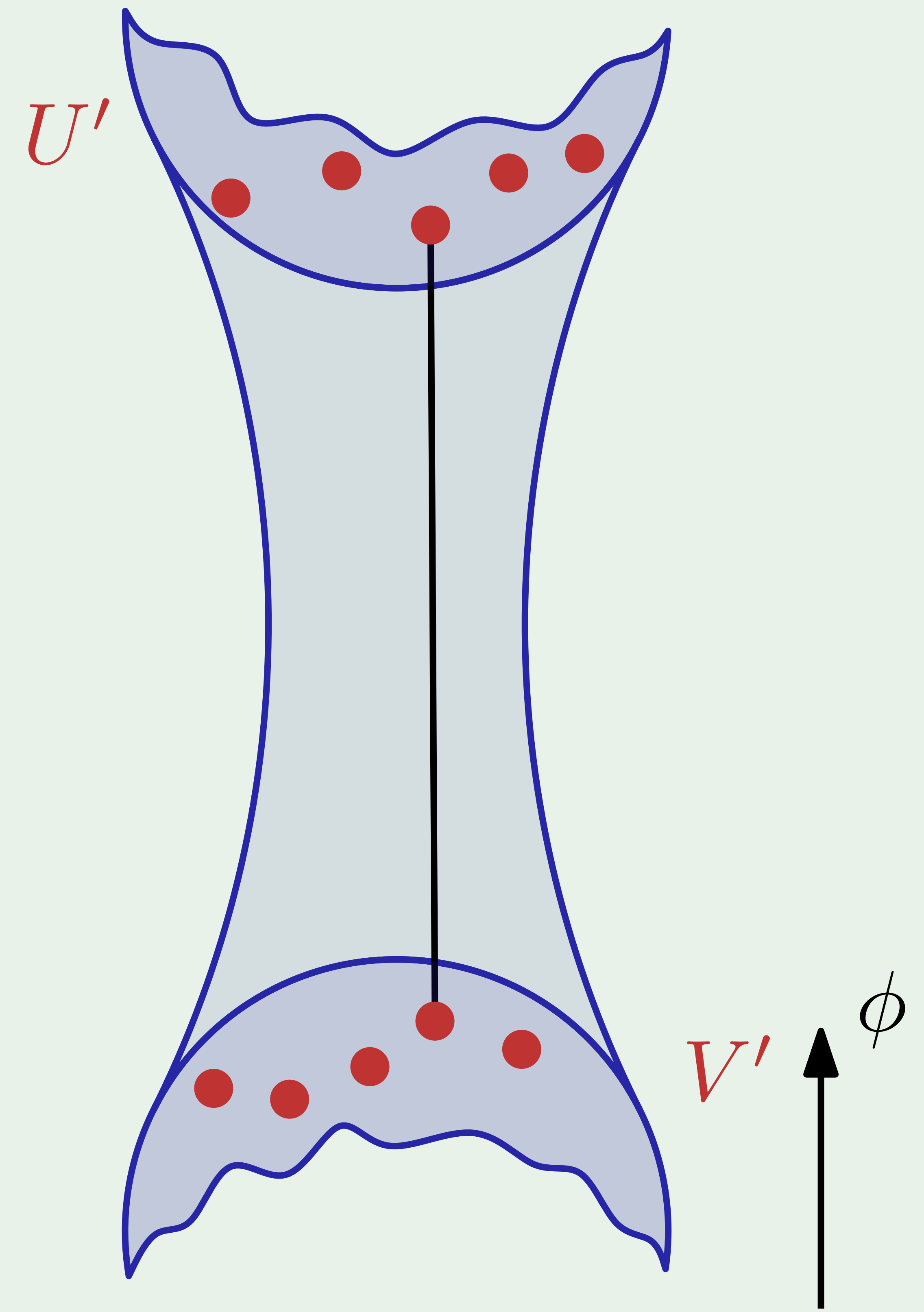

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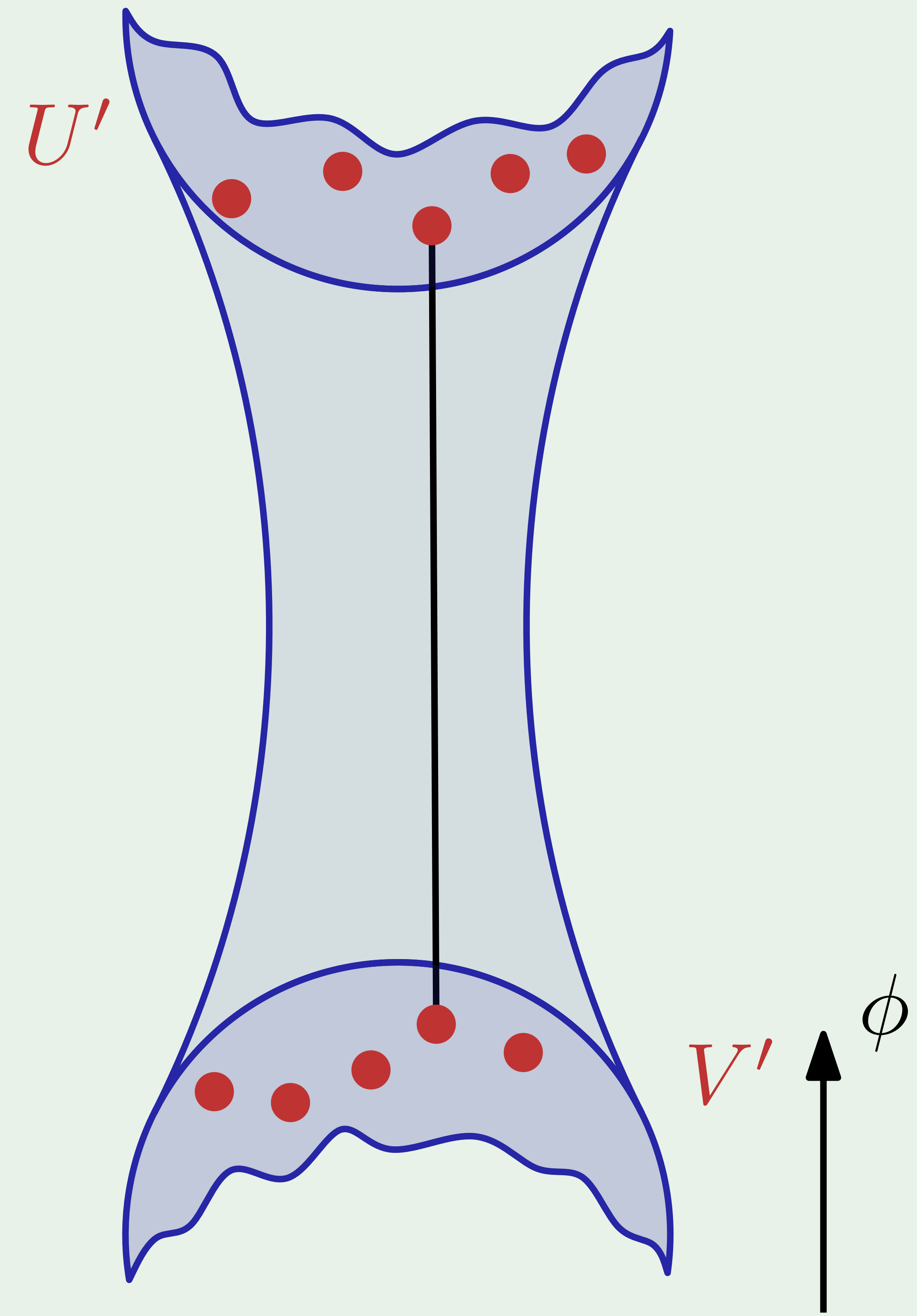

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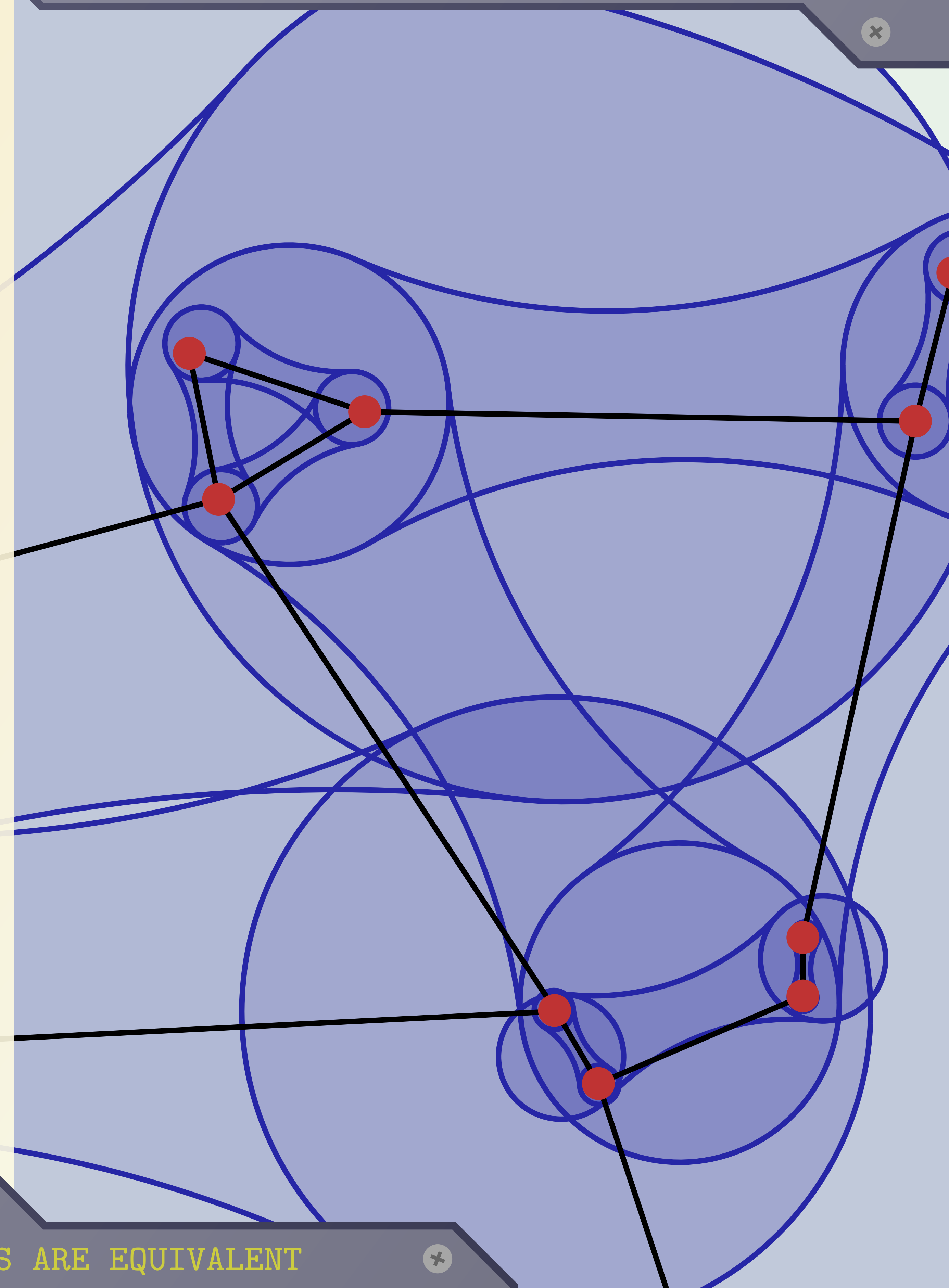

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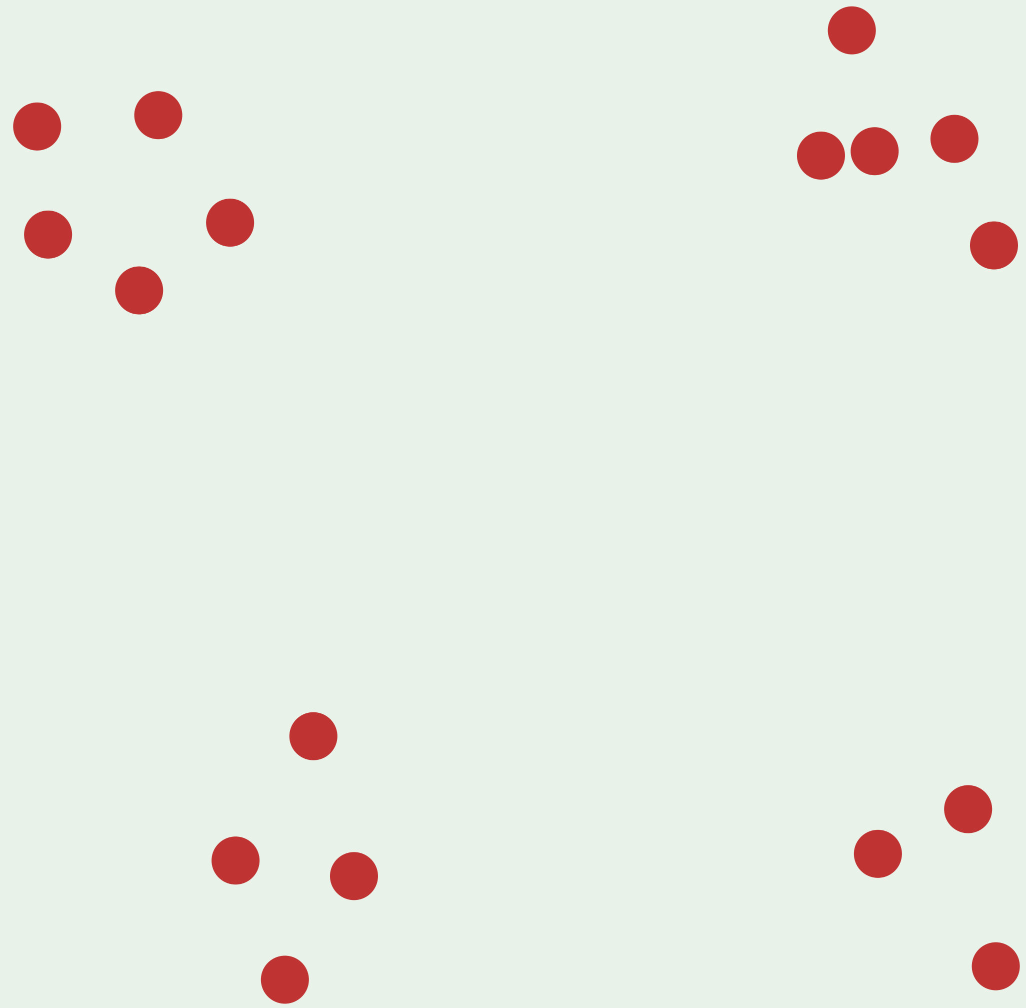

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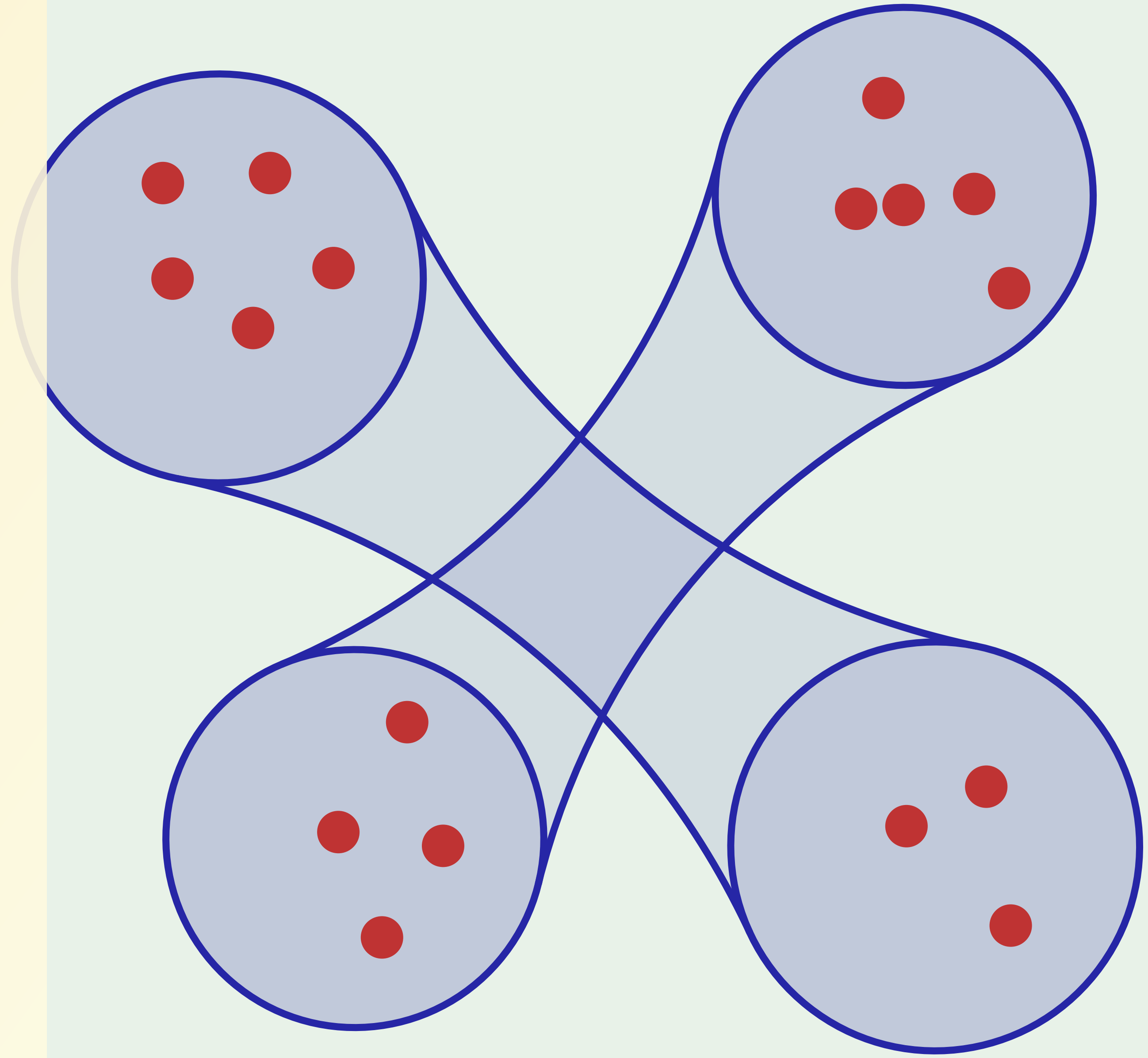

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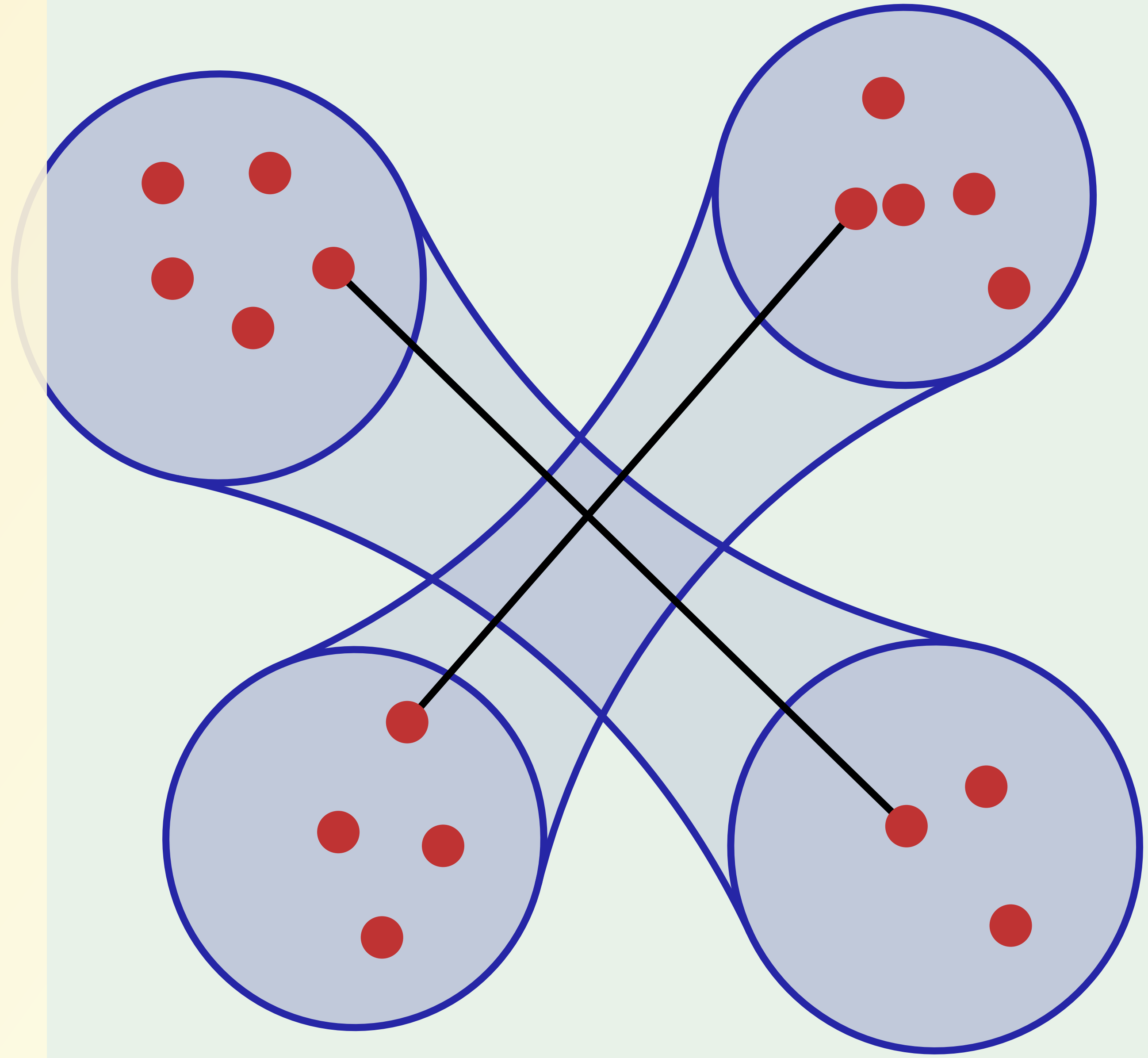

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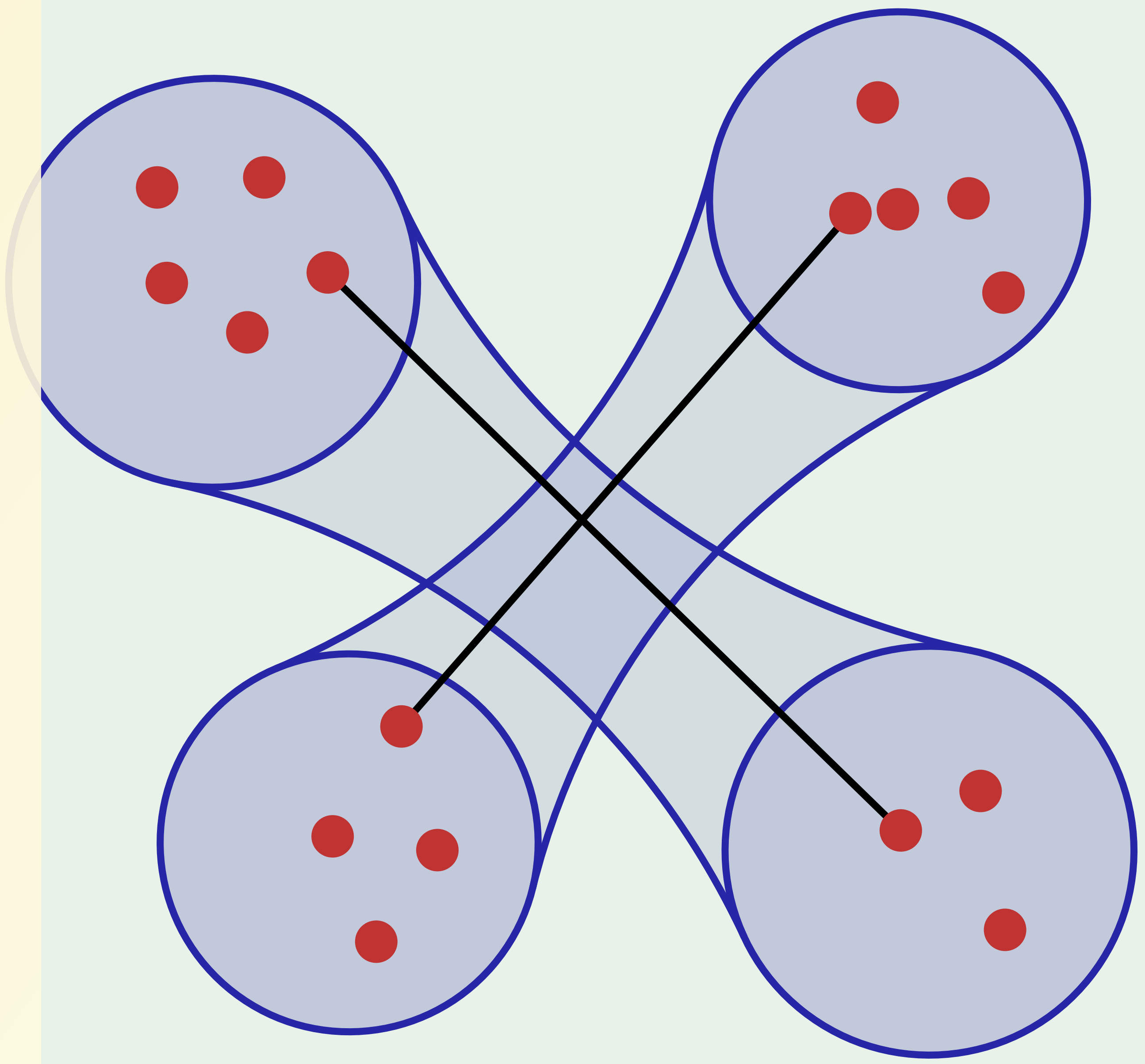
Unfortunately, G need not be planar.

Claim: any edge e of G crosses at most $O(1)$ edges of length $\Omega(|e|)$.



EXTRACTING THE MST

Start with P and G.

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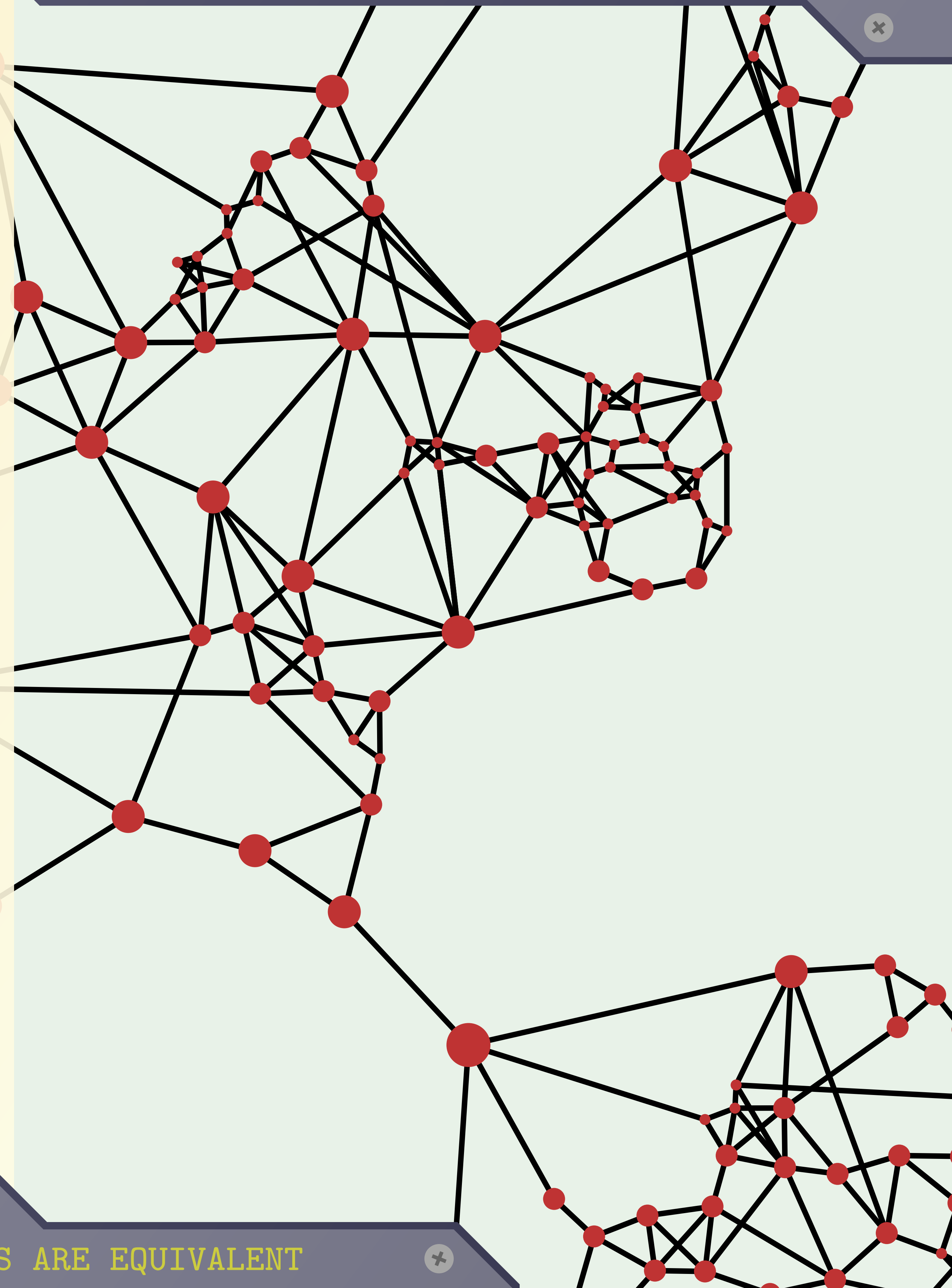
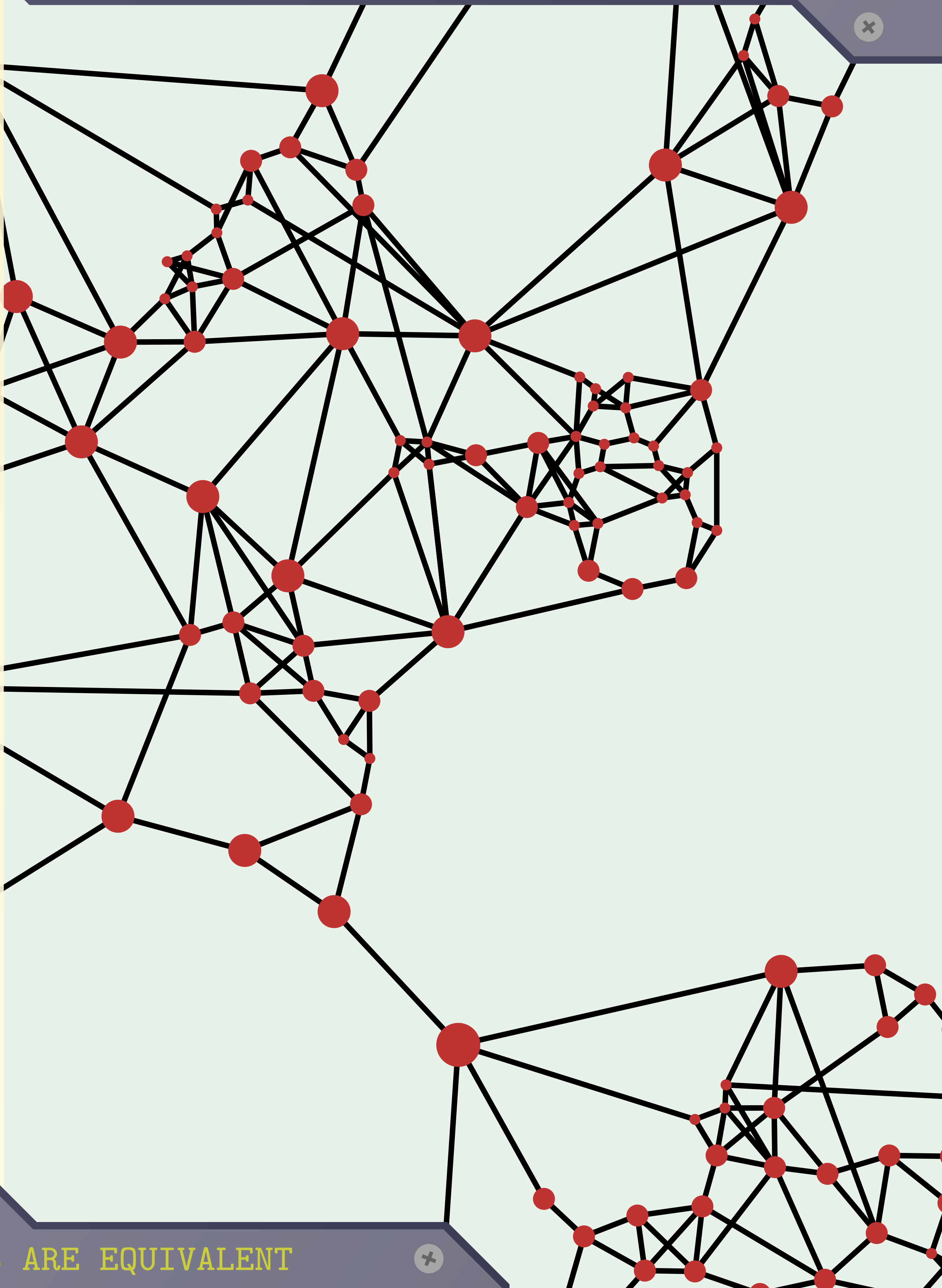


Start with P and G.

Remember we have a quadtree T on P.



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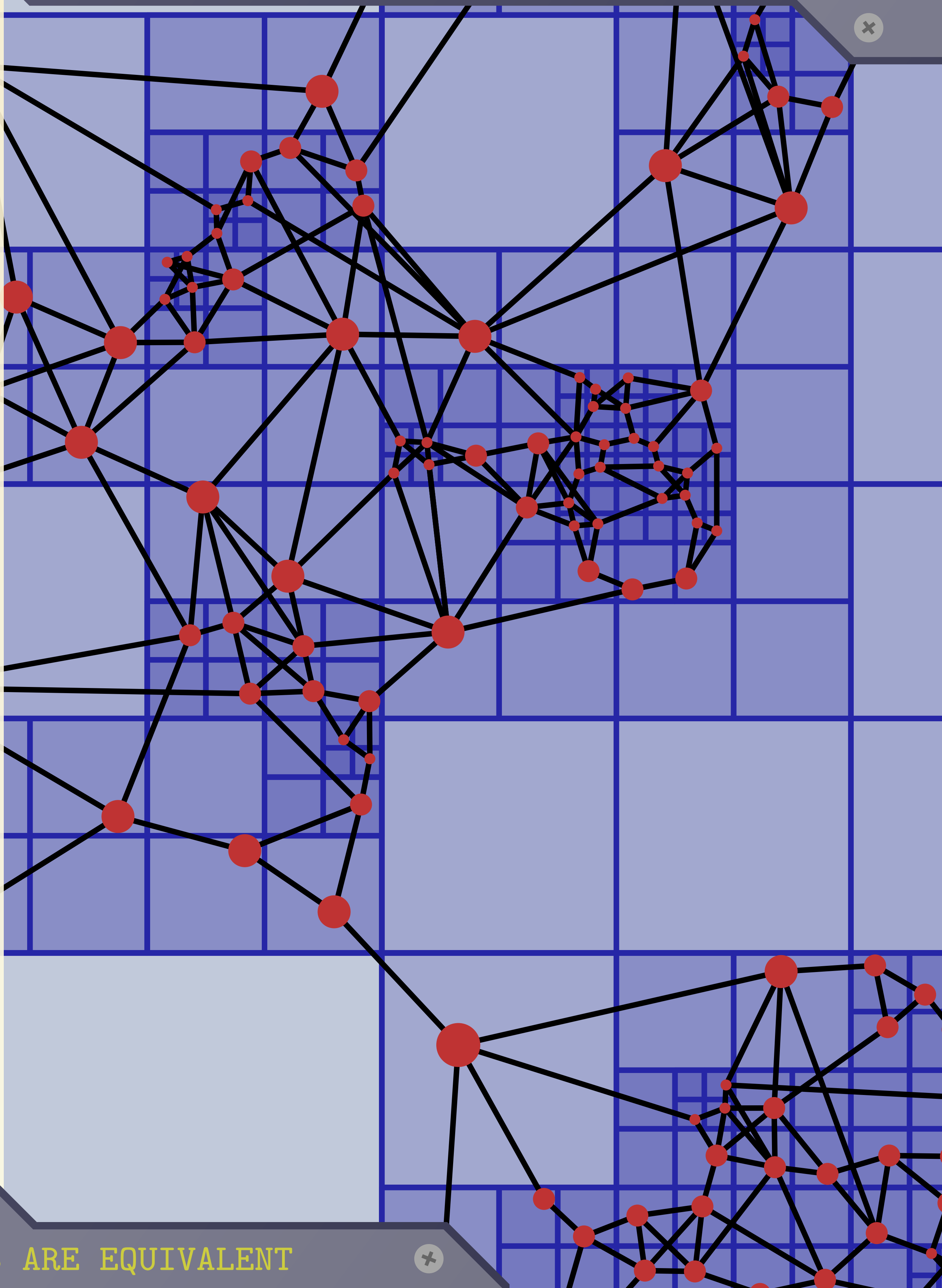

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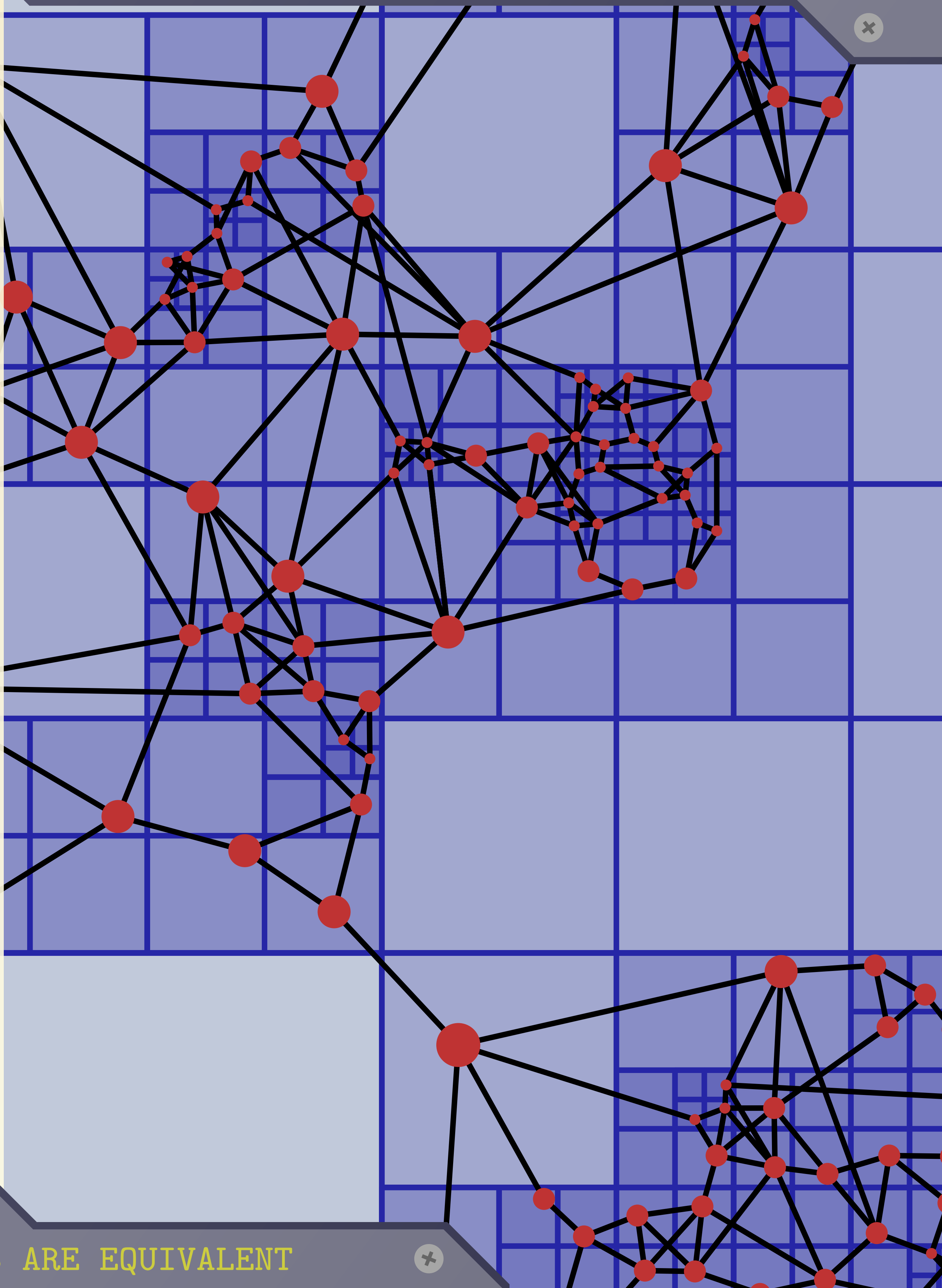
We can process the edges of G by increasing length, Borůvka-style.



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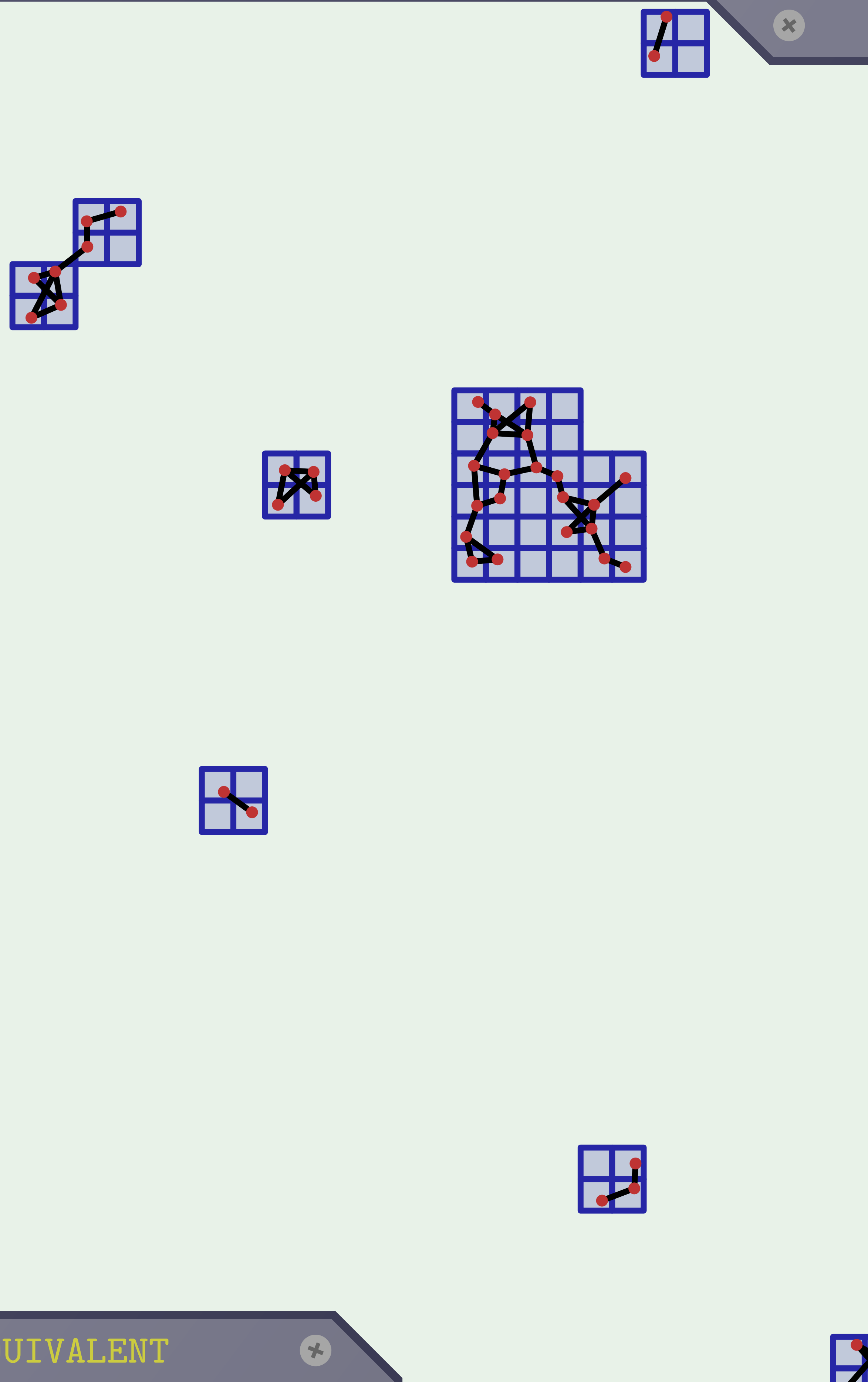

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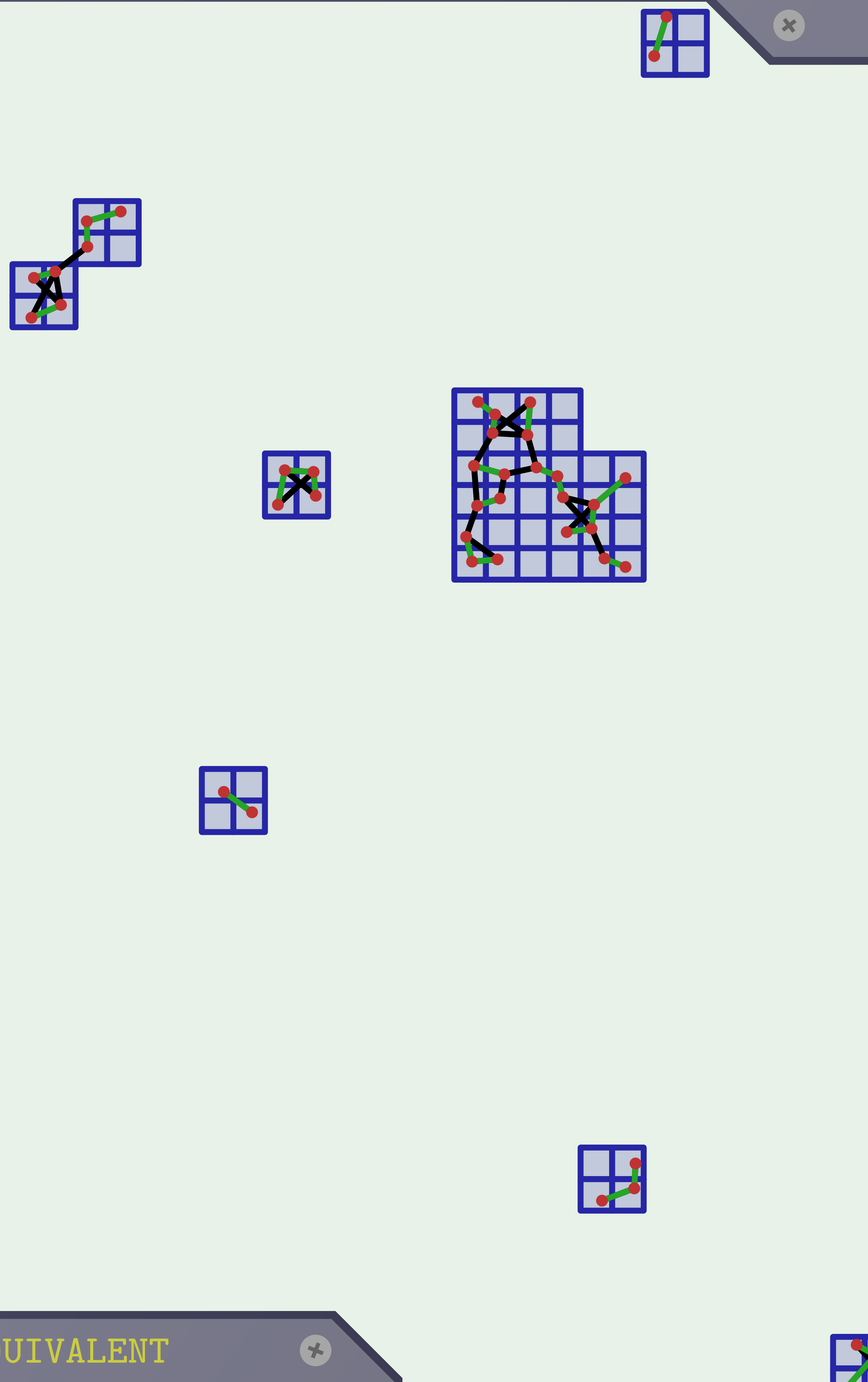

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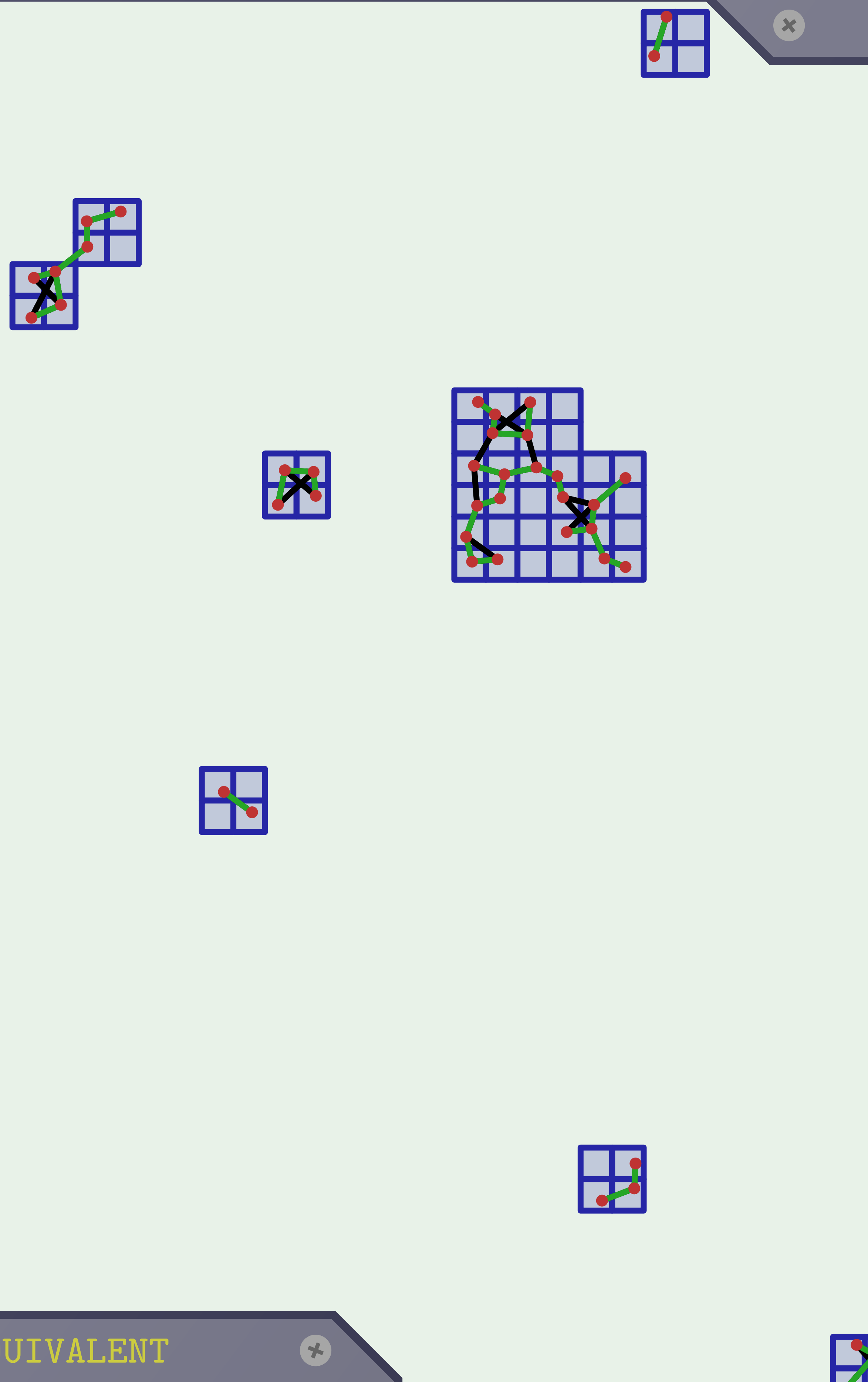

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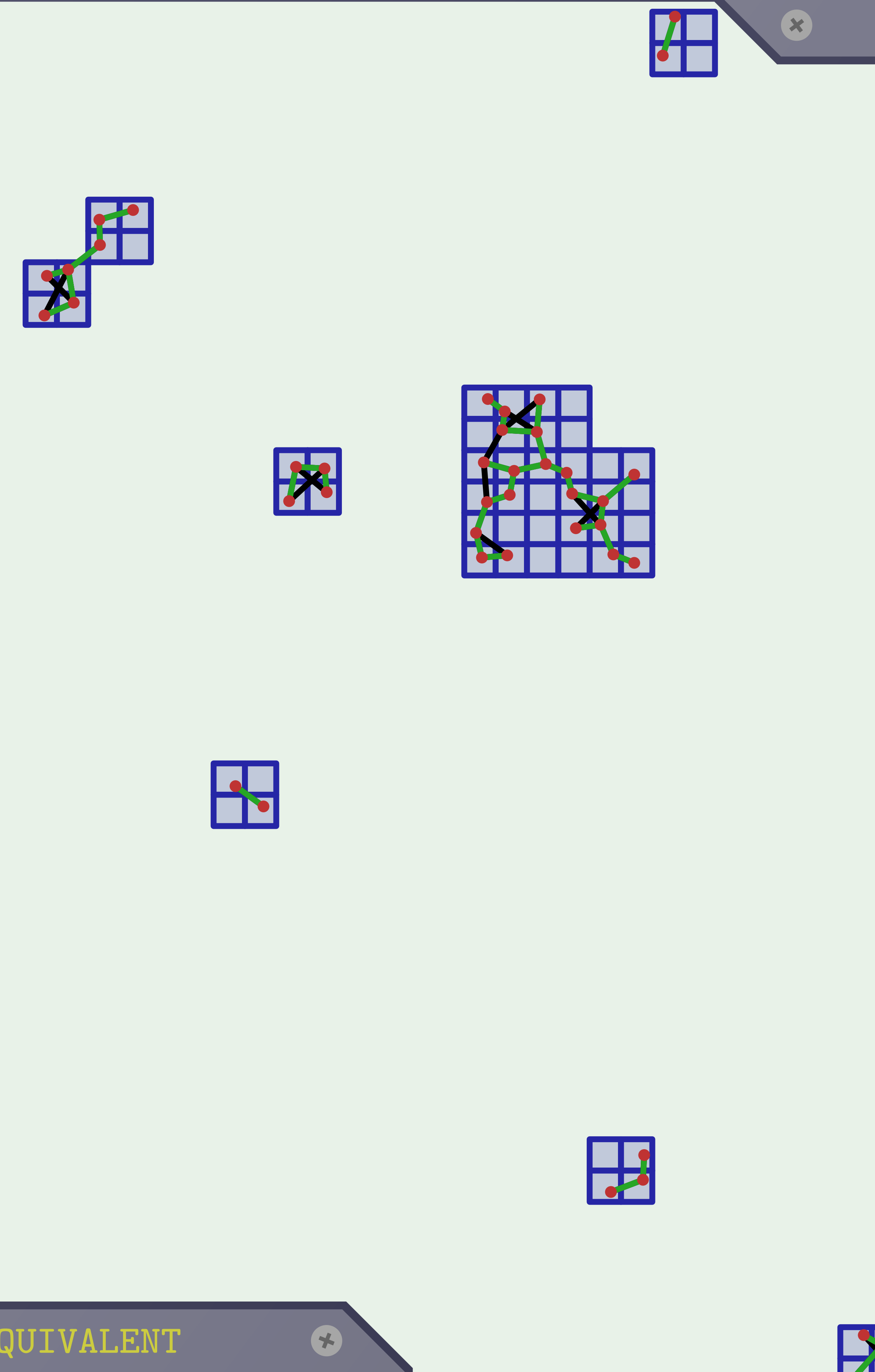

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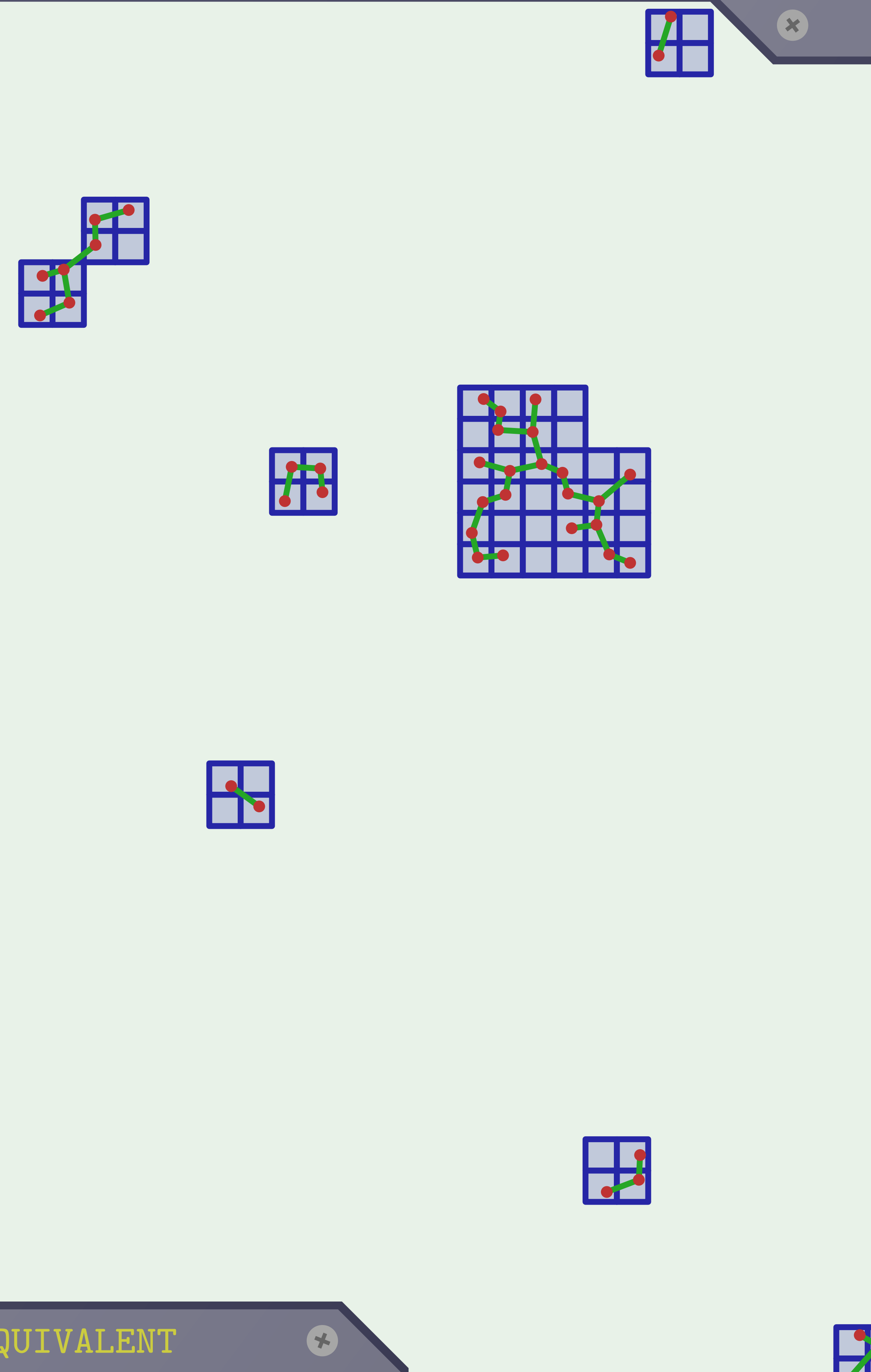

EXTRACTING THE MST

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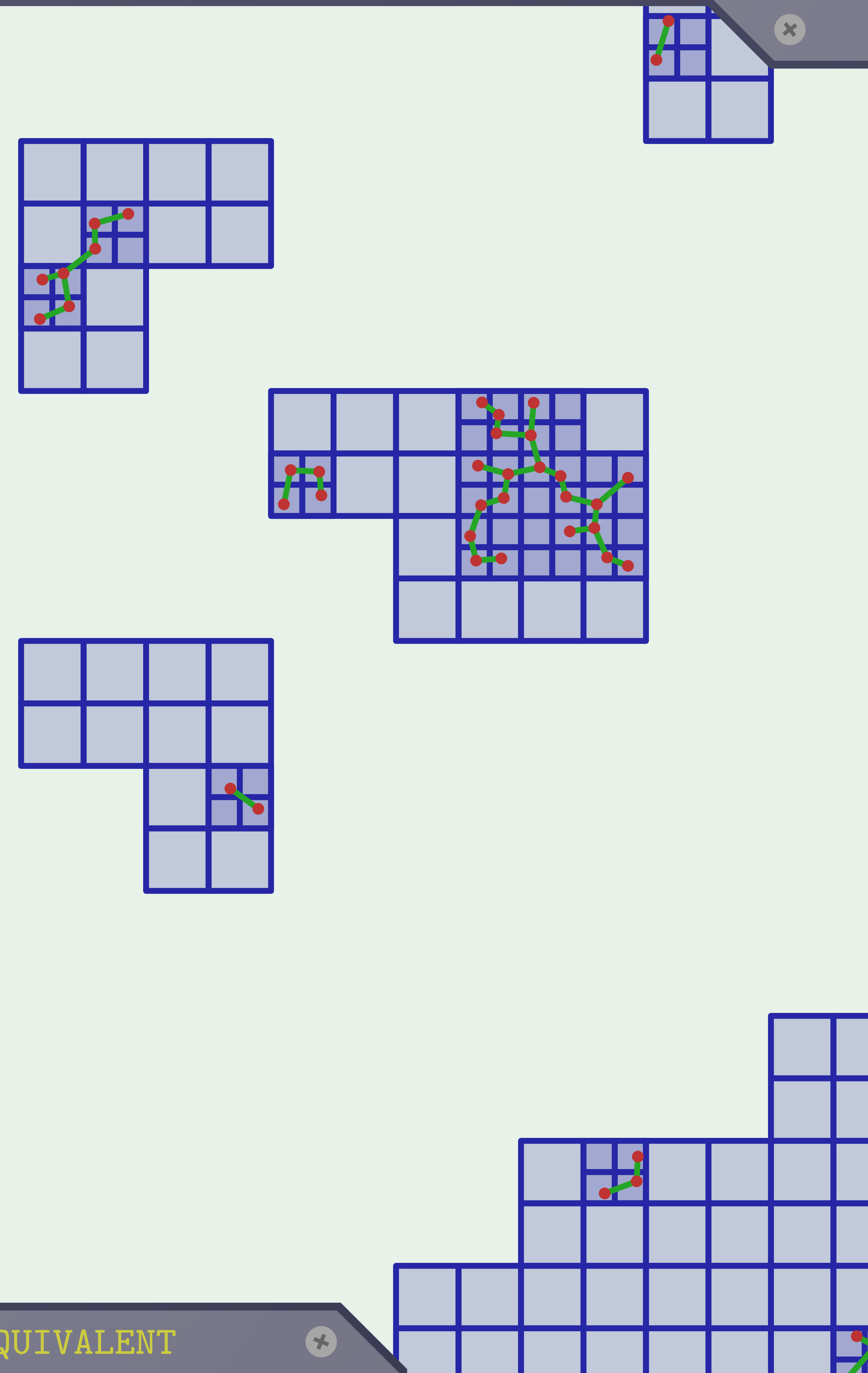

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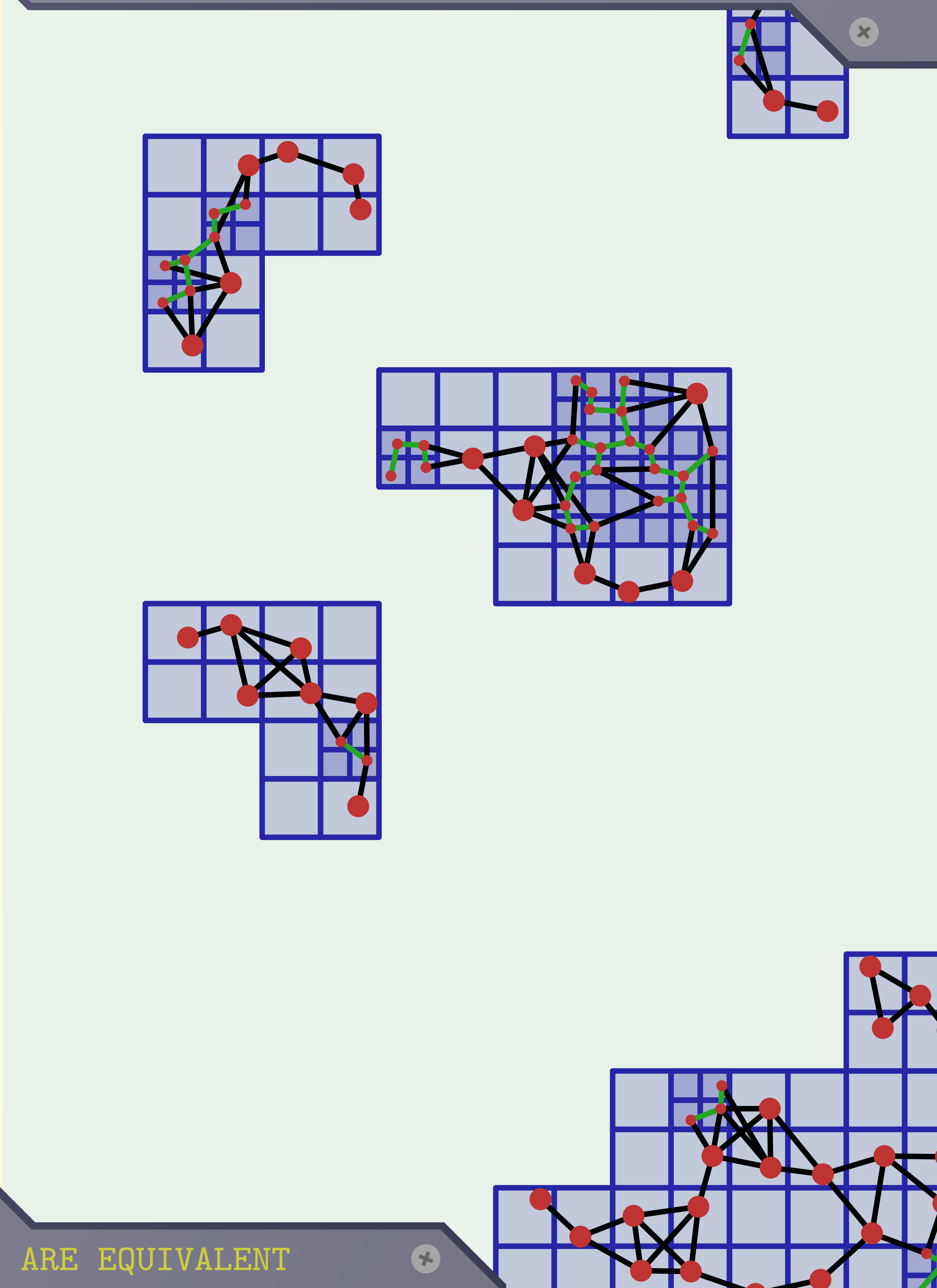

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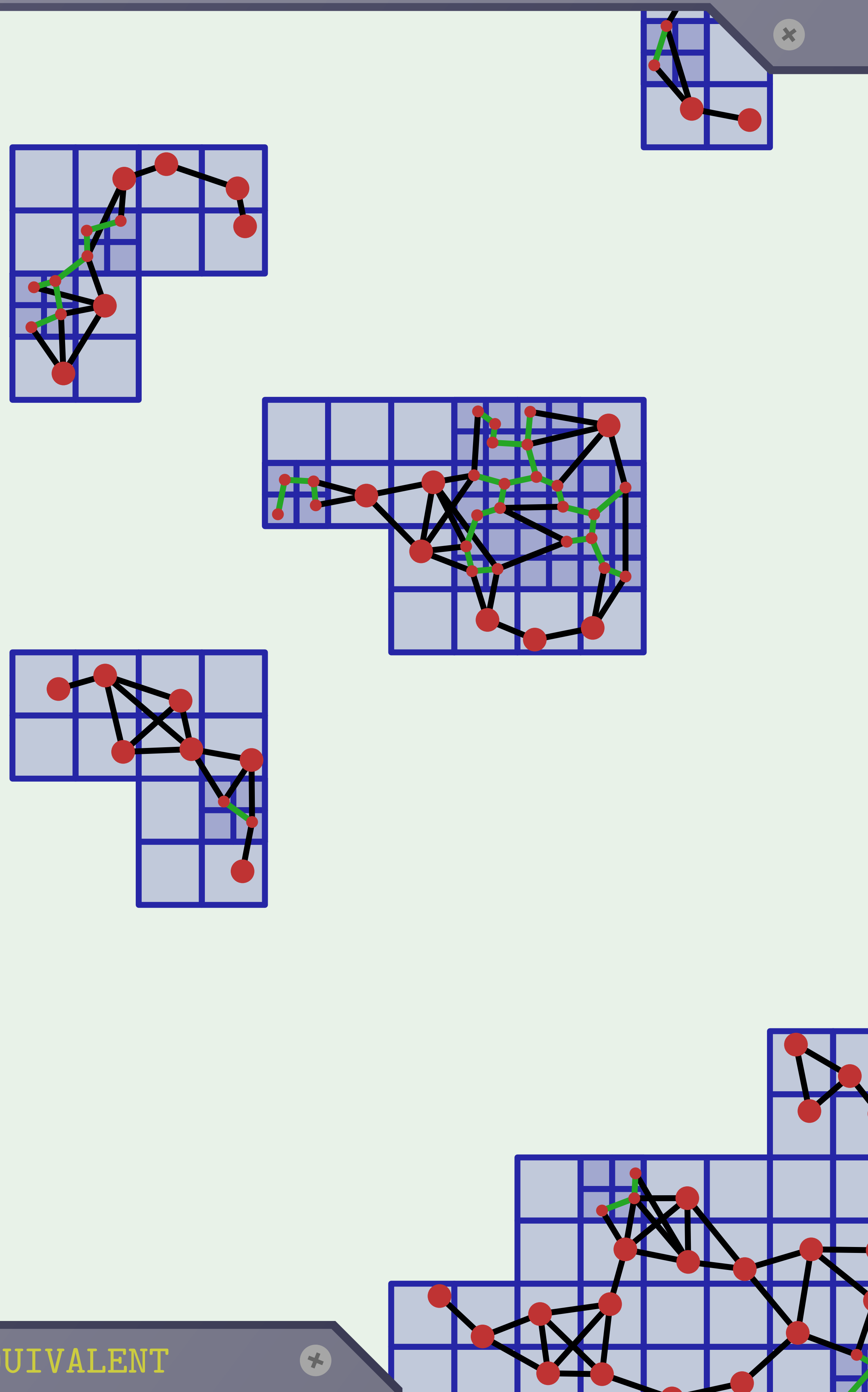

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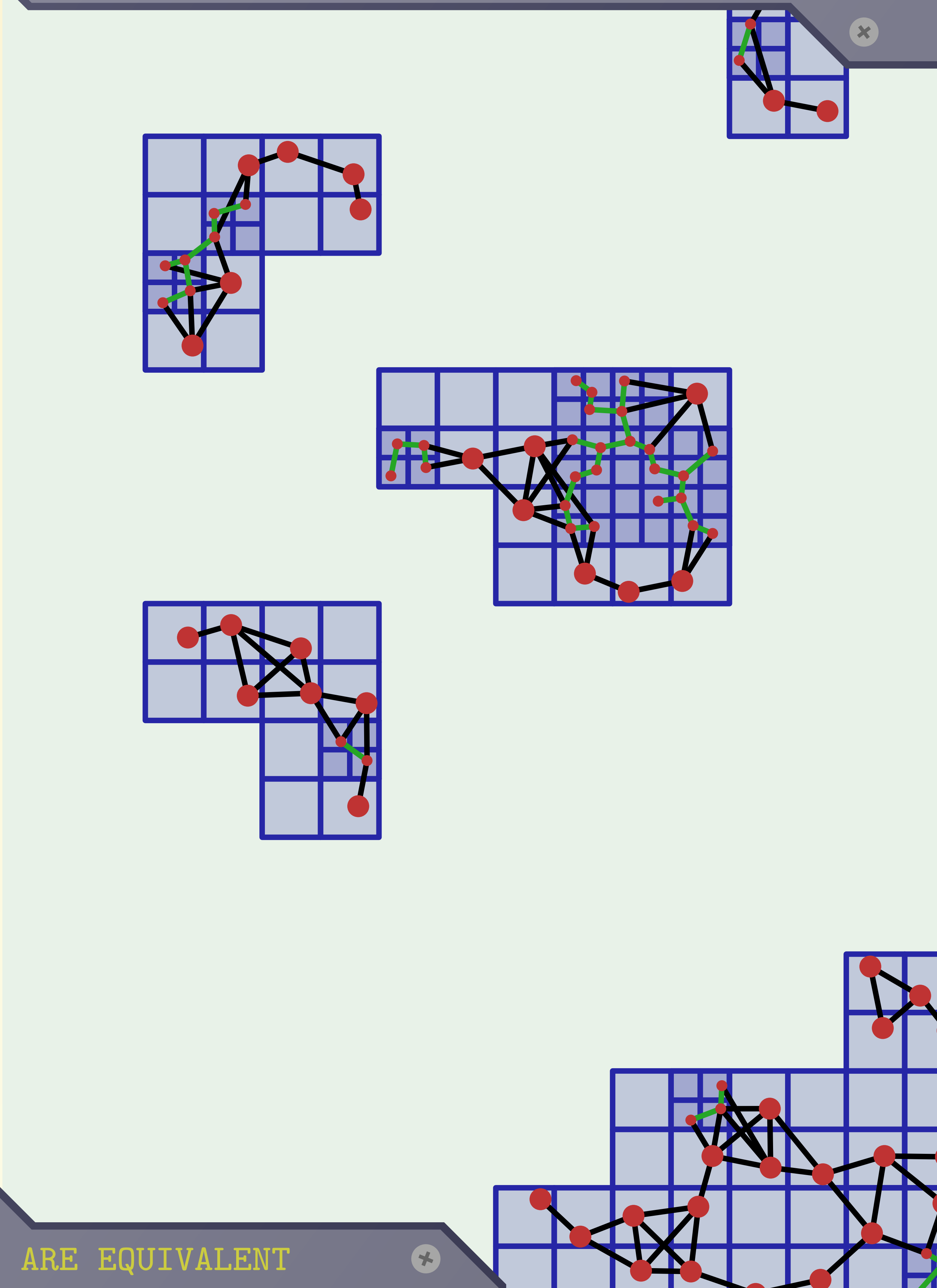

Start with P and G.

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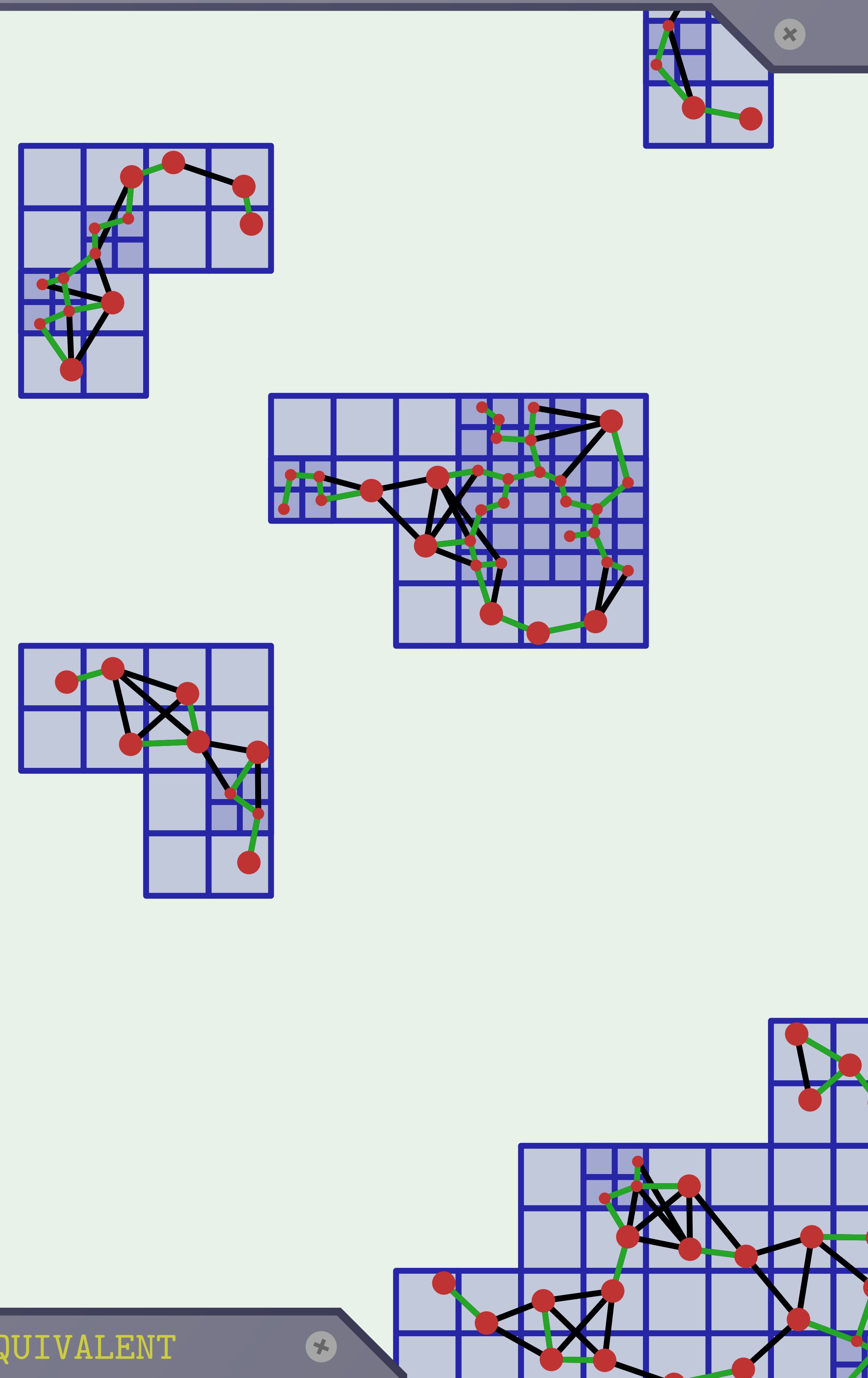

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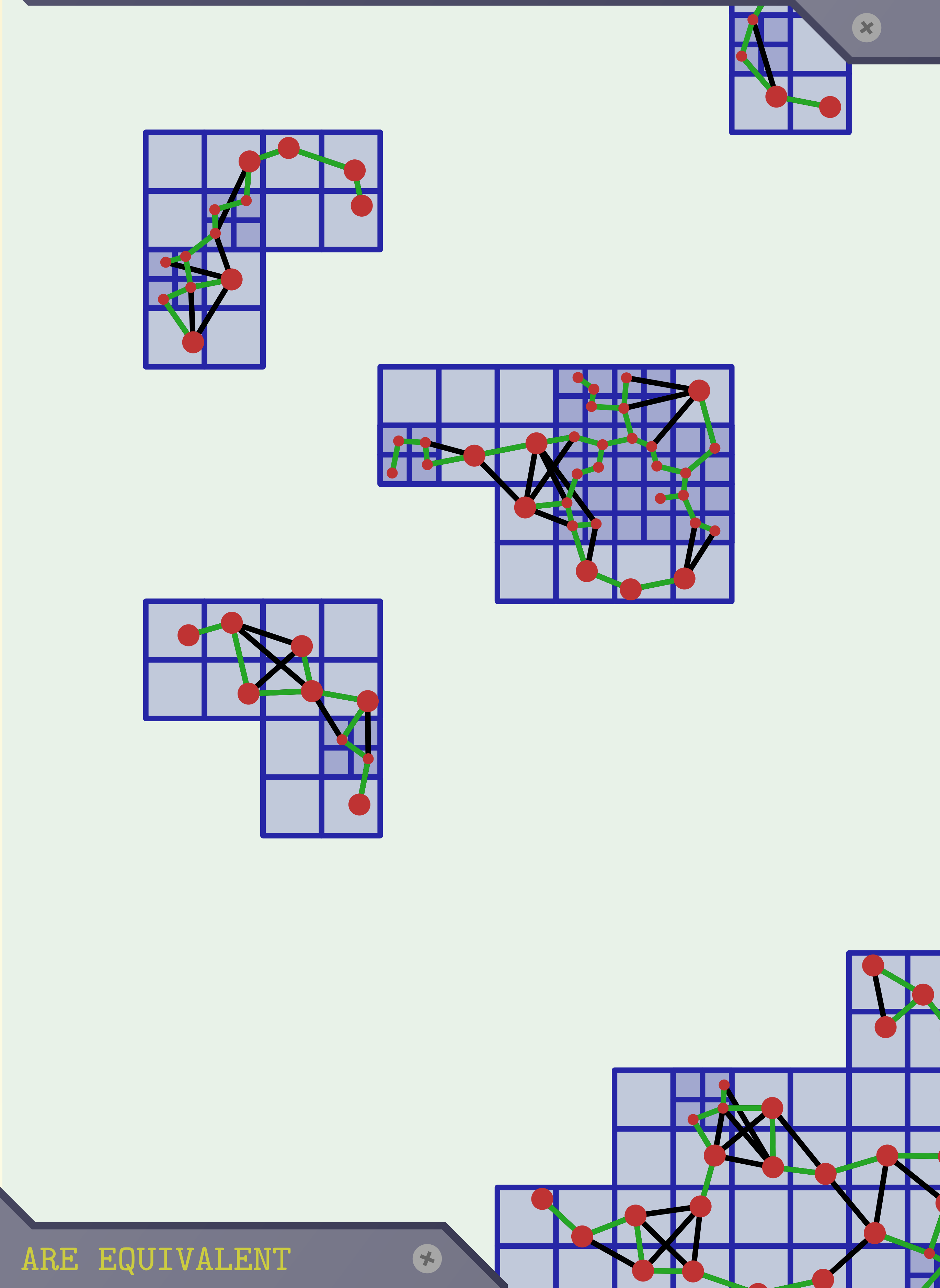

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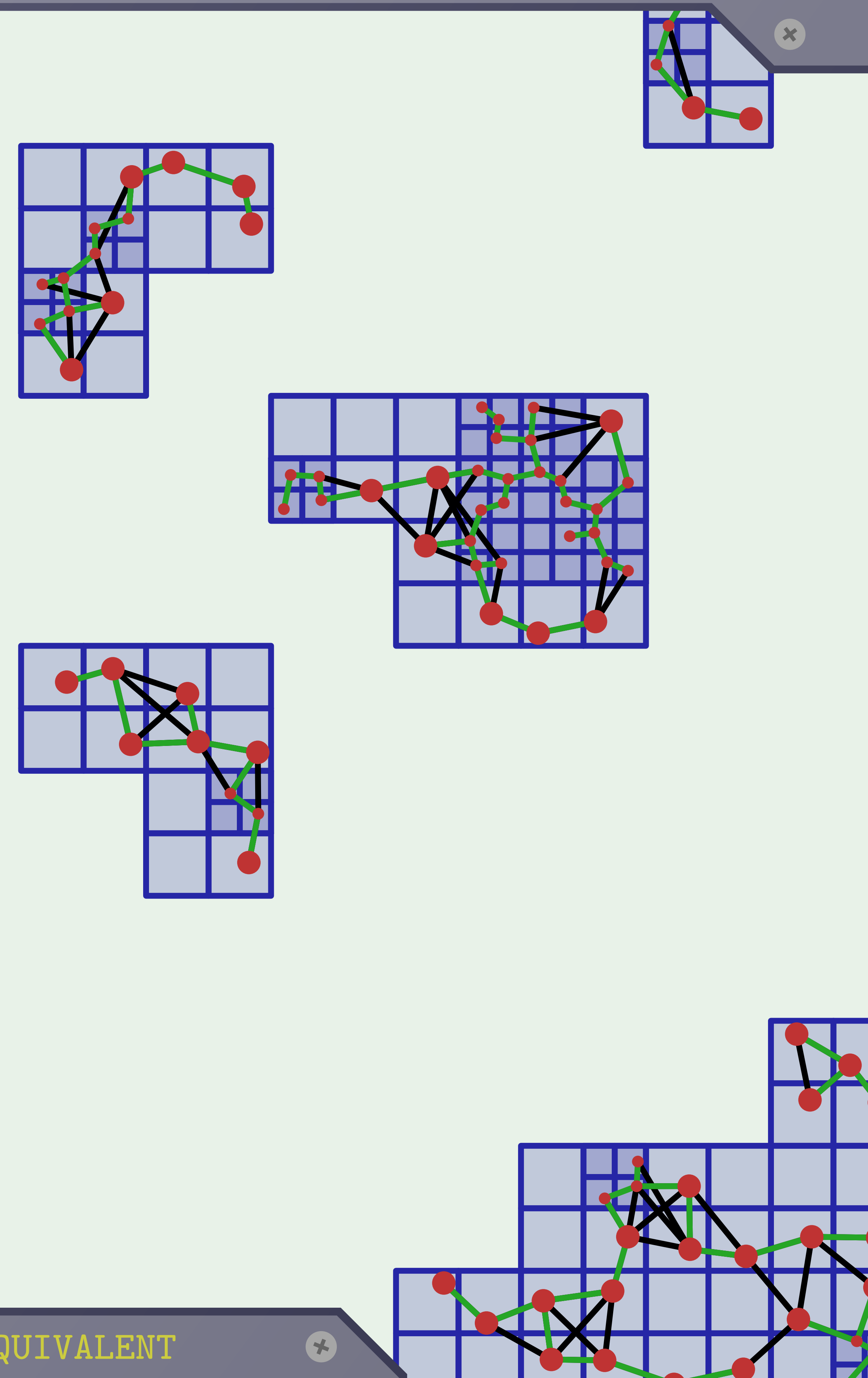

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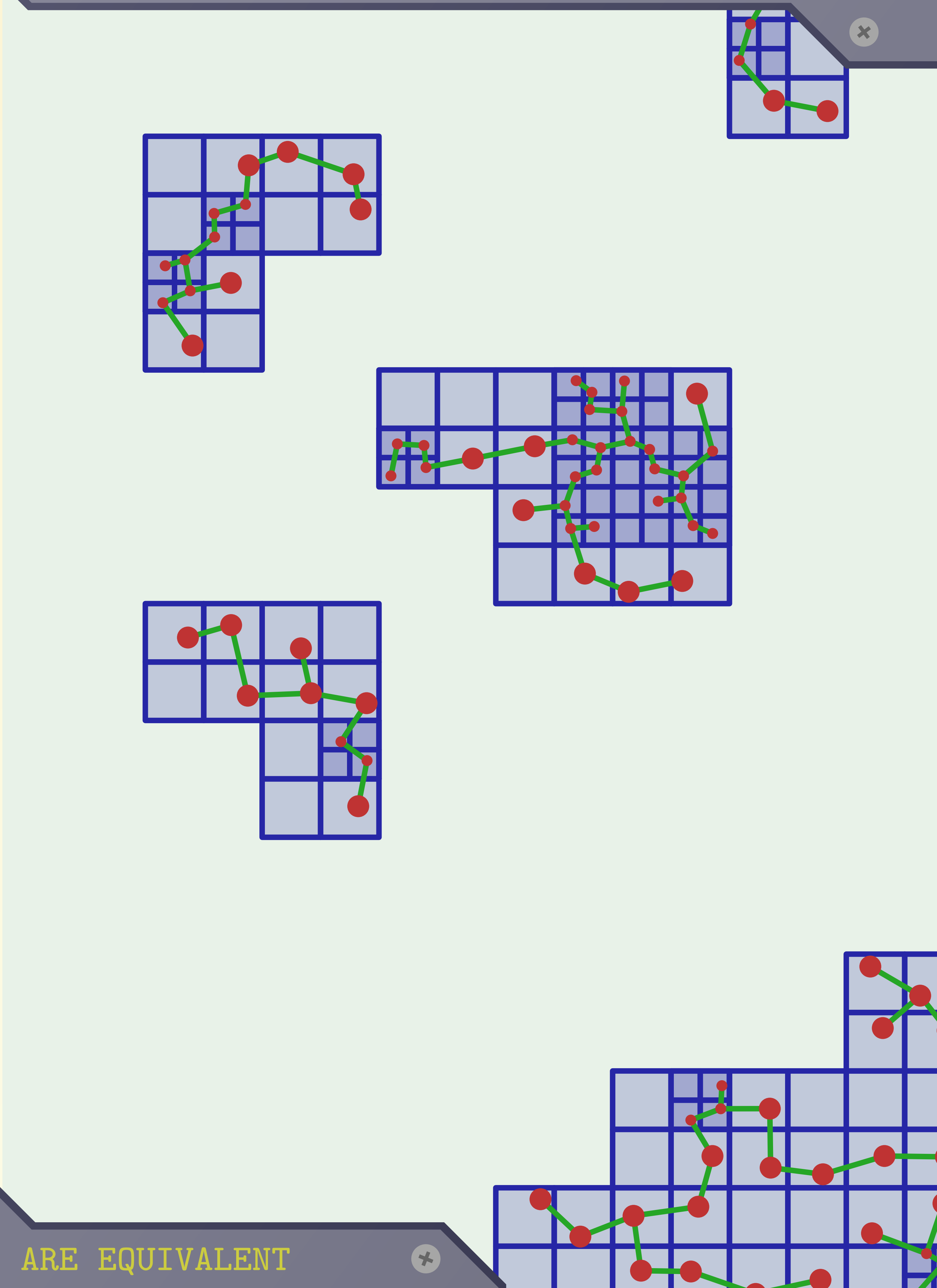

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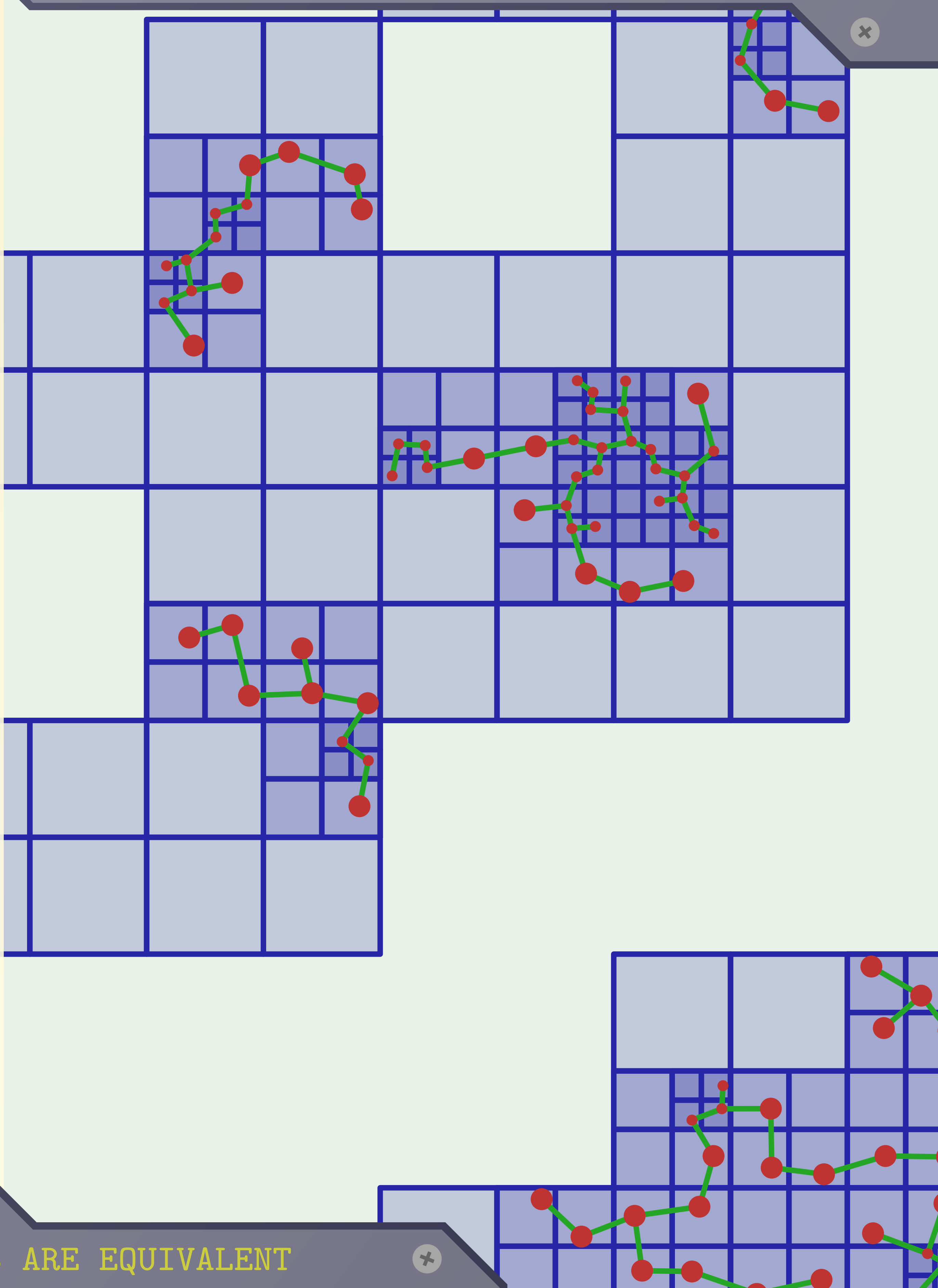

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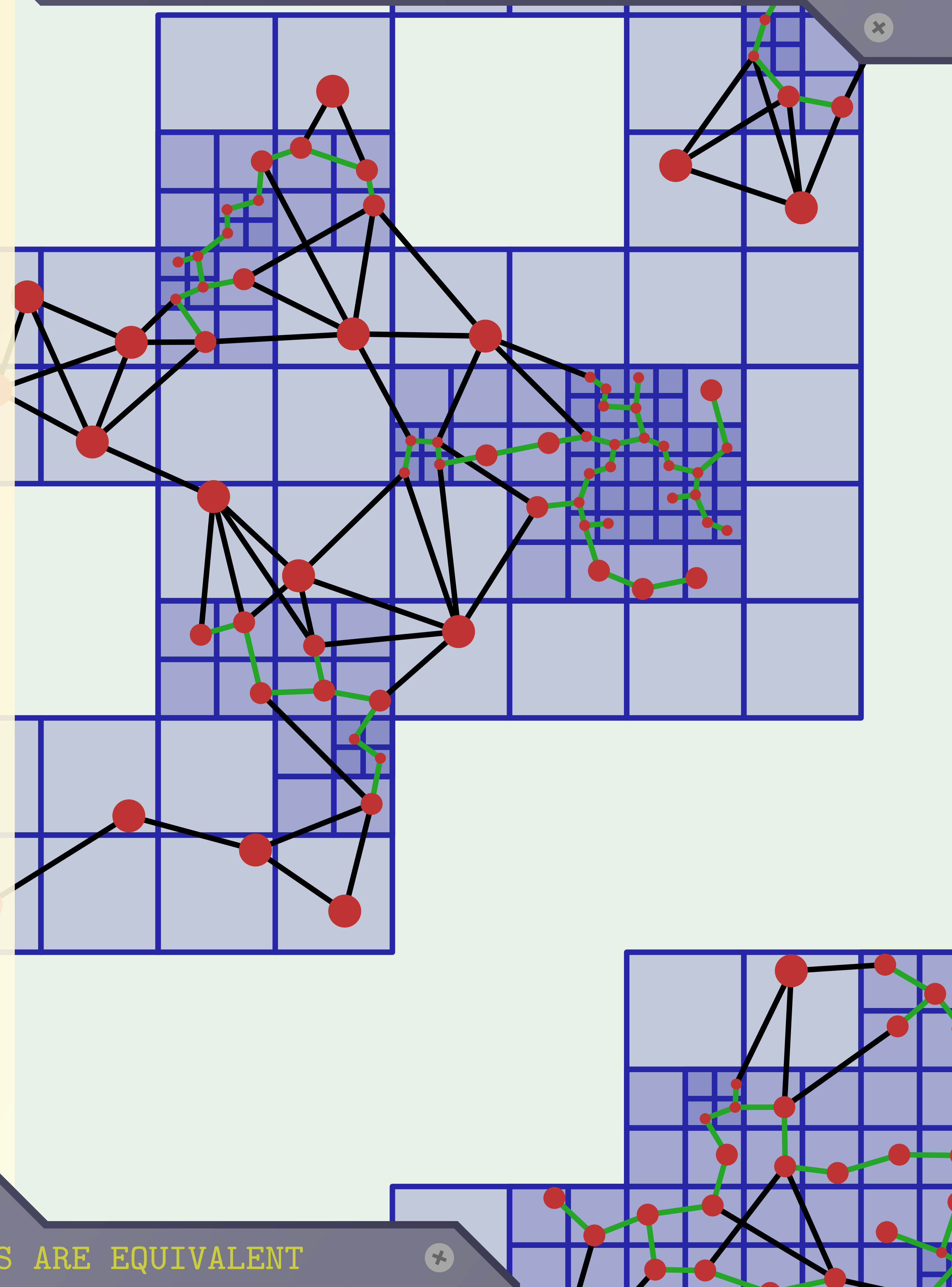

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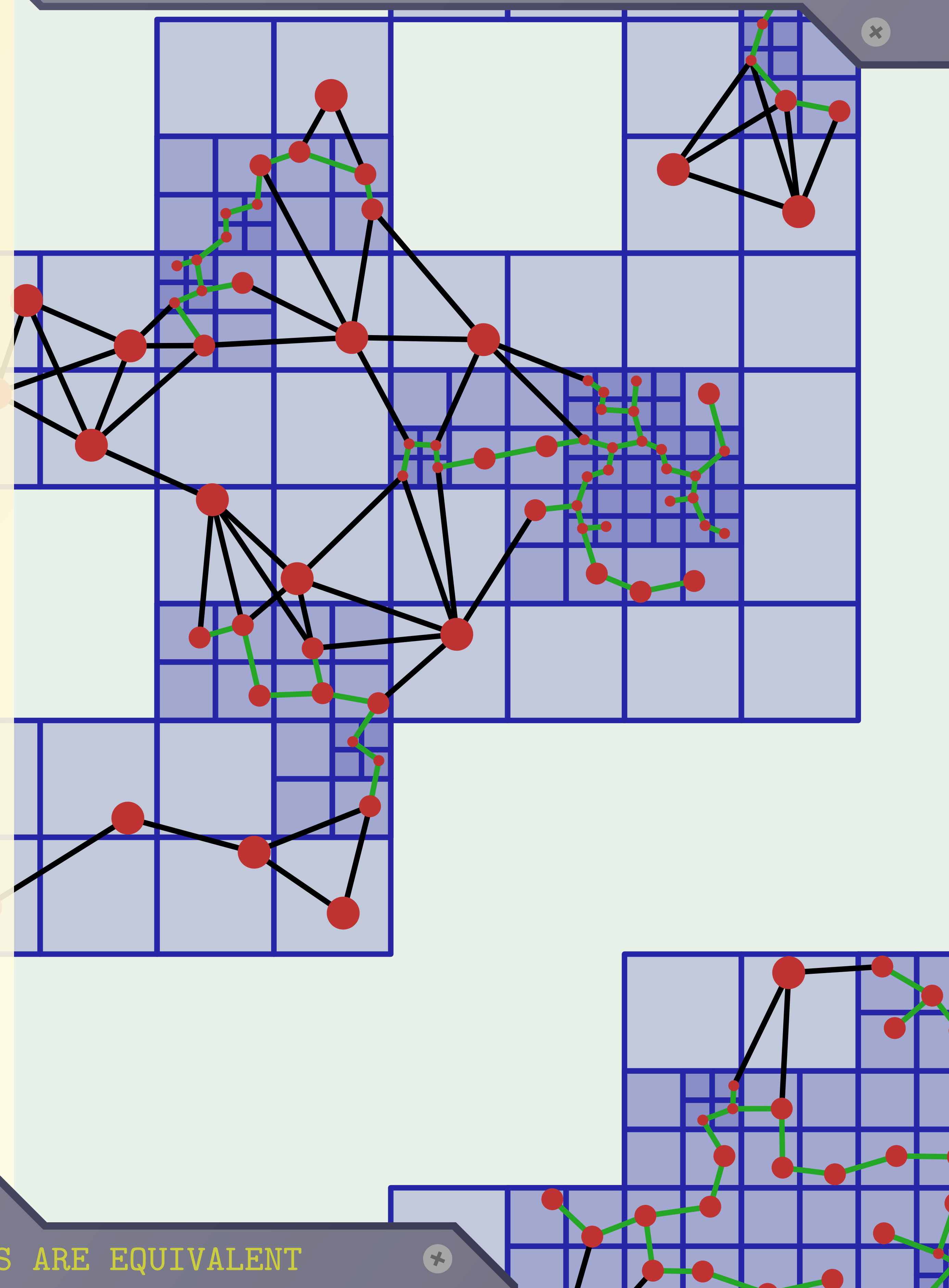

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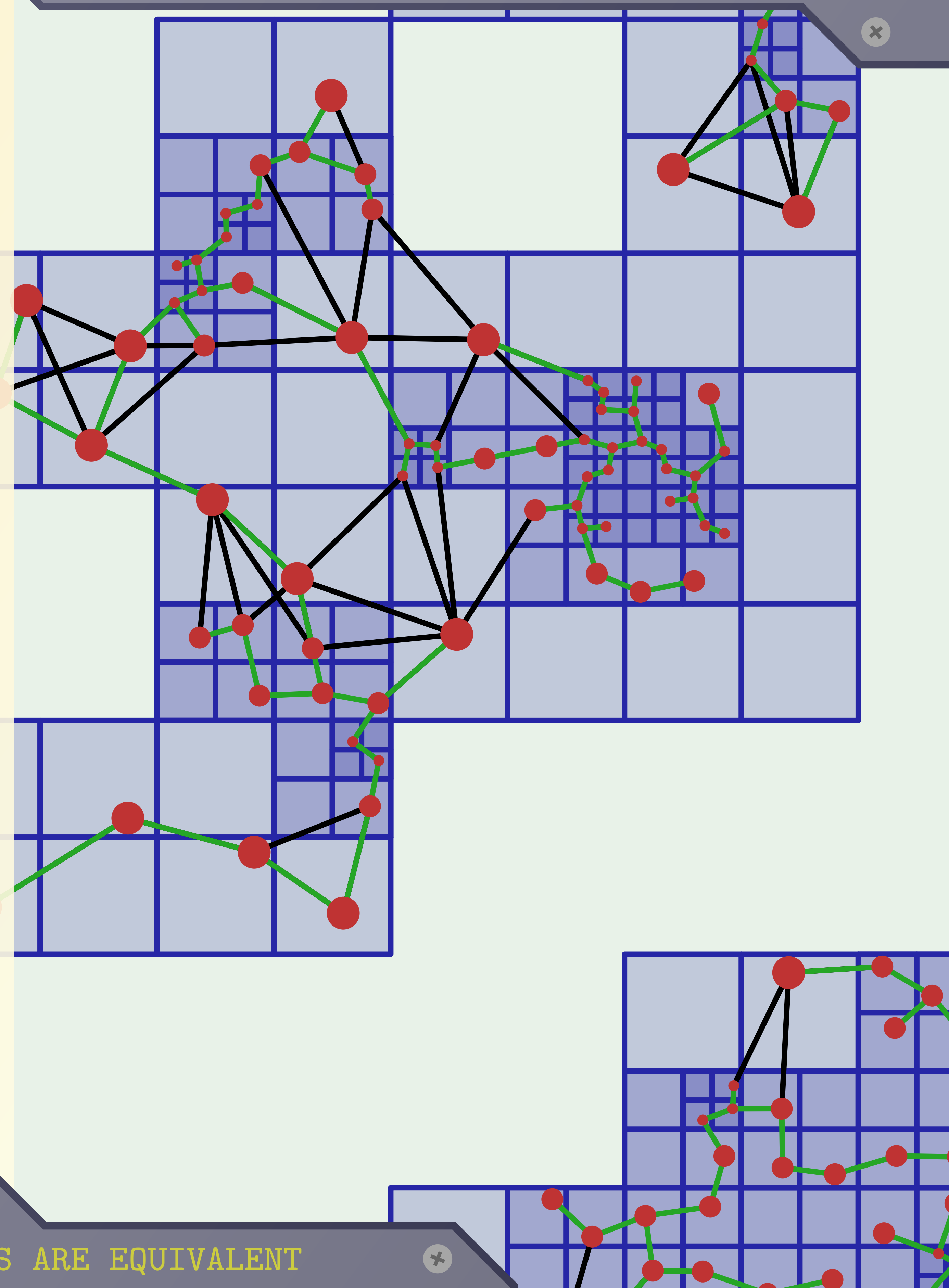

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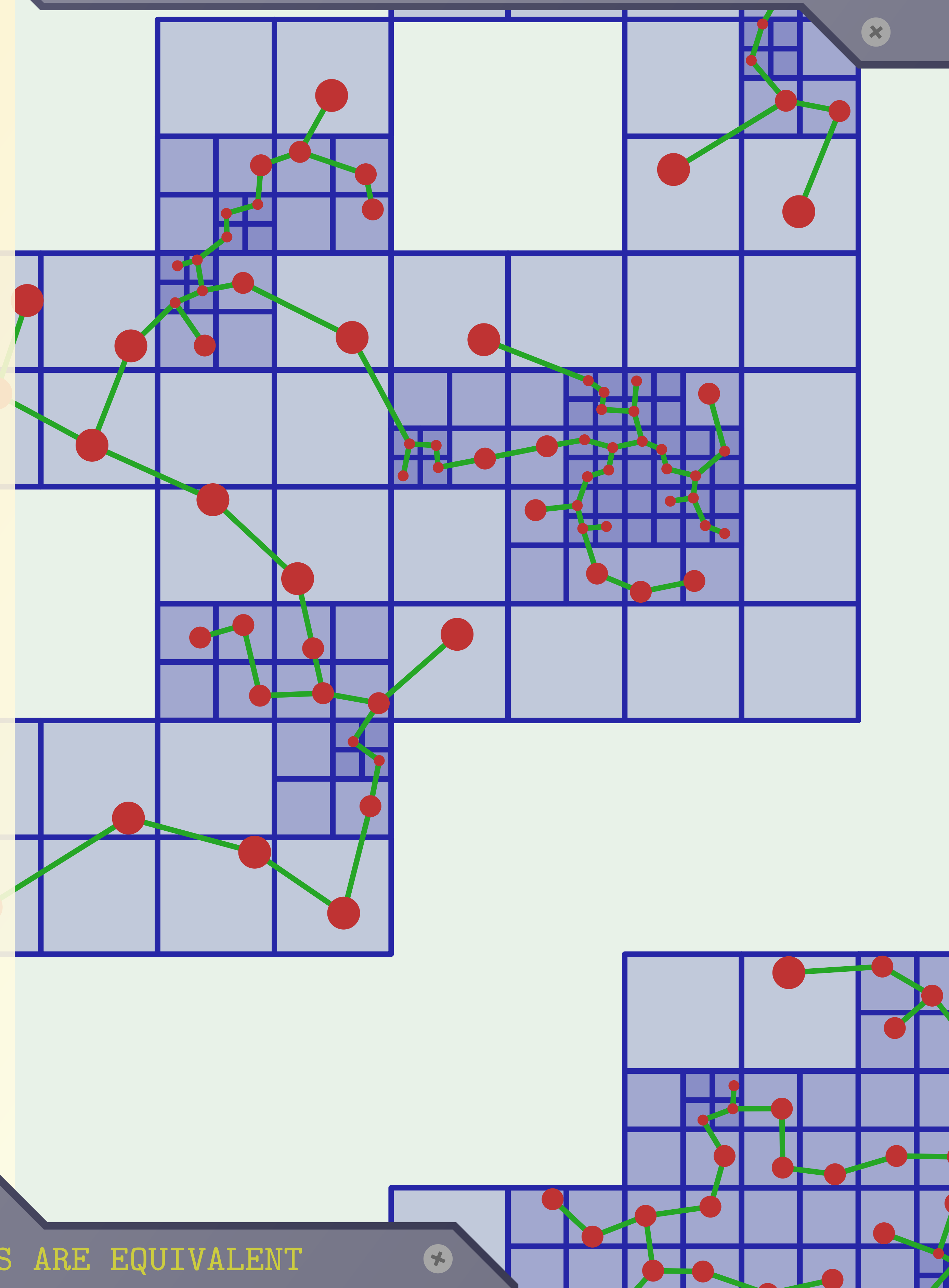

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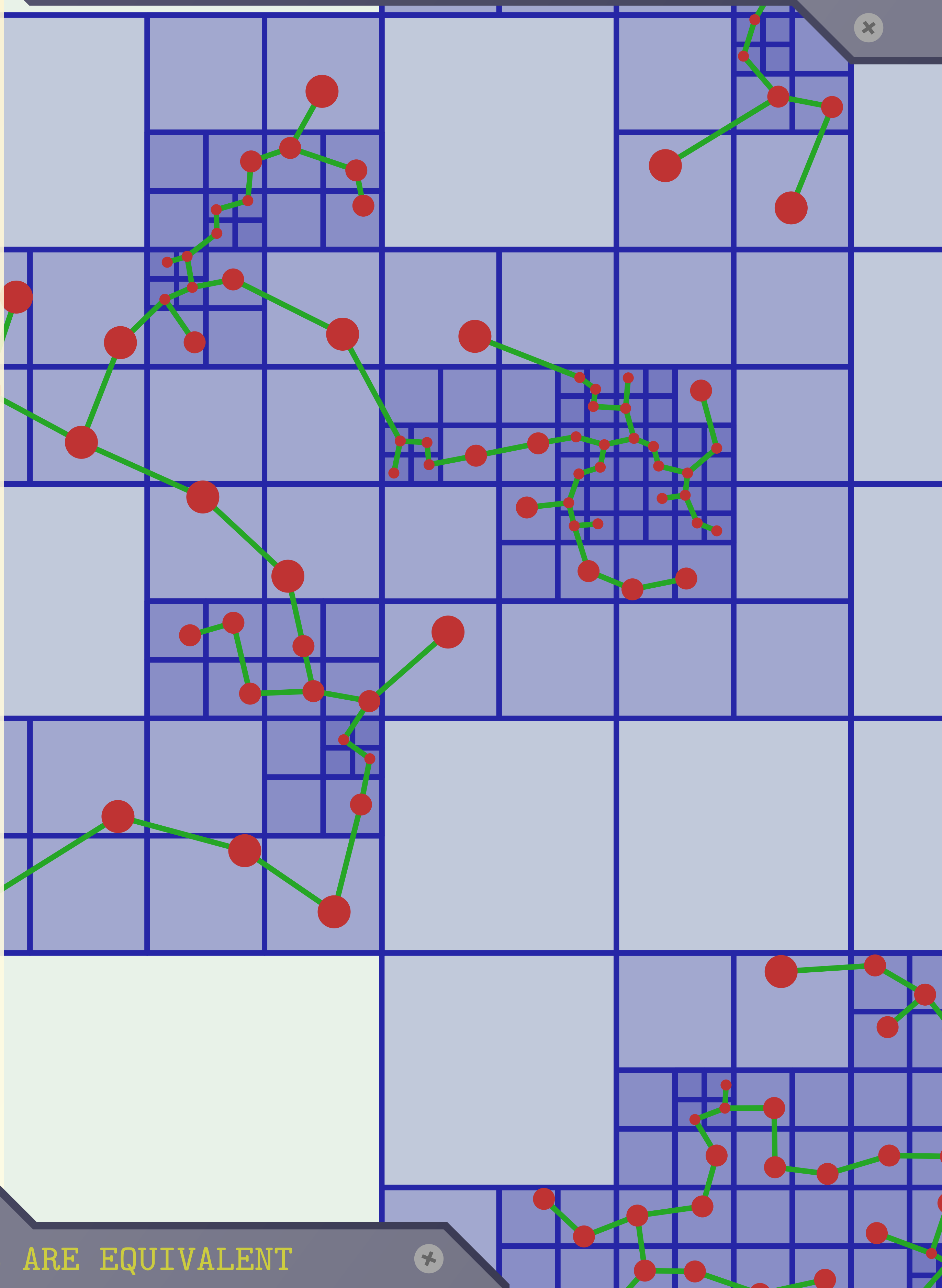

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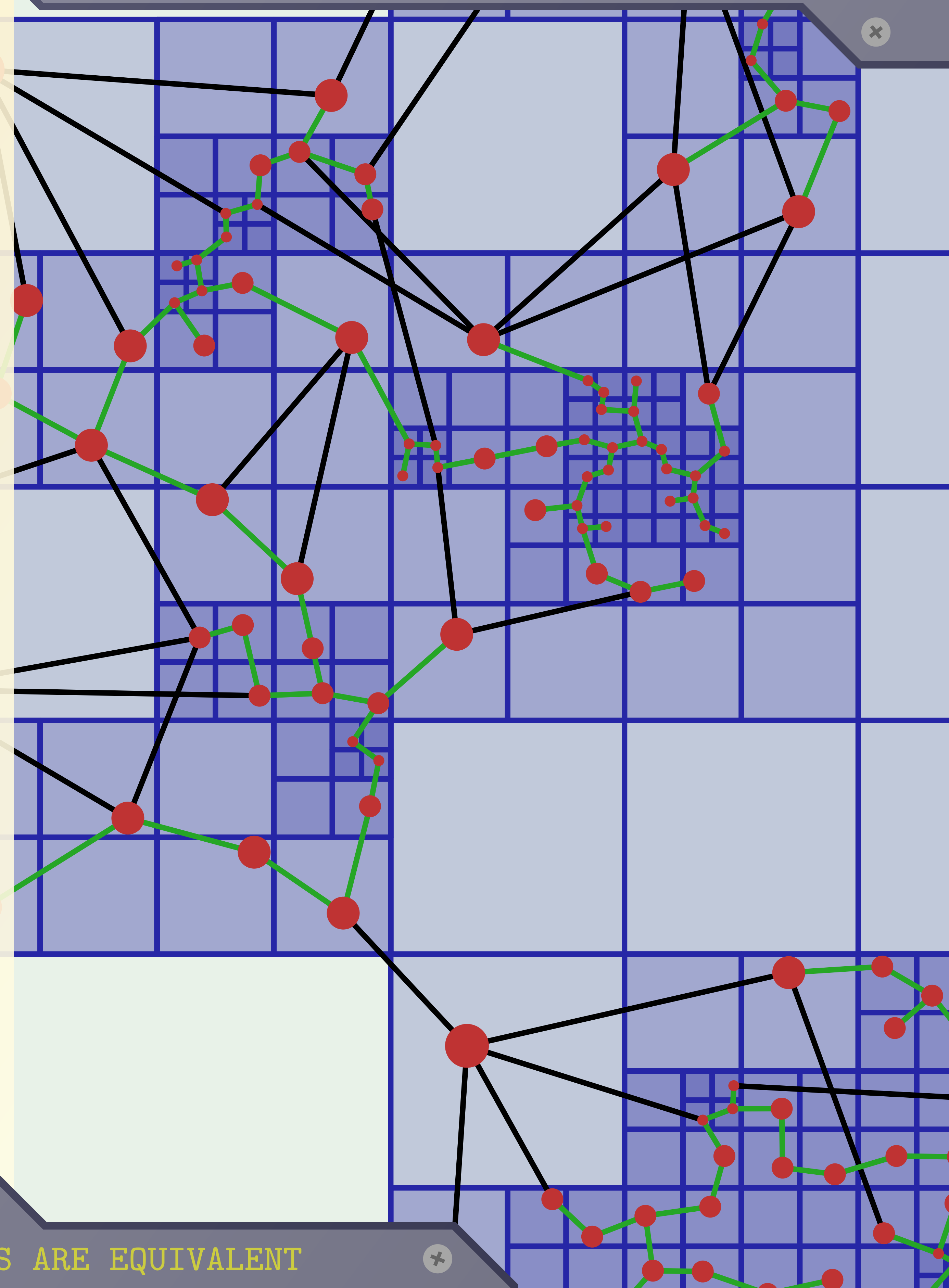

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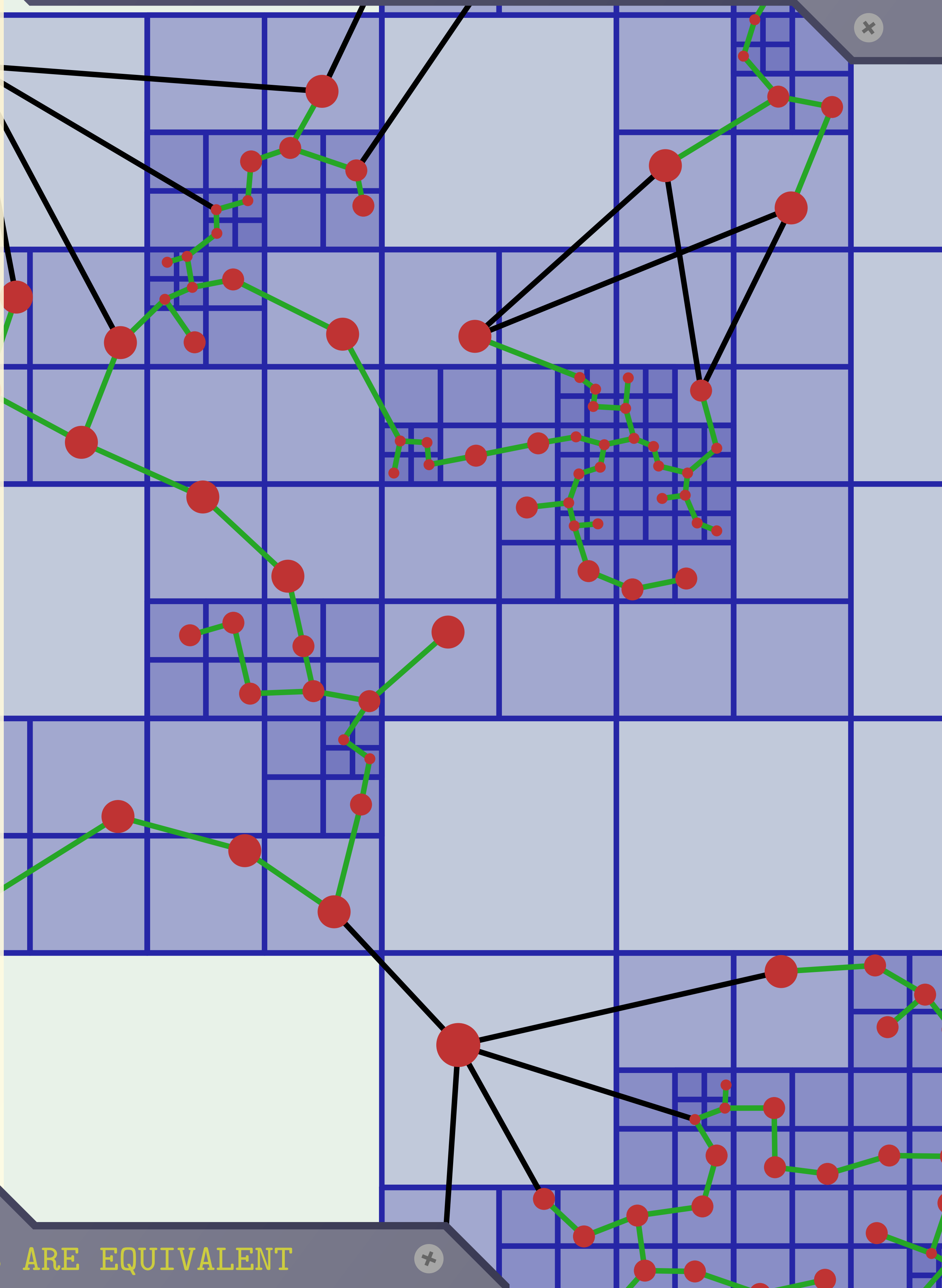

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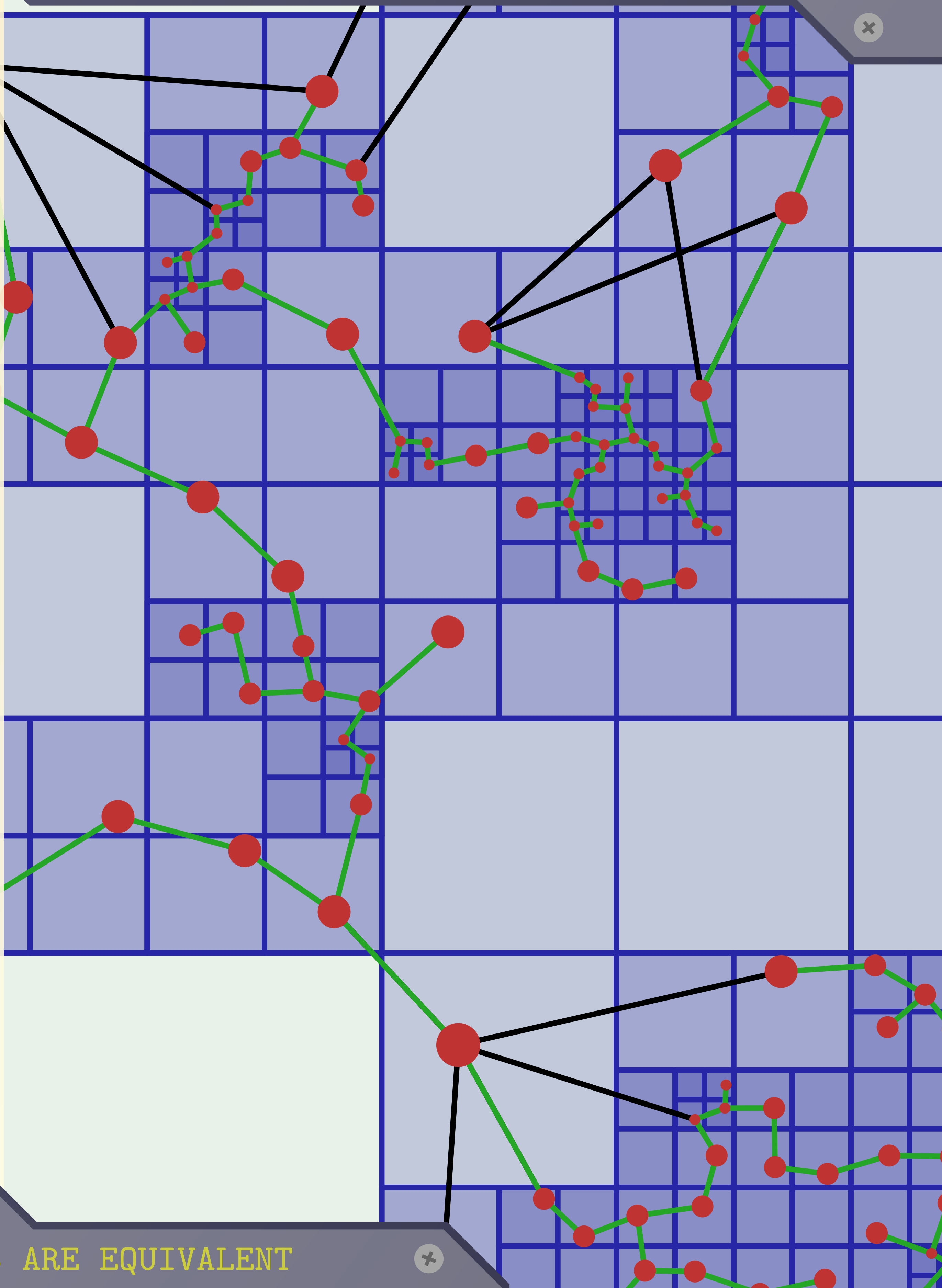

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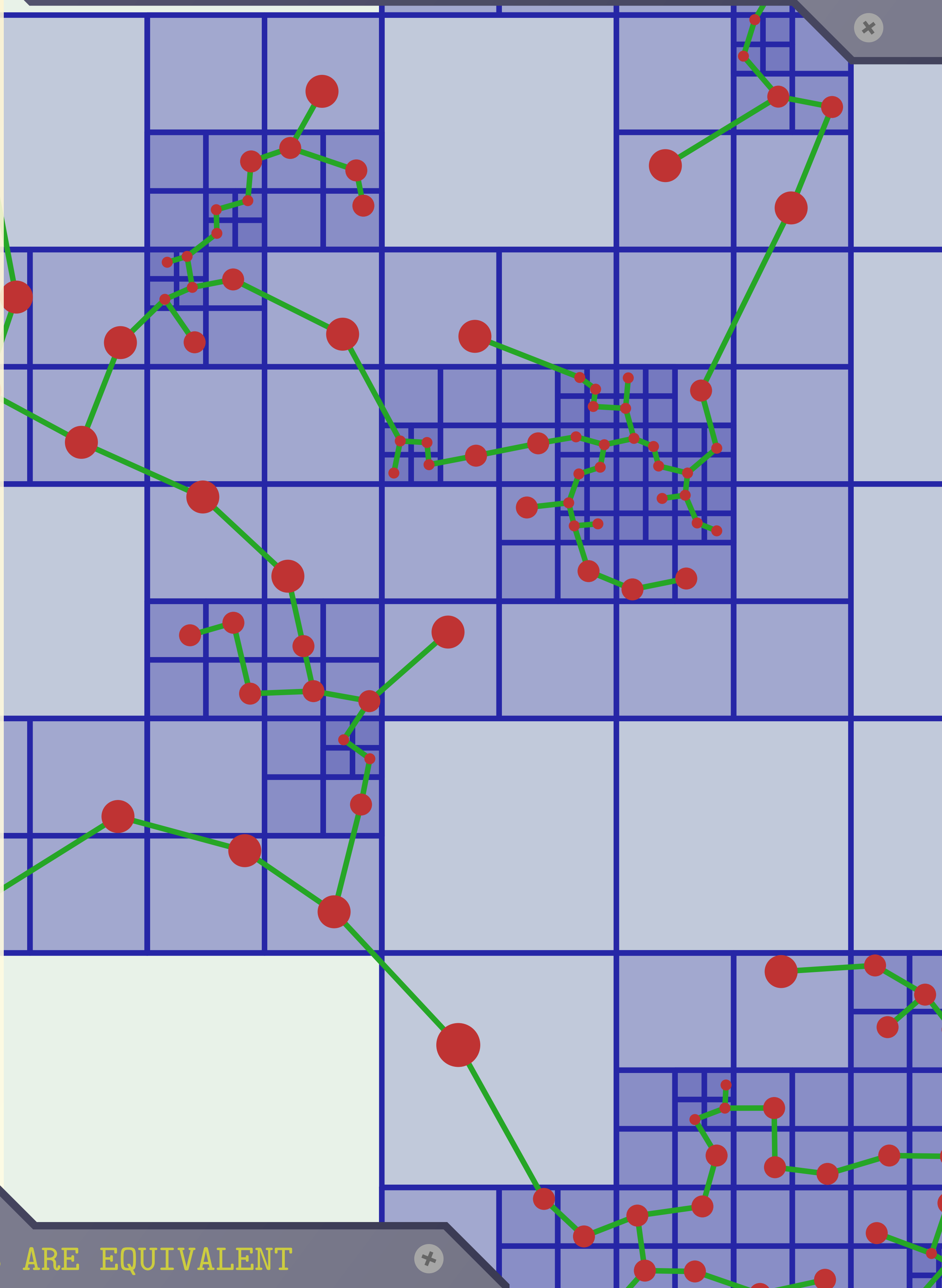

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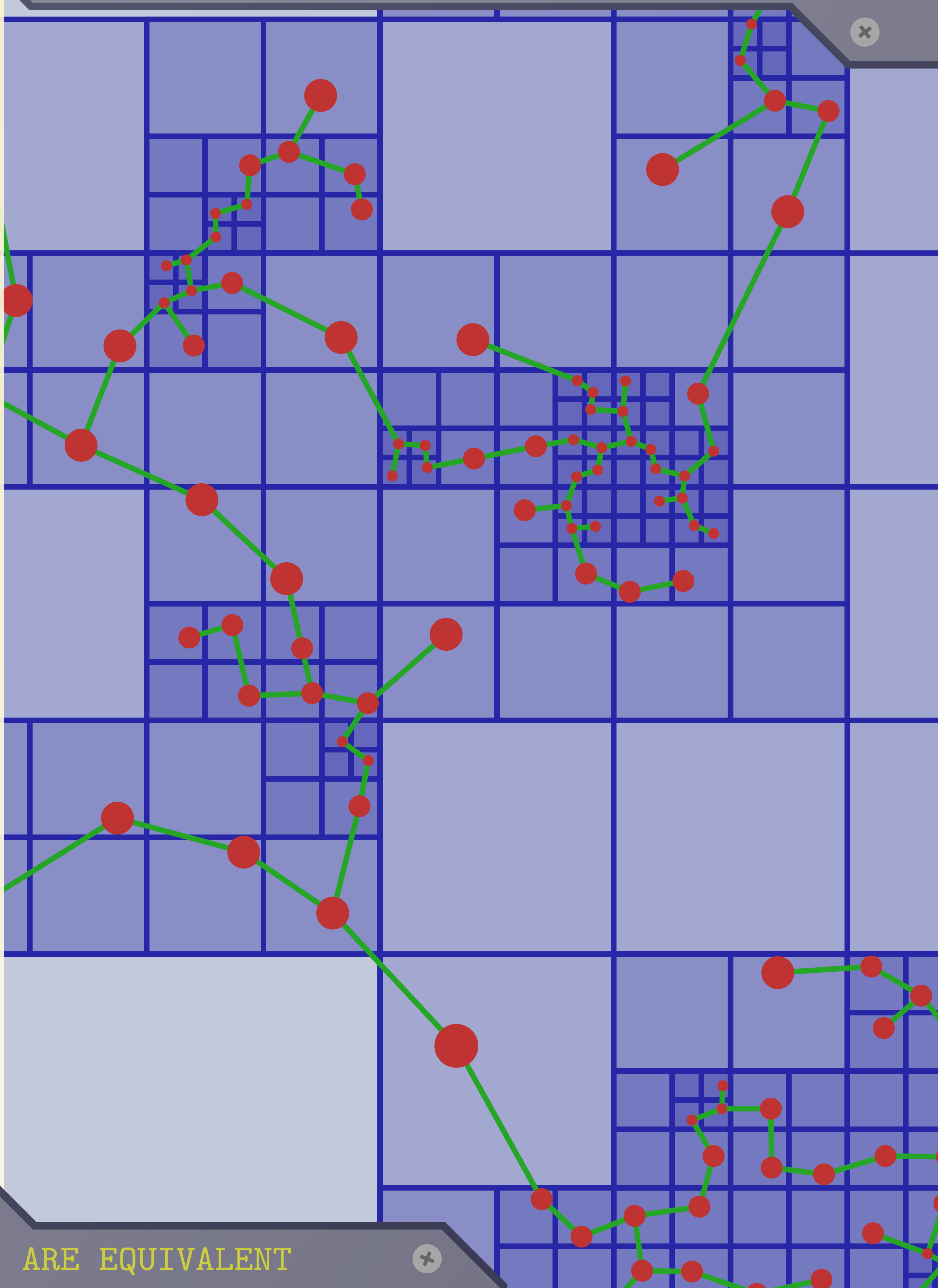

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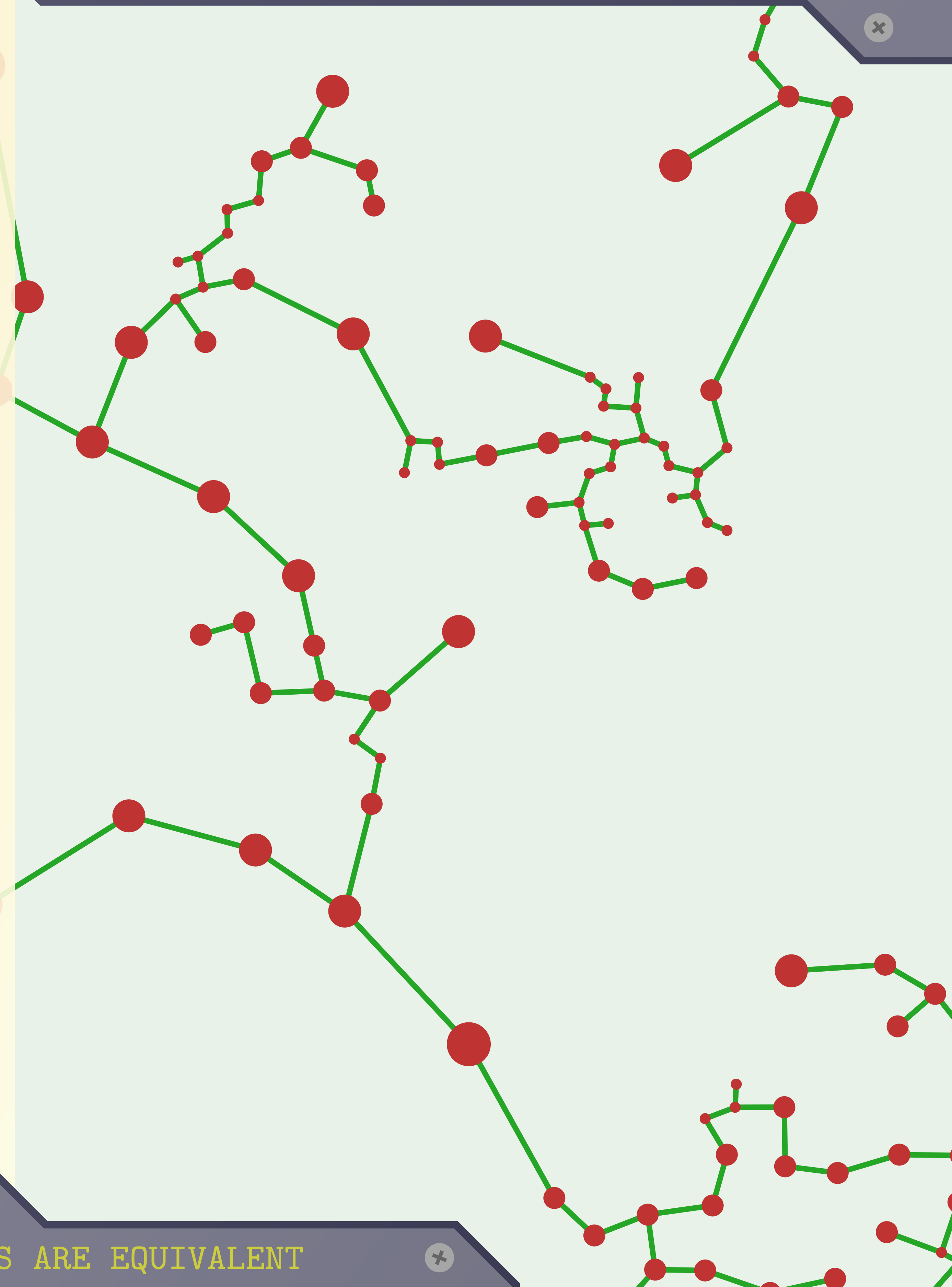

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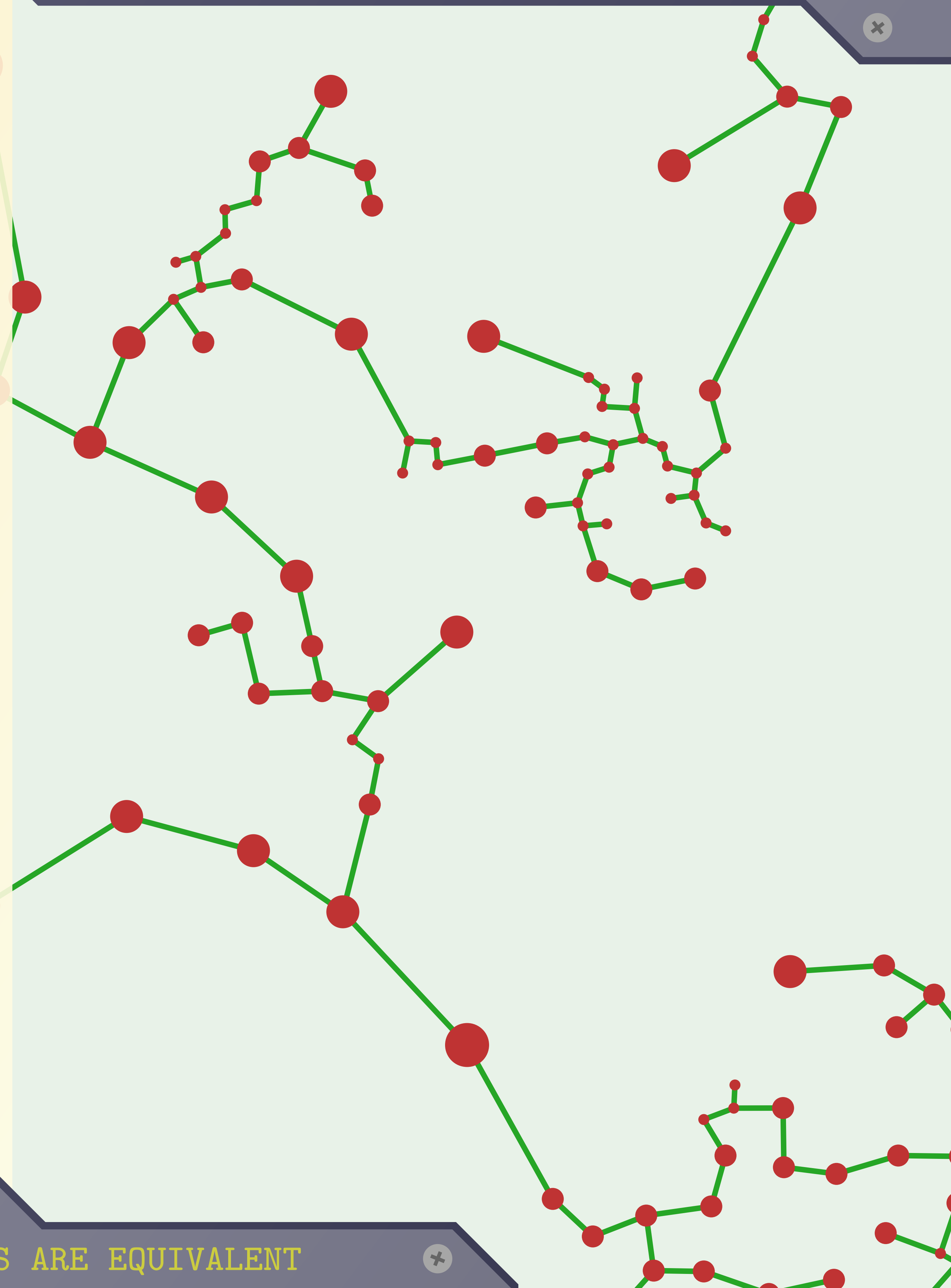
Done!



Given a quadtree on a set of points P, we can compute the MST of P in linear time.

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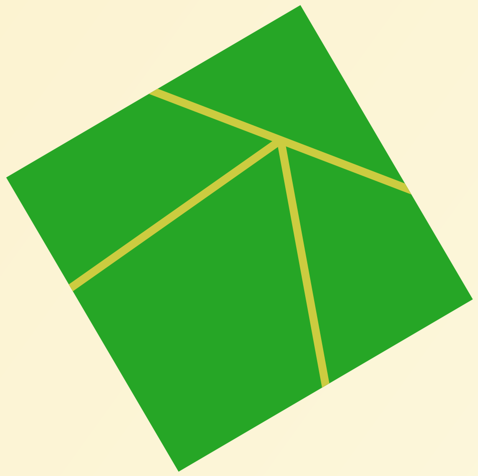
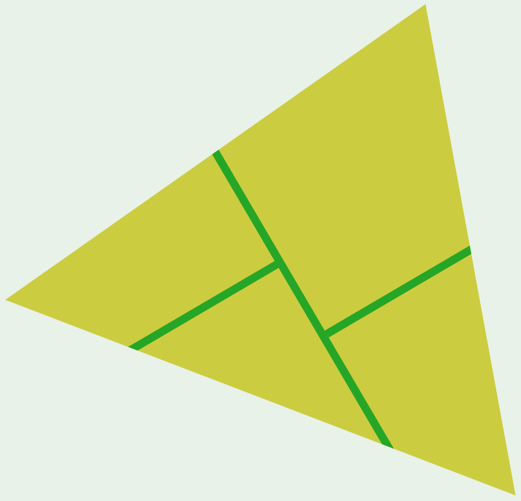
Many major proximity structures on planar point sets can be derived from each other in linear deterministic time.

THANK *YOU!*





ANY QUE*STIONS?*