UNIONS OF ONIONS

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Universiteit Utrecht

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Freie Universität Berlin

CHAPTER I

THE PROBLEM

PREPROCESSING PARADIGM

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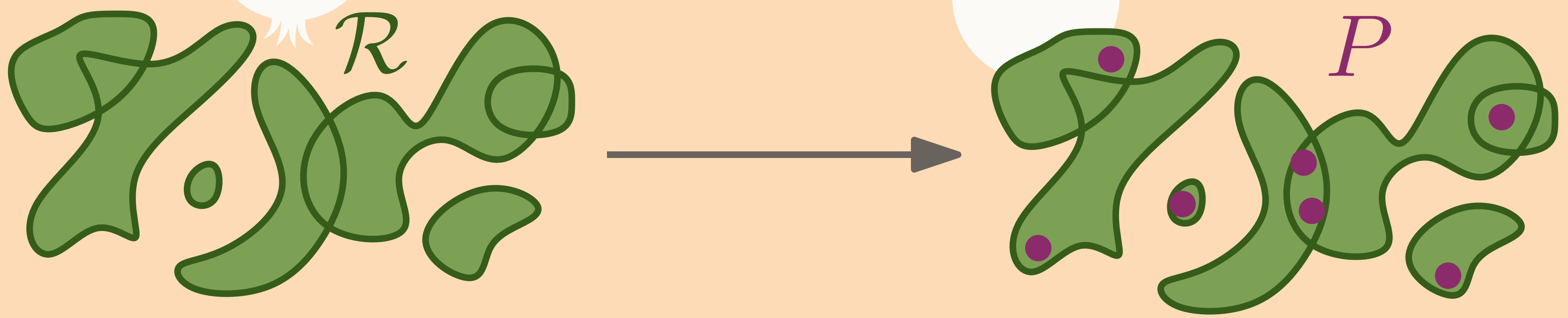

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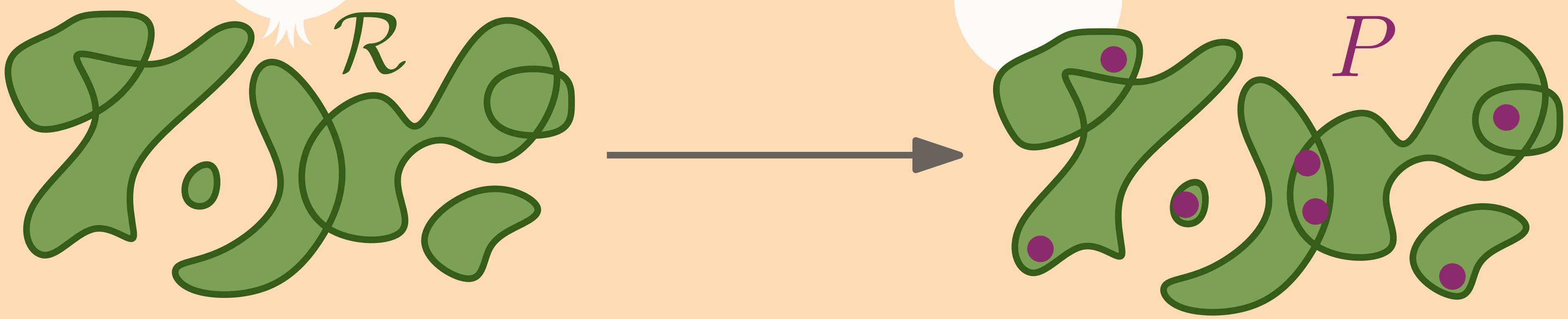

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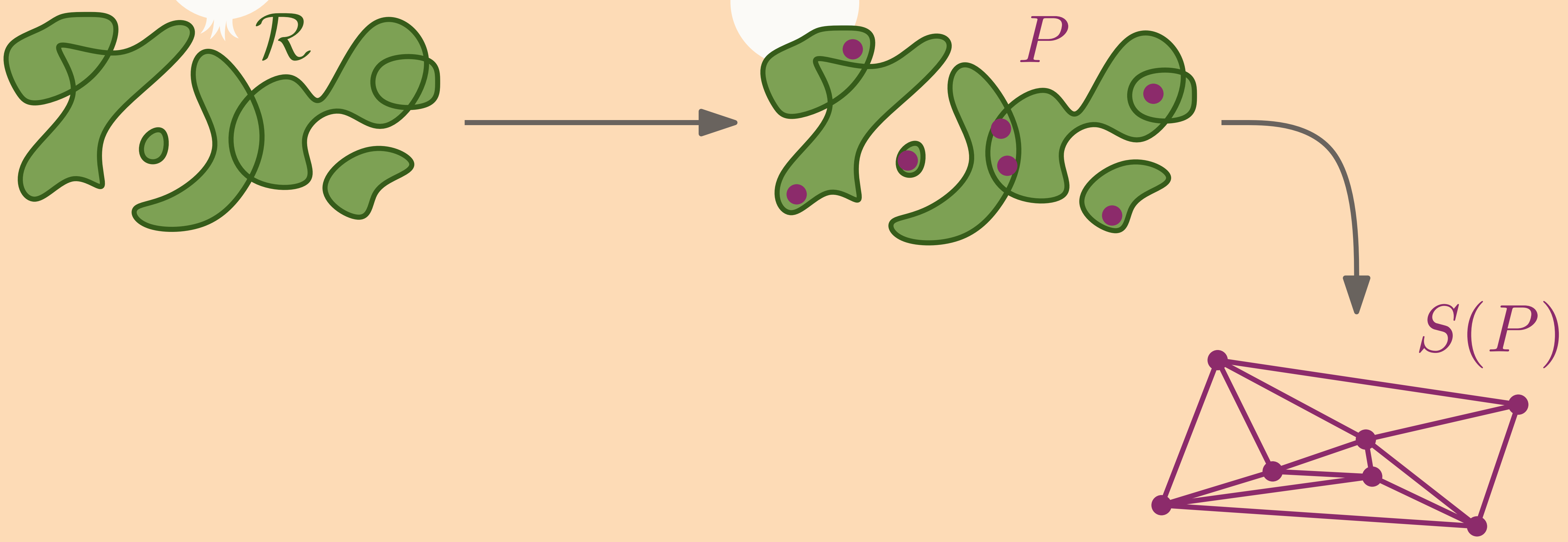

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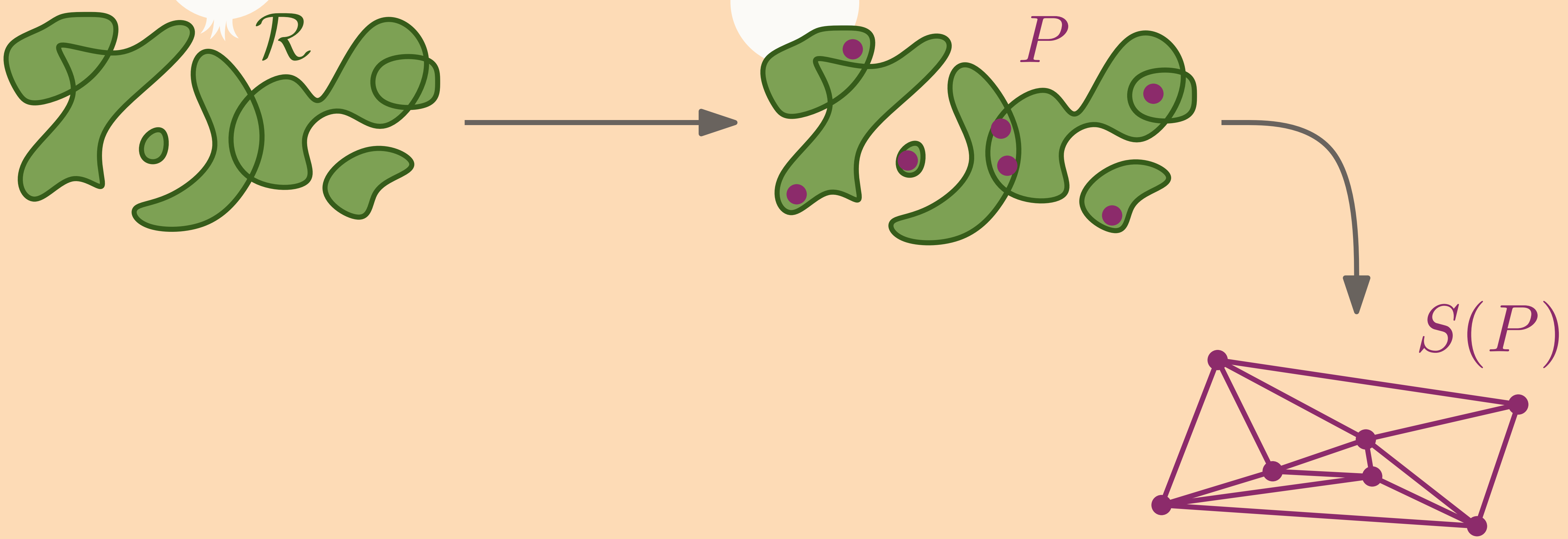
But time is precious!



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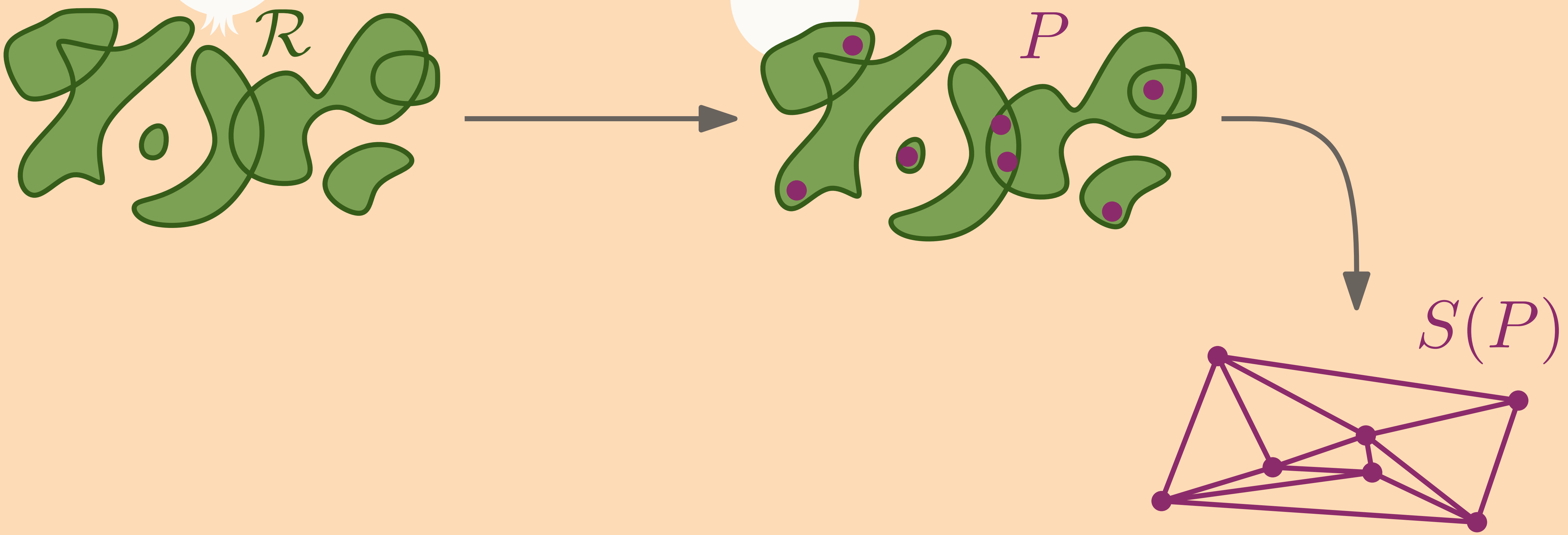
QUESTION: *Can we do some of the computation of $S(P)$ before we know P?*



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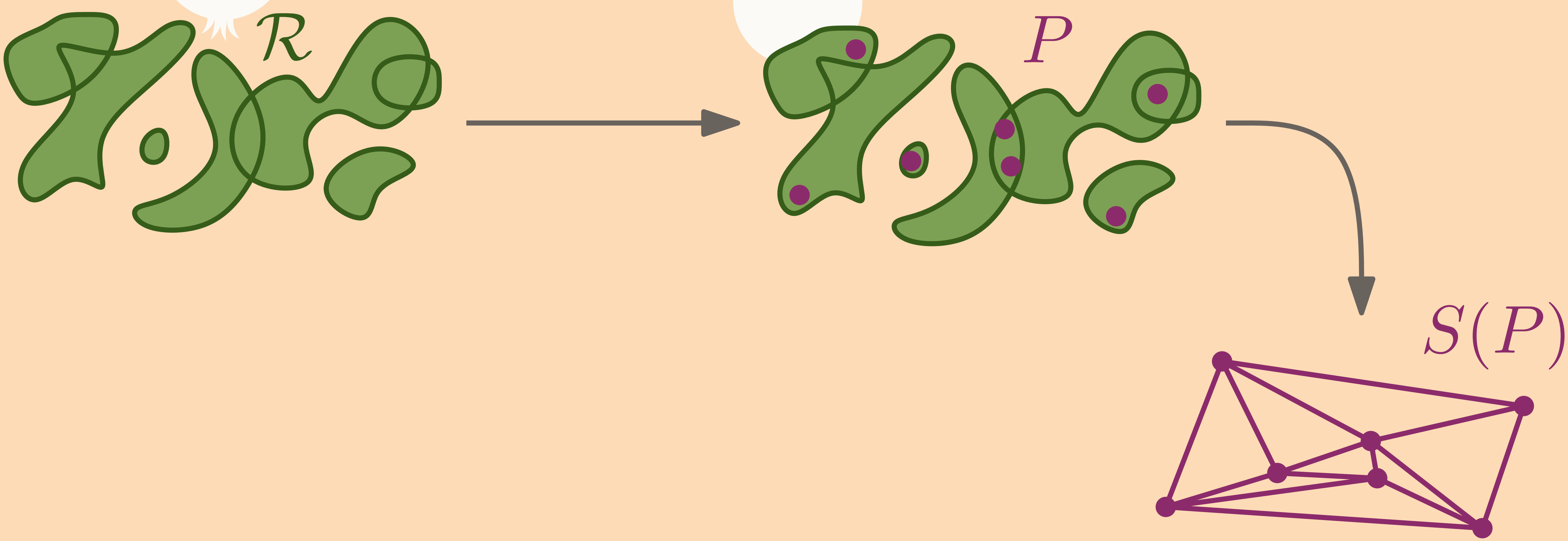
First, compute an intermediary structure $H(\mathcal{R})$.



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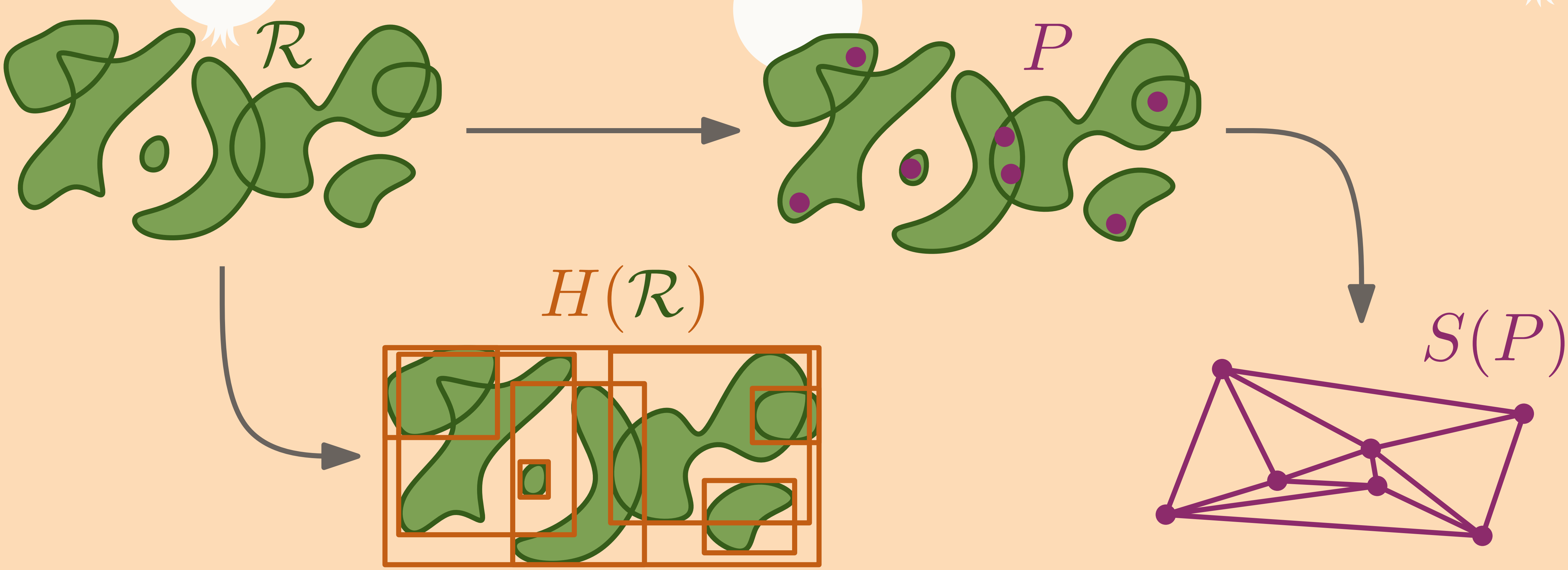

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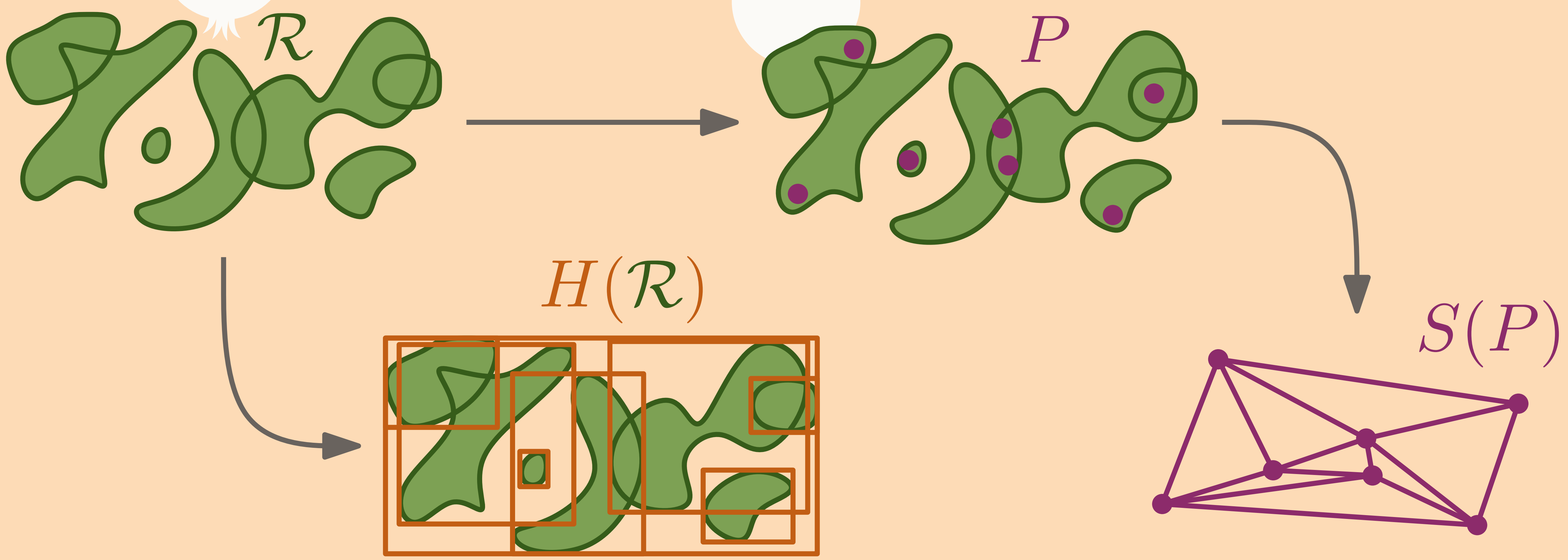

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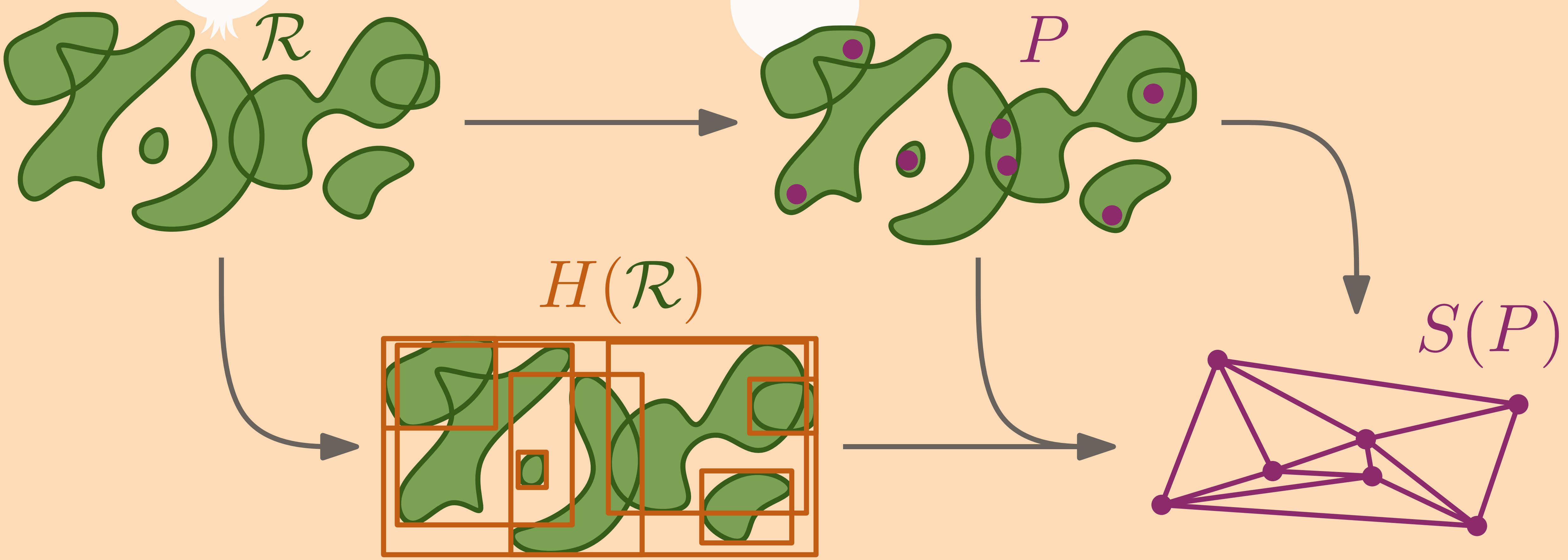


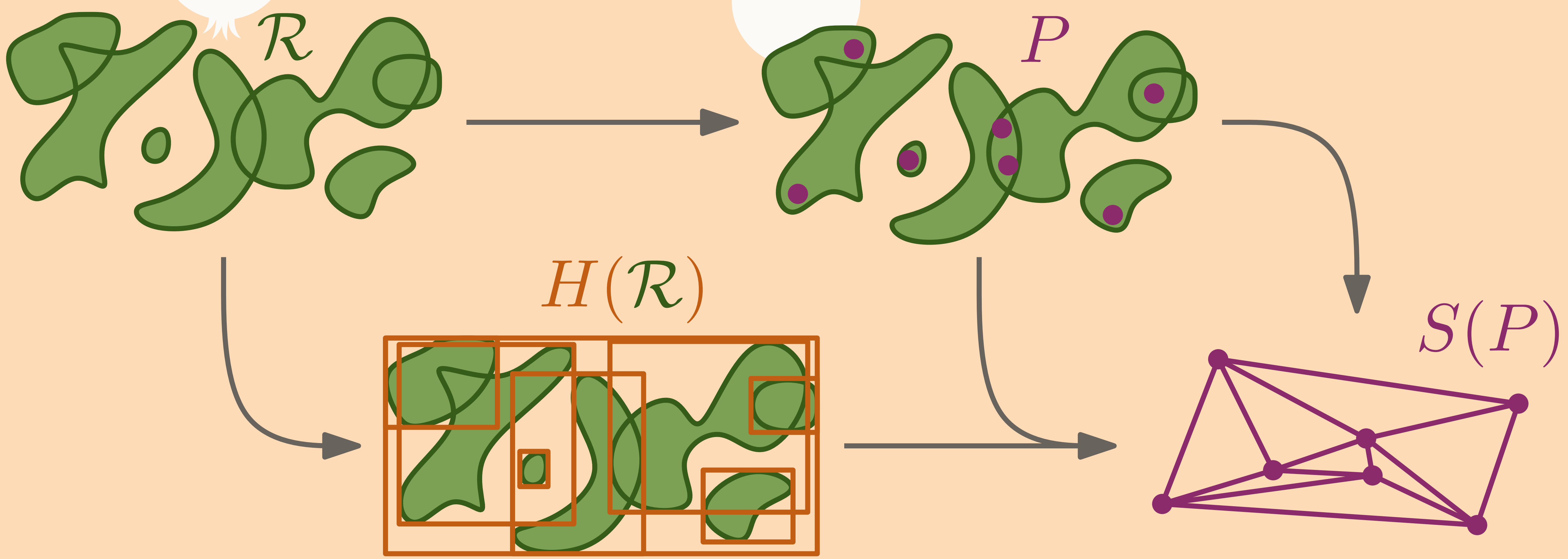
PREPROCESSING PARADIGM

Linear time algorithms are known for:

- Triangulations [Held & Mitchell]
- Delaunay triangulations [L & Snoeyink]

Superlinear, but $o(n \log n)$:

- Convex hull (lines) [Ezra & Mulzer]



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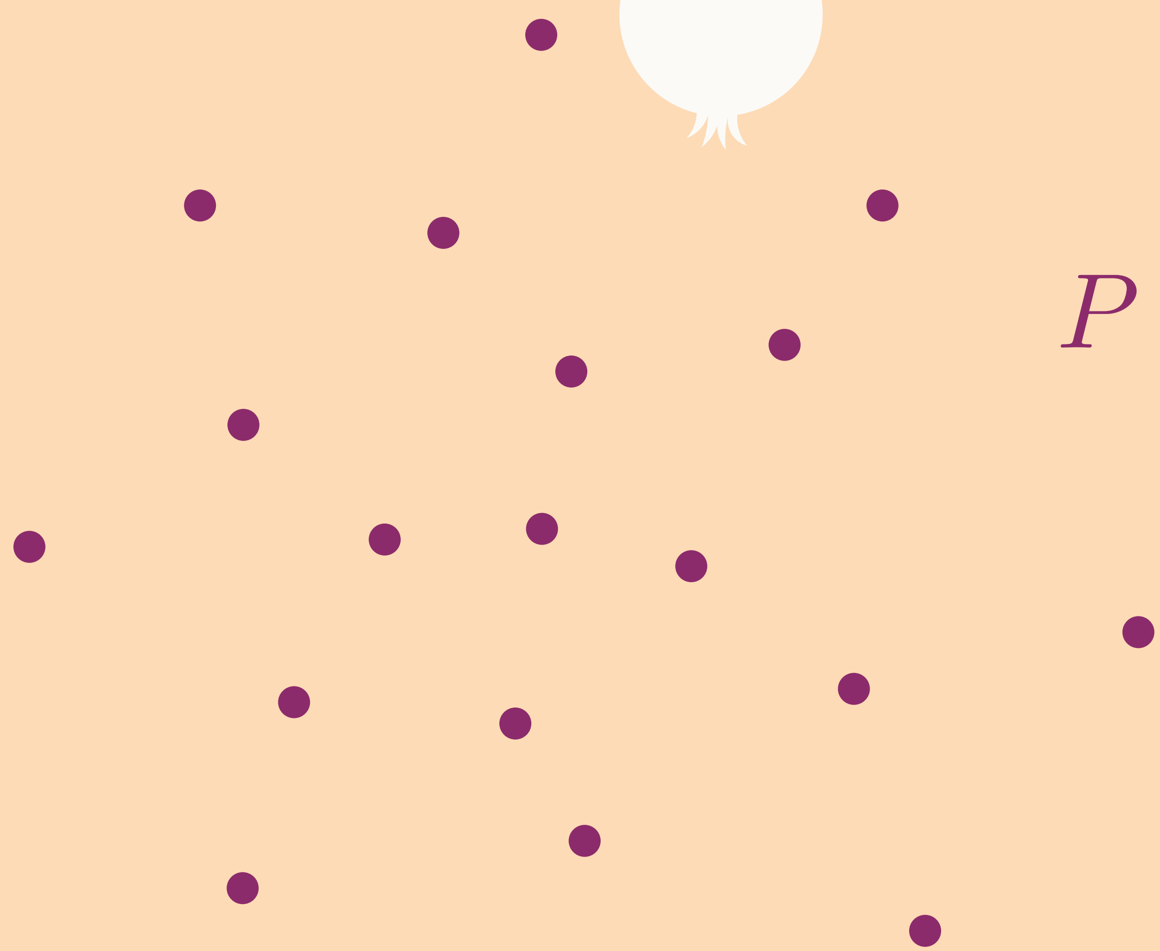

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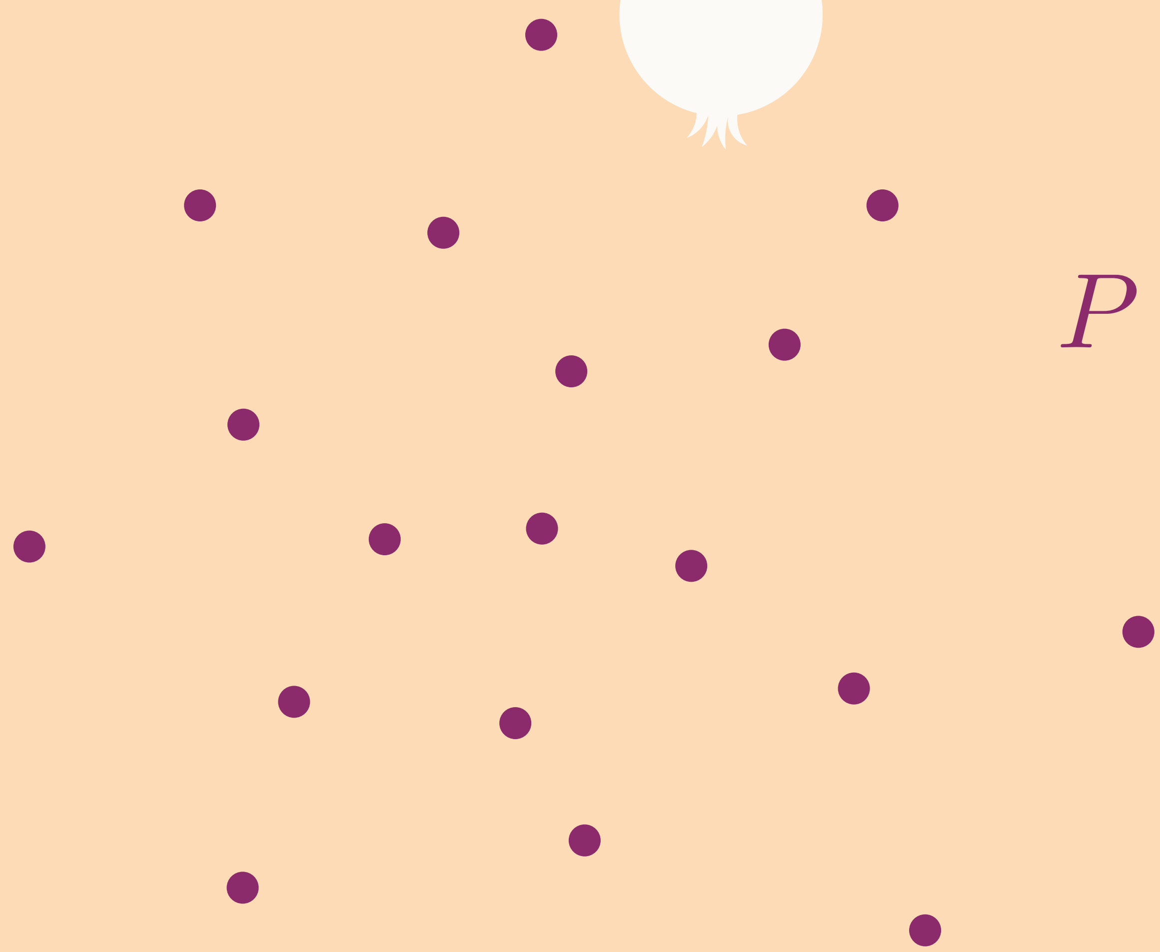
DEFINITION: The *onion* of P is the convex hull of P, plus the onion of the rest of the points.



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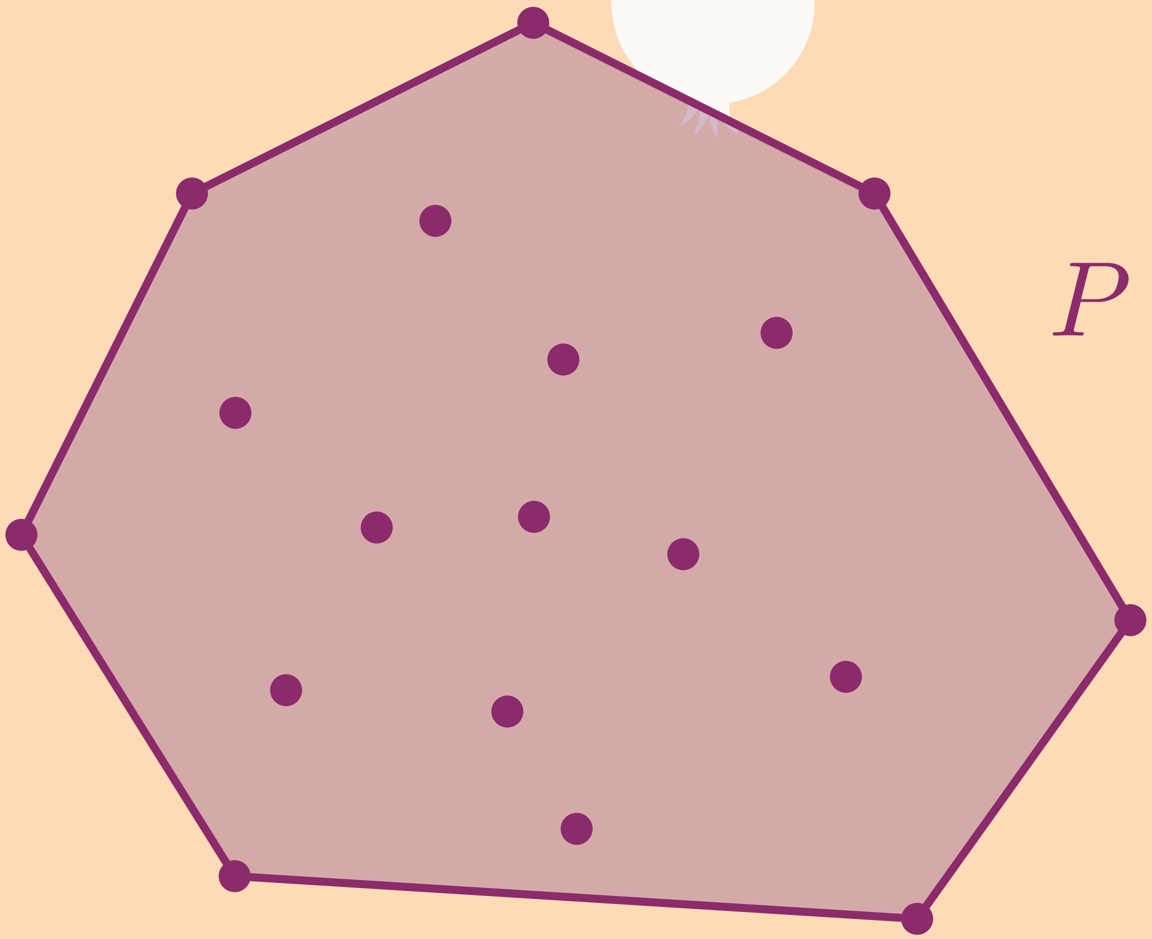

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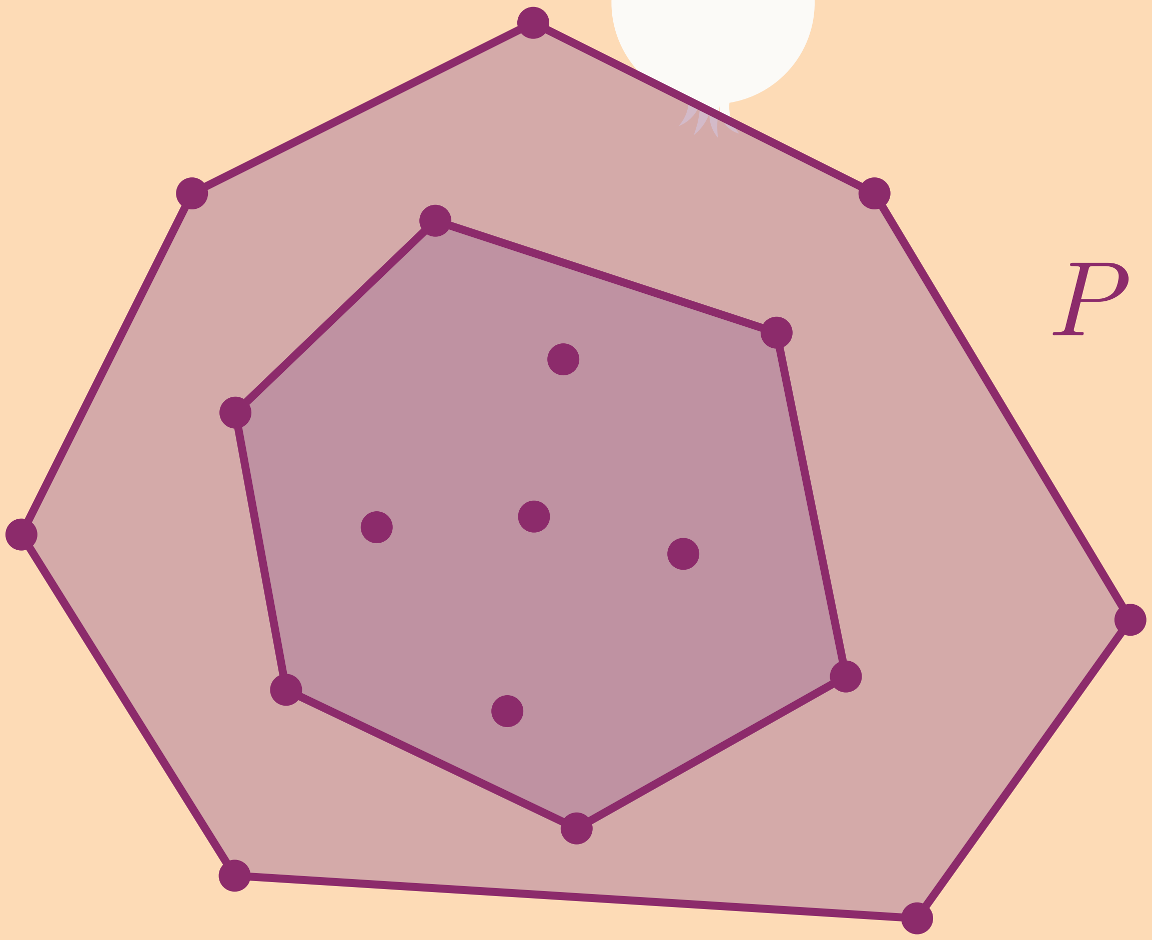

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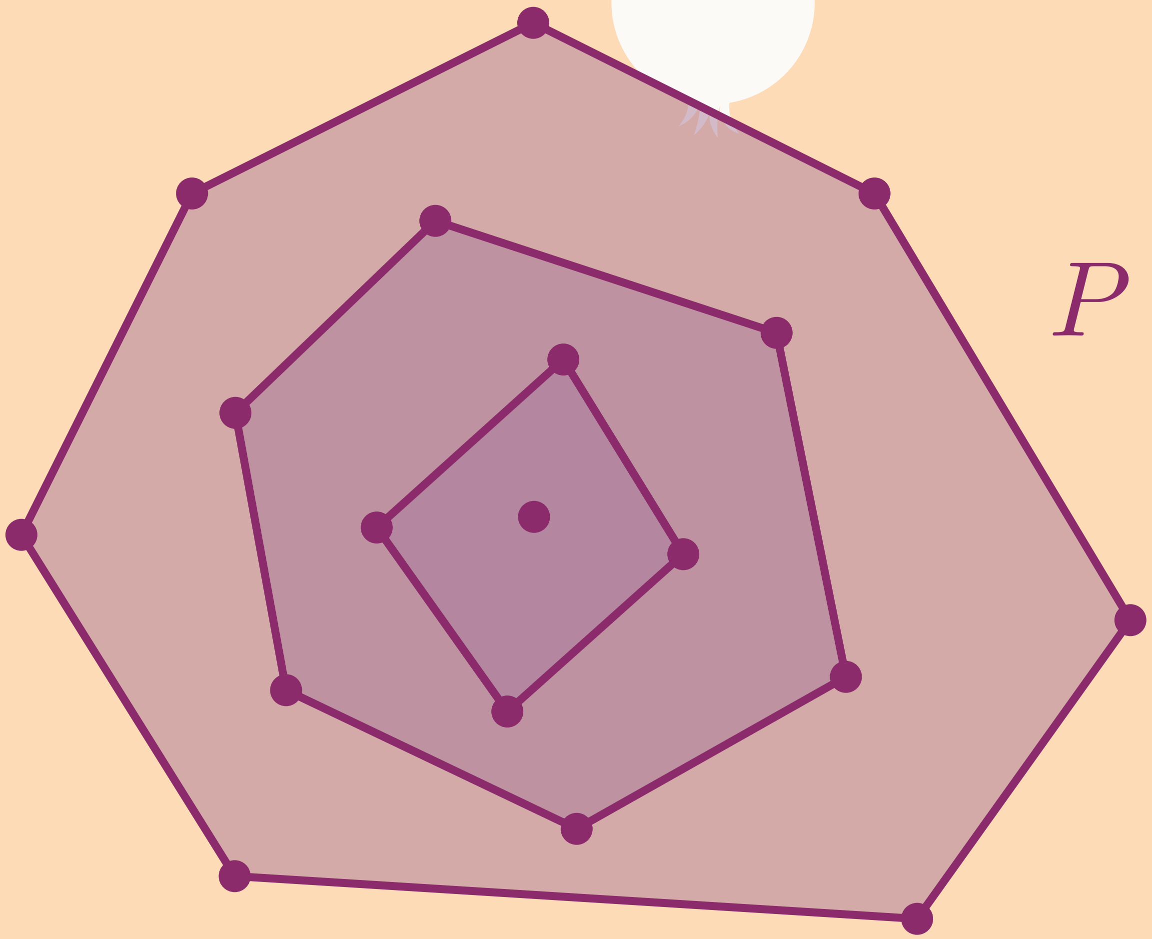

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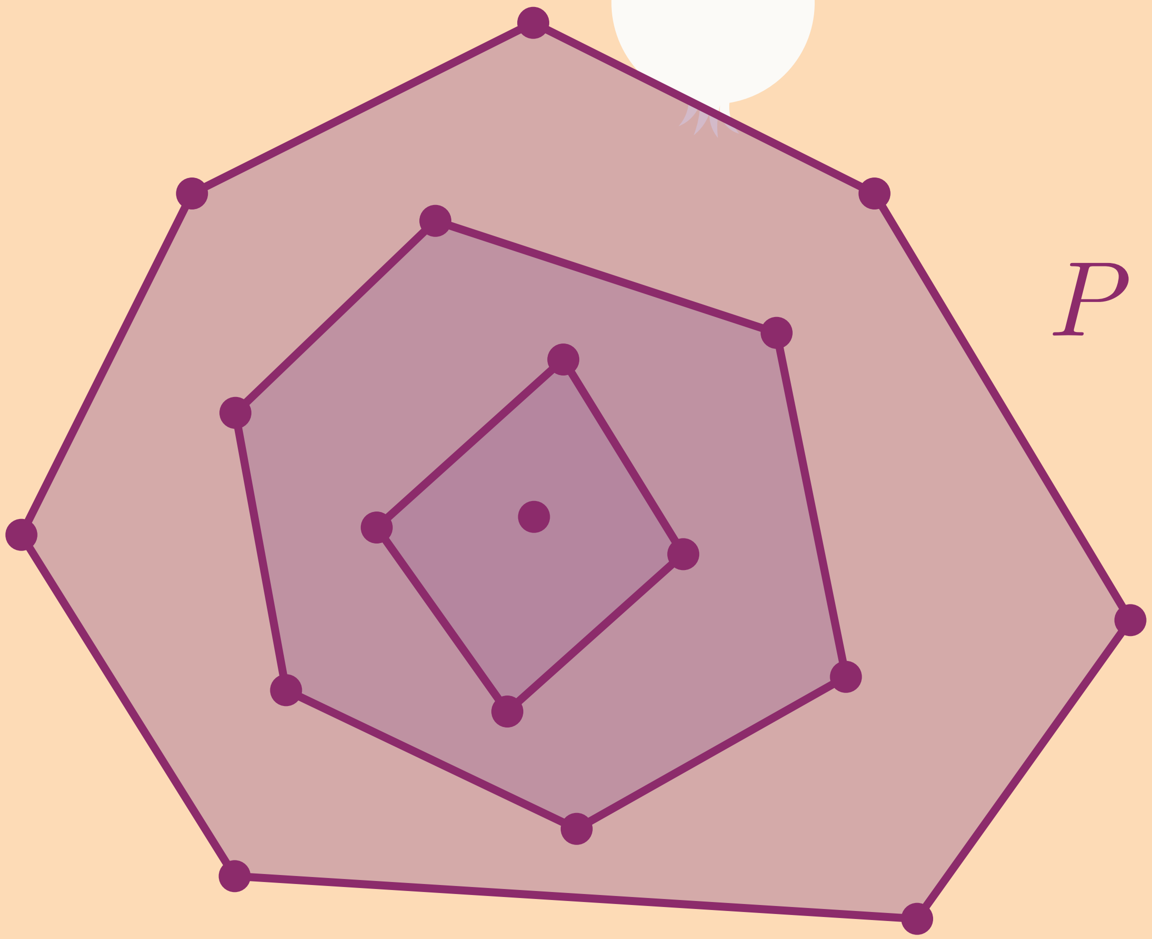
Onions have many useful and tasty applications.



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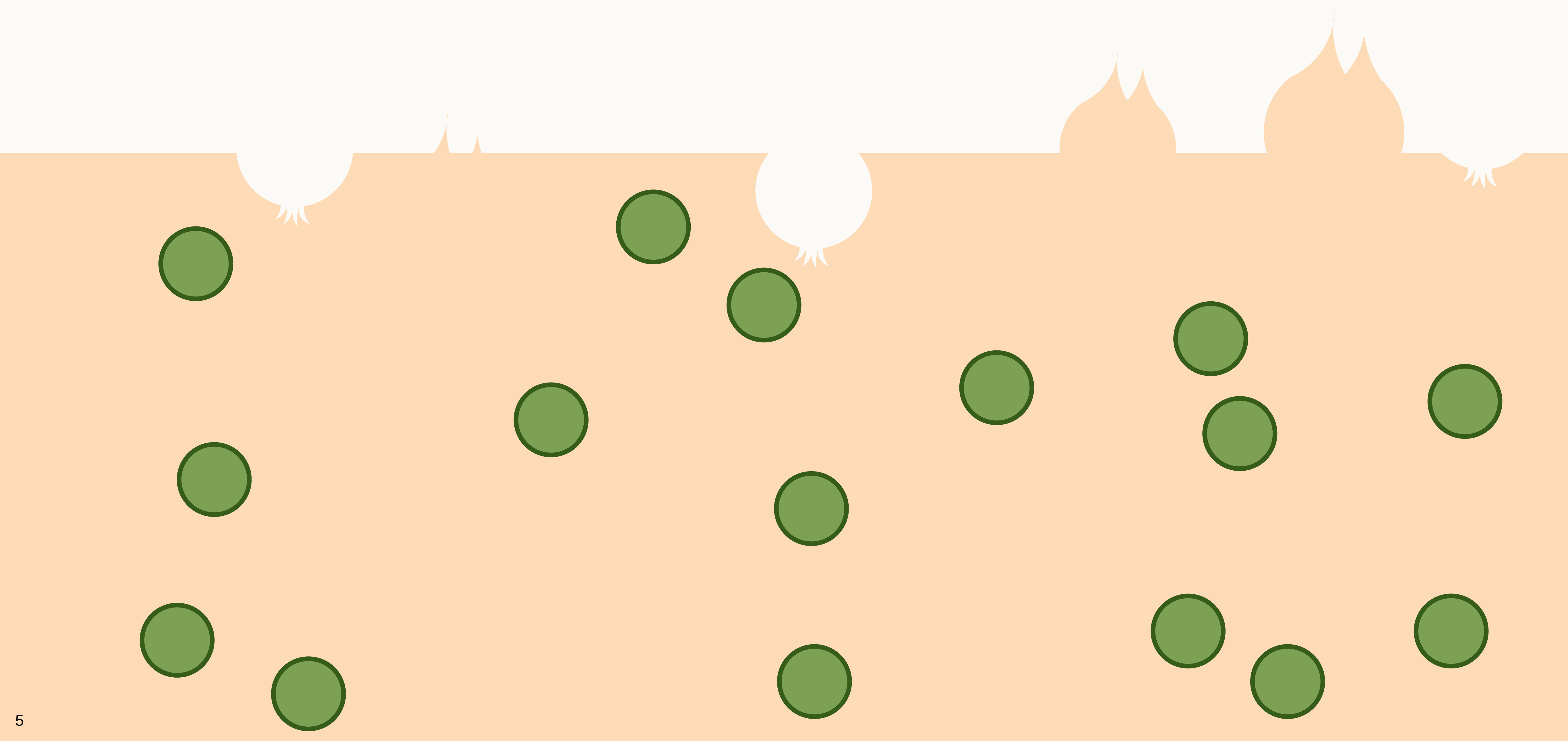

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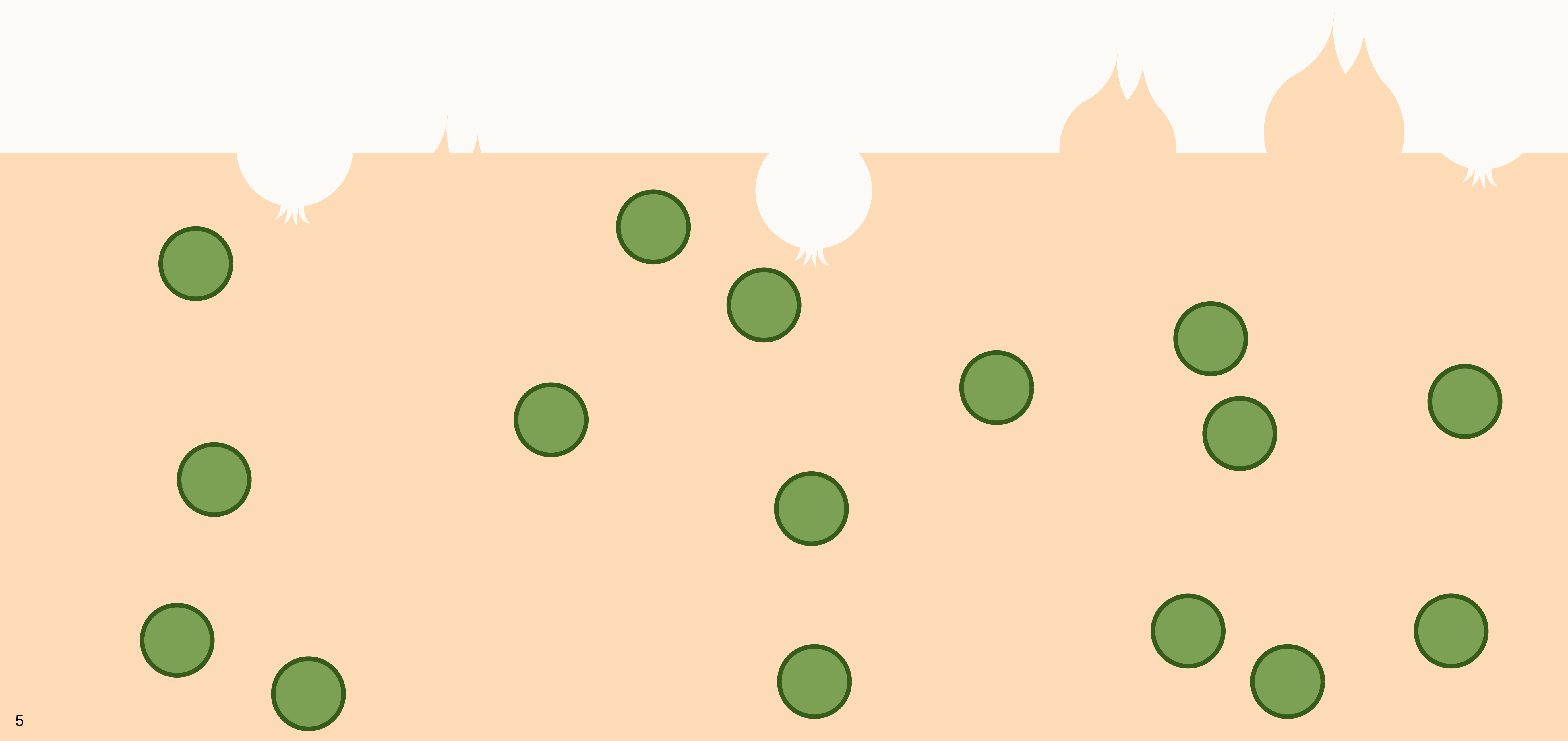

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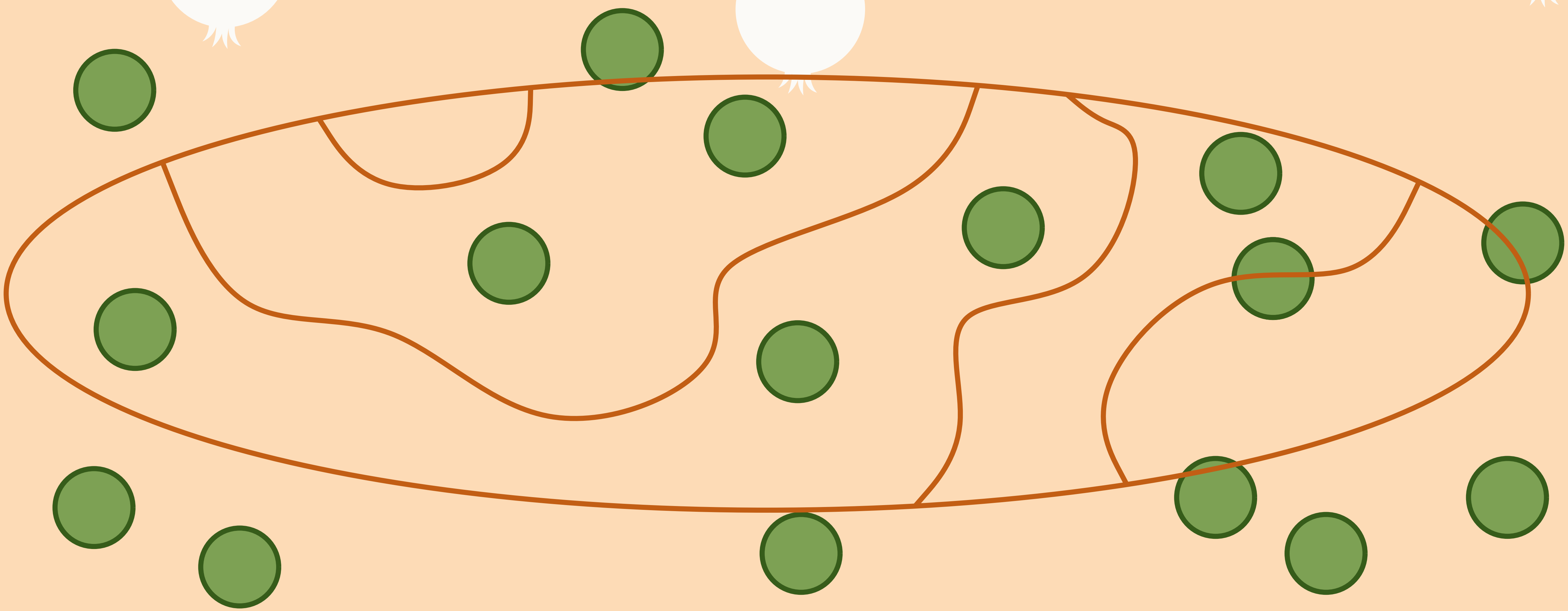

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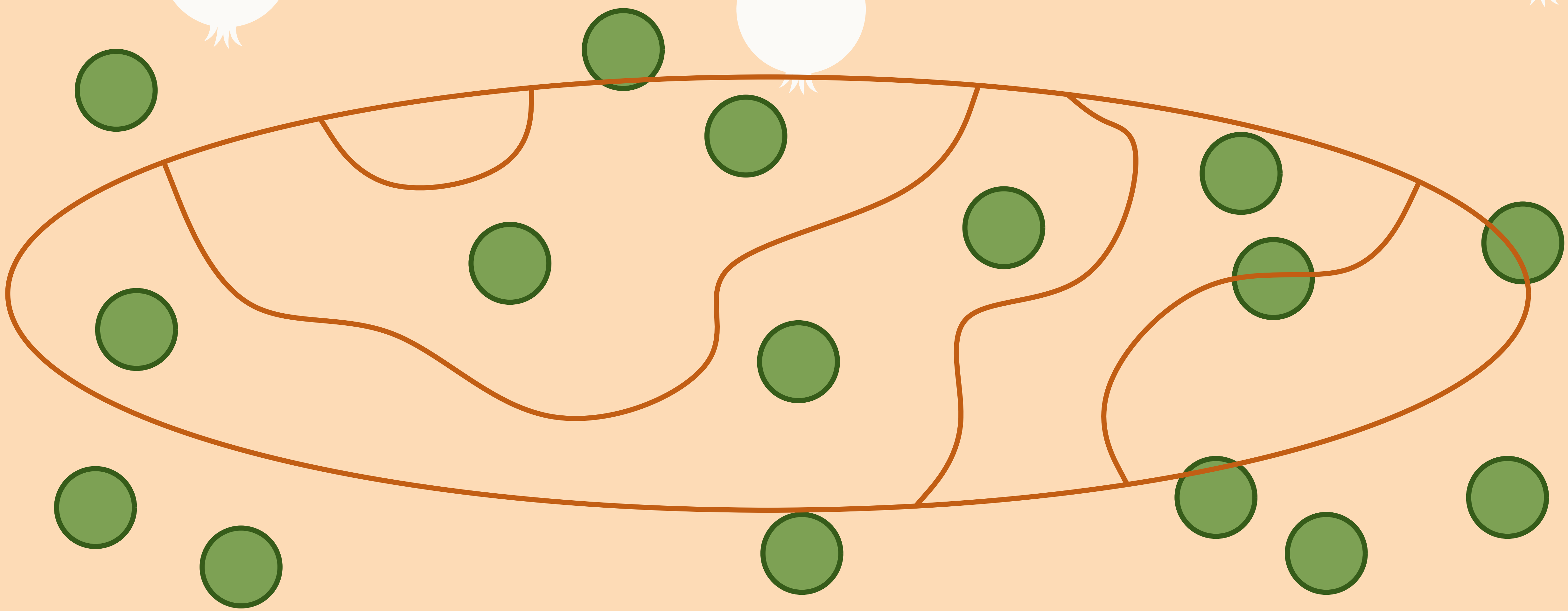

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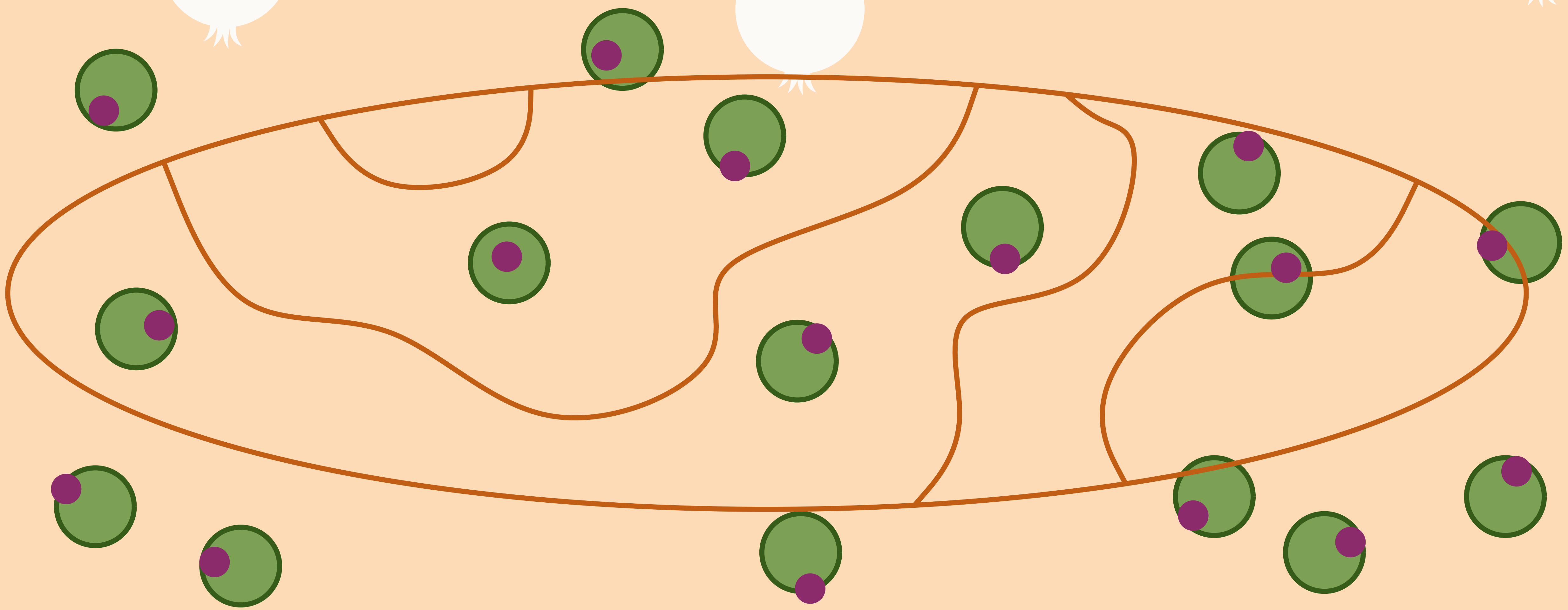

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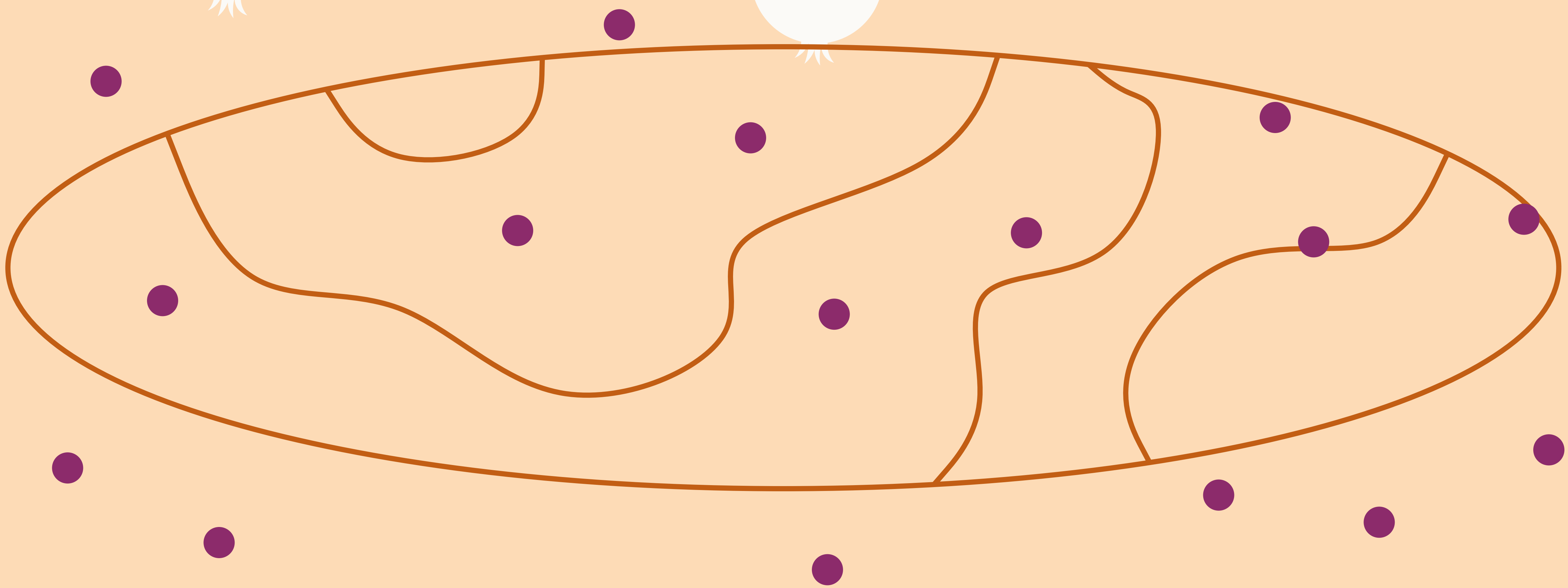

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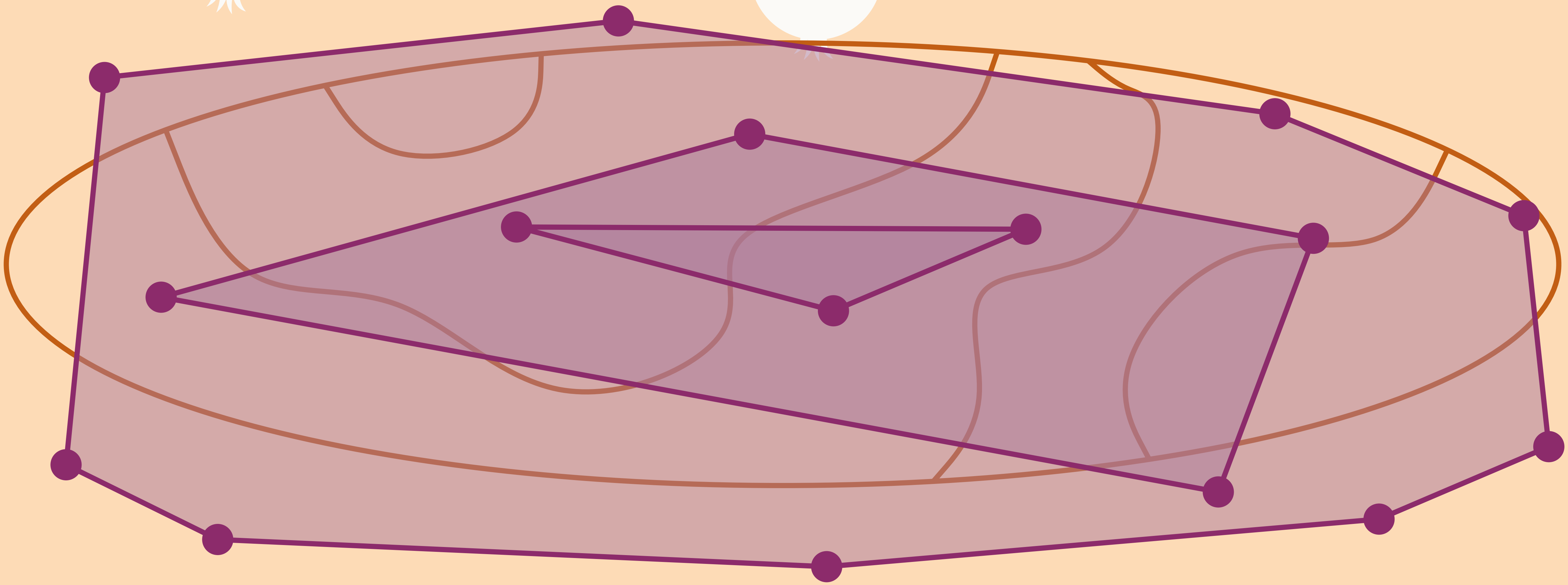

CHAPTER II

THE SOLUTION

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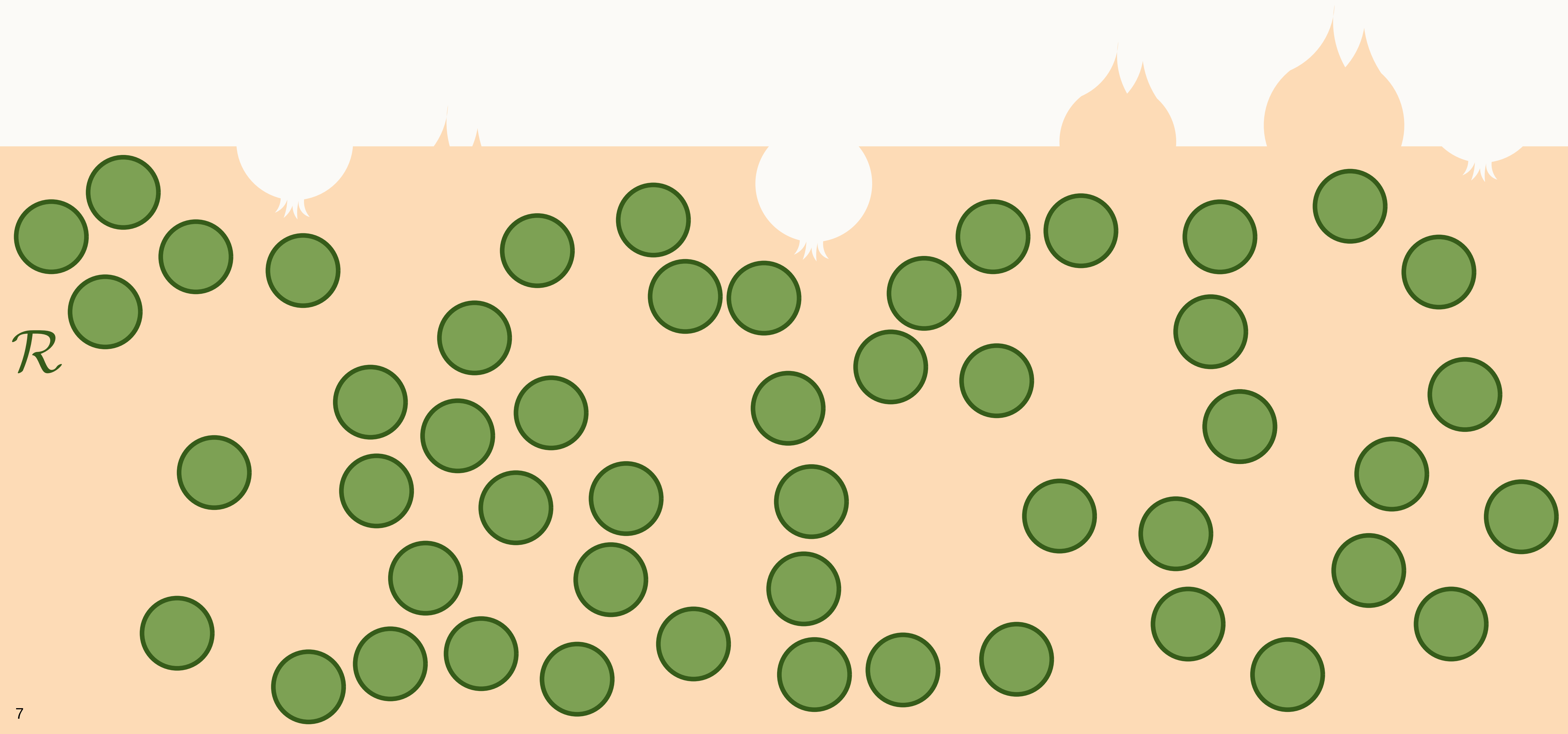

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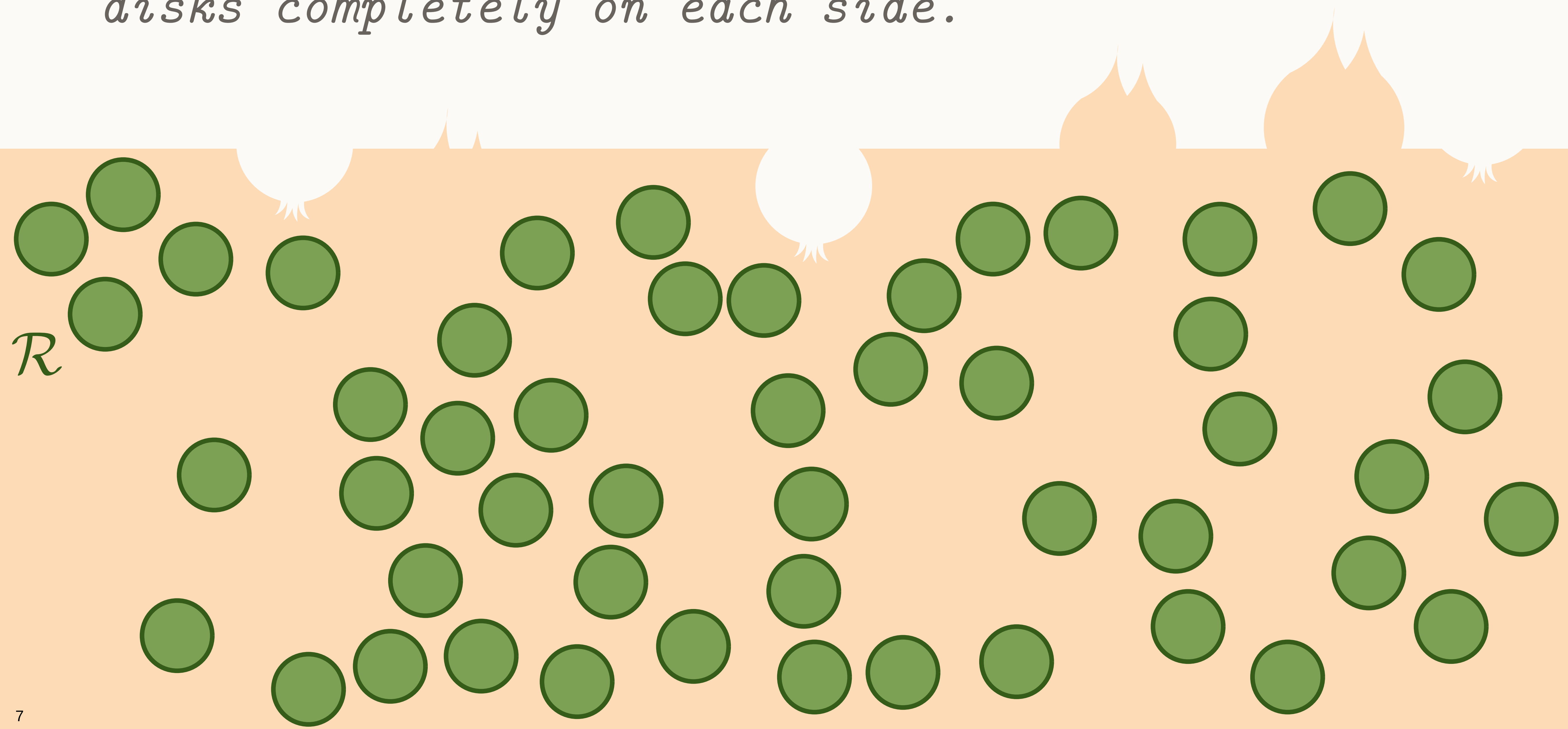
LEMMA: *There exists a line that intersects at most $\sqrt{n \log n}$ disks and has at most half of the disks completely on each side.*



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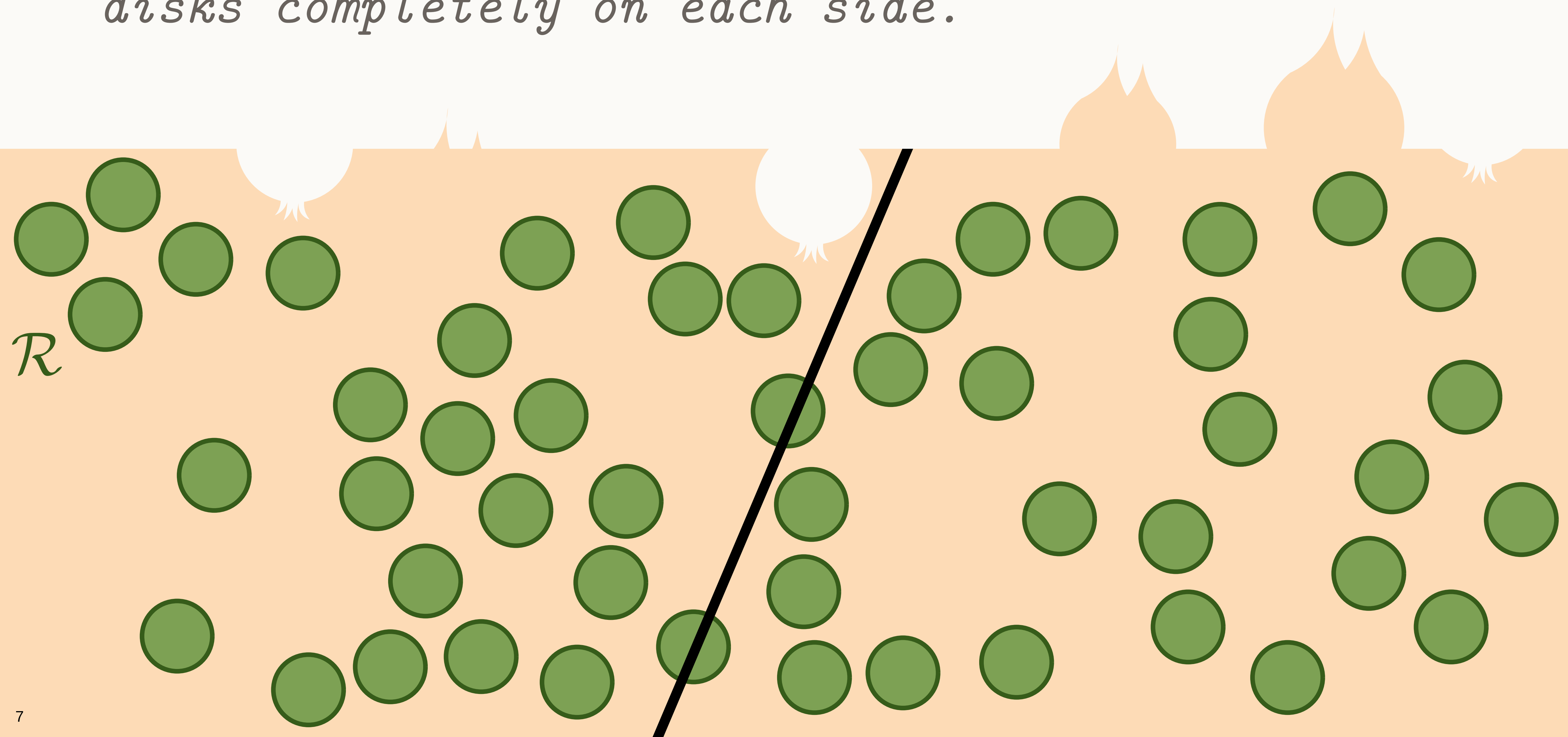

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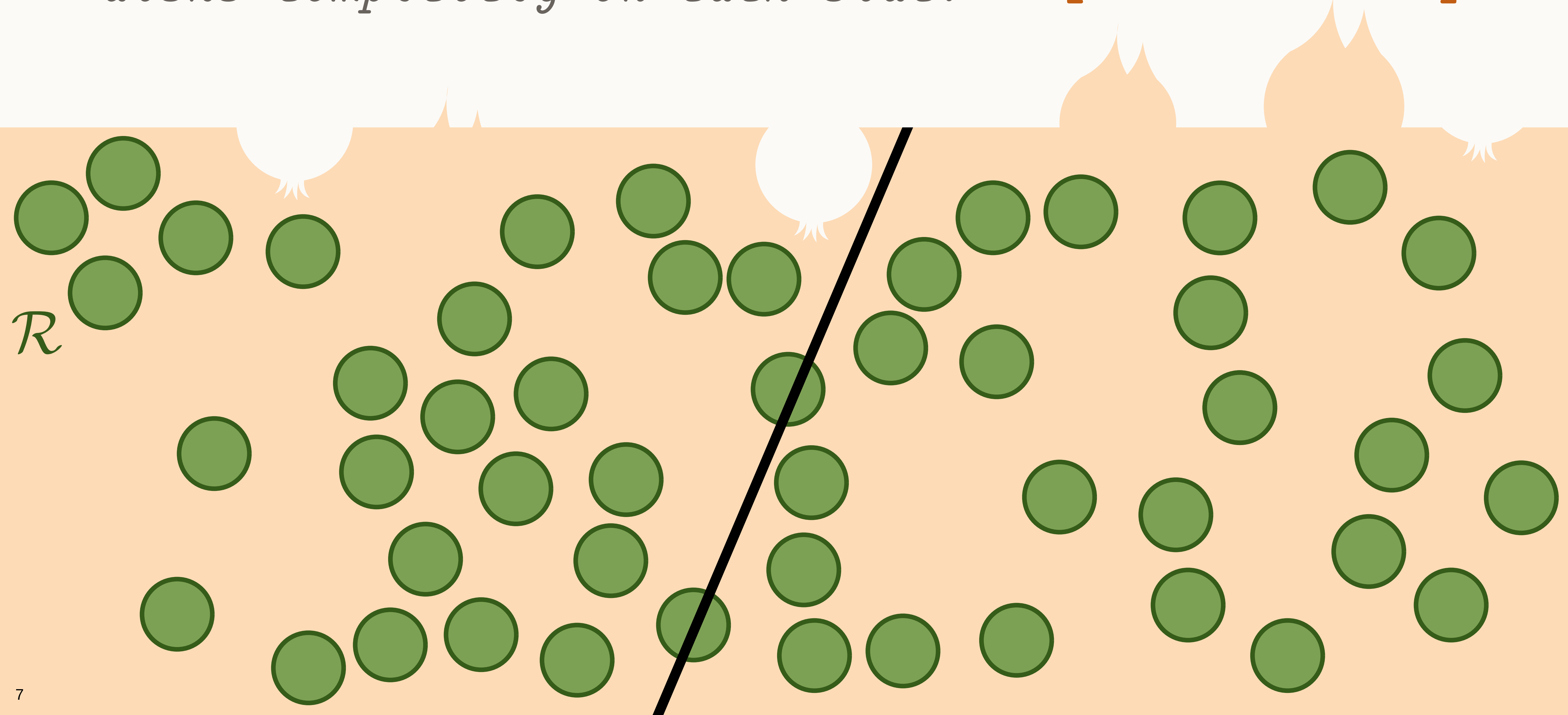
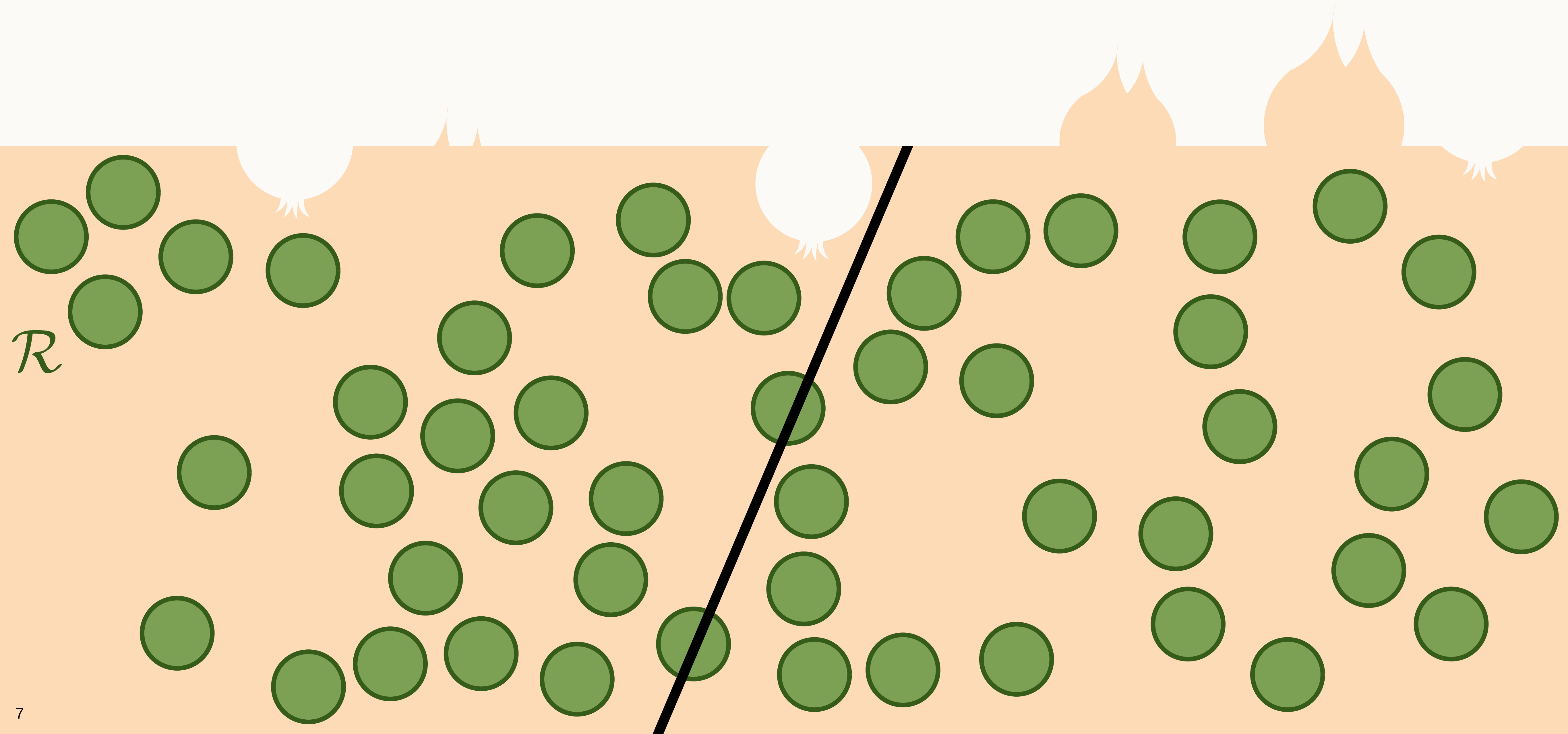


PREPROCESSING ALGORITHM

Consider a set $\mathcal{R}$ of n disjoint unit disks.

We apply the lemma recursively, and get a binary space partition tree T.



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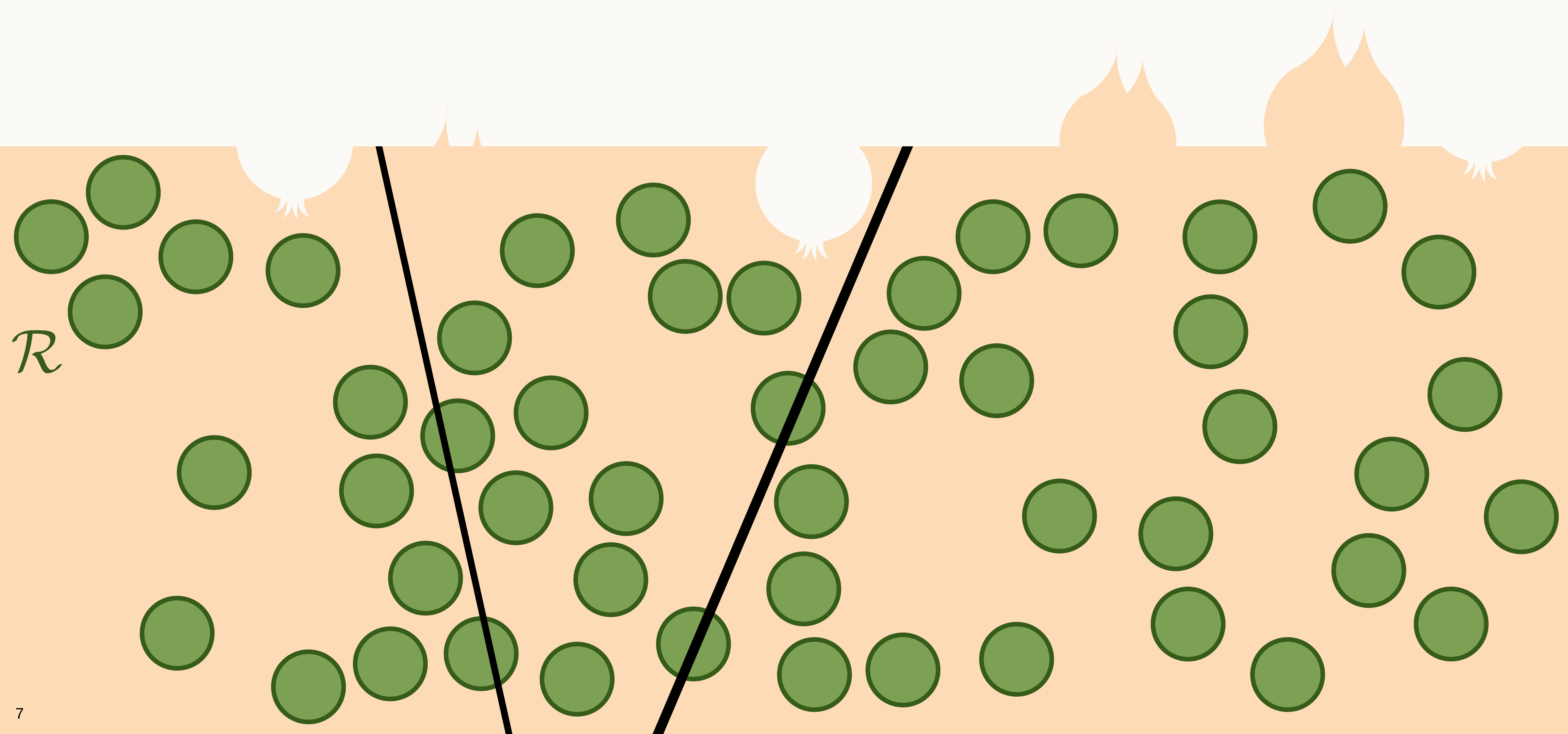

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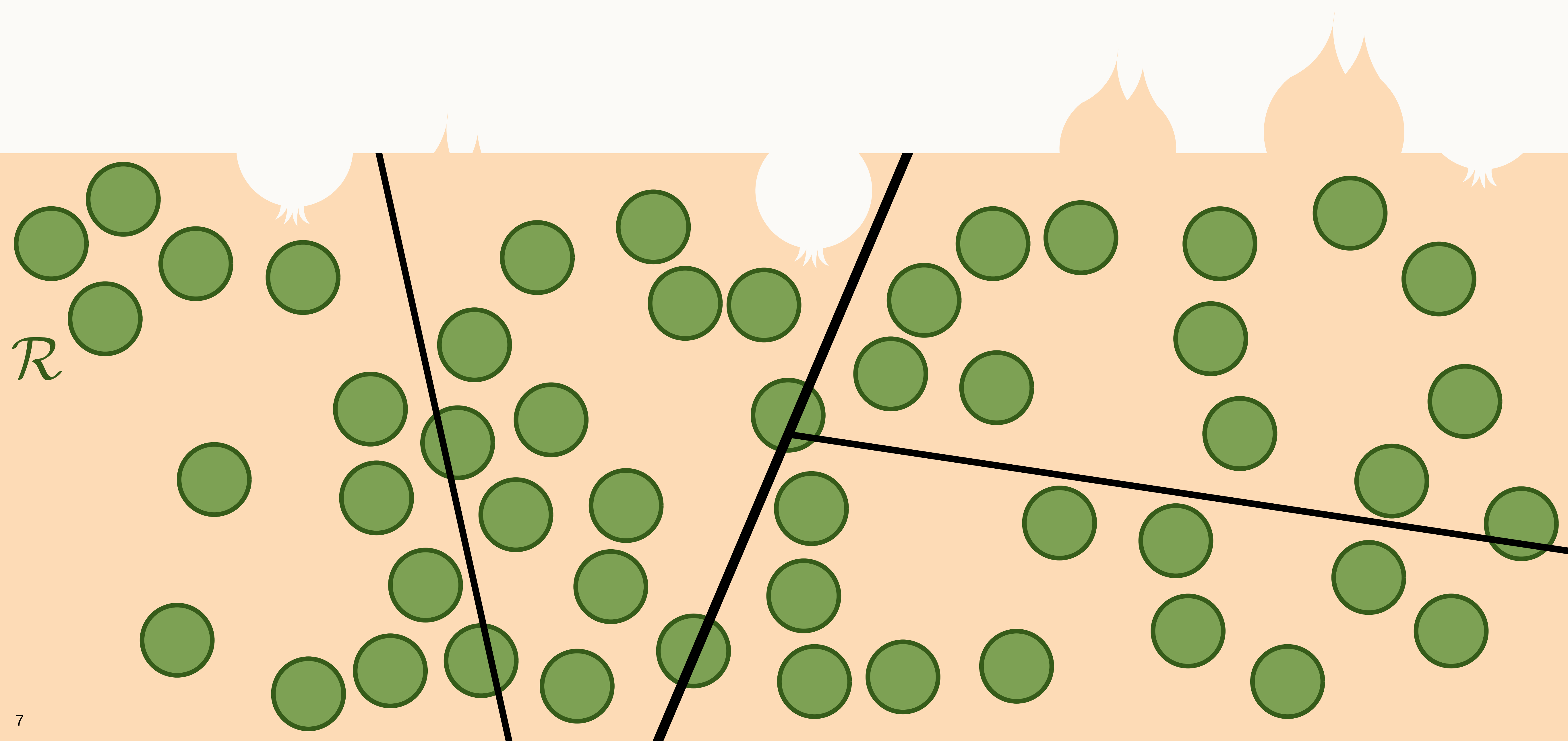

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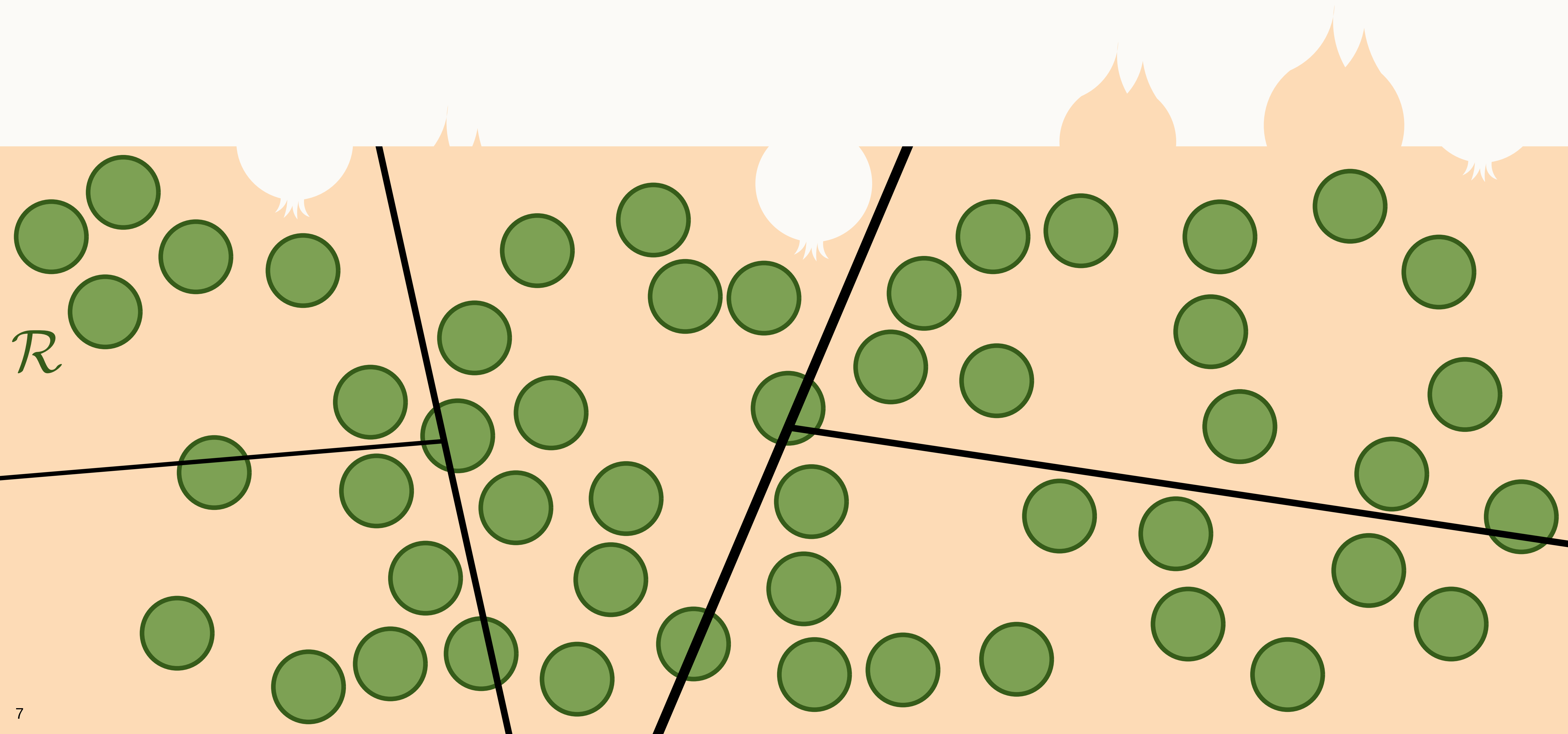

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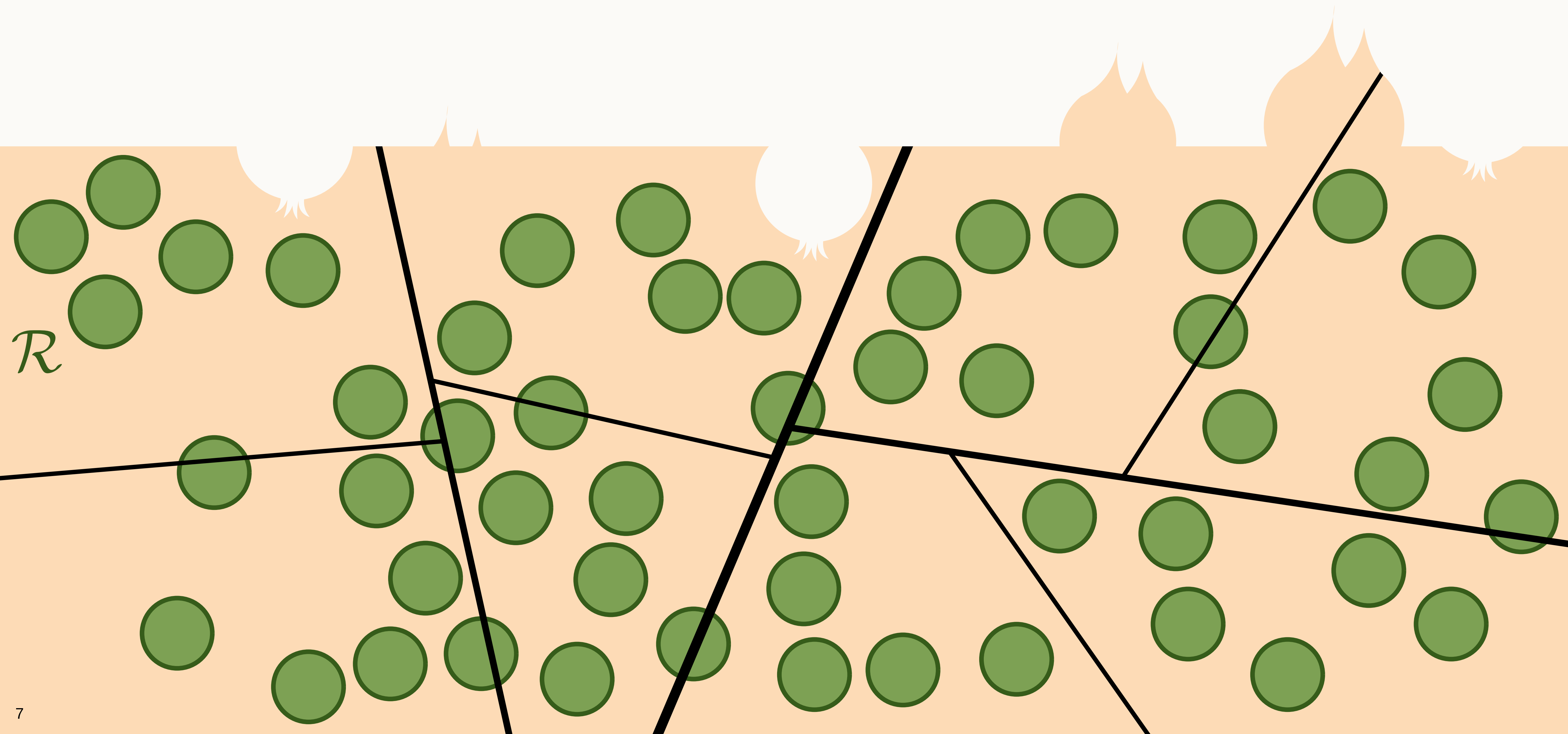

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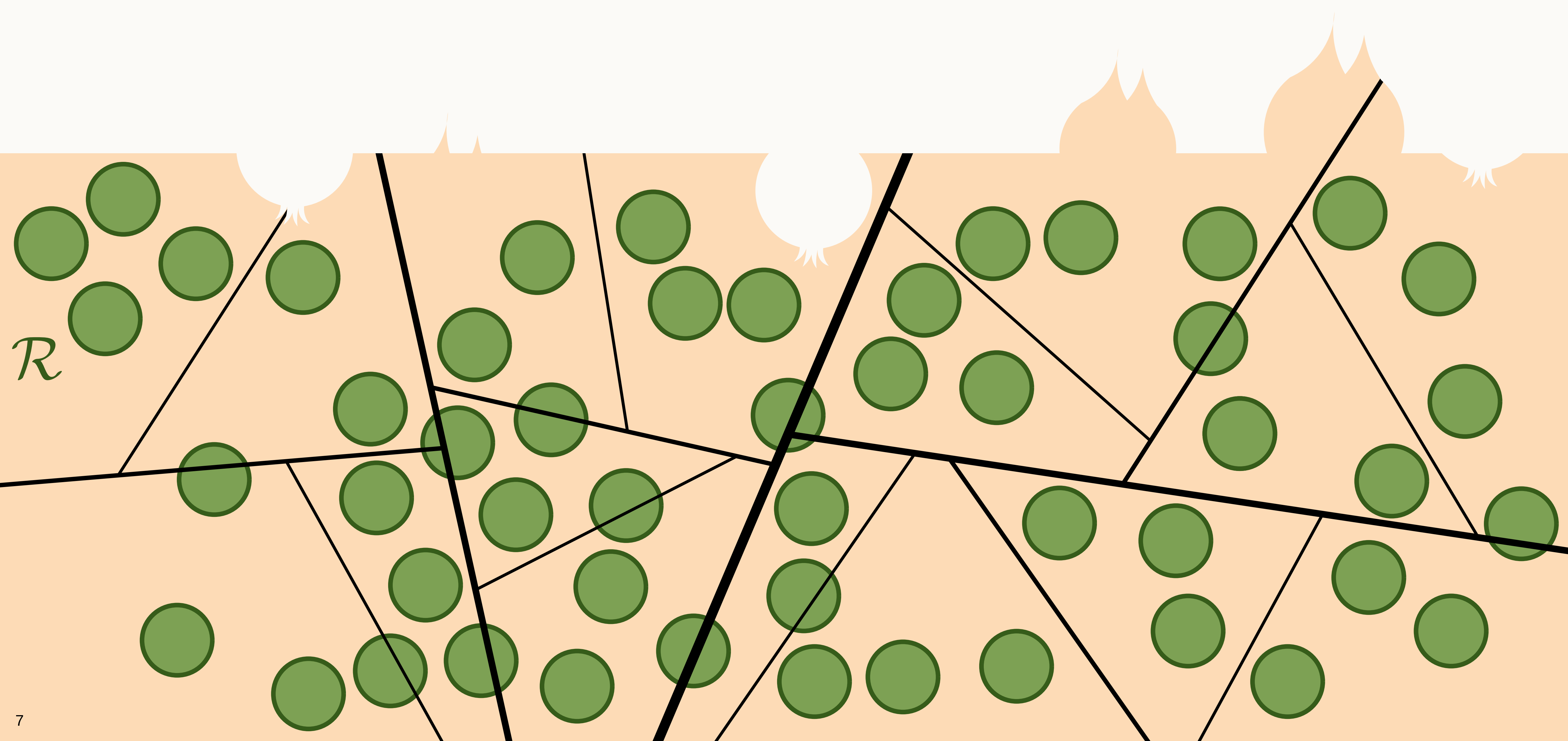

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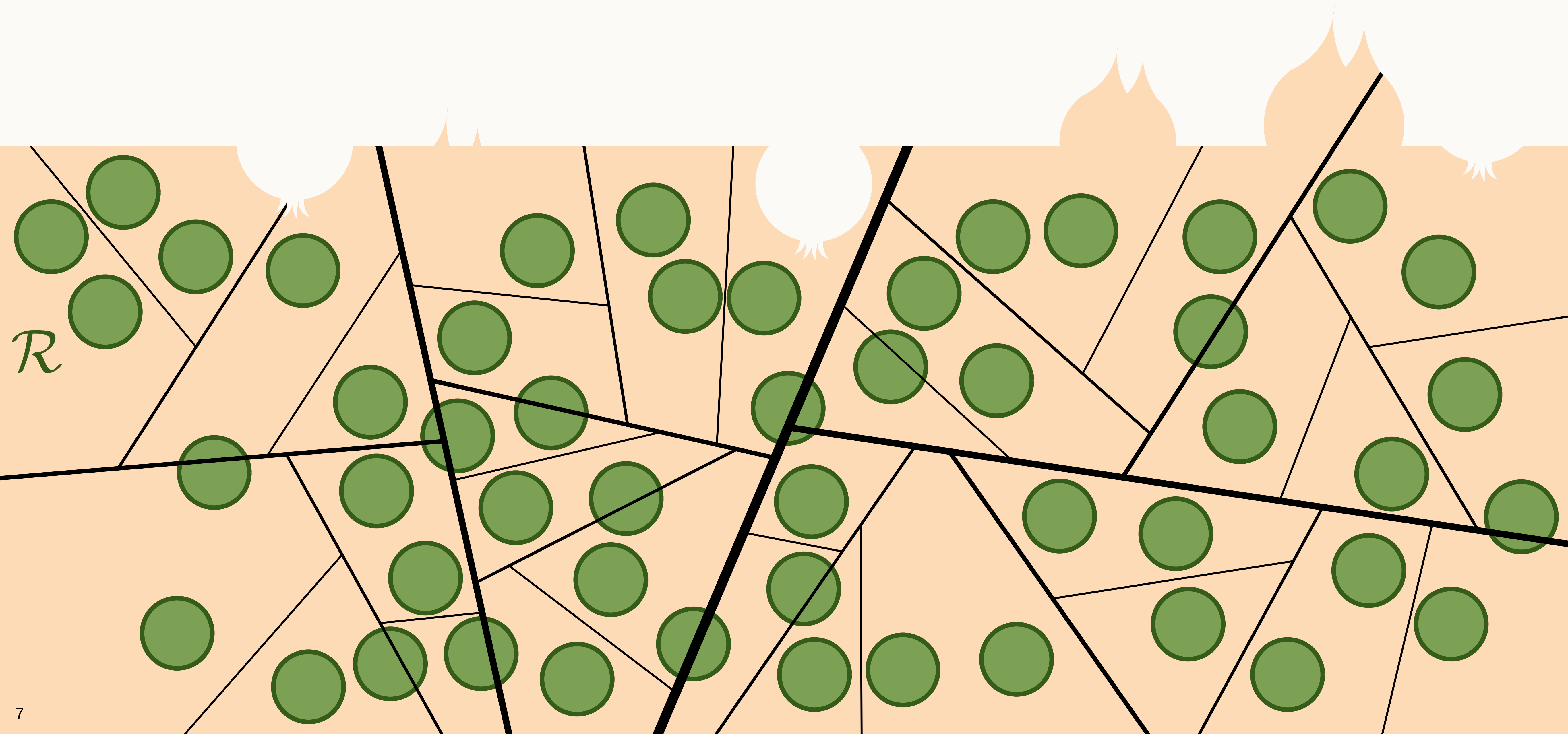

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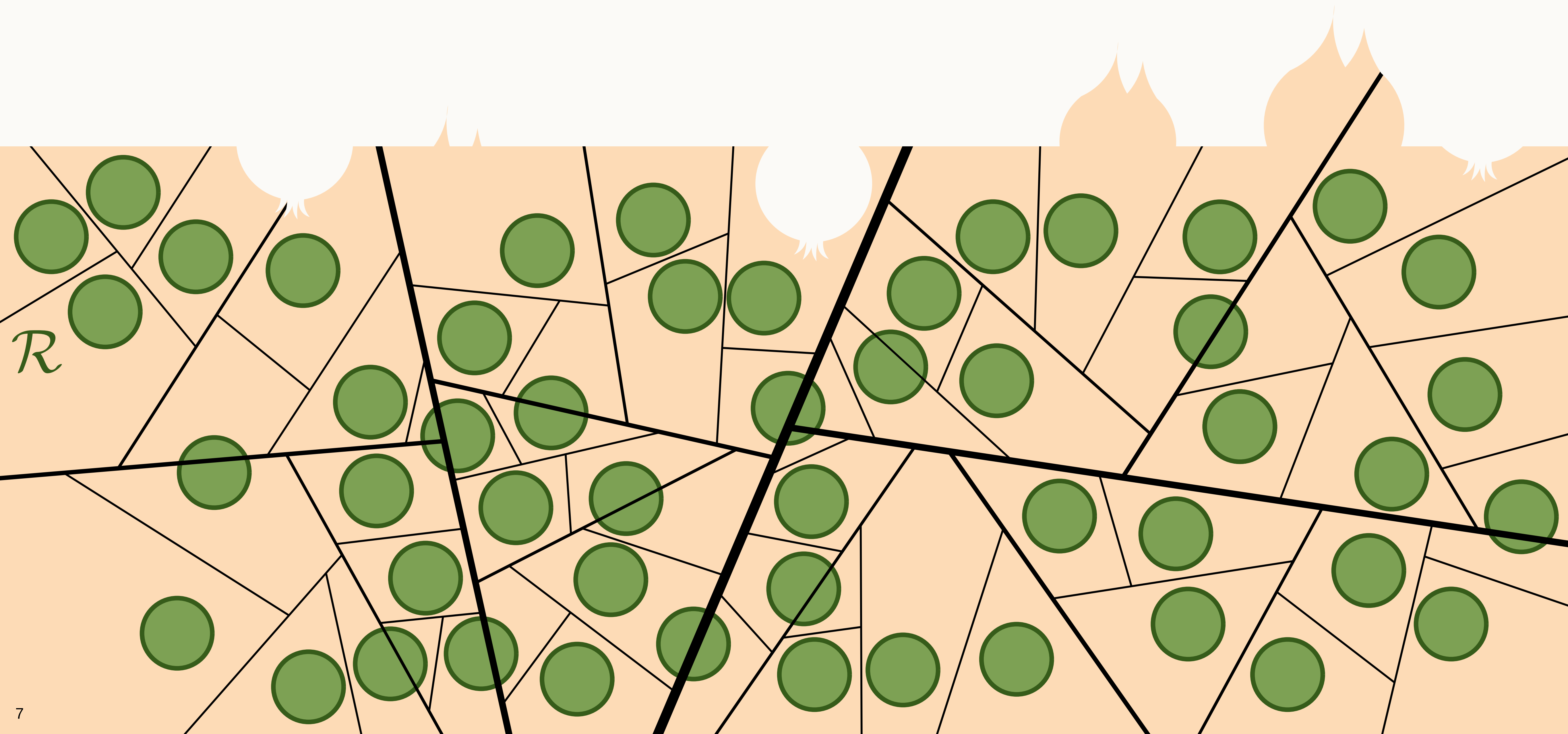

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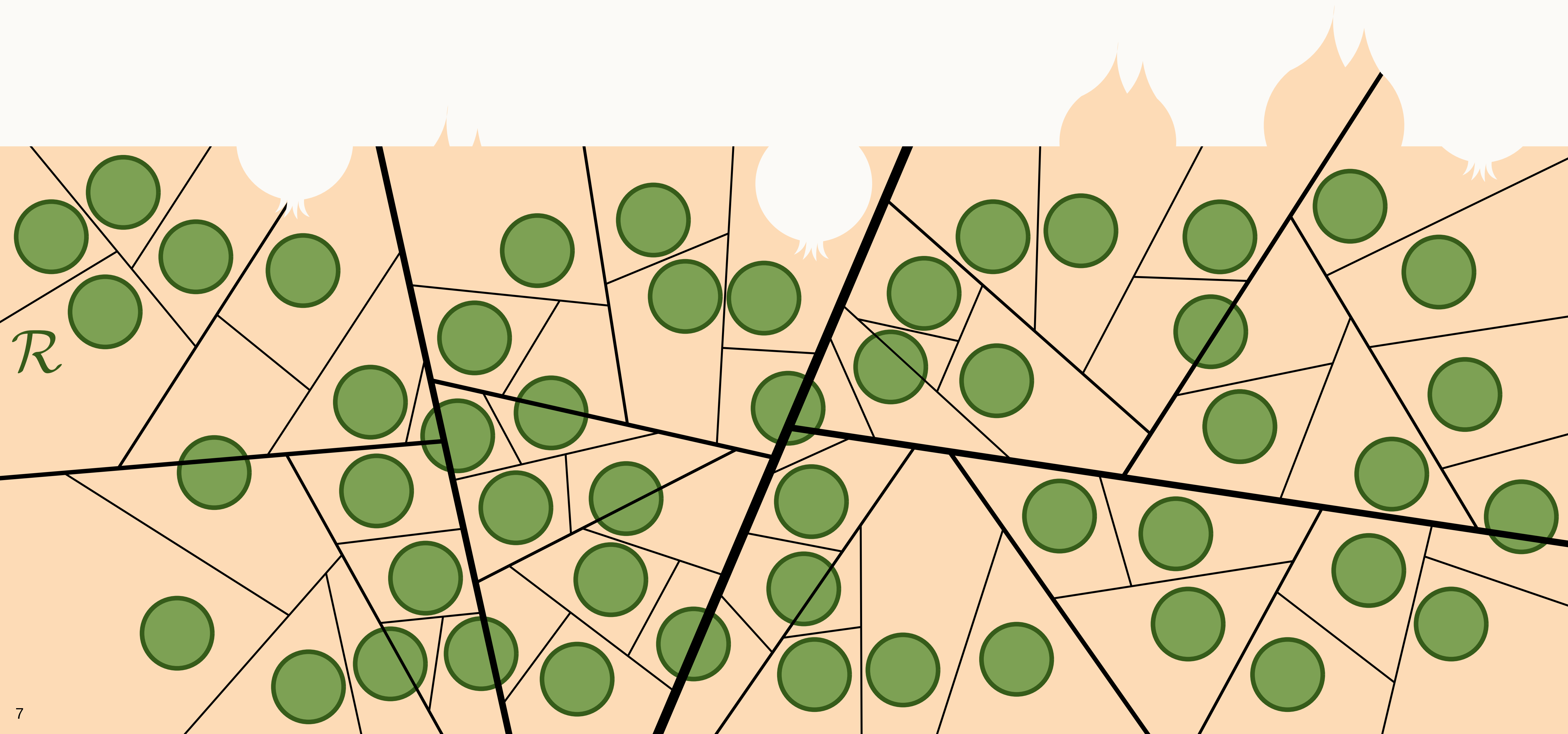

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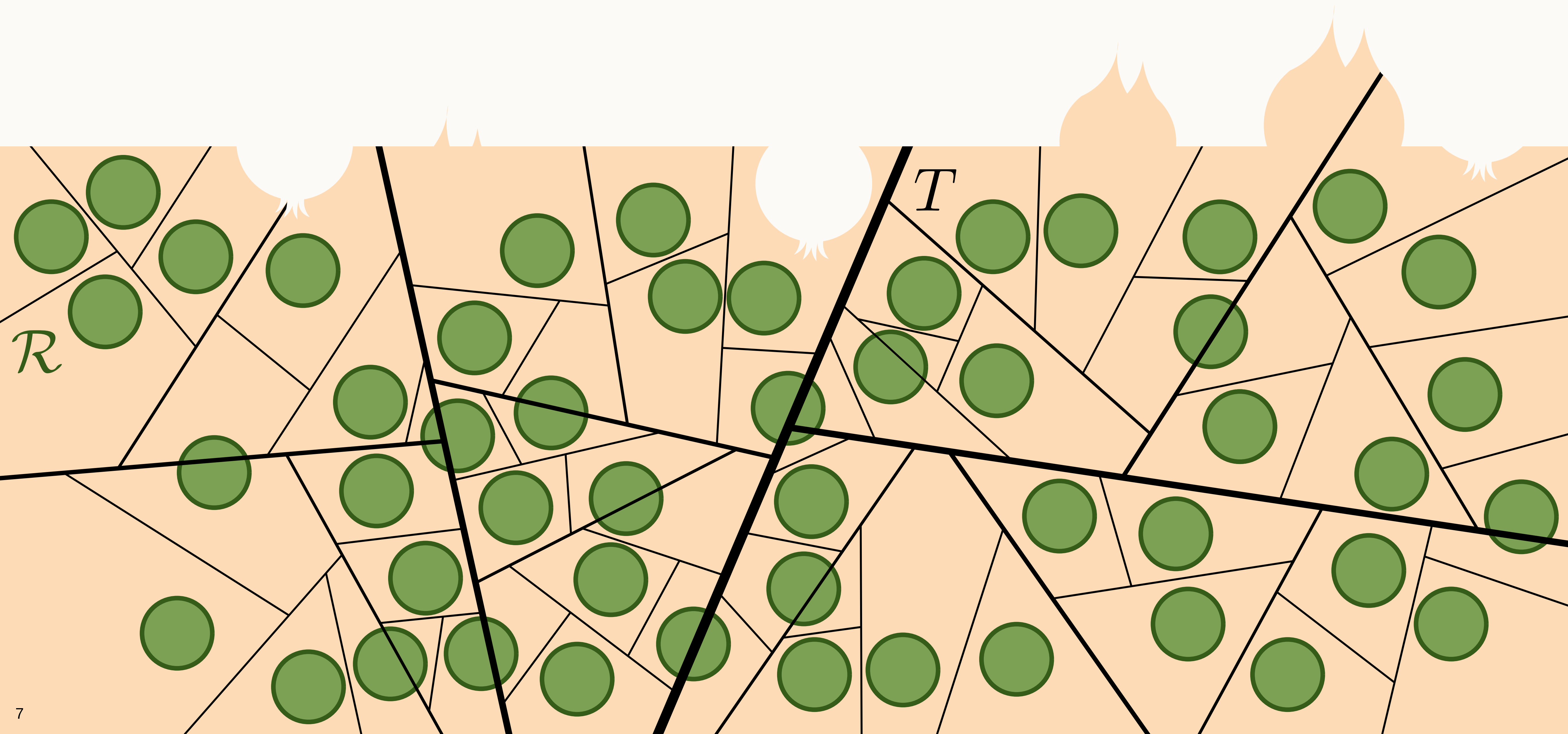

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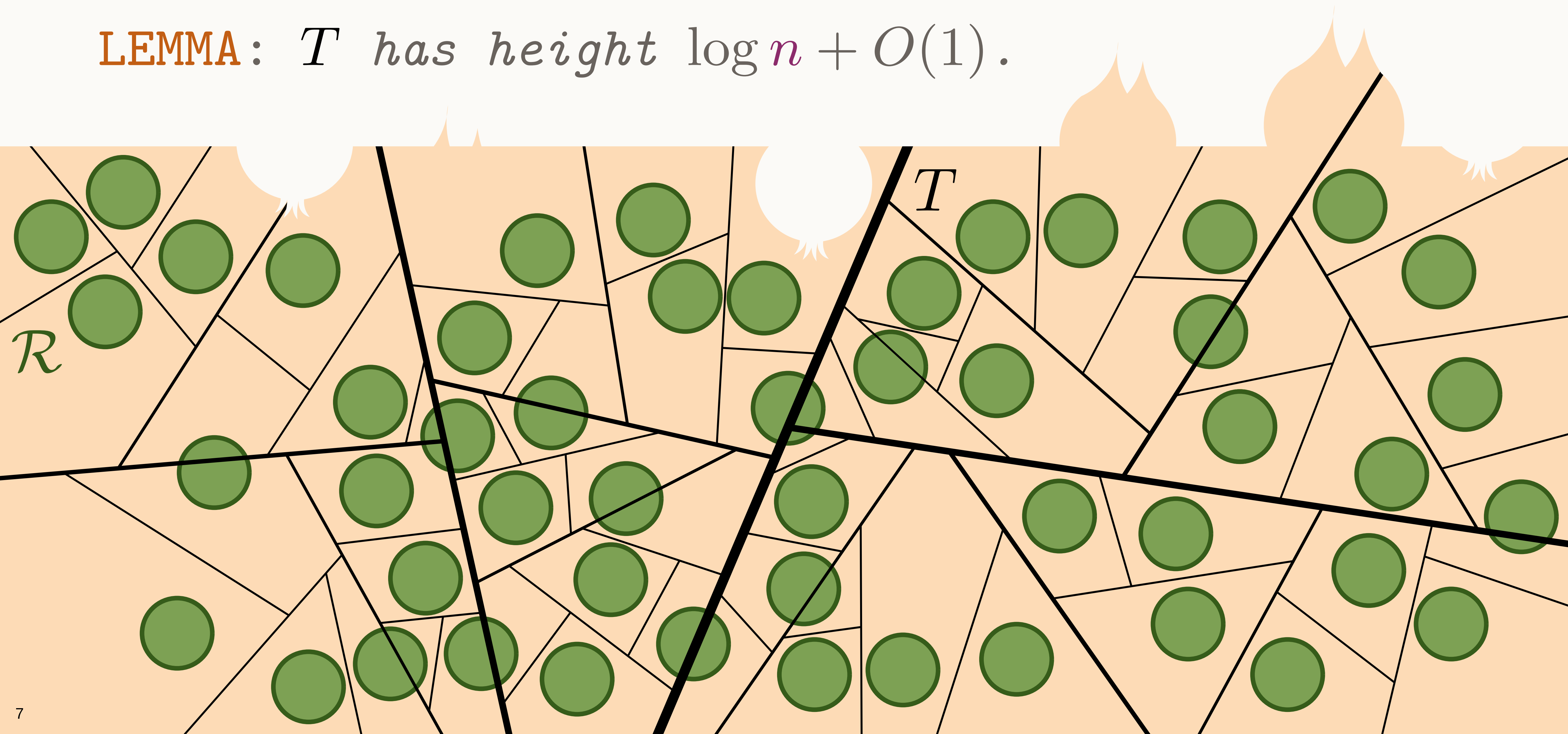
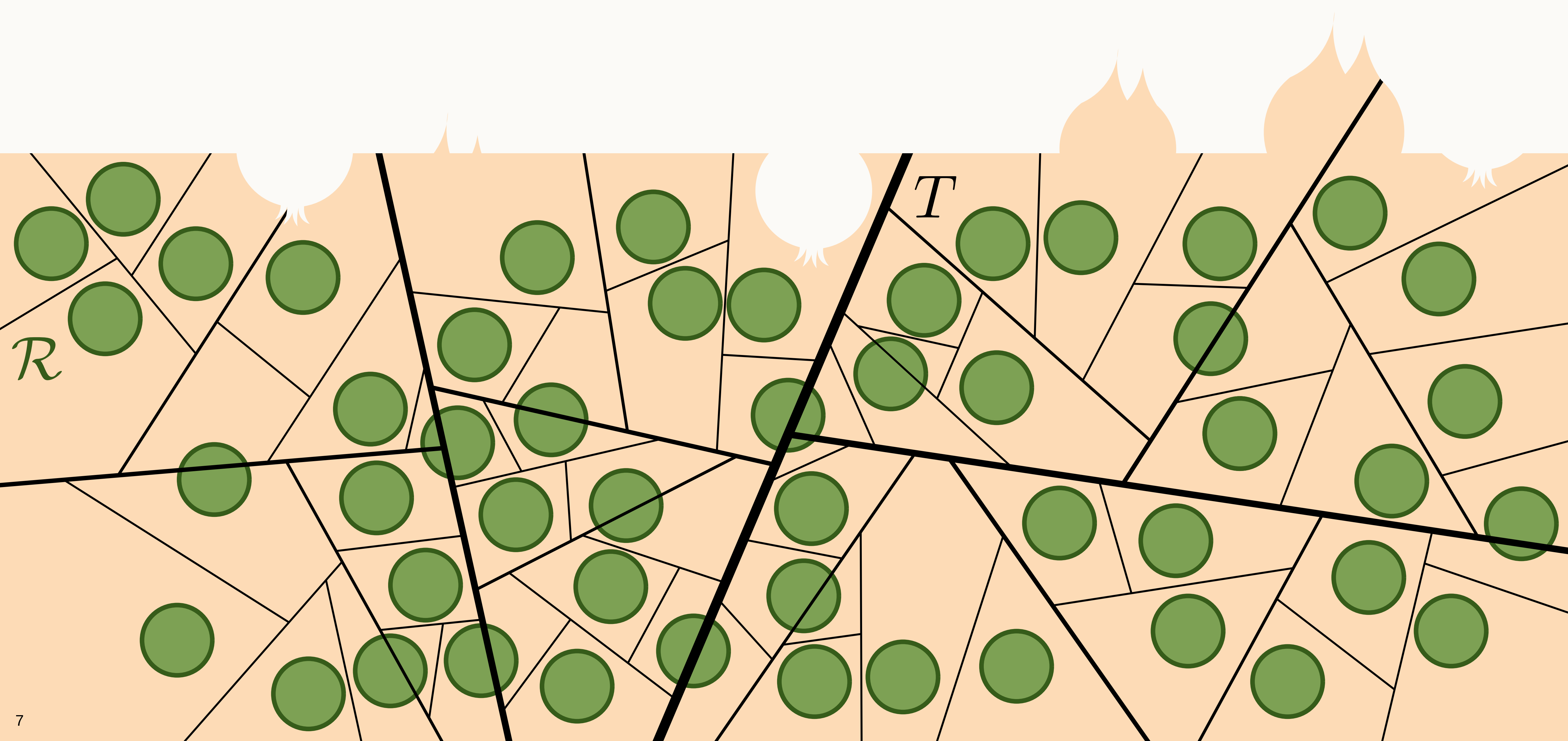
LEMMA: T *has height* $\log n + O(1)$.



PREPROCESSING ALGORITHM

HINT: $T = H(\mathcal{R})$ *is our magical structure!*



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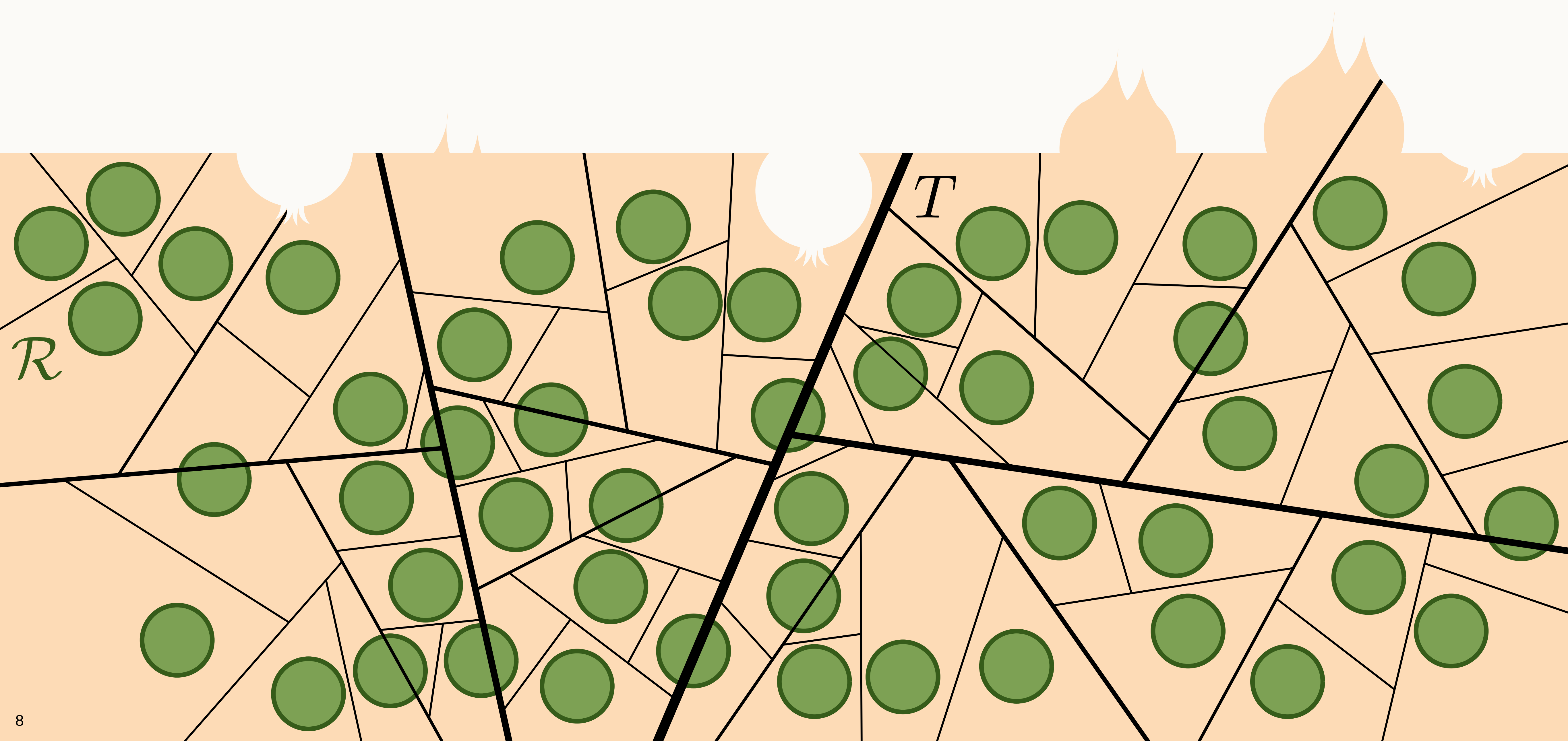

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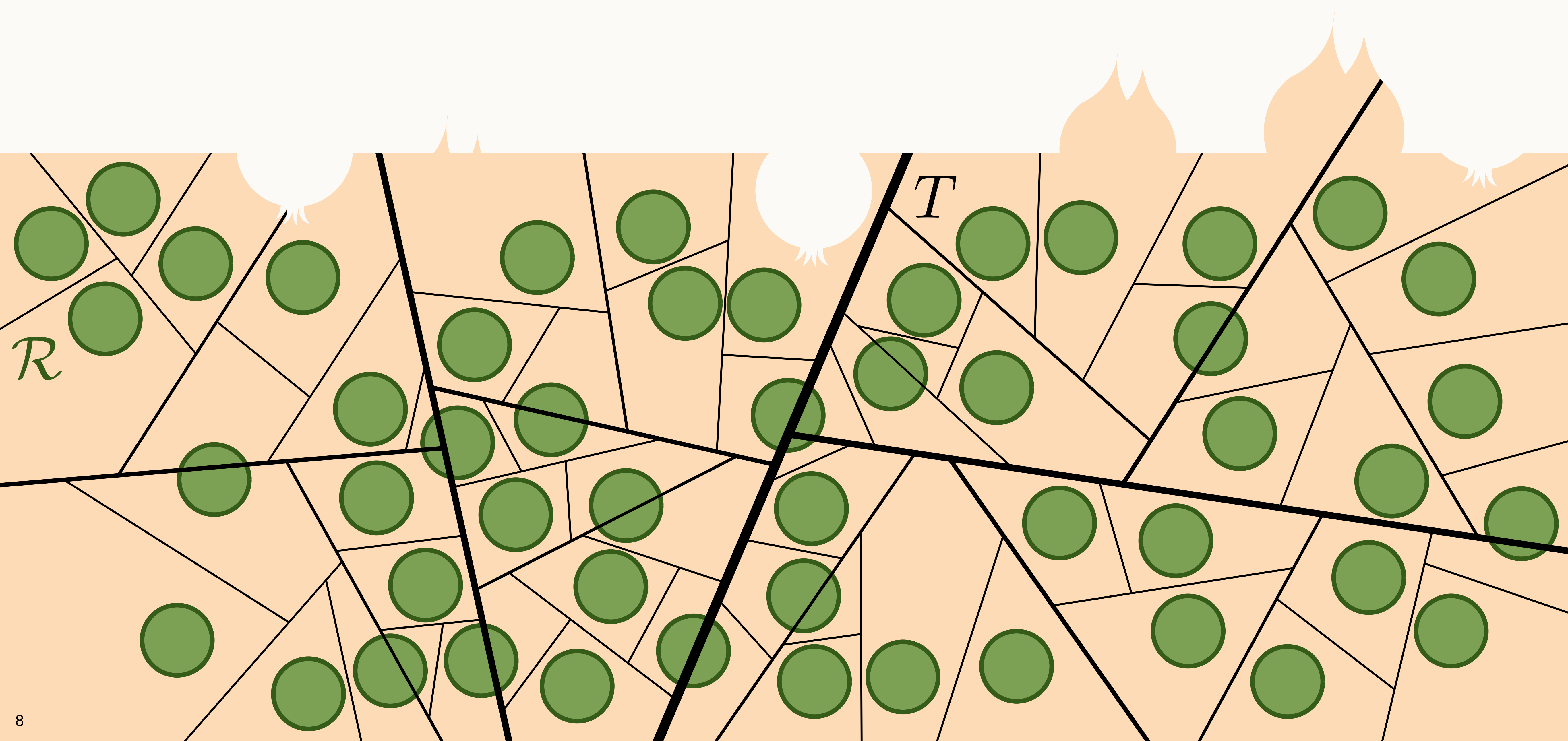

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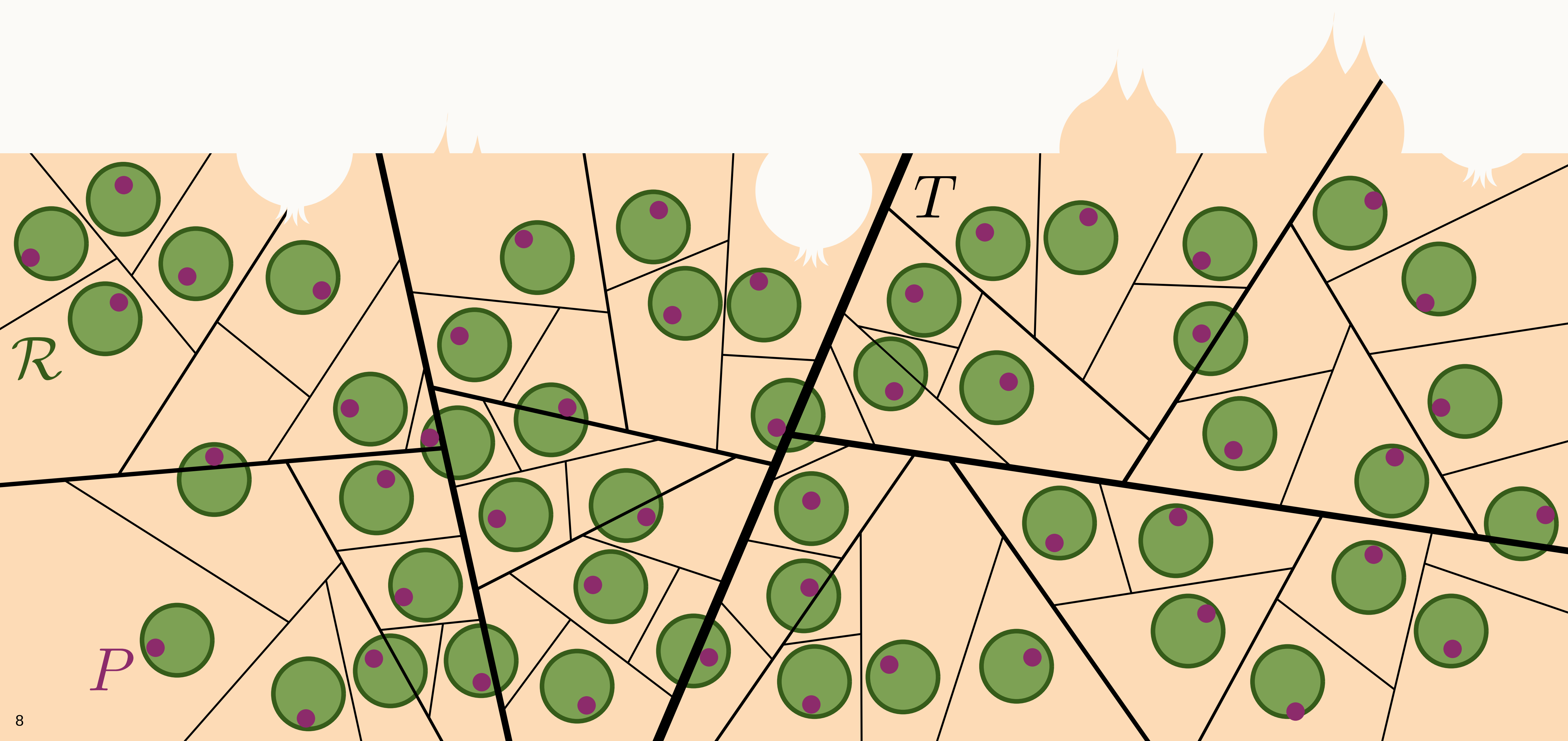

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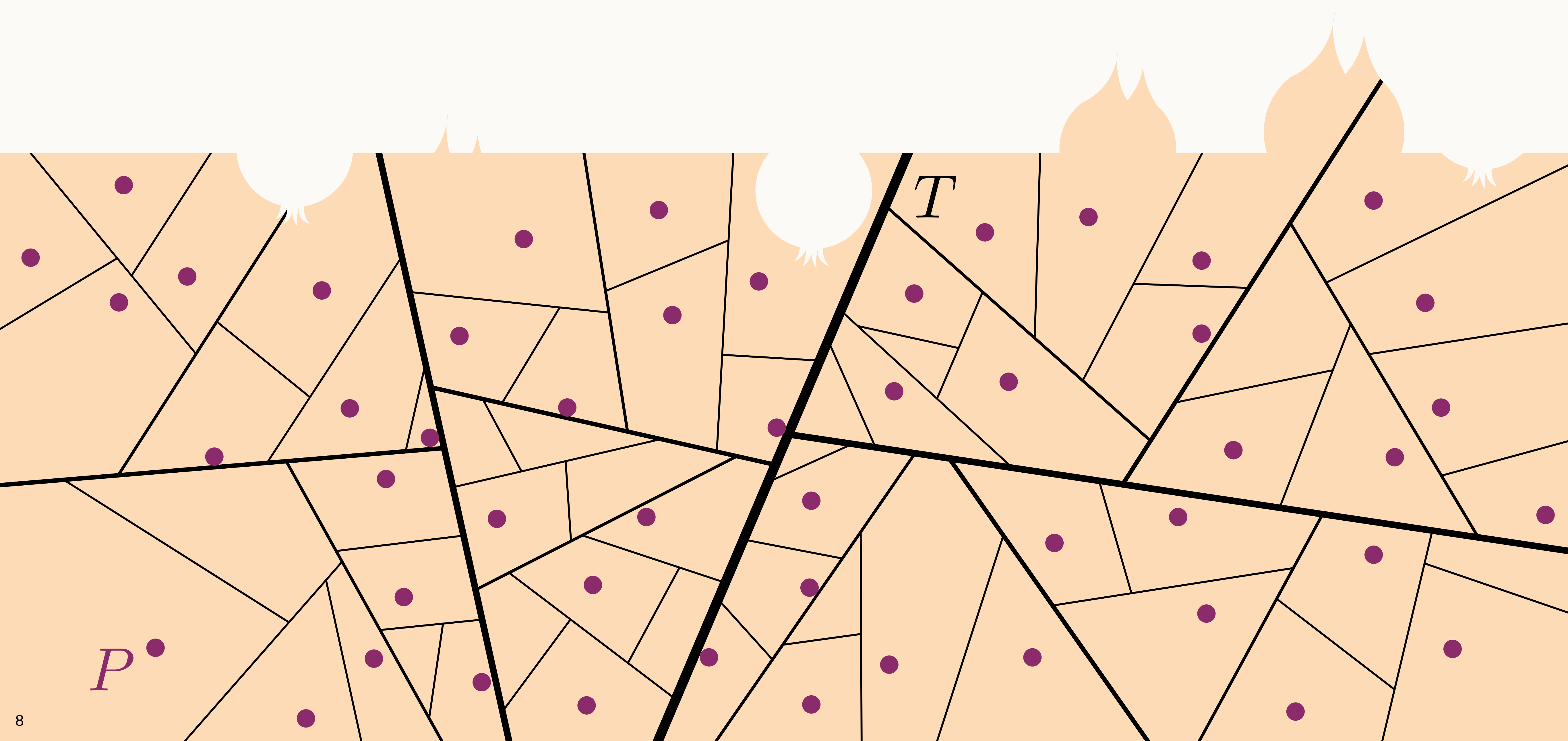

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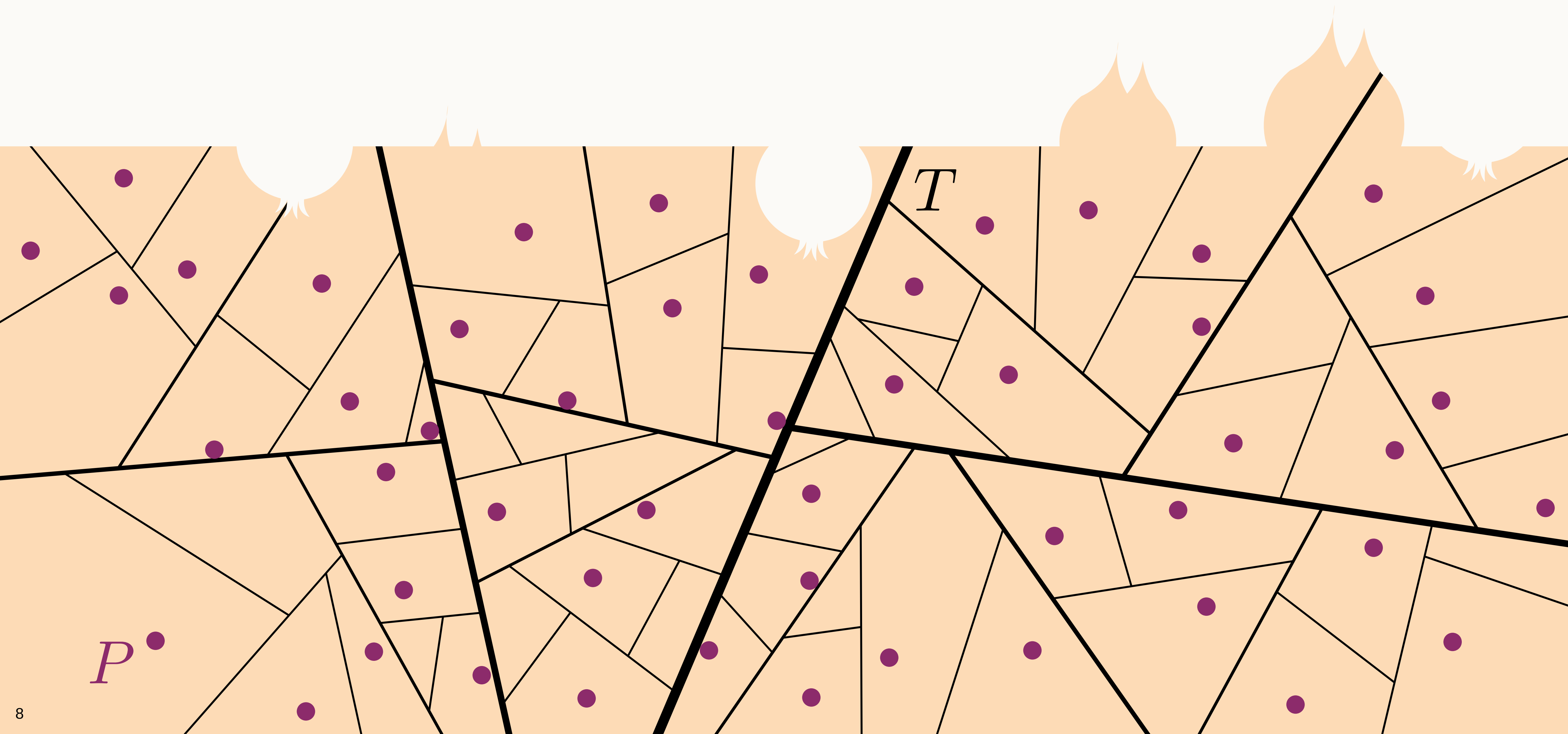

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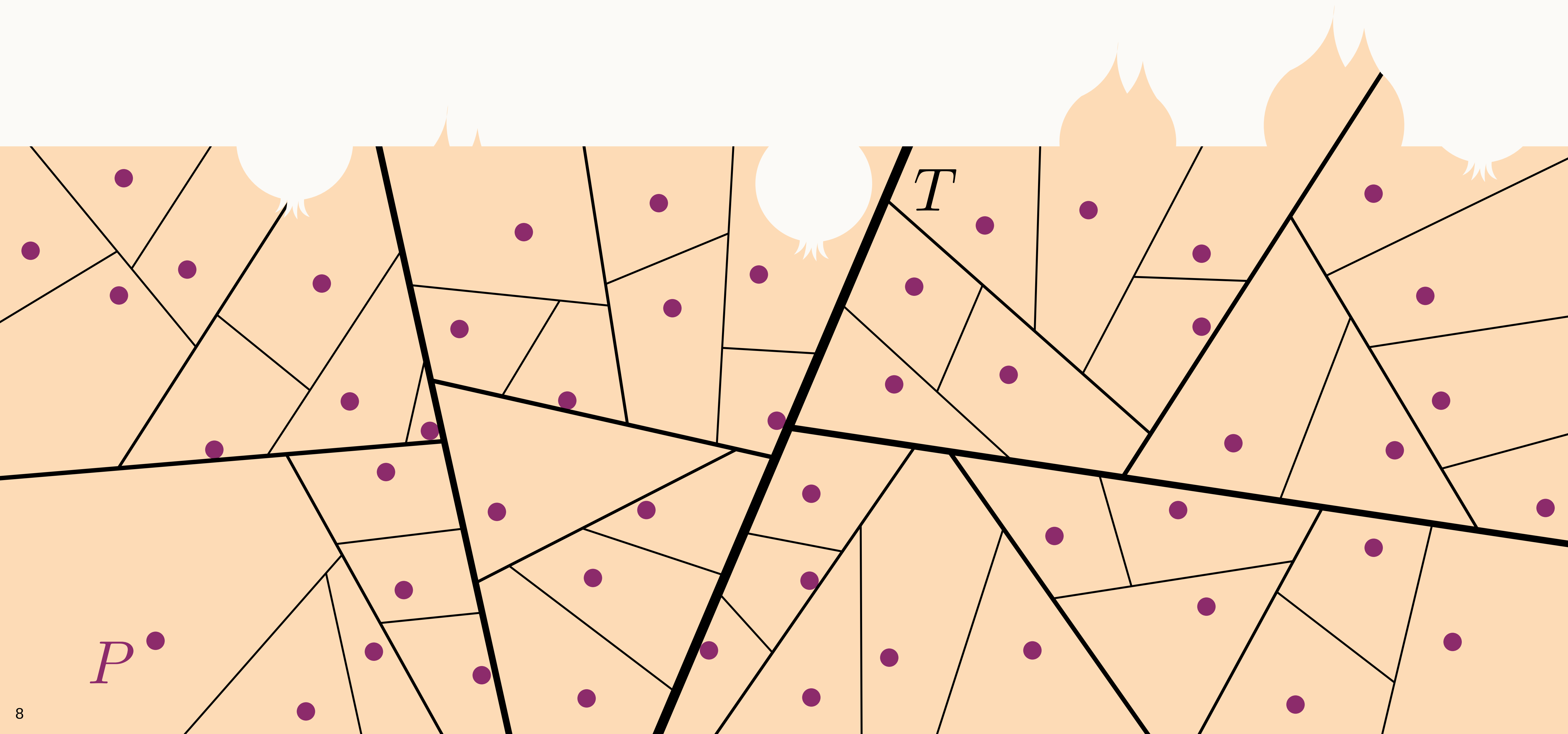

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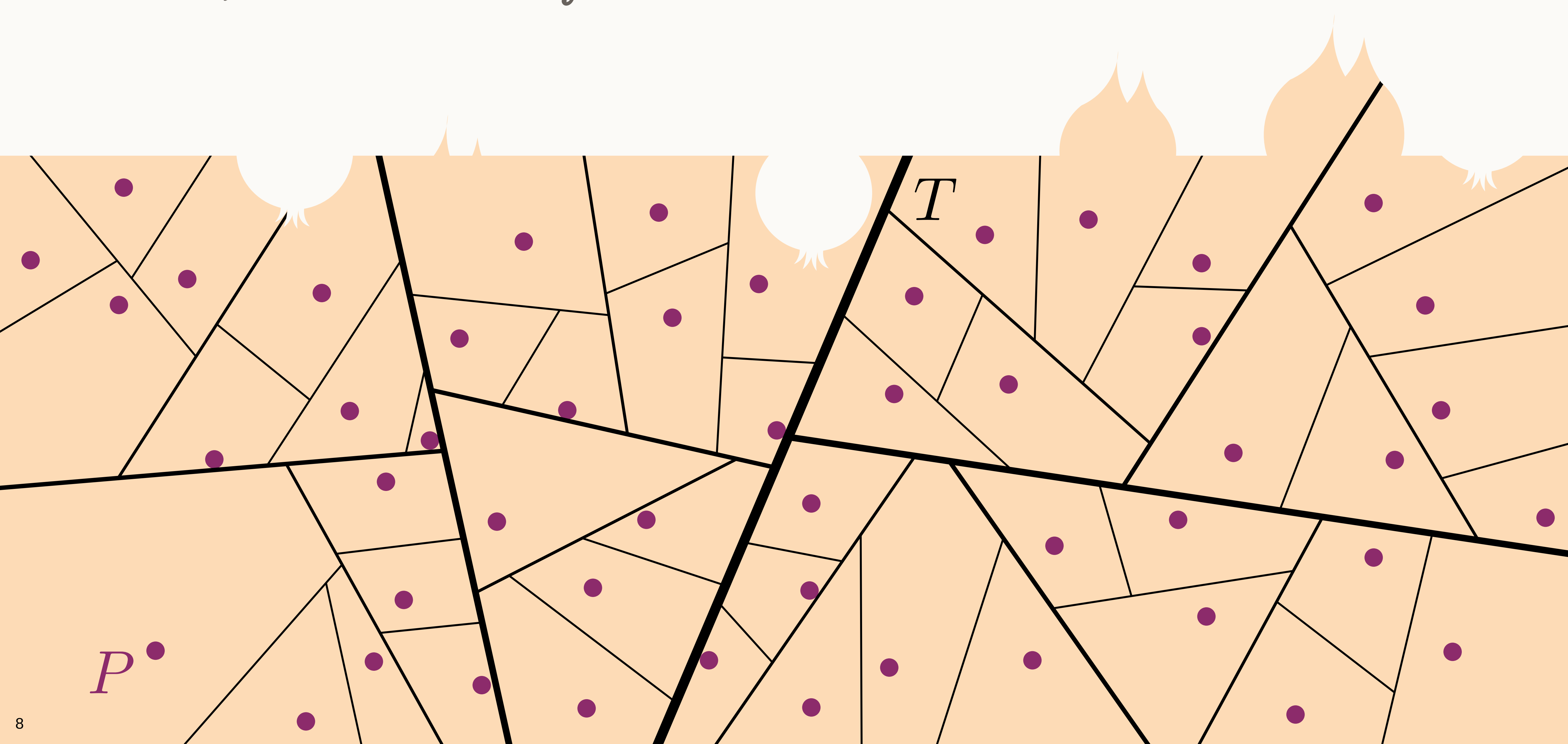

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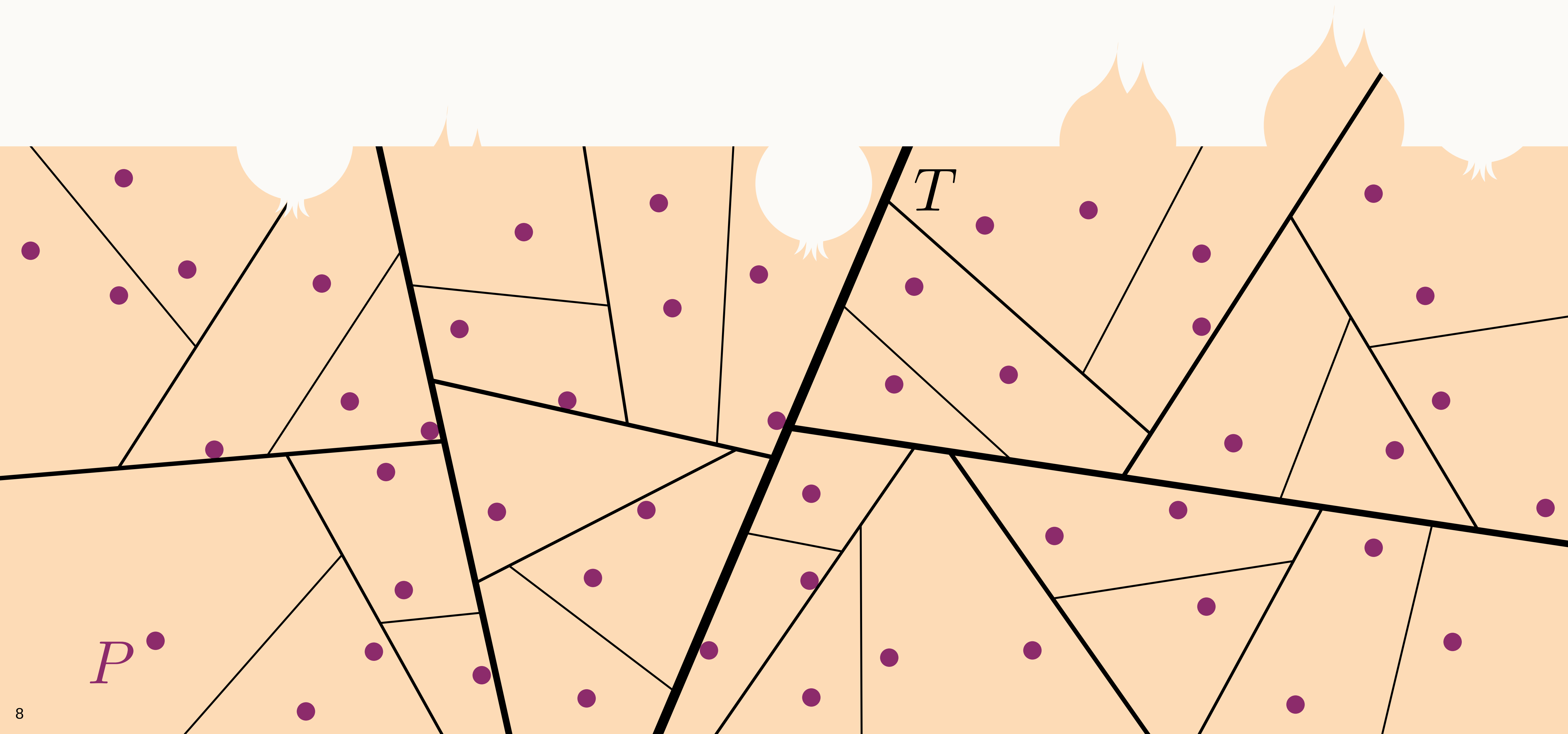

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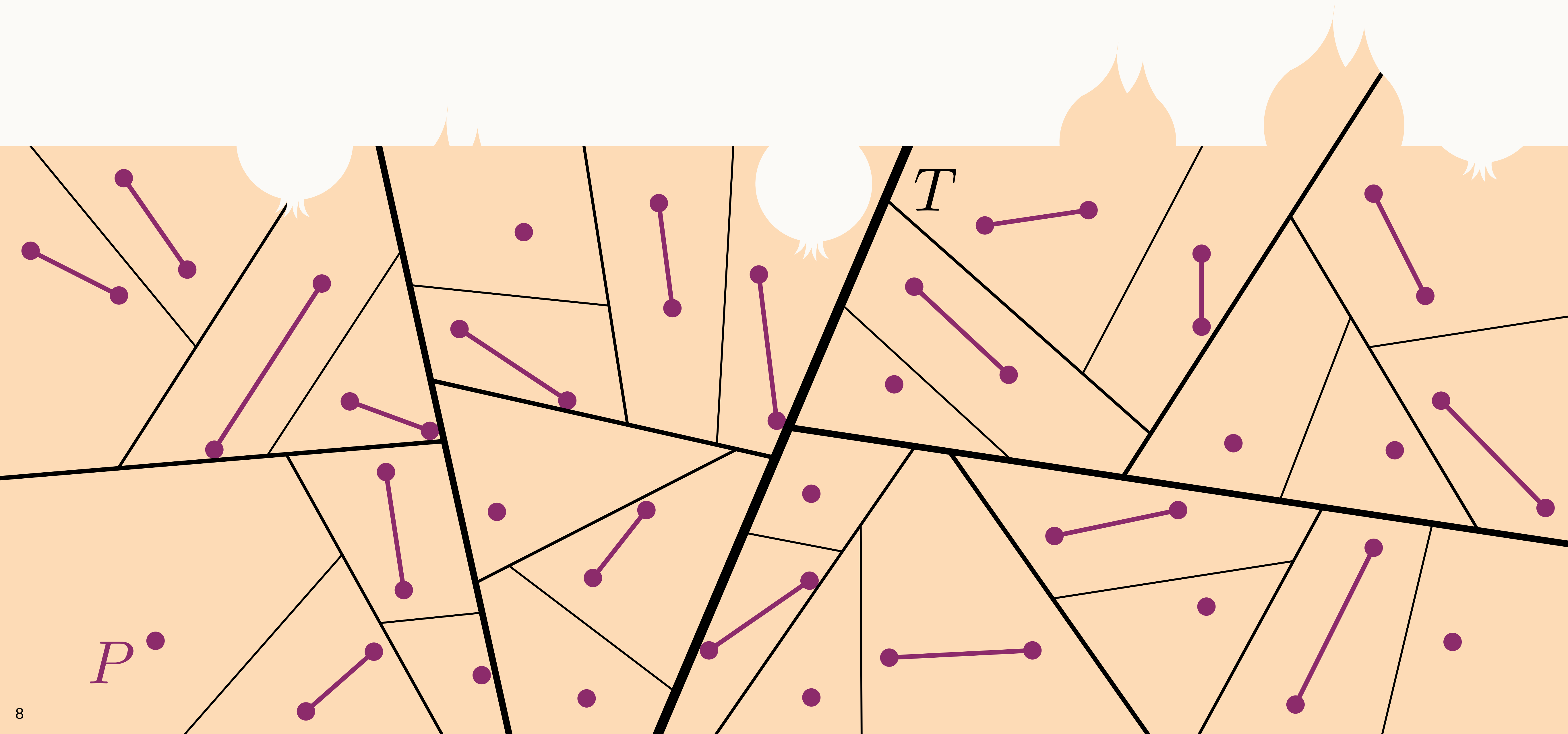

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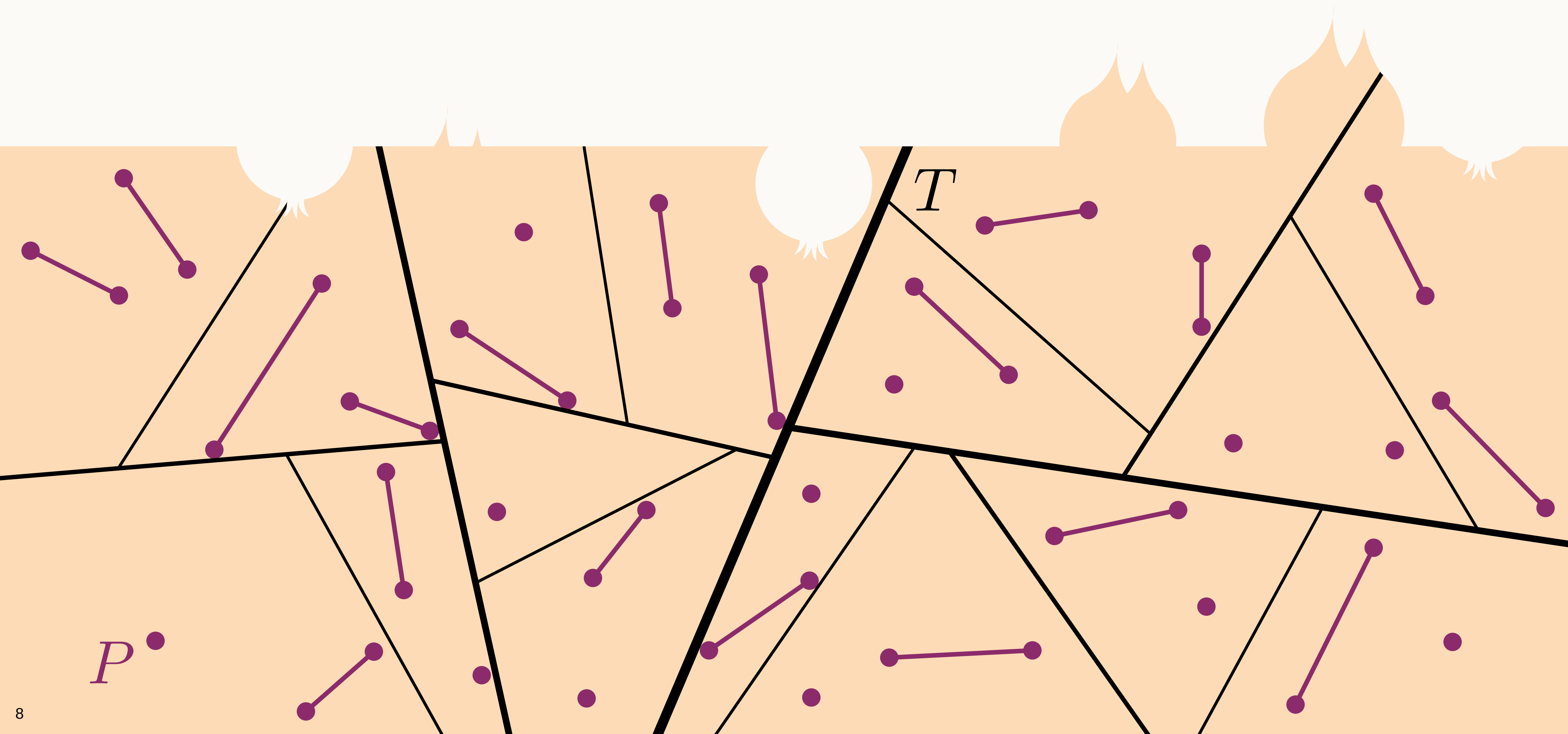

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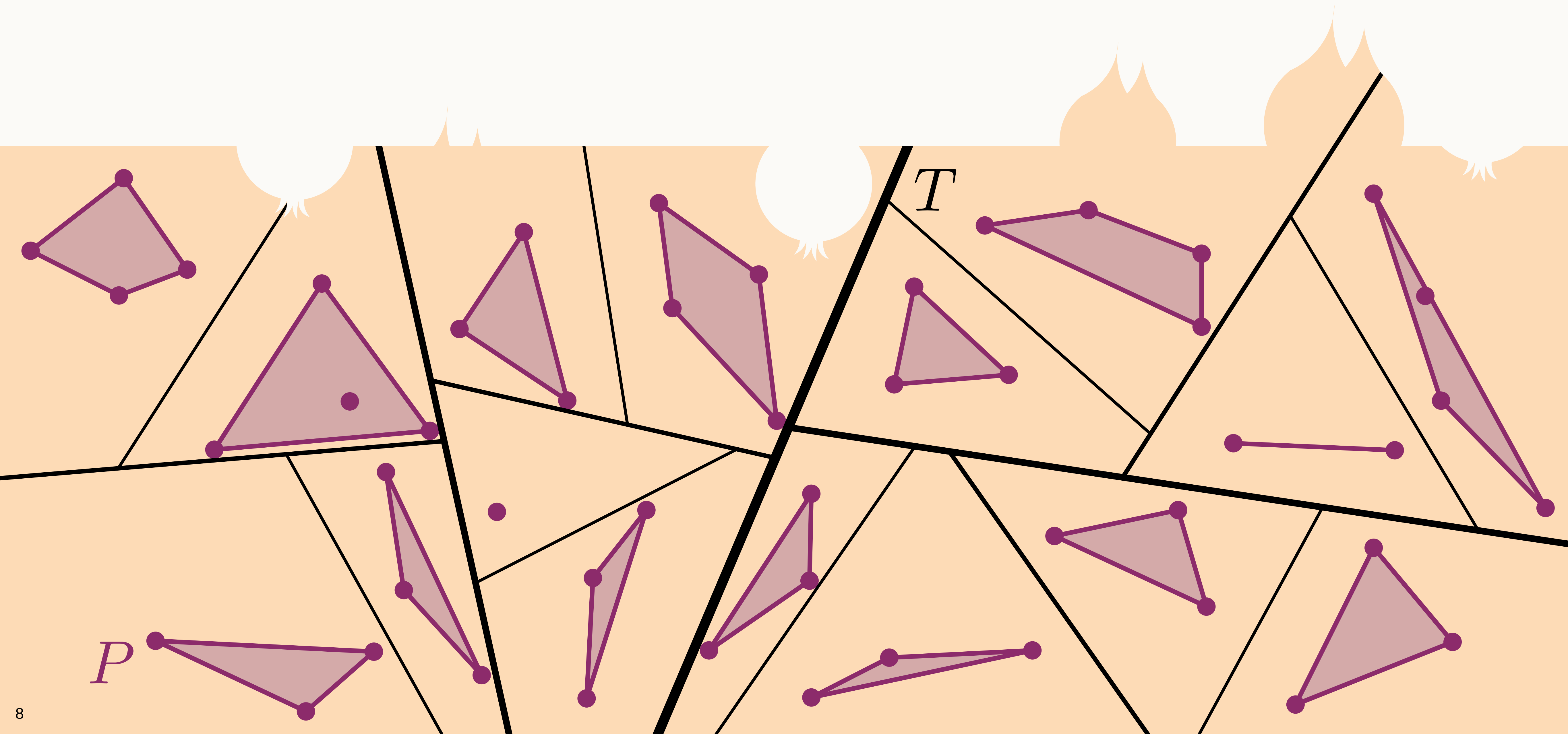

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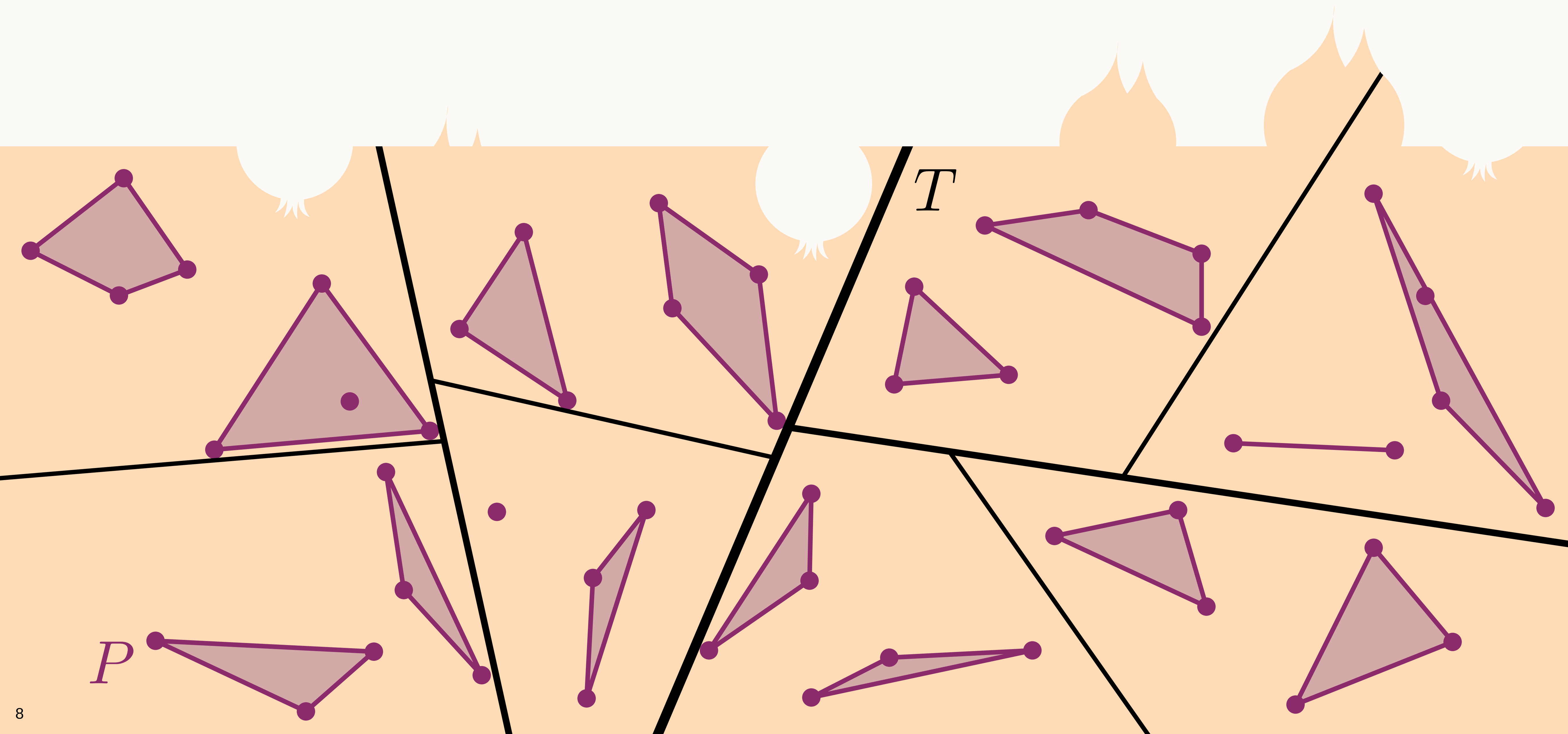

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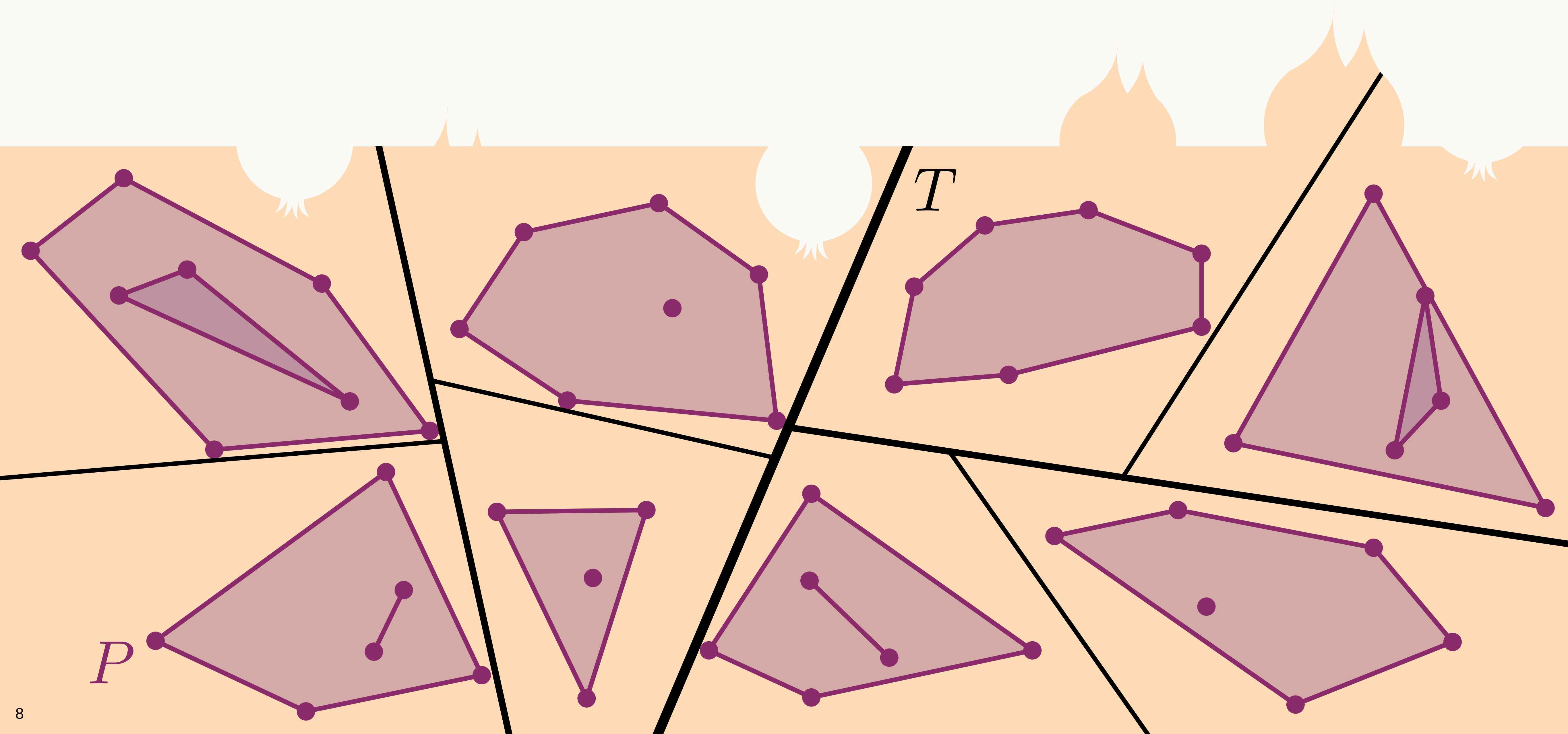

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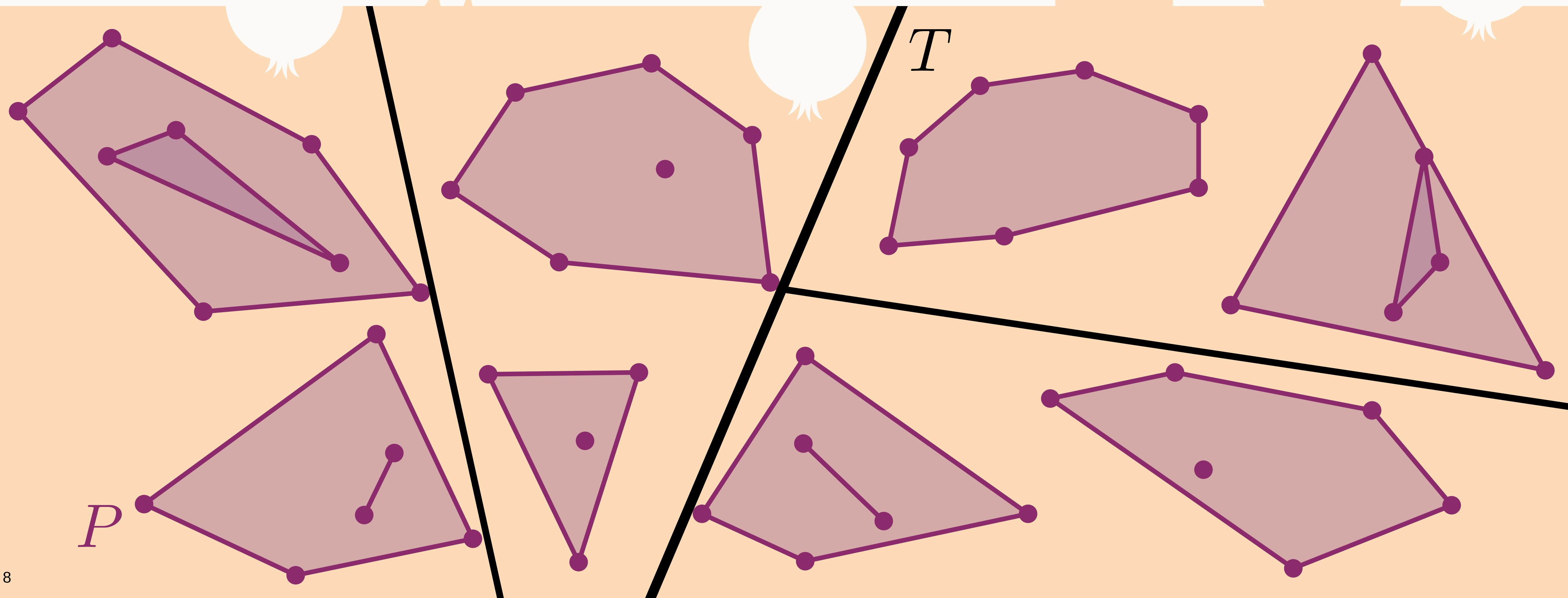
First, locate P in $\mathcal{R}$ and T.

Remove any empty leaves of T.

Then, recursively unite the onions.



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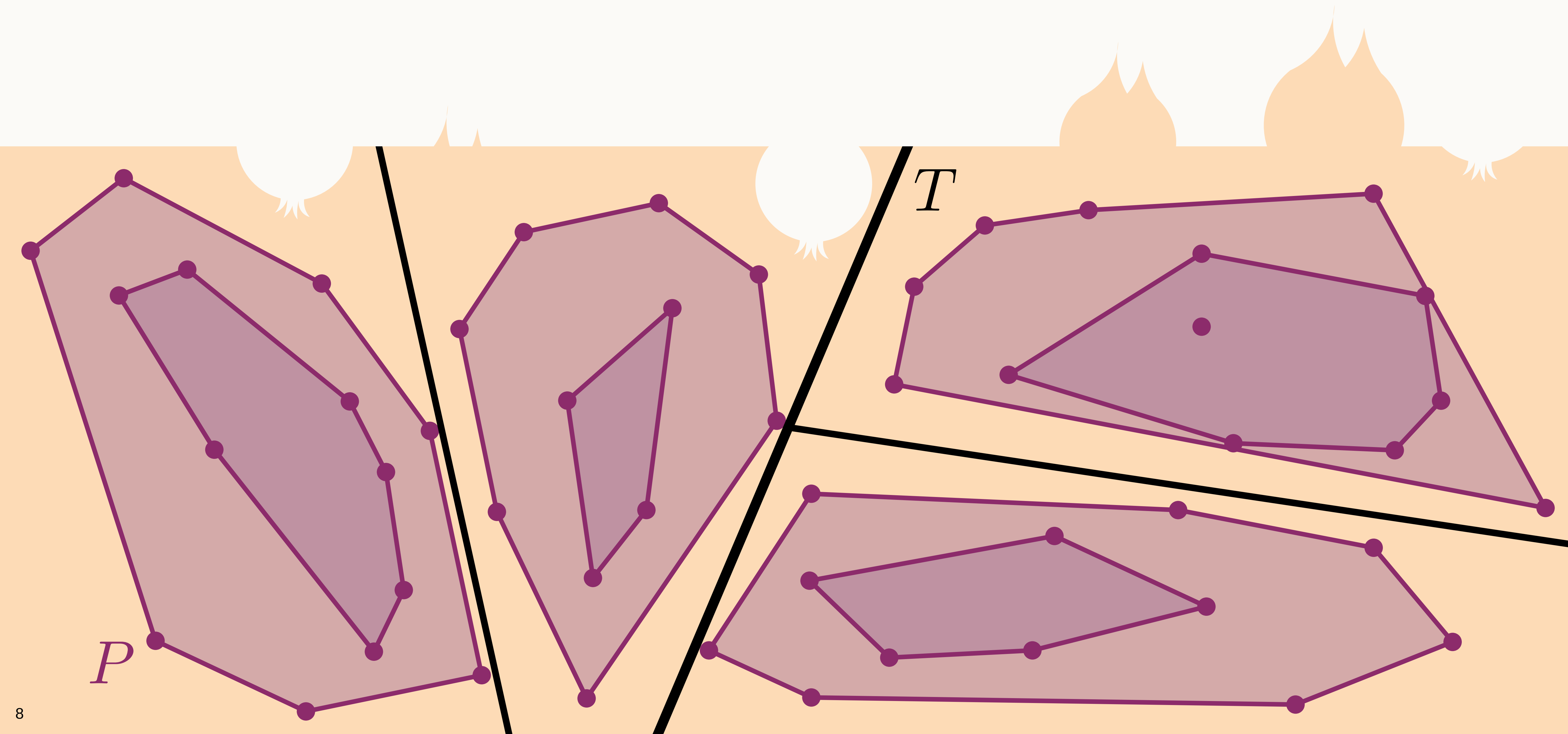

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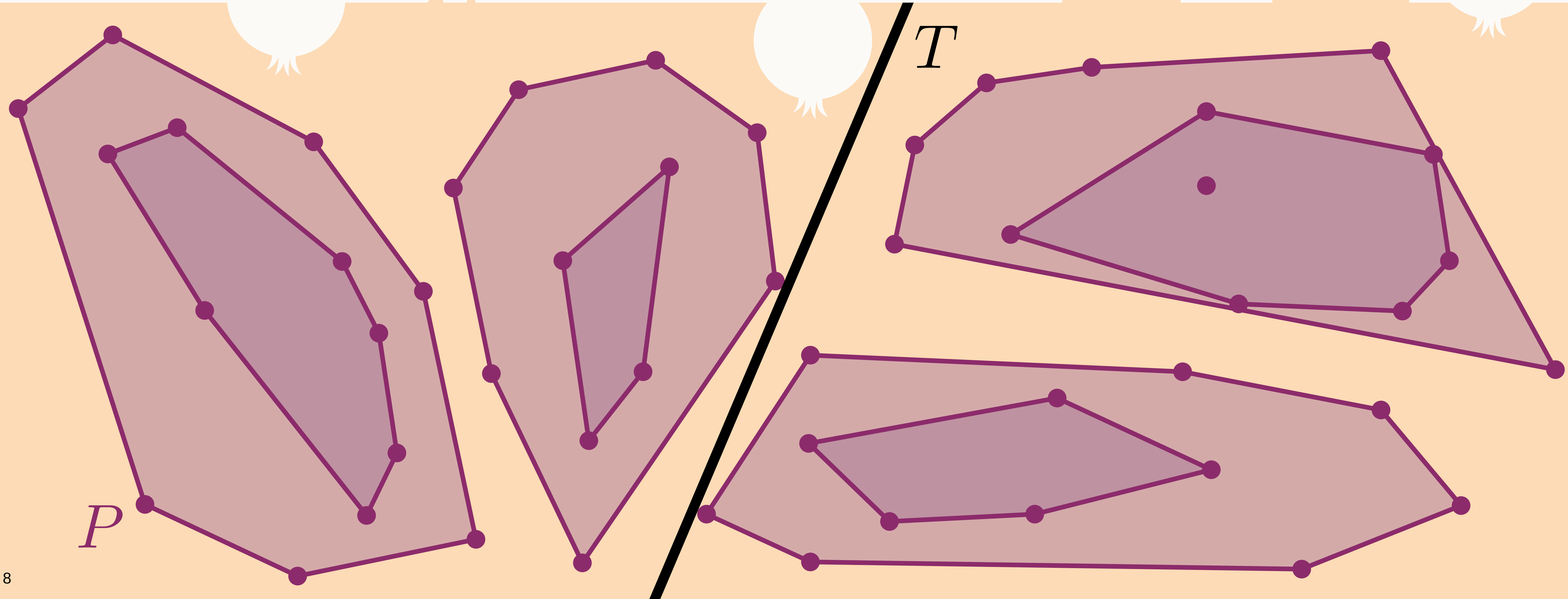

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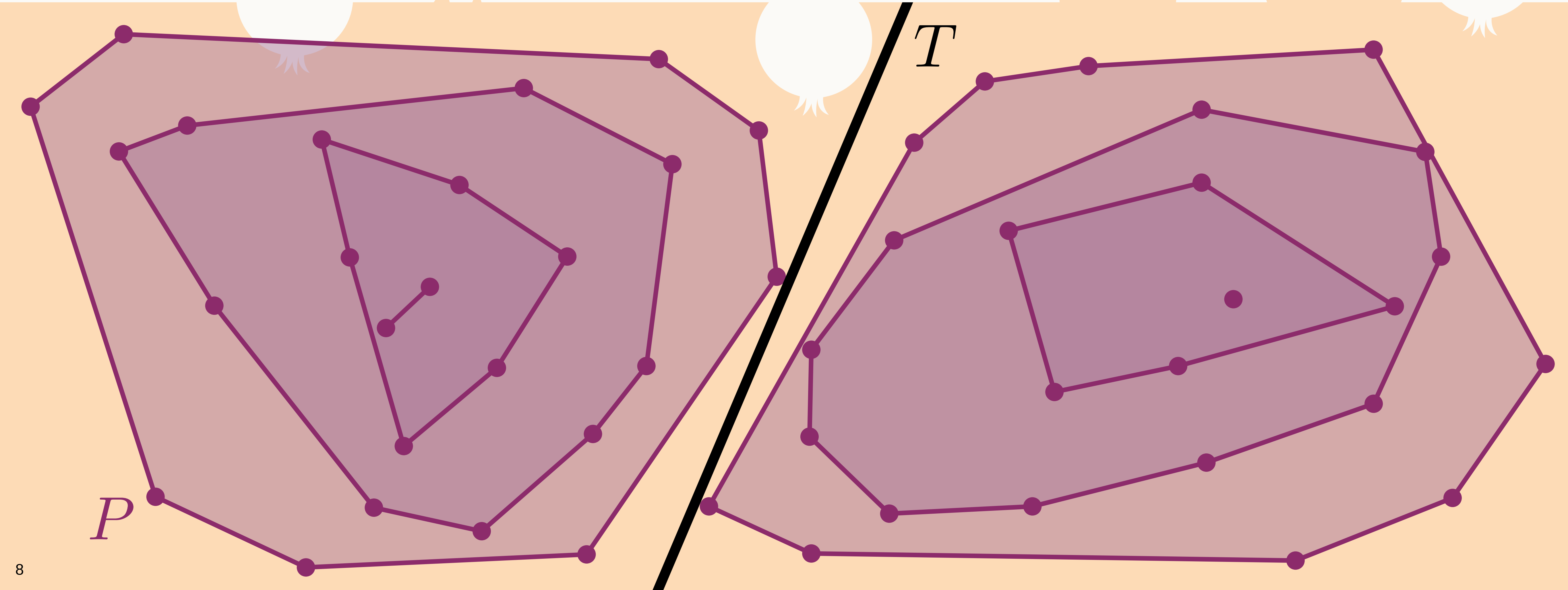

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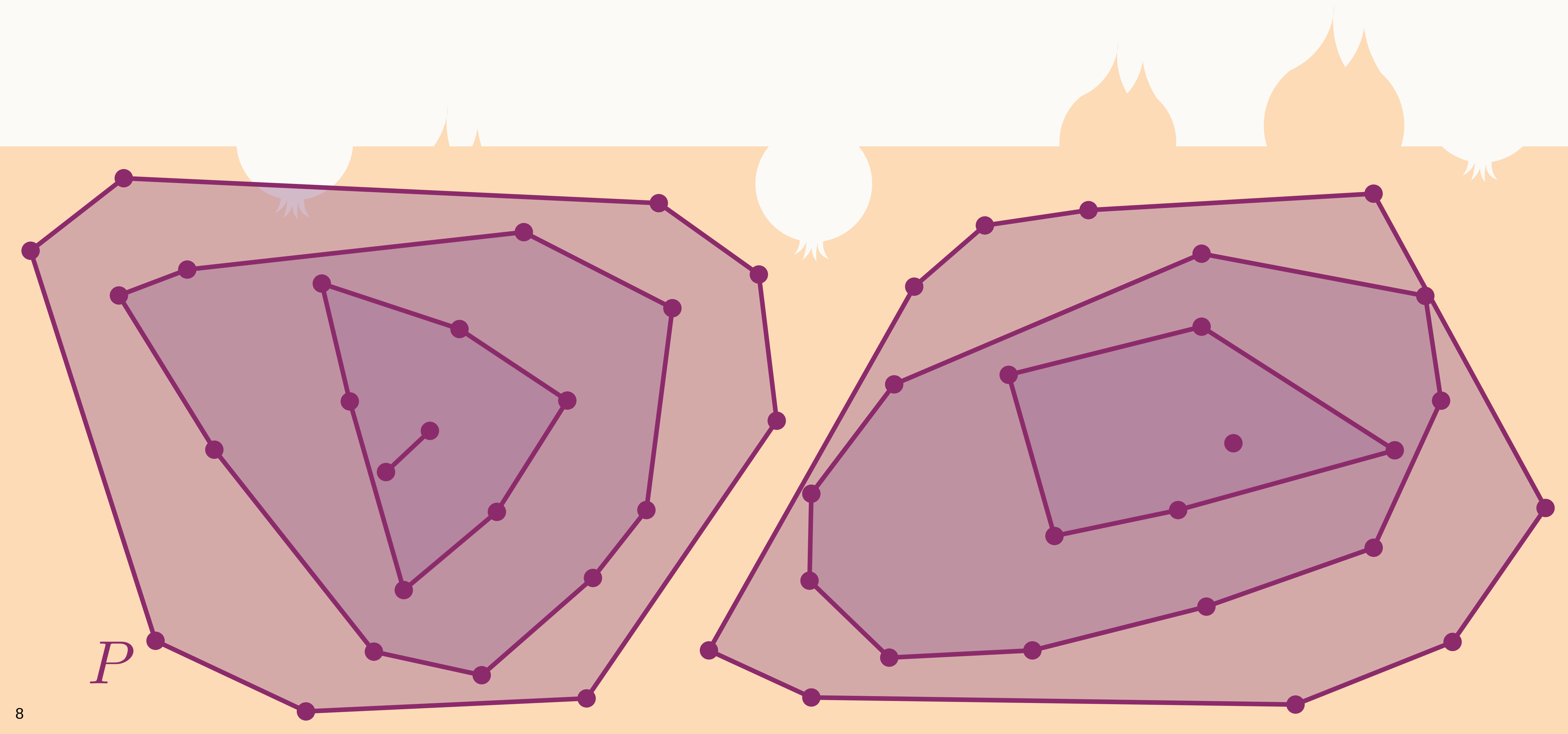

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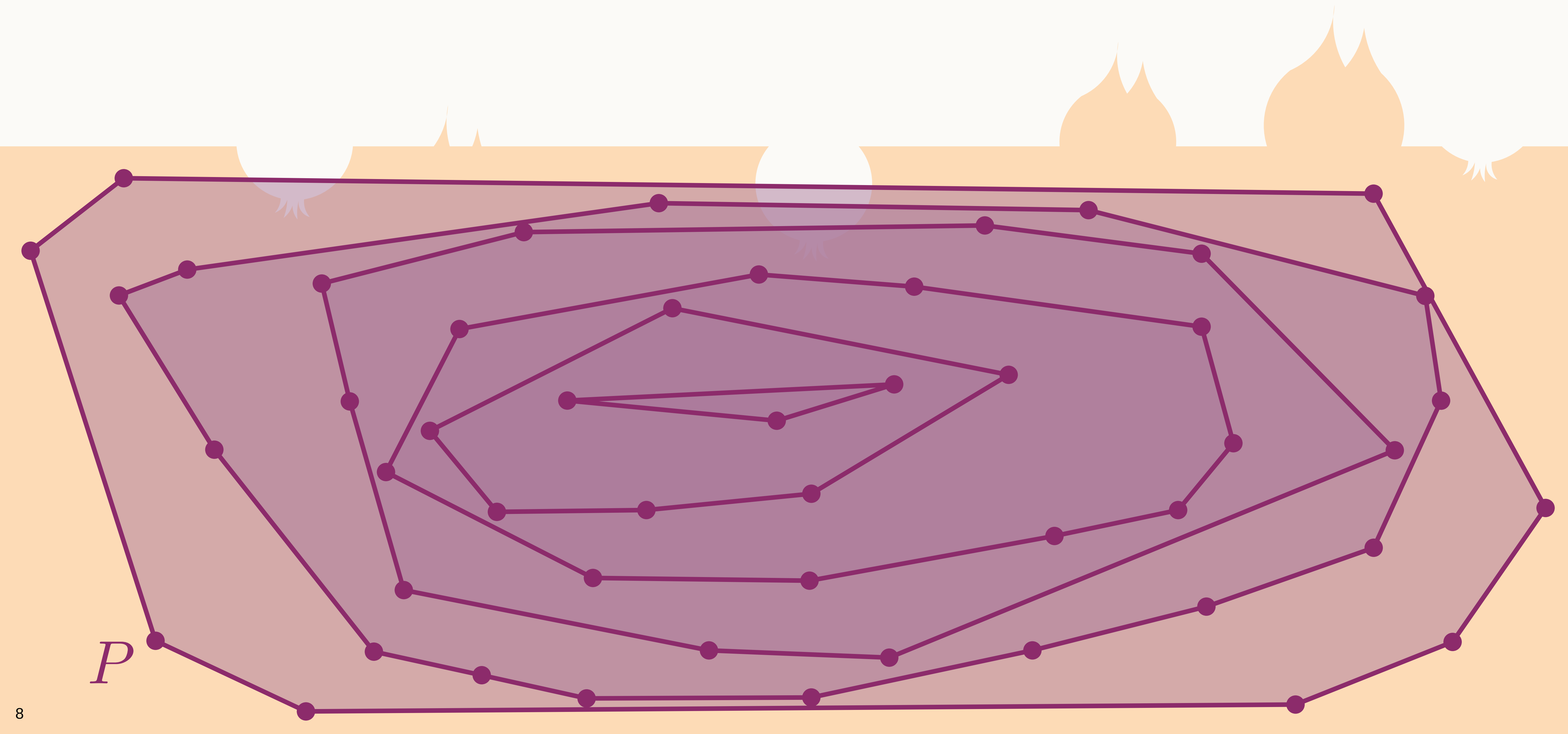

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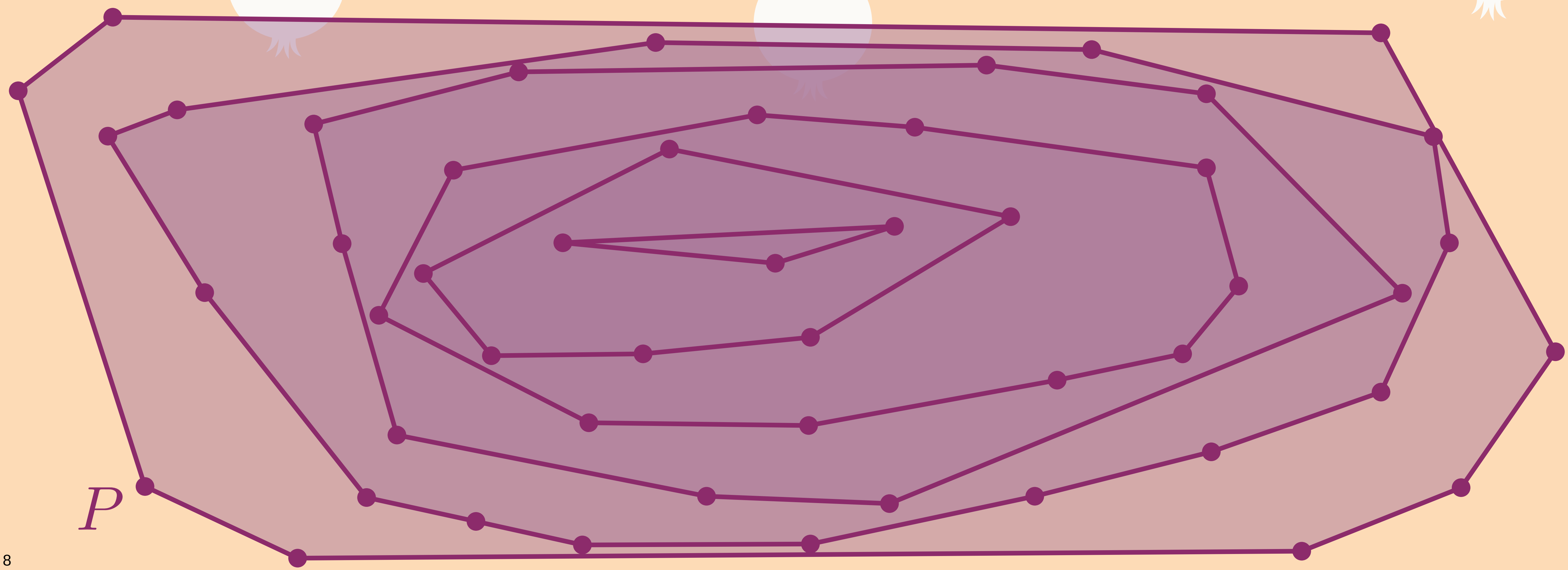
Done!



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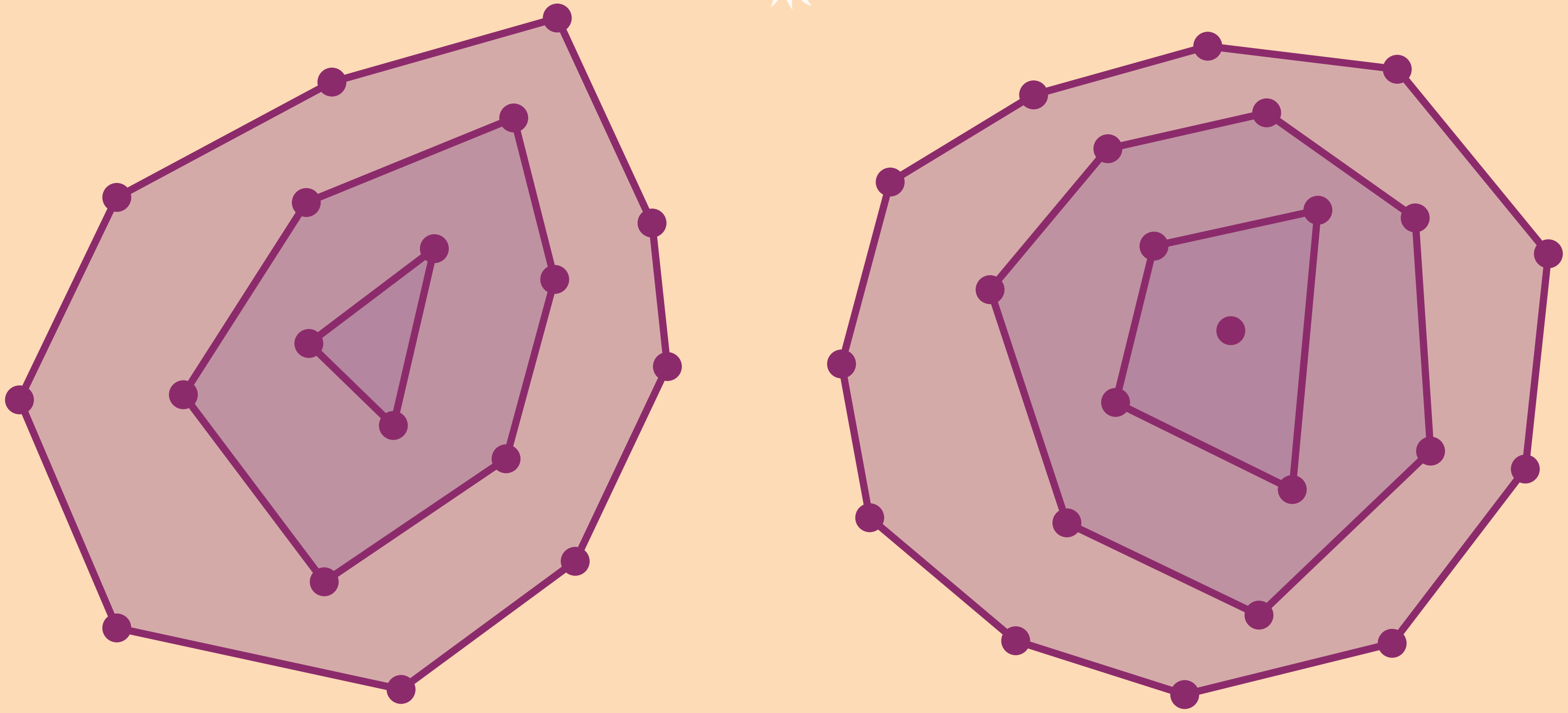

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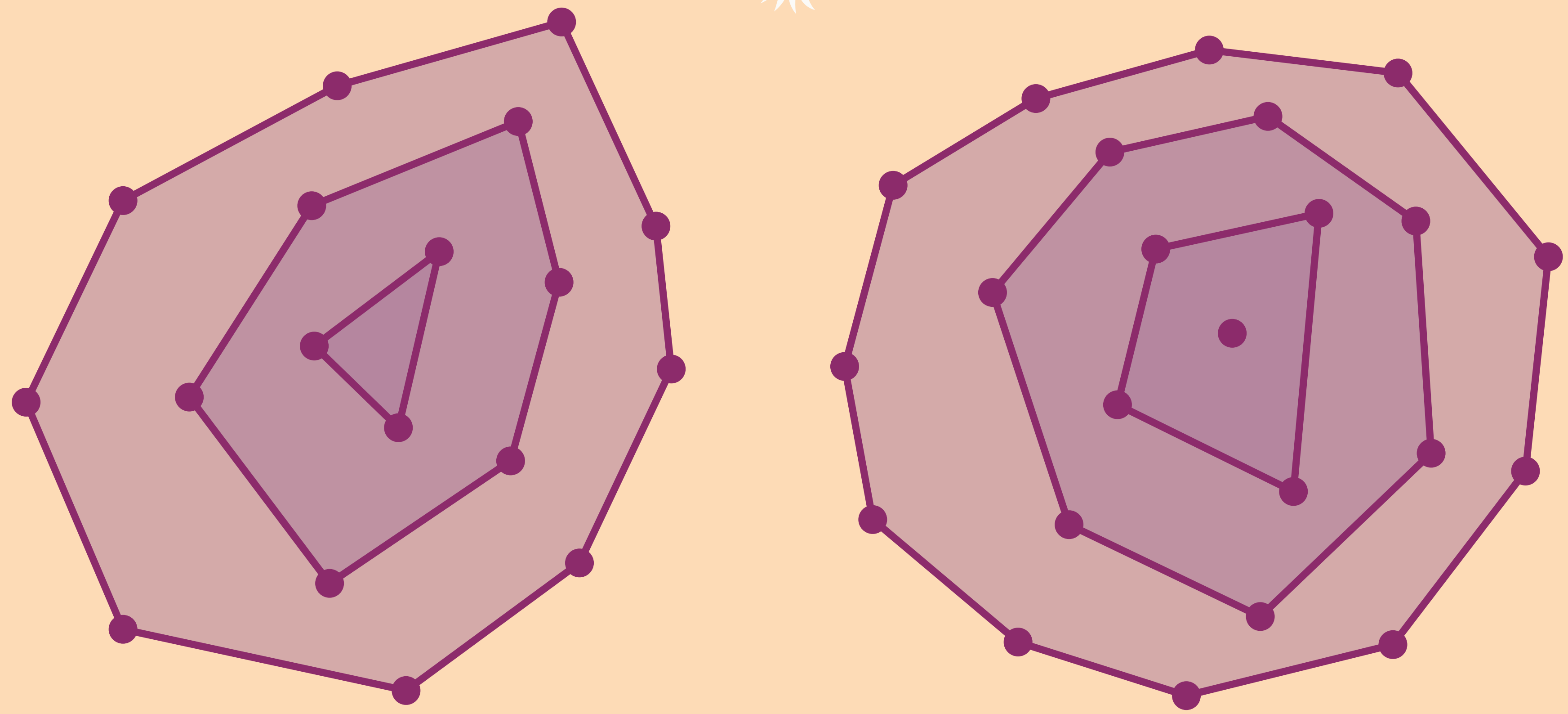

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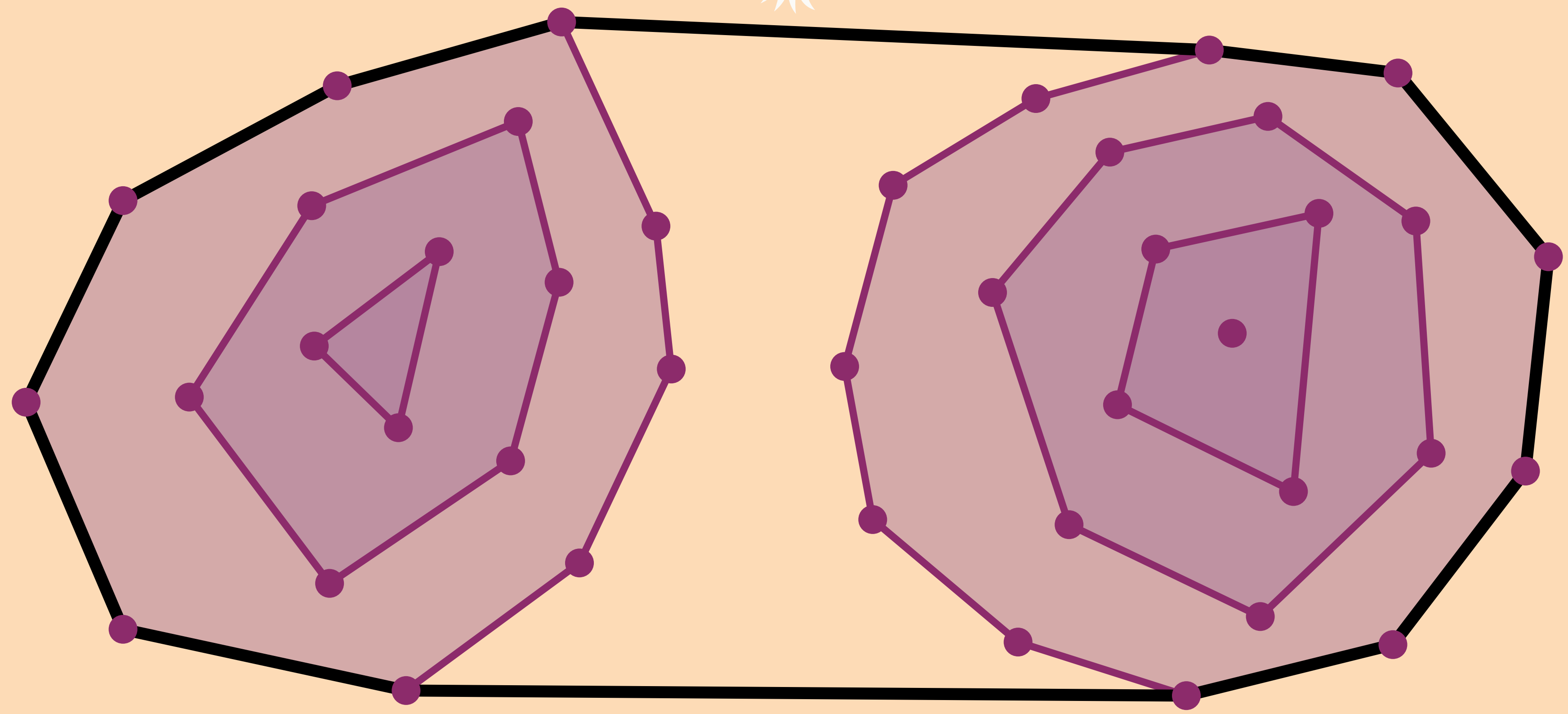

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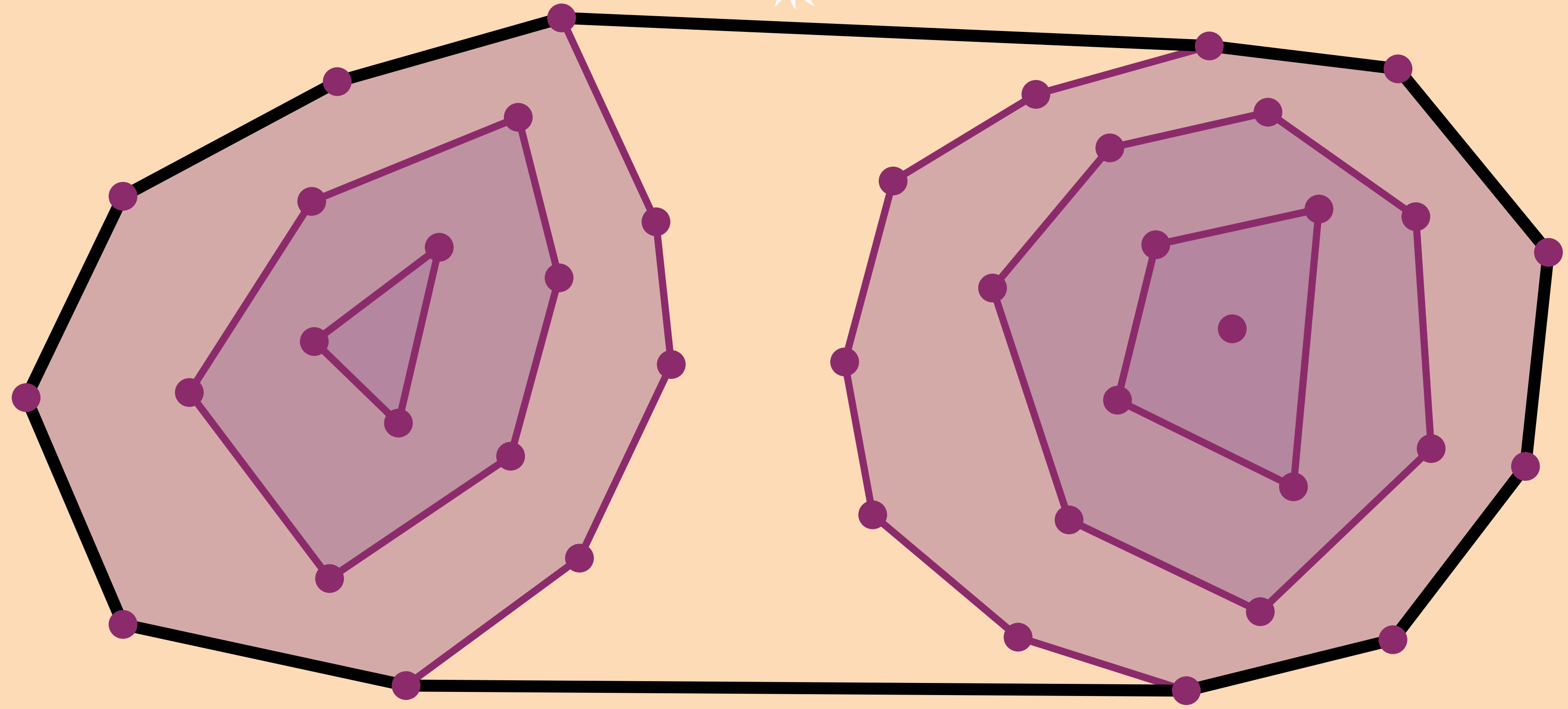

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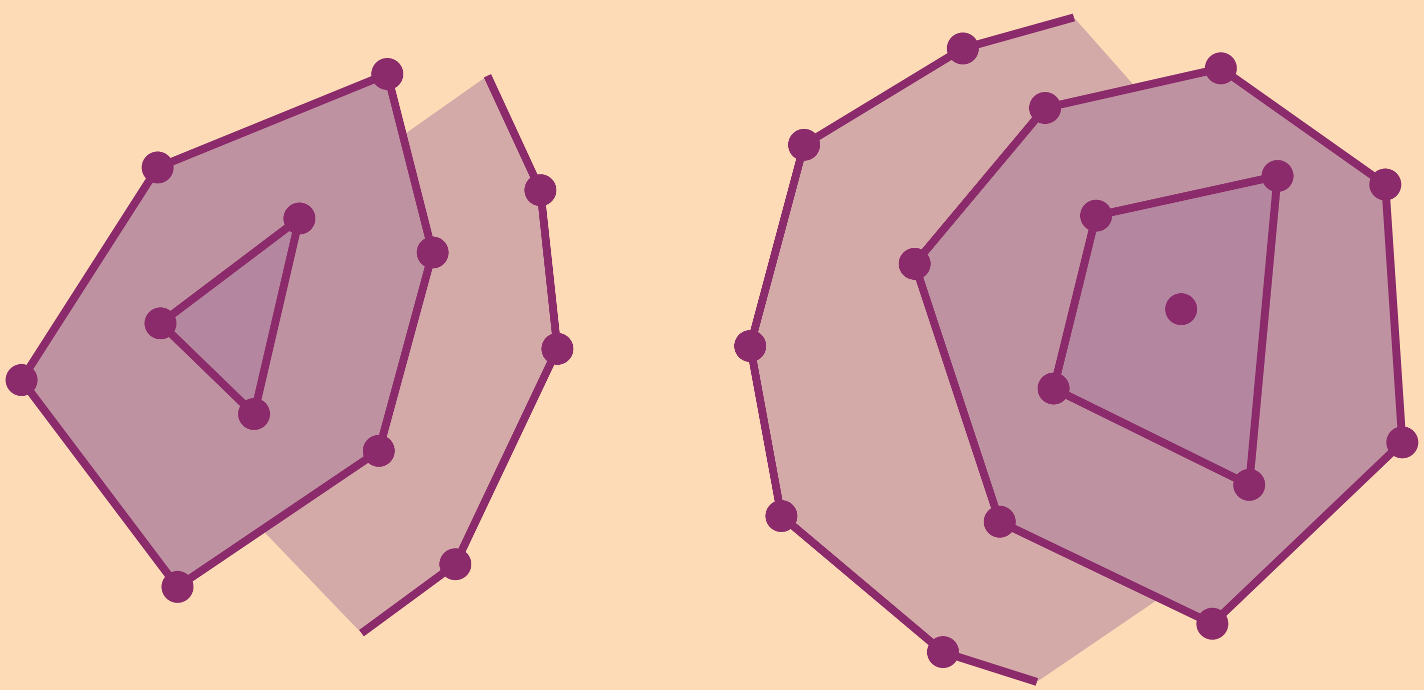

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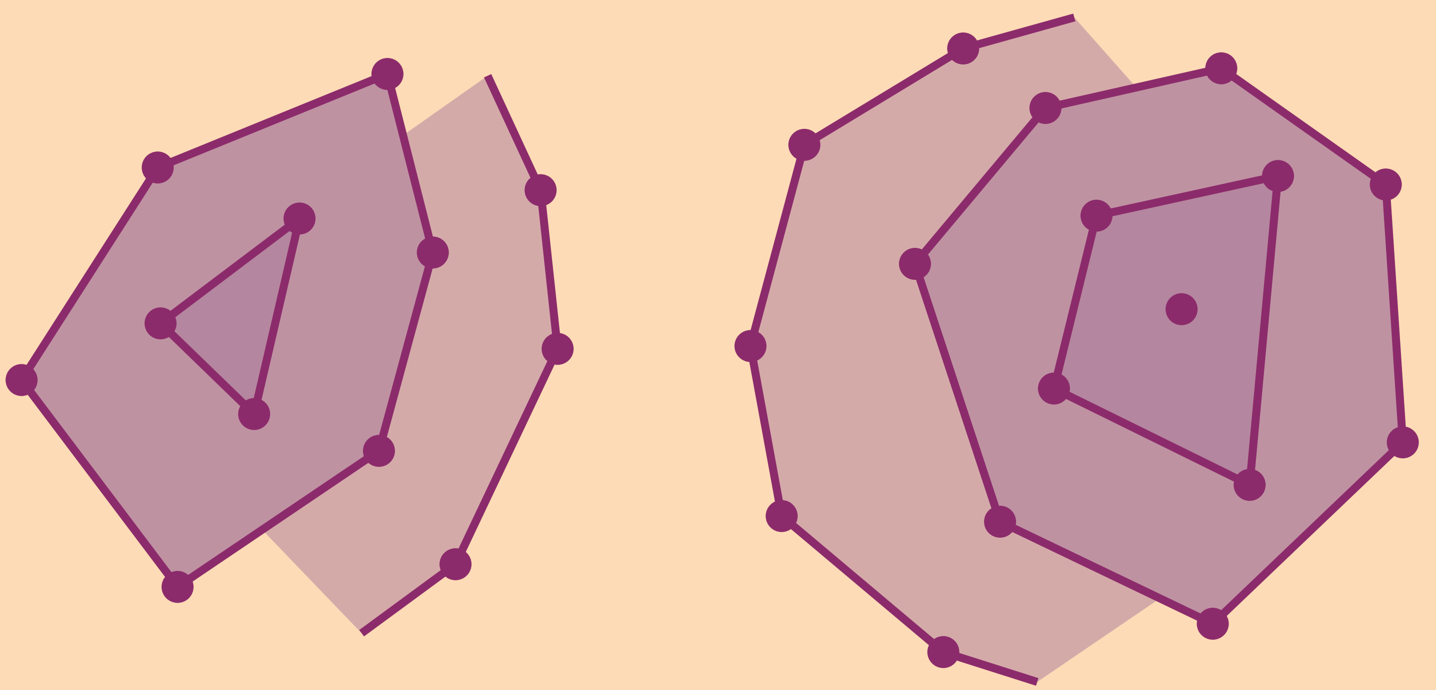
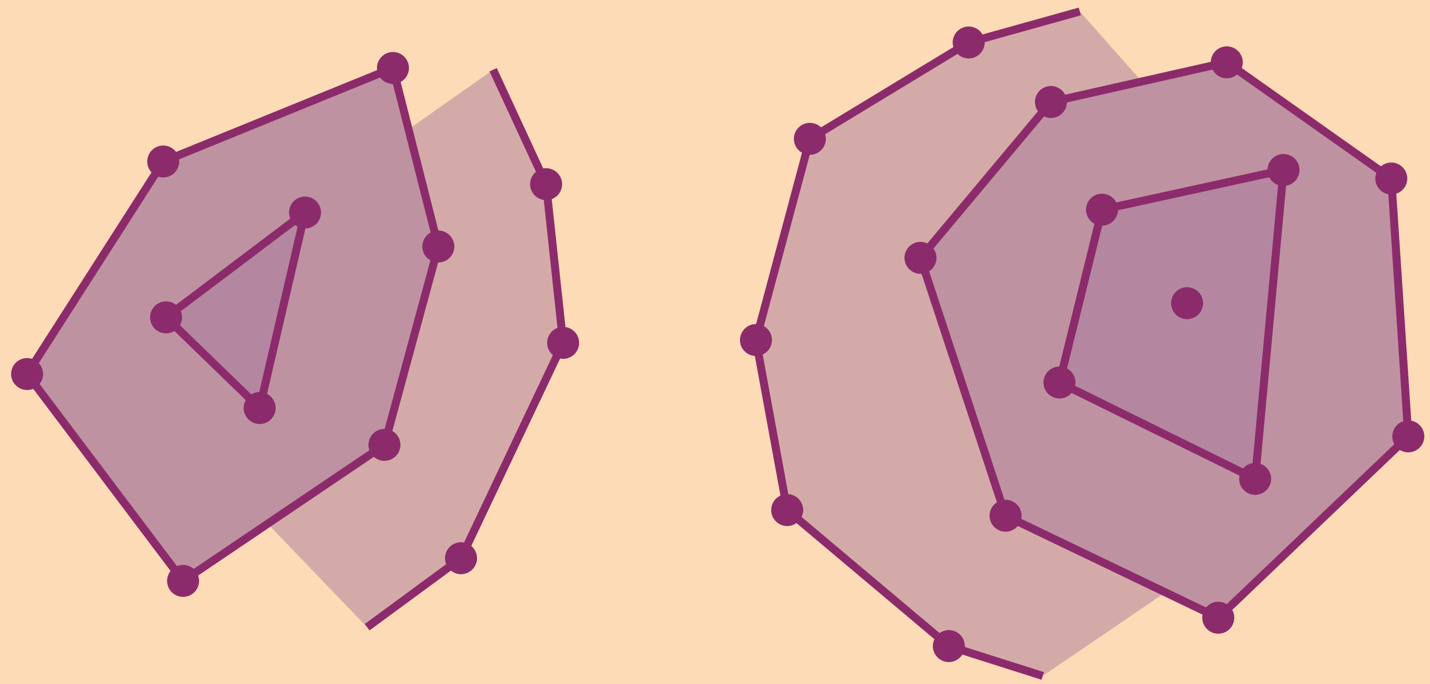
Unfortunately, what is left are no longer proper onions...

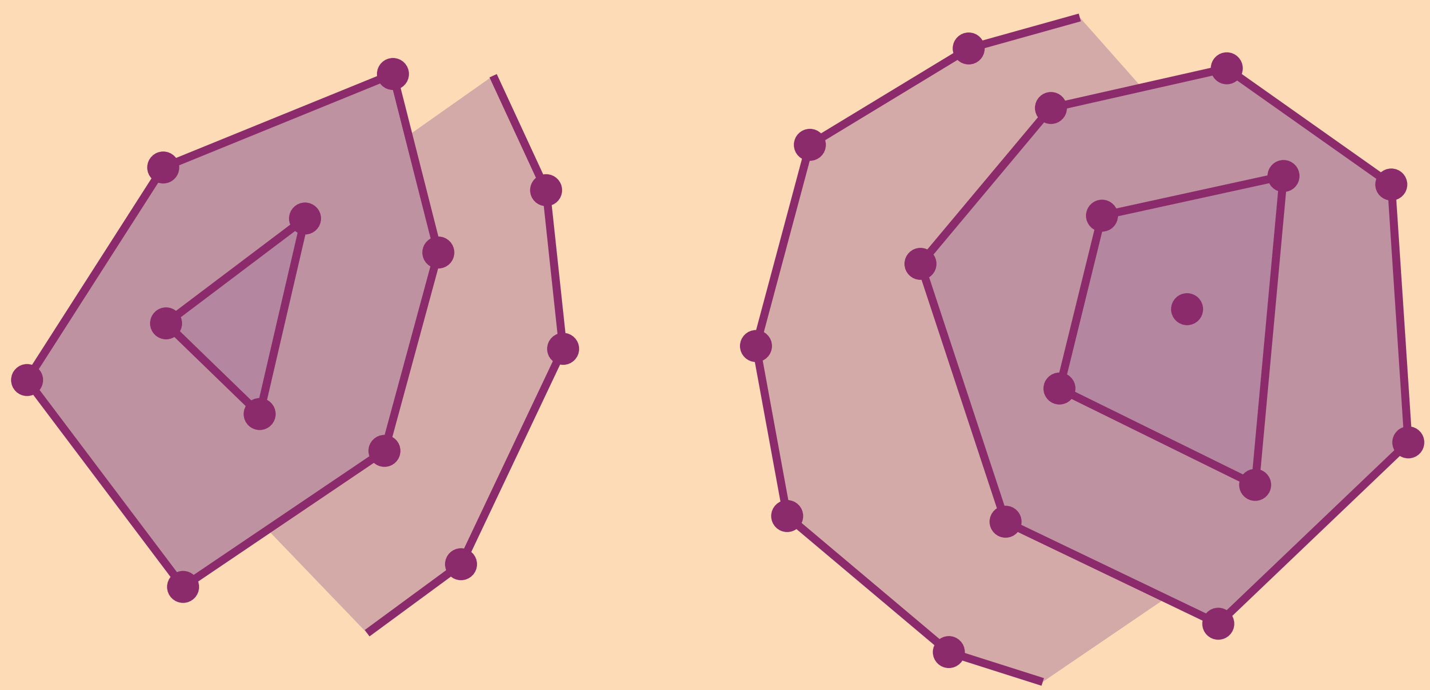


ONION RESTAURATION & UNIFICATION



ONION RESTAURATION & UNIFICATION

Compute convex hulls of the individual onions.



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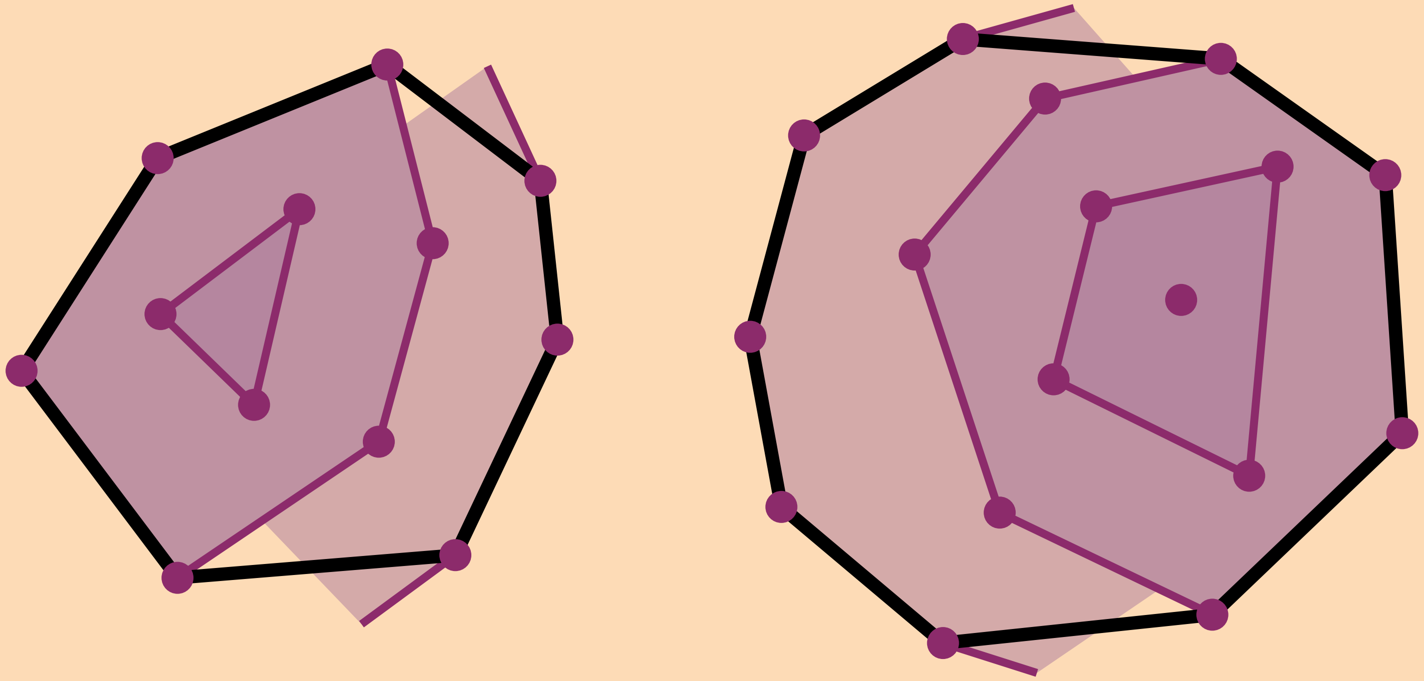

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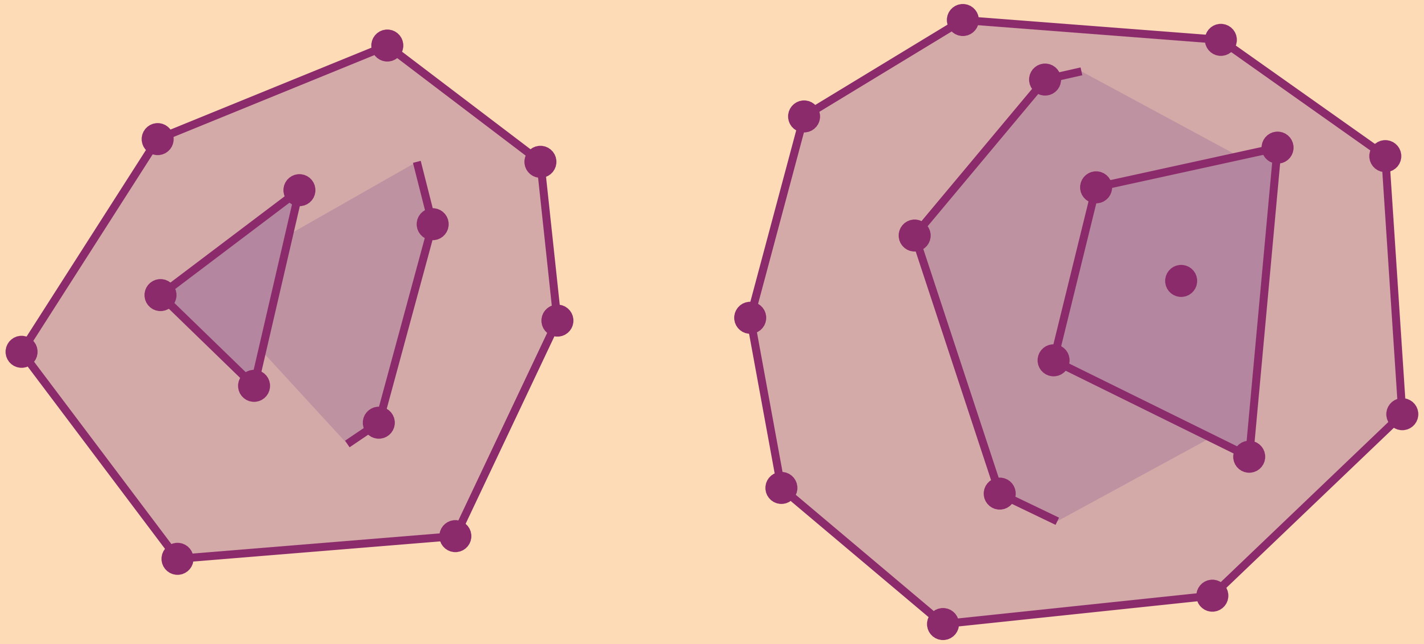

ONION RESTAURATION & UNIFICATION

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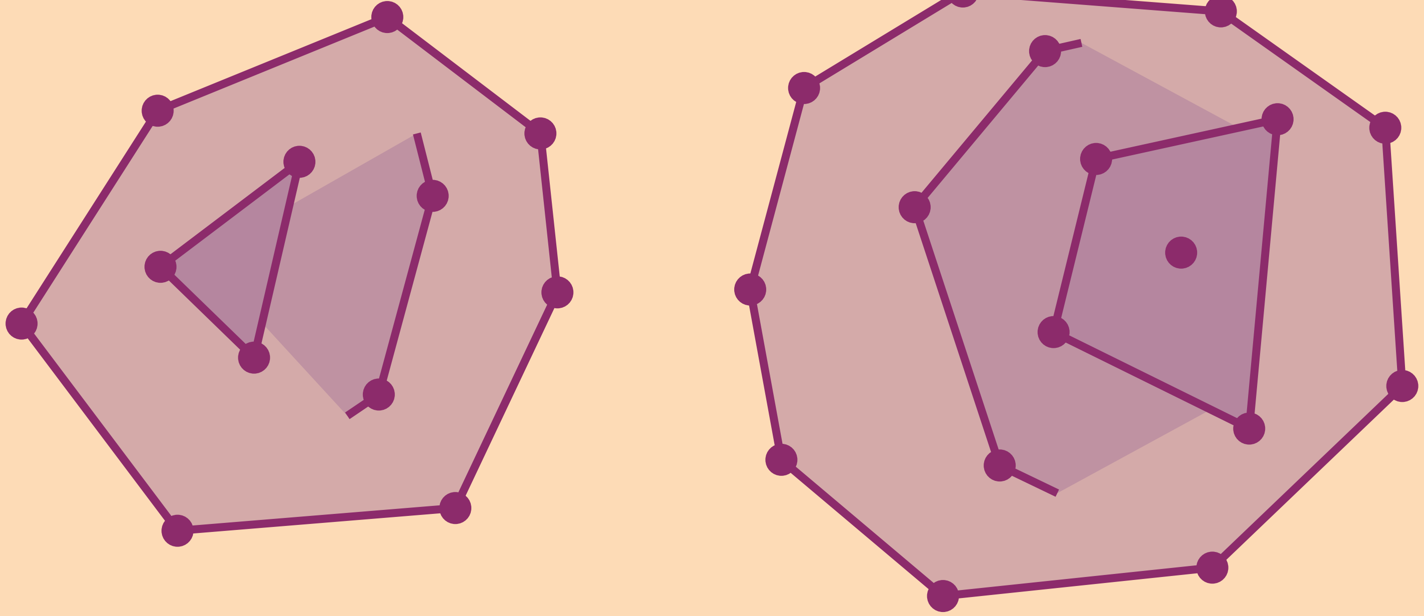
Recurse once again.



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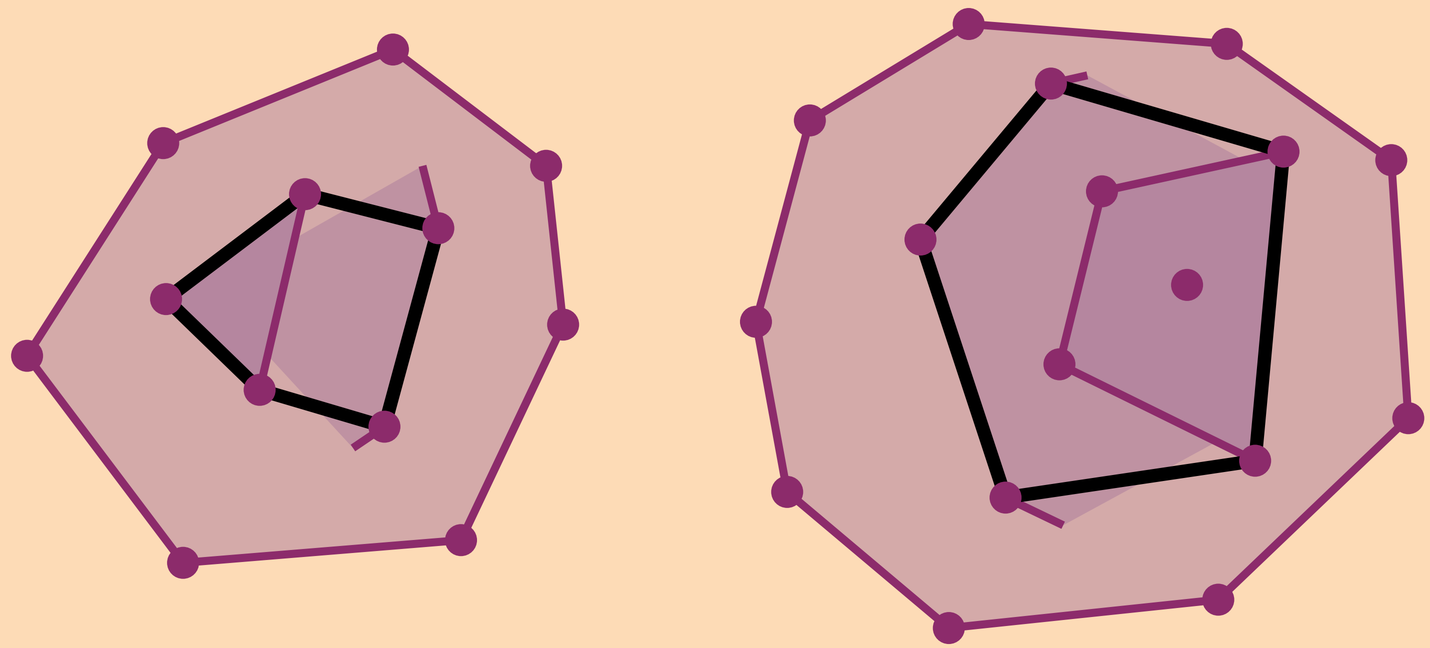

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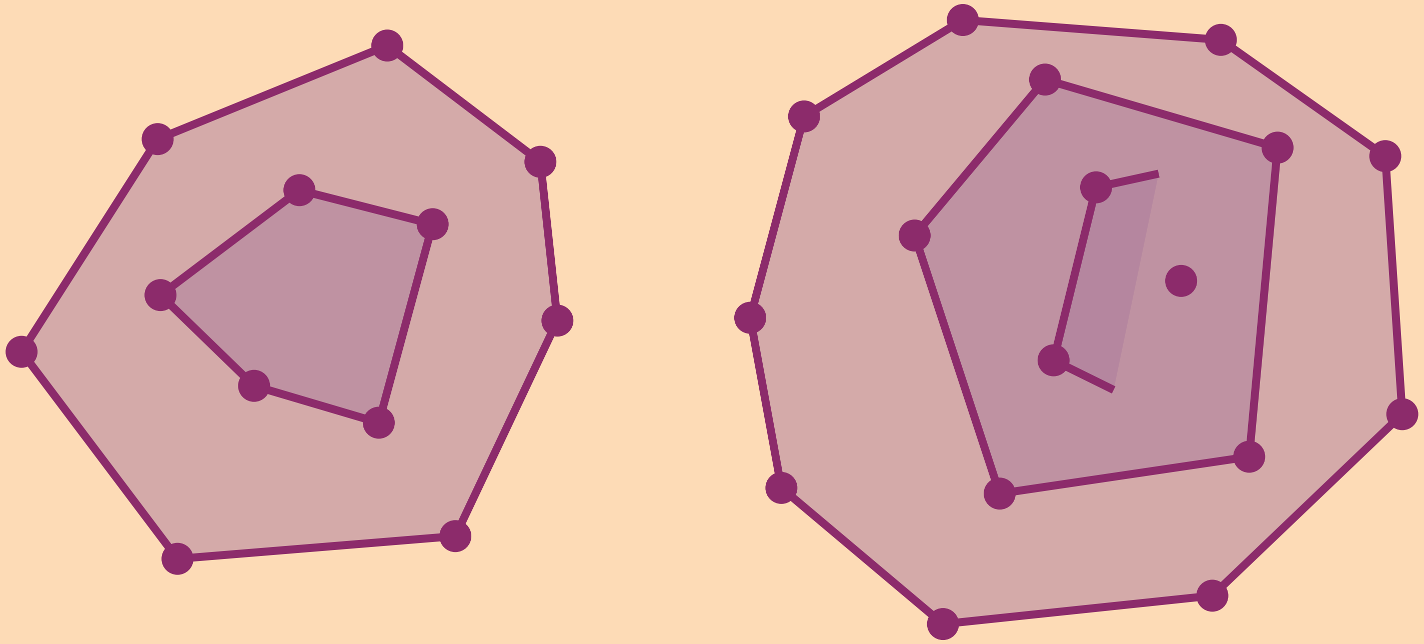

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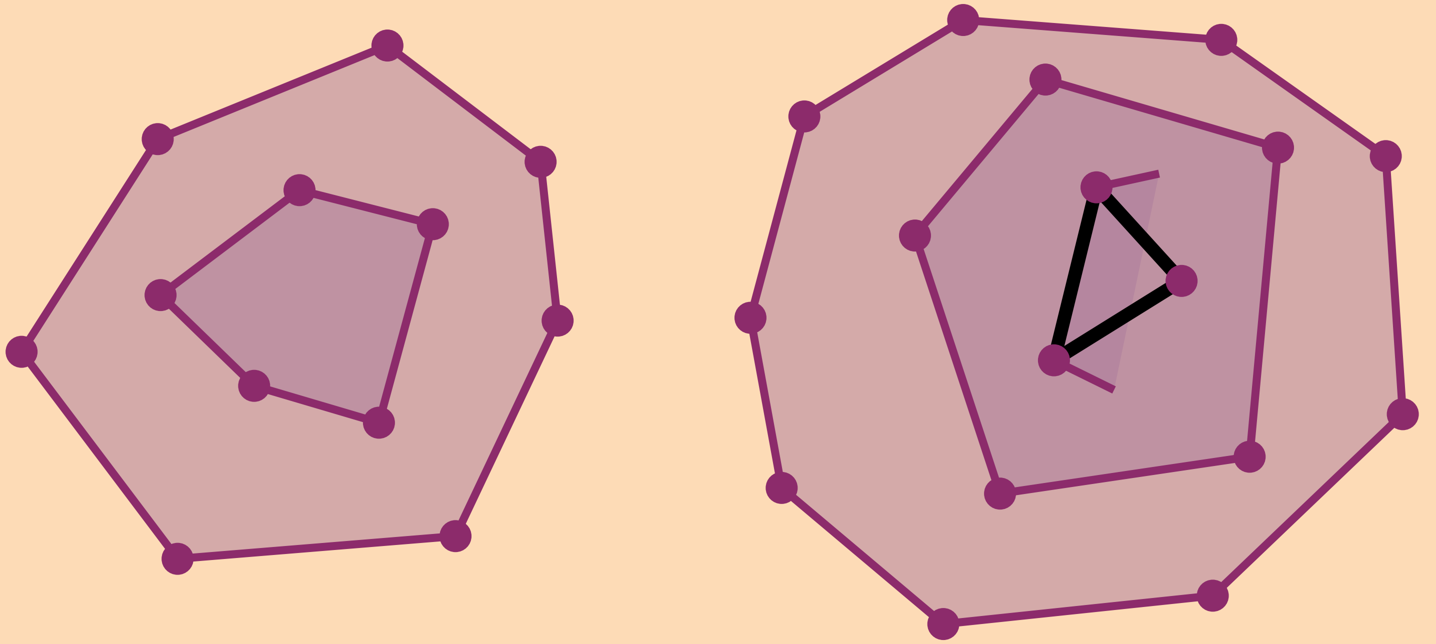

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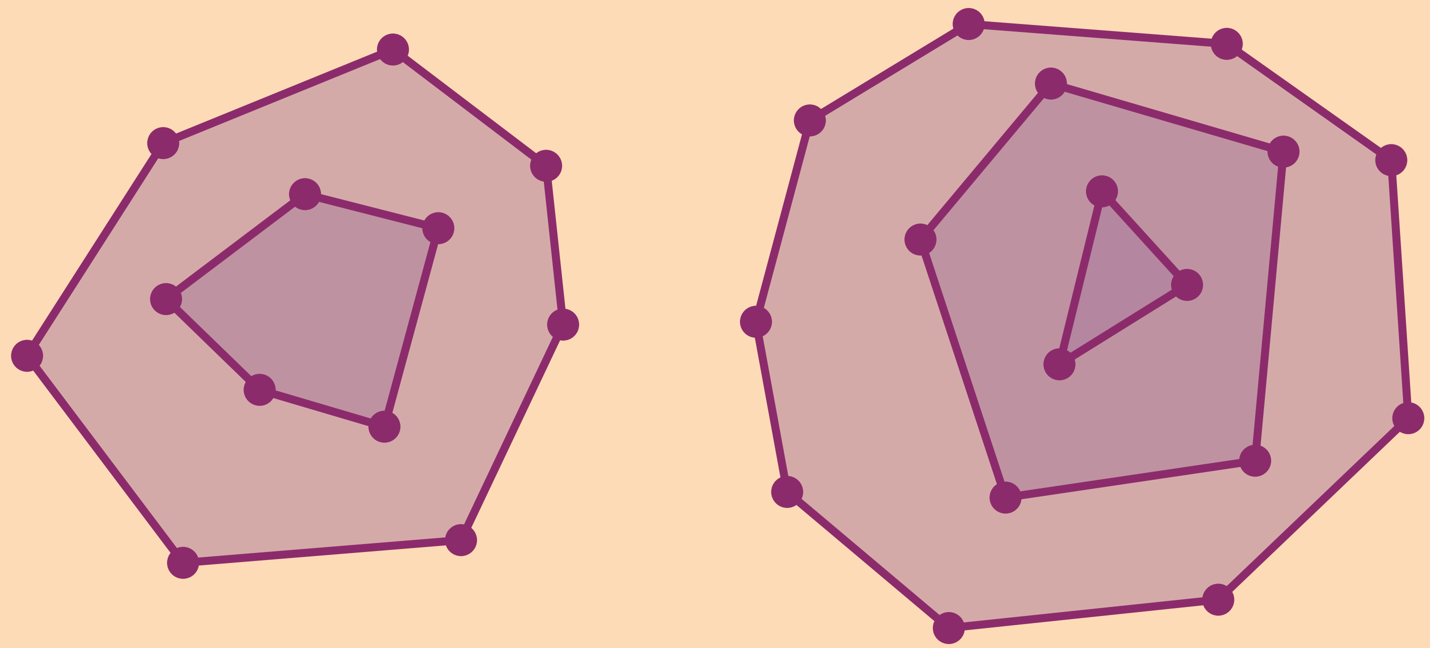

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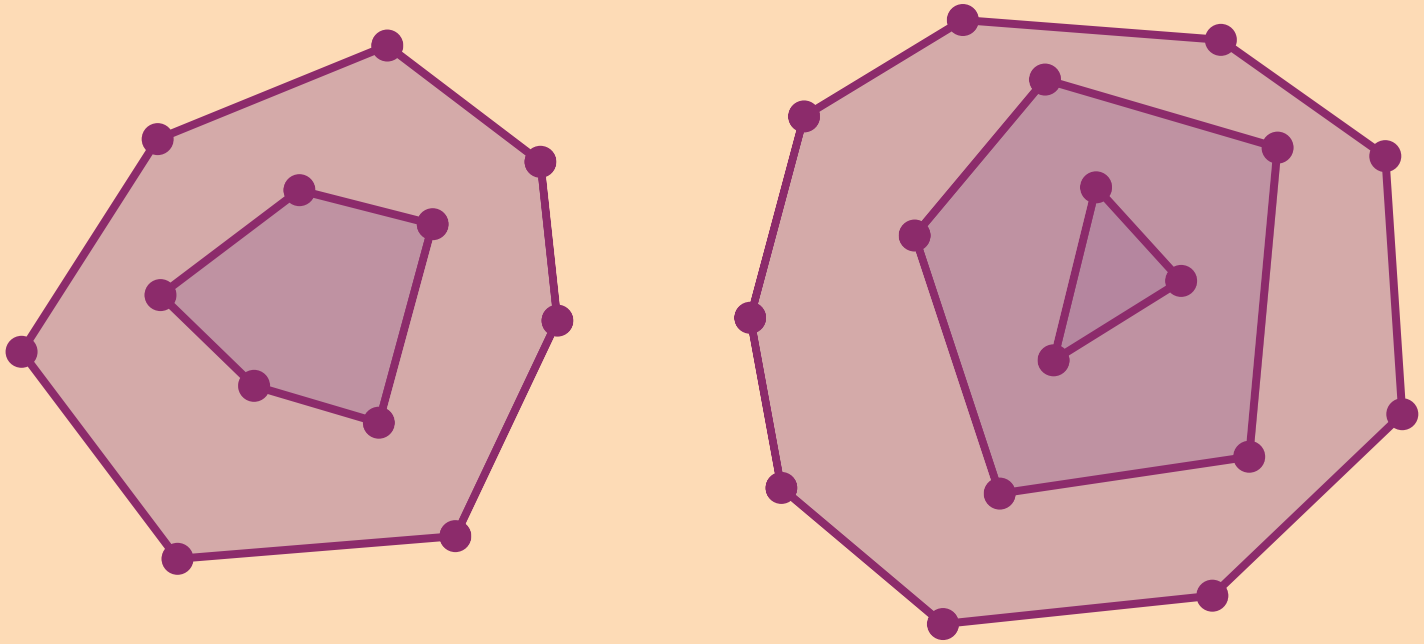

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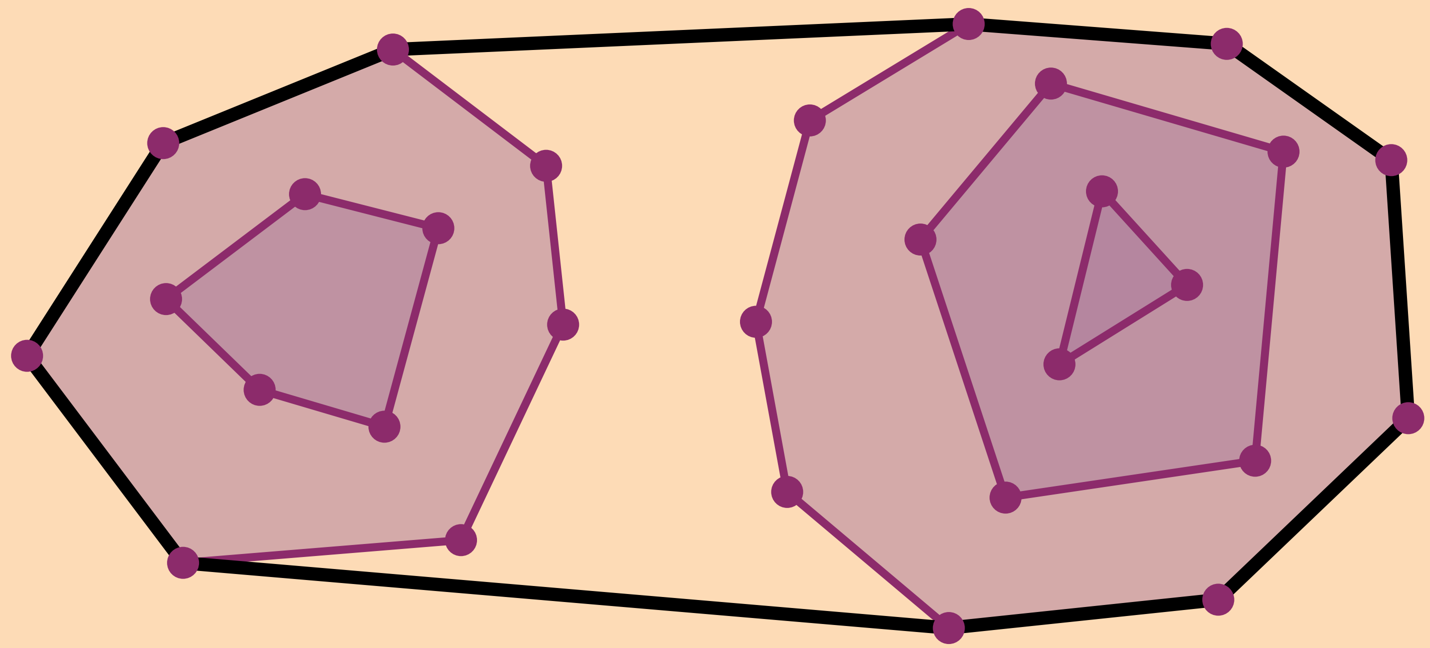

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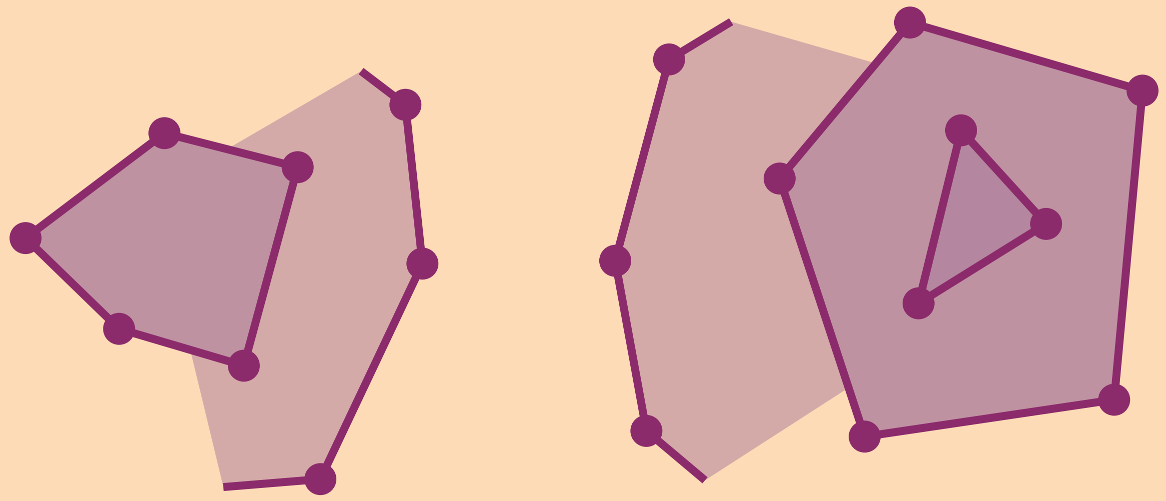

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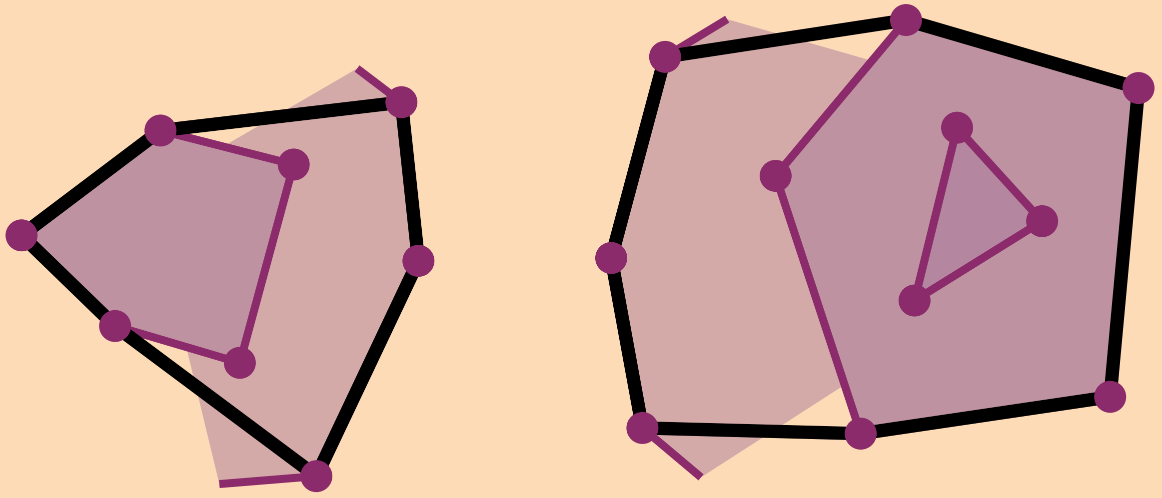

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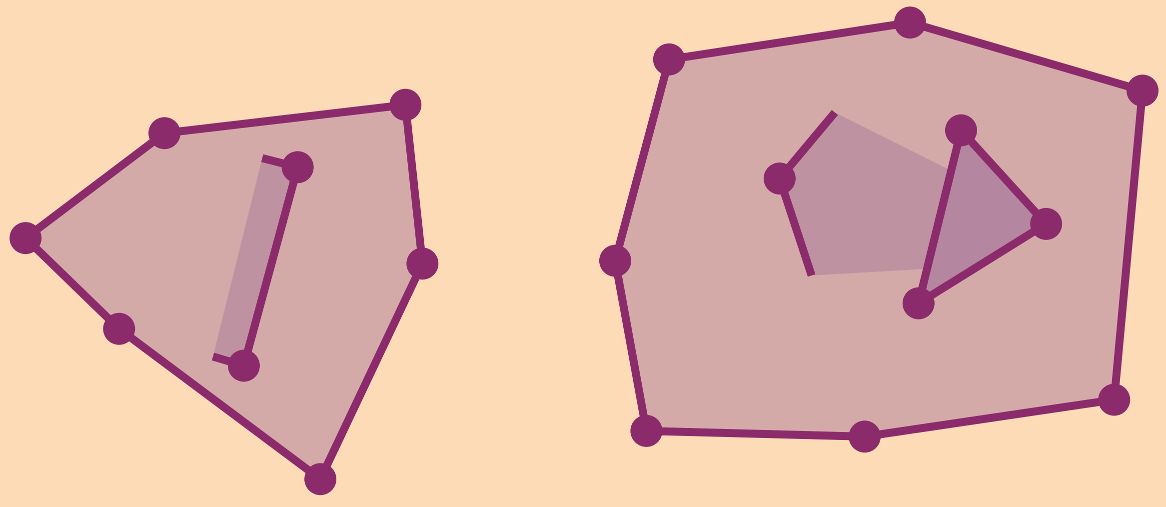

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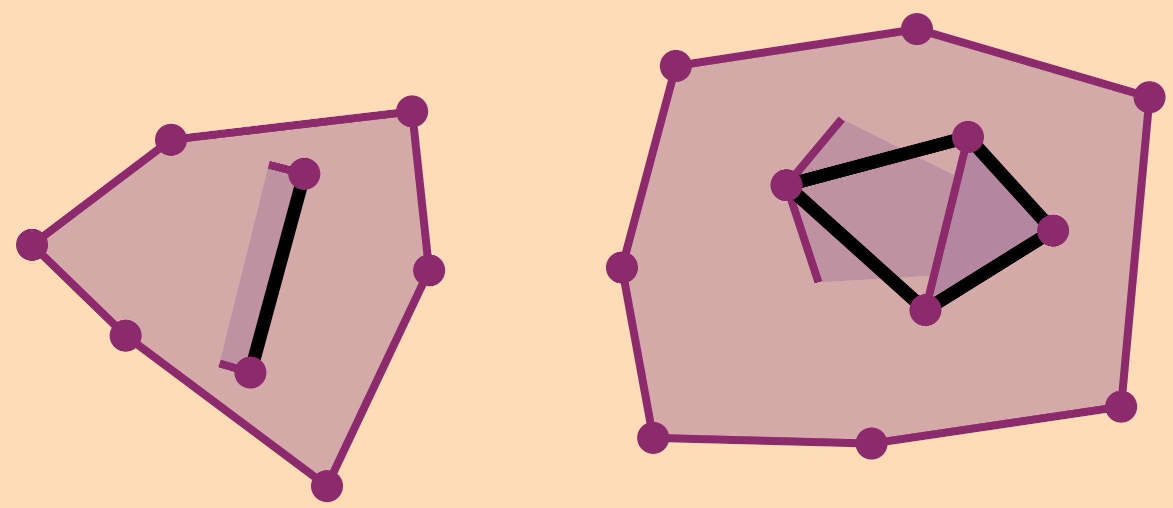

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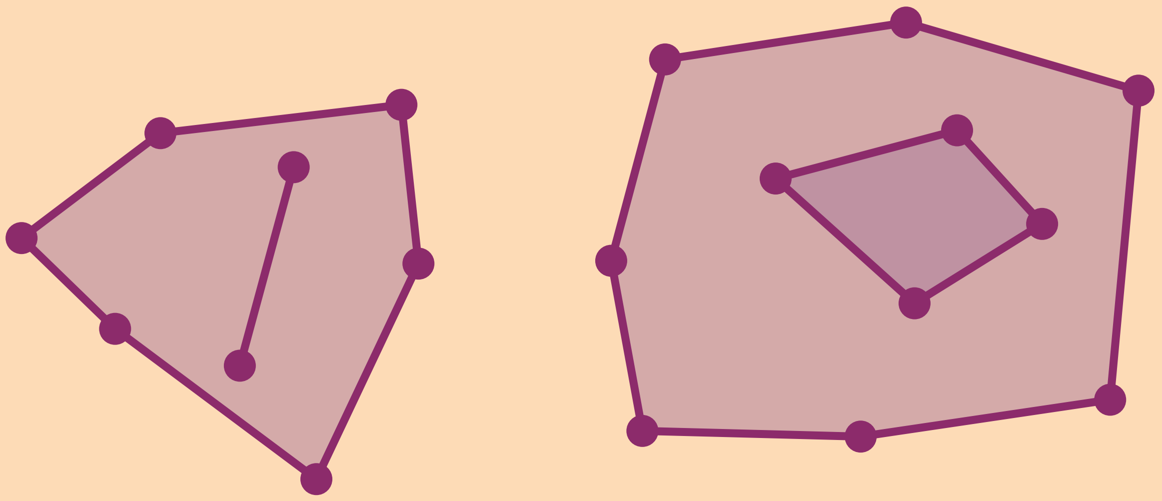

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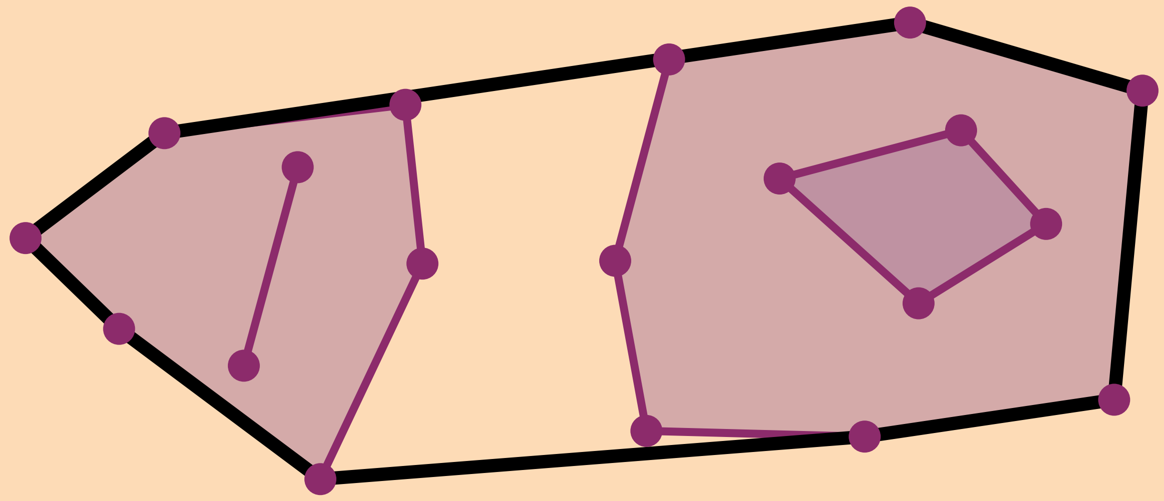

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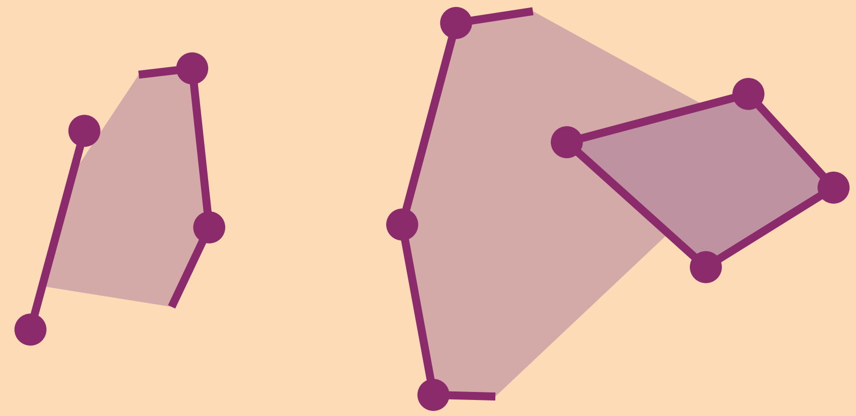

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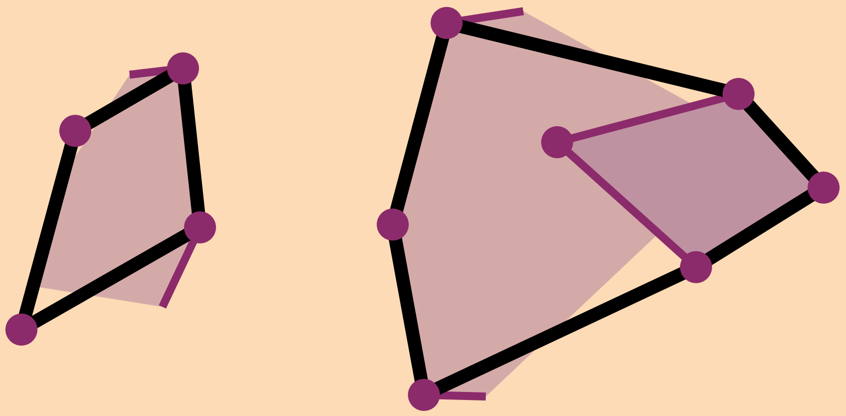

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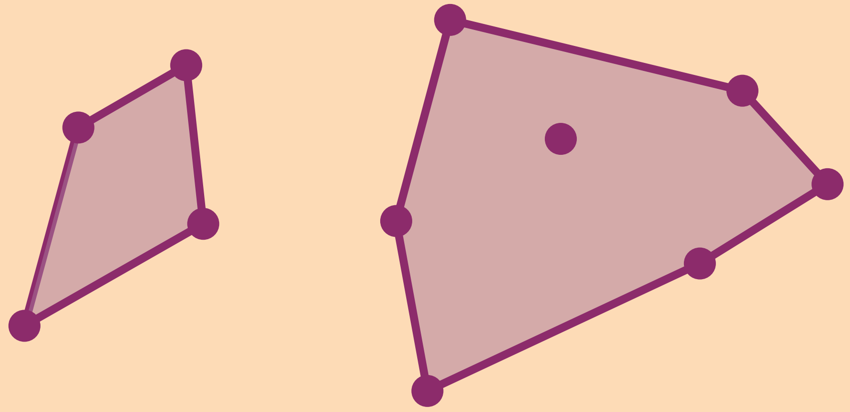

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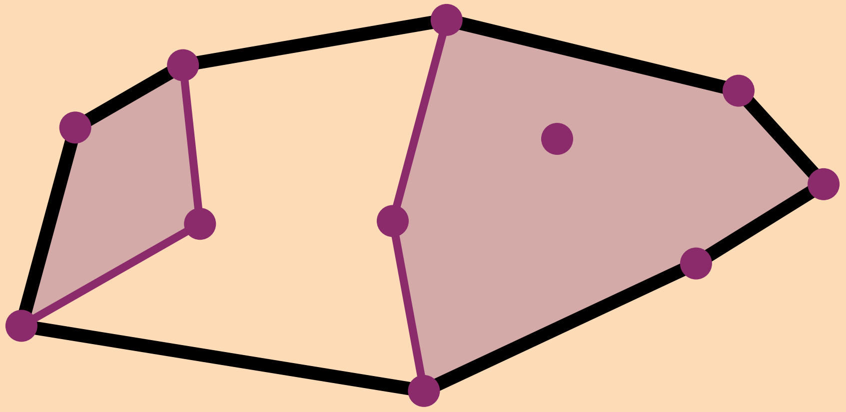

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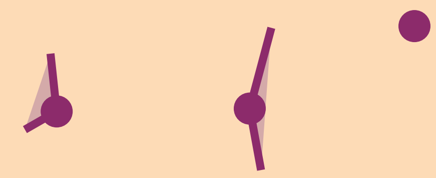

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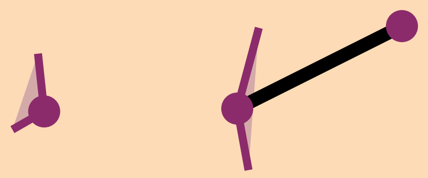

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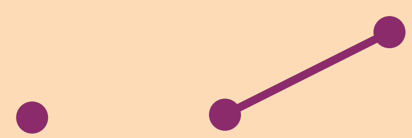

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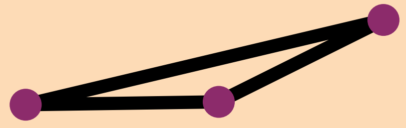

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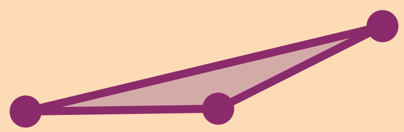

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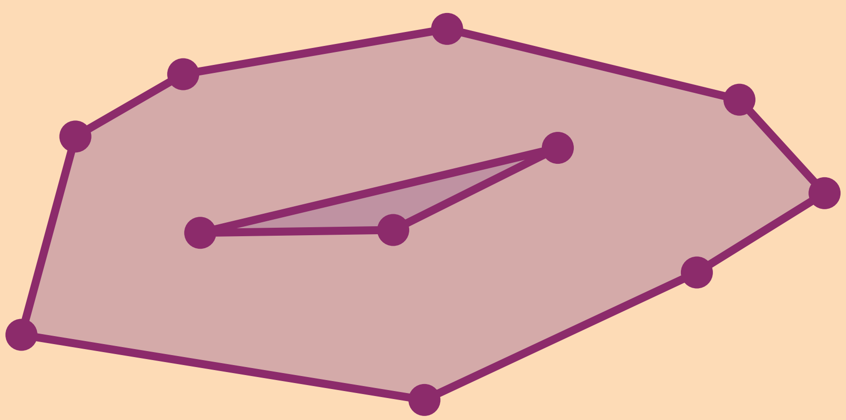

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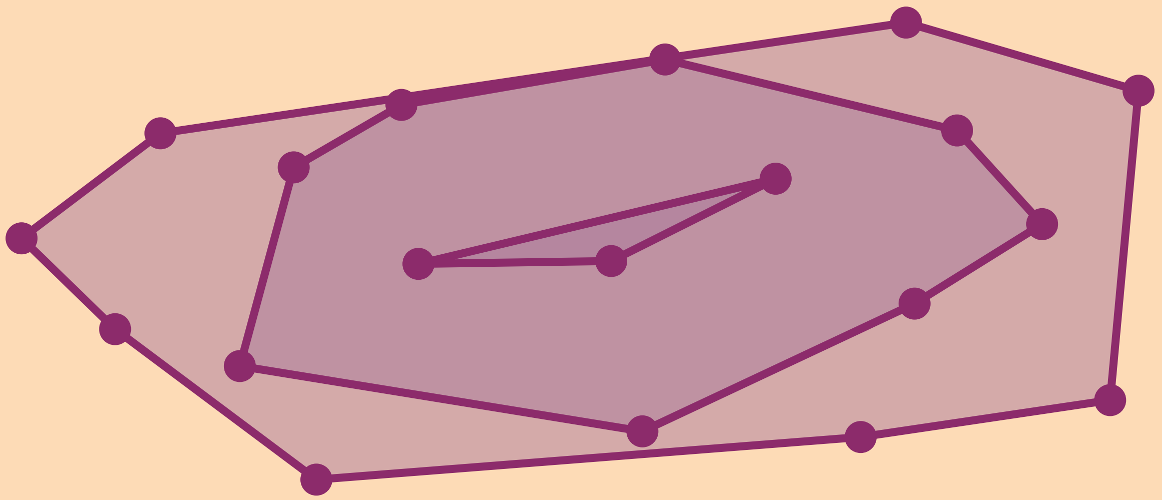

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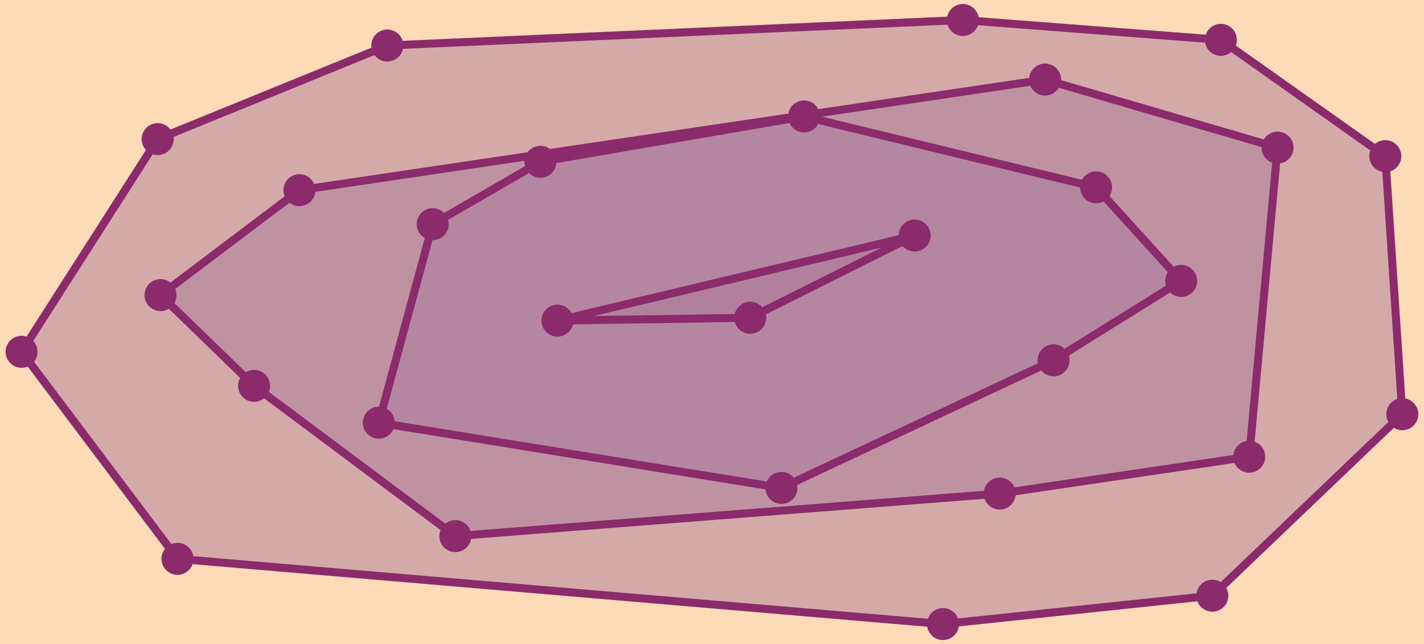

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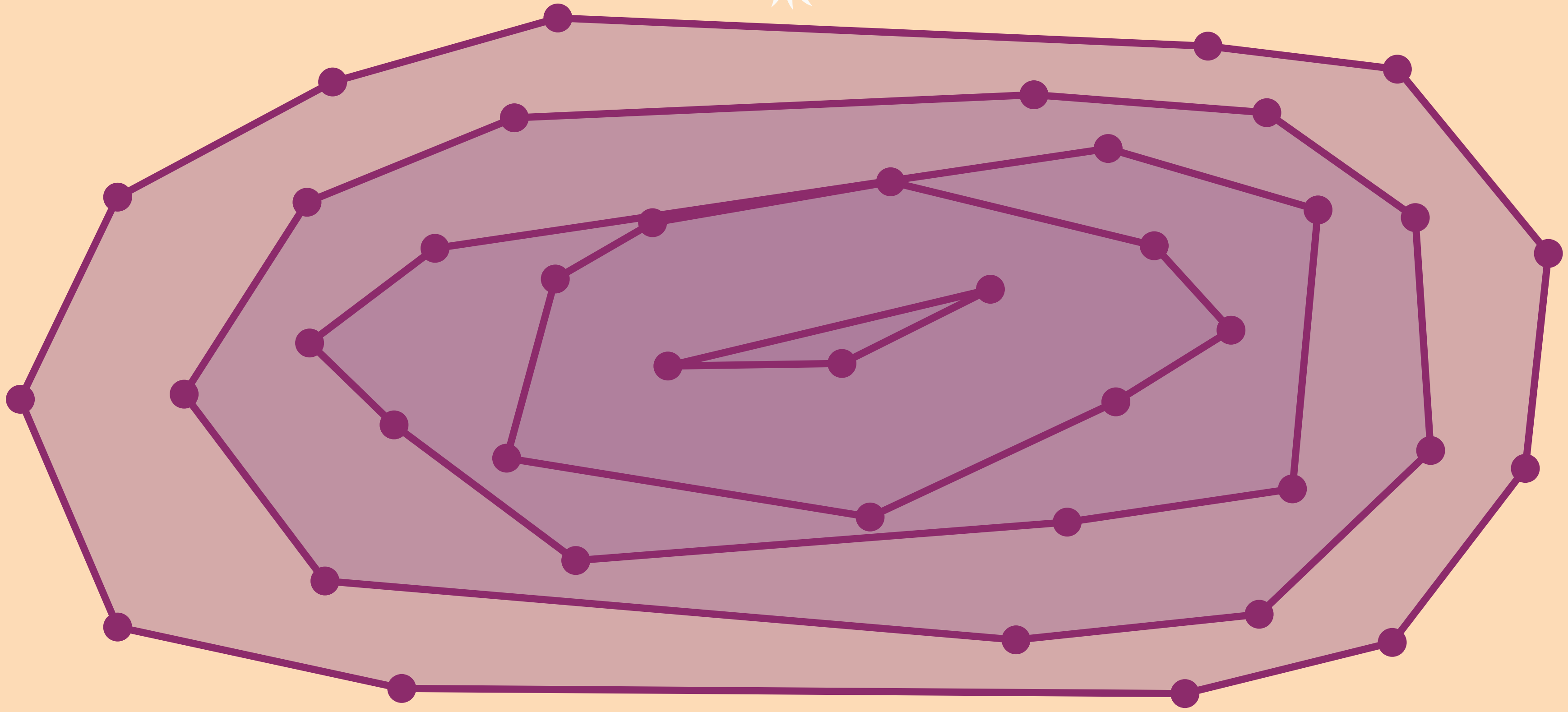

ONION RESTAURATION & UNIFICATION

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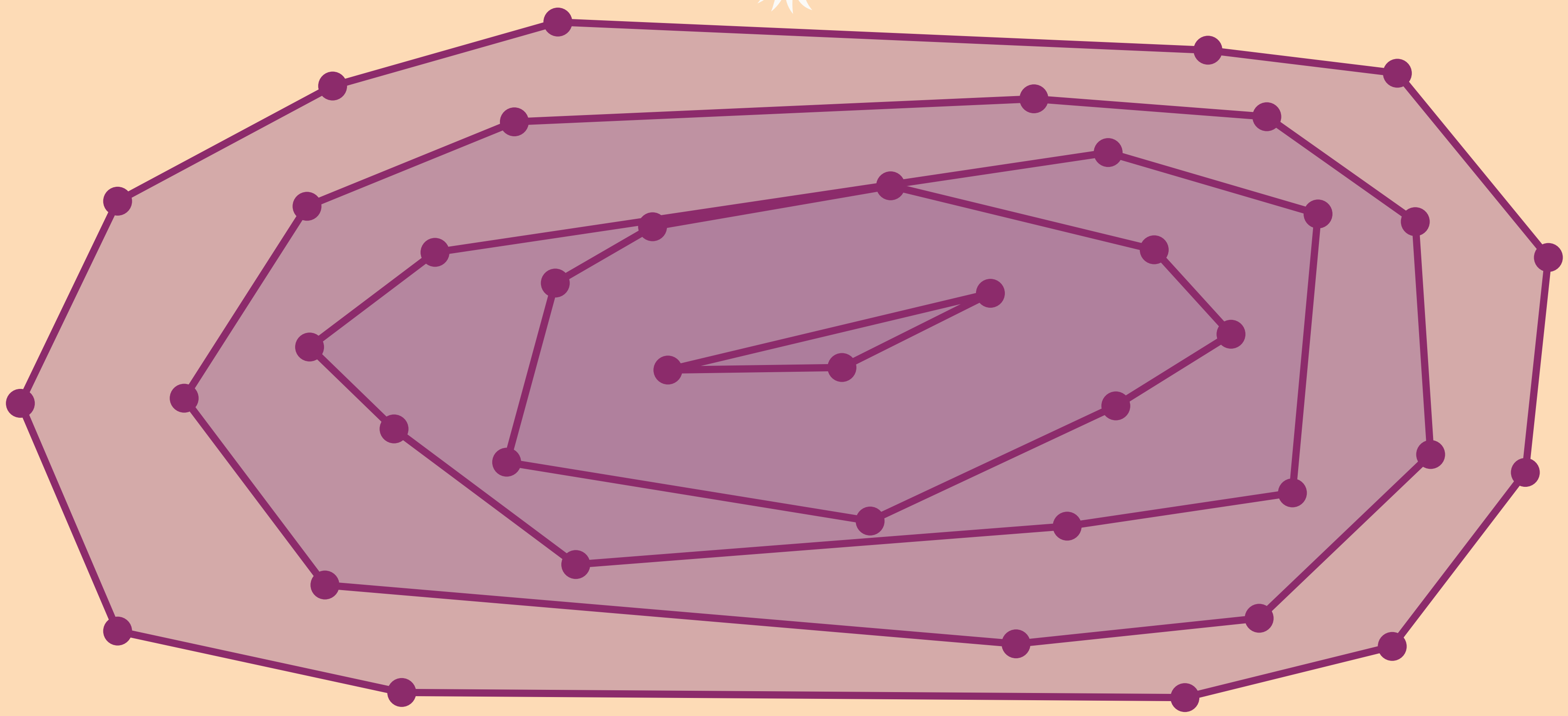
Recurse once again.

Etc...

That took $O(k^2 \log n)$ time to compute k layers.



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THE FINAL RESULT

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So, we can compute $H(\mathcal{R})$ in $(n \log n)$ time.

We can unite a pair of onions in $O(k^2 \log n)$ time.

So, we can reconstruct an onion in $O(n \log k)$ time.

THANK YOU!

any questions?