Linear-size Universal Point Sets for One-Bend Drawings

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Universal point sets

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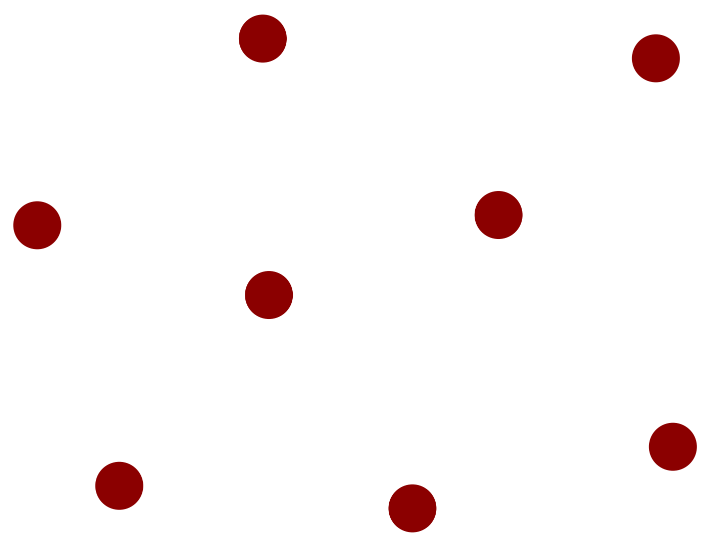

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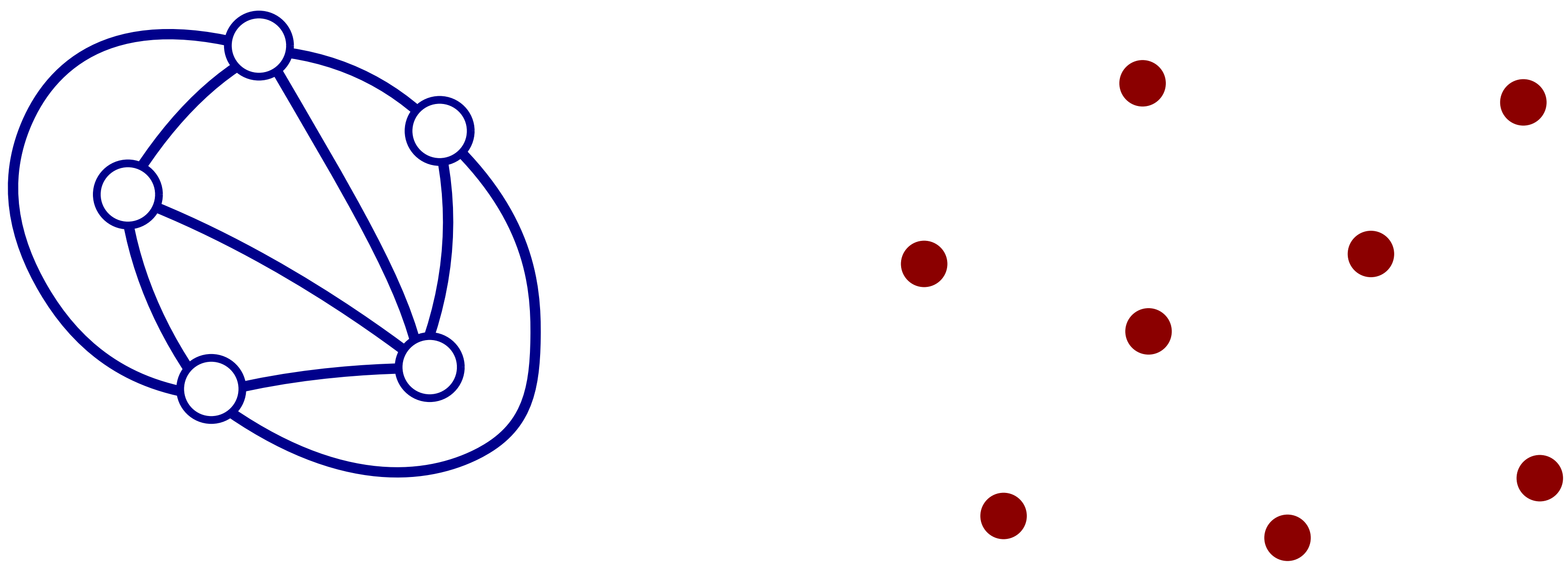

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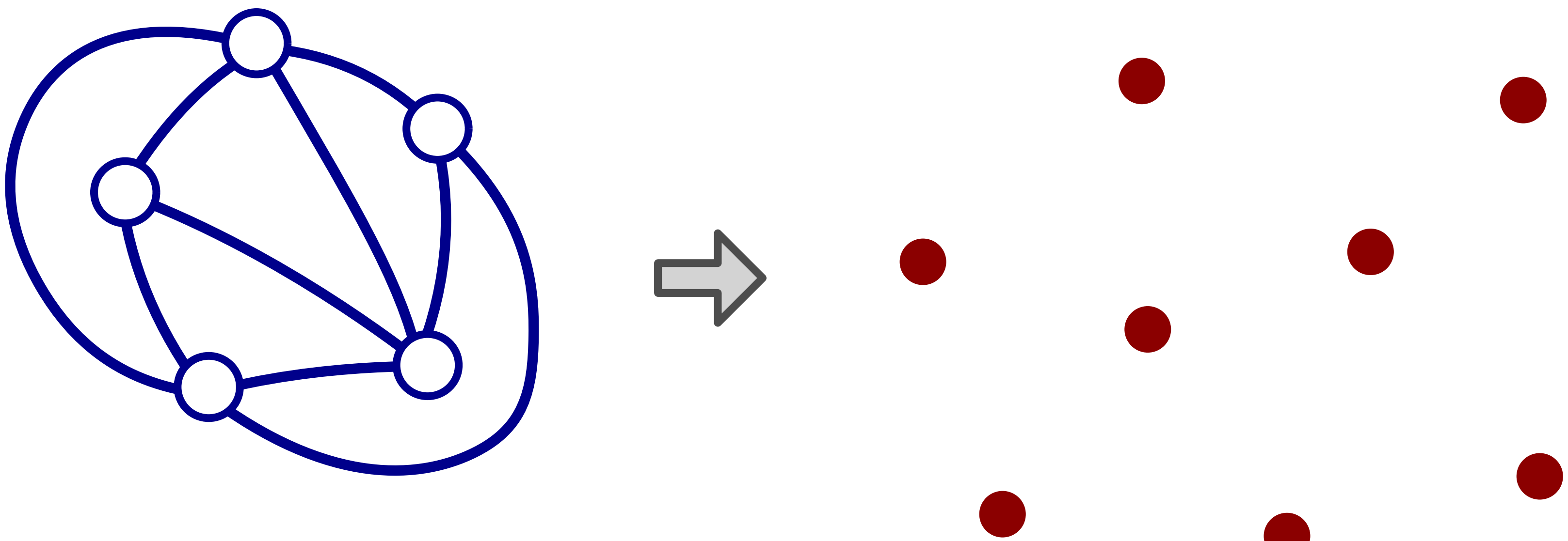

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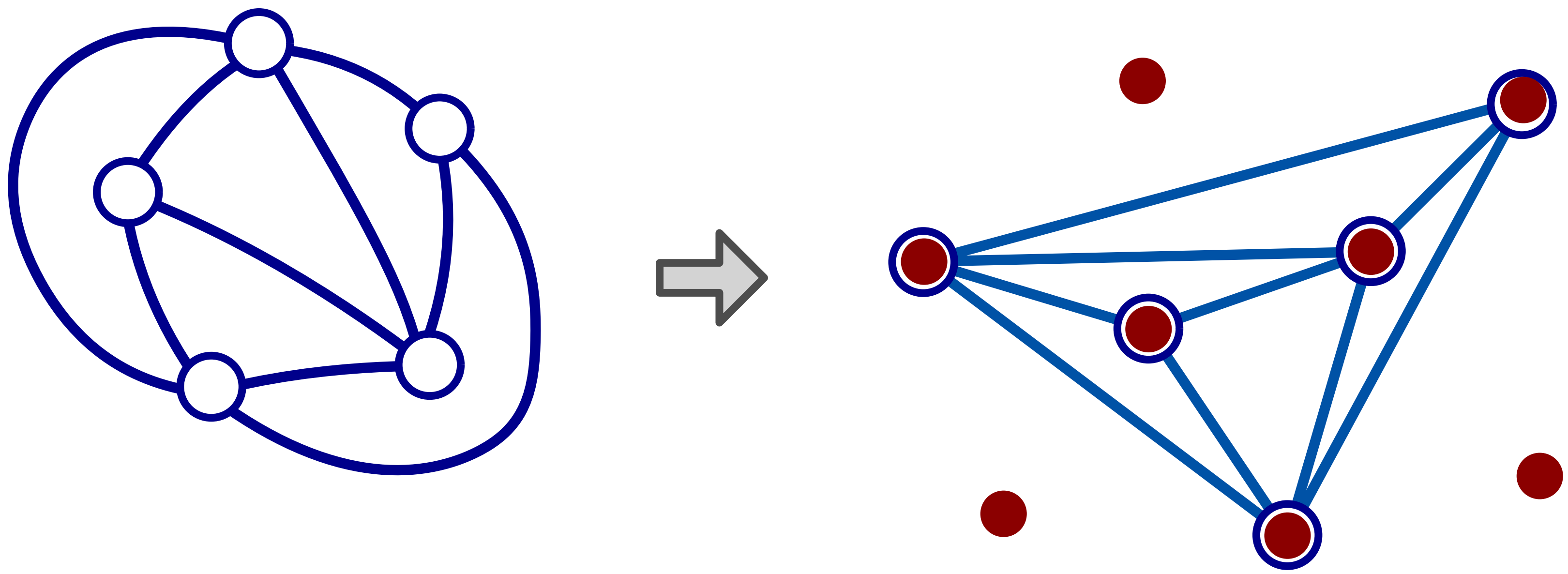

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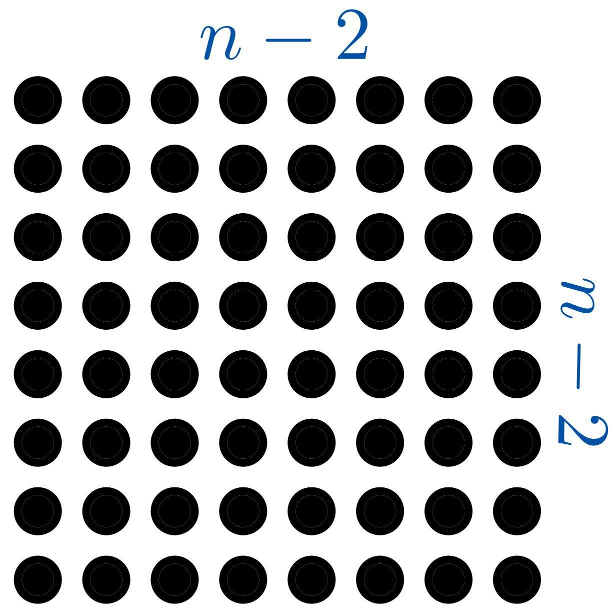
An $(n-1) \times (n-1)$ section of the integer lattice is n-universal.



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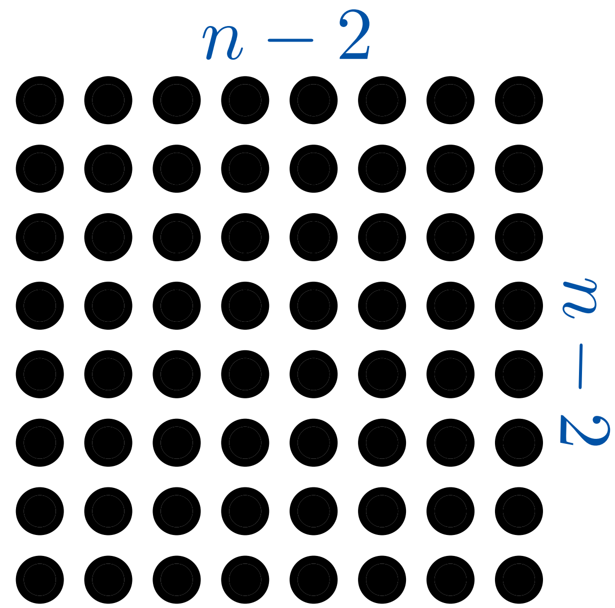

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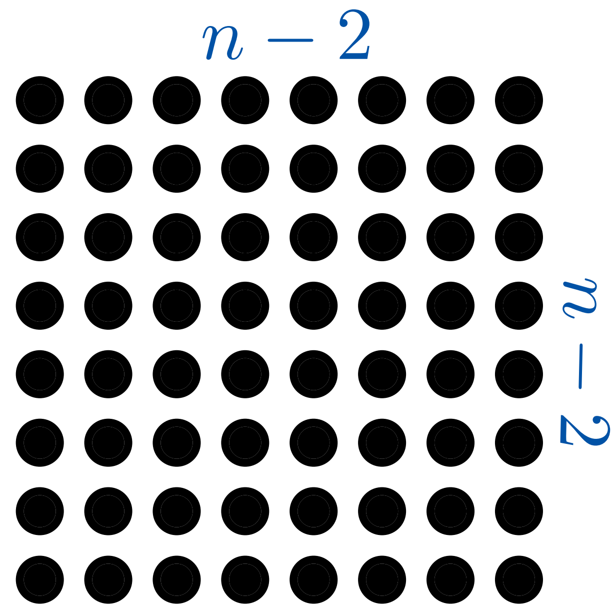

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Frati & Patrignani (2008): If a rectangular section of the integer lattice is n-universal, it must contain at least $n^2/9$ points.

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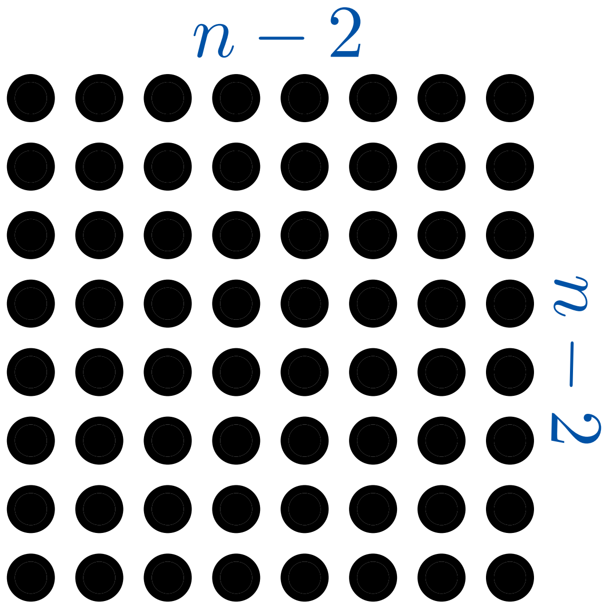

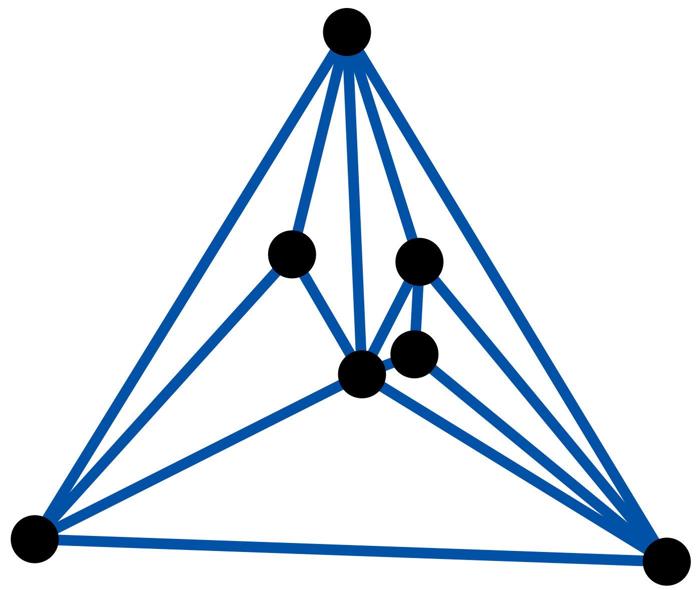

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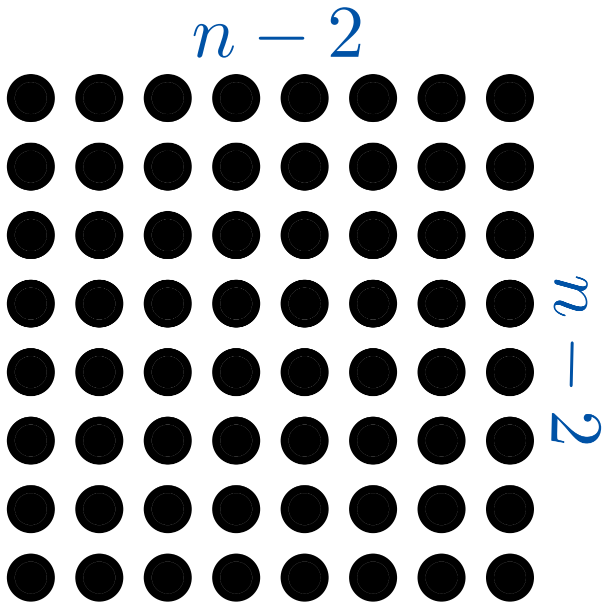

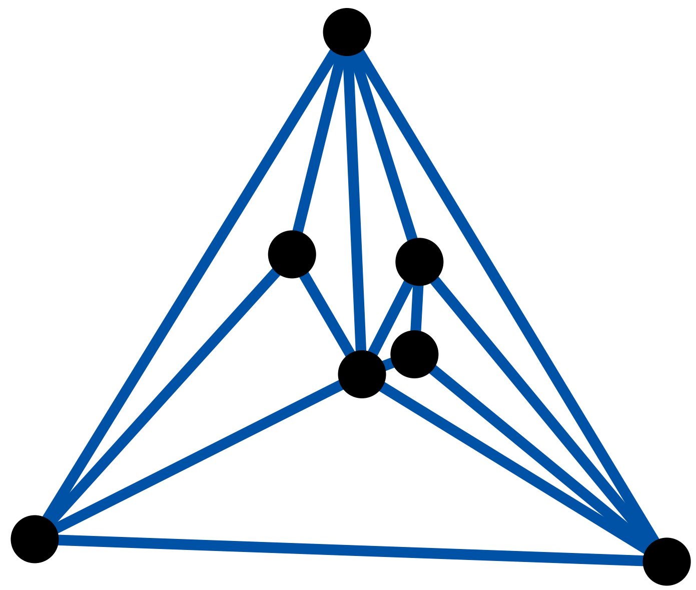

Open Problem: Find n-universal point sets of size $o(n^2)$.

k-Bend Universal Point Sets

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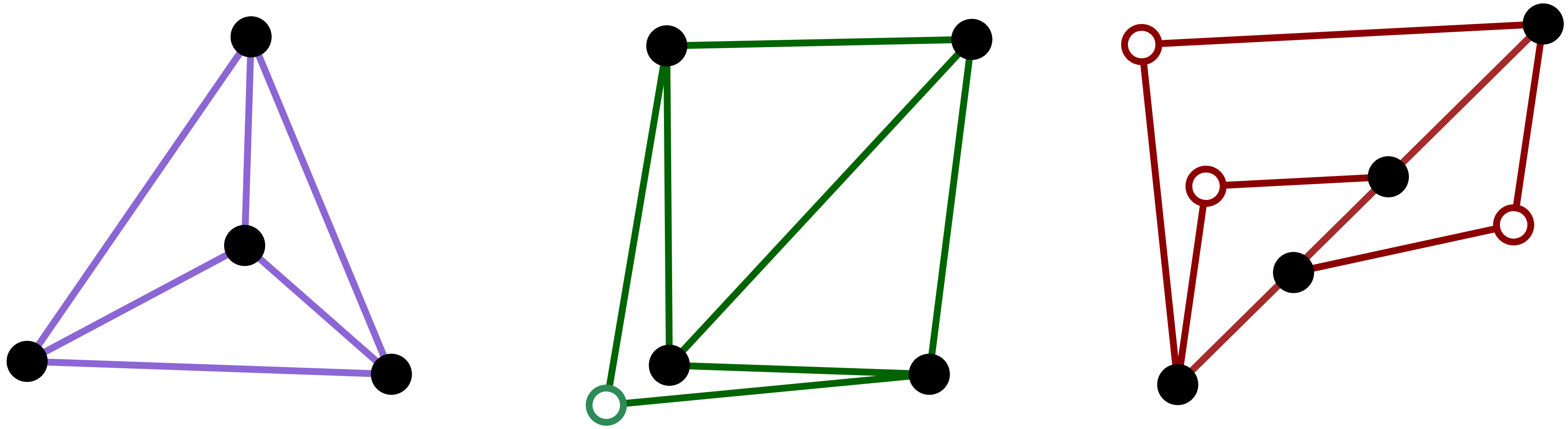

For example, every set of 4 points in the plane is 1-bend 4-universal, because K_4 embeds on every 4-element point set with 1-bend edges.

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Everett et al. (2010): $\forall n \in \mathbb{N}\ \exists S_n \subseteq \mathbb{R}^2$ such that $|S_n| = n$, every n-vertex planar graph has a 1-bend embedding in which **all vertices** are mapped into S_n. (**Bends not in S_n.**)

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Theorem (GD 2015). $\forall n \in \mathbb{N}\ \exists S_n \subseteq \mathbb{R}^2 : |S_n| \leq 6n$ such that every n-vertex planar graph admits a 1-bend embedding in which **all vertices and bend points** are mapped into S_n.

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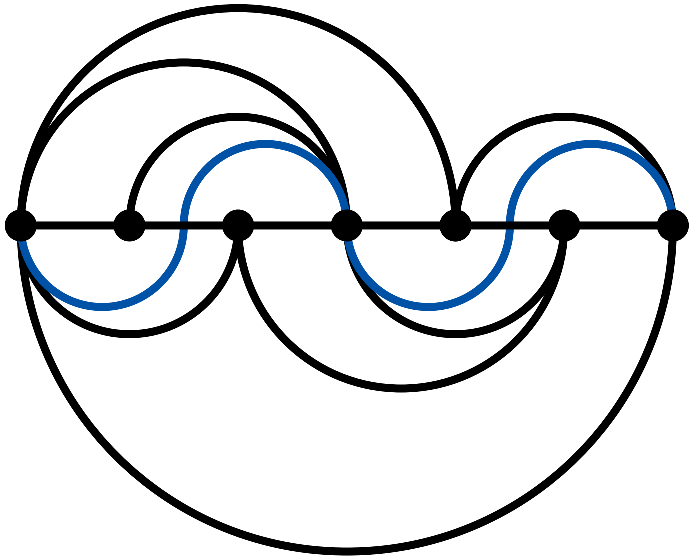

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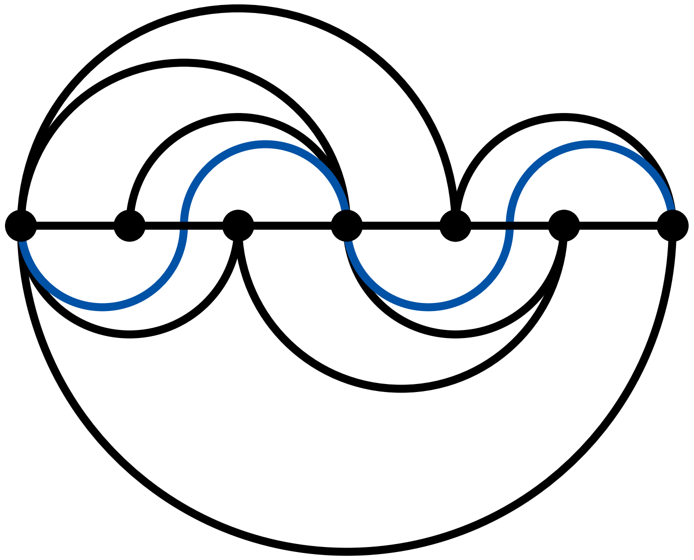

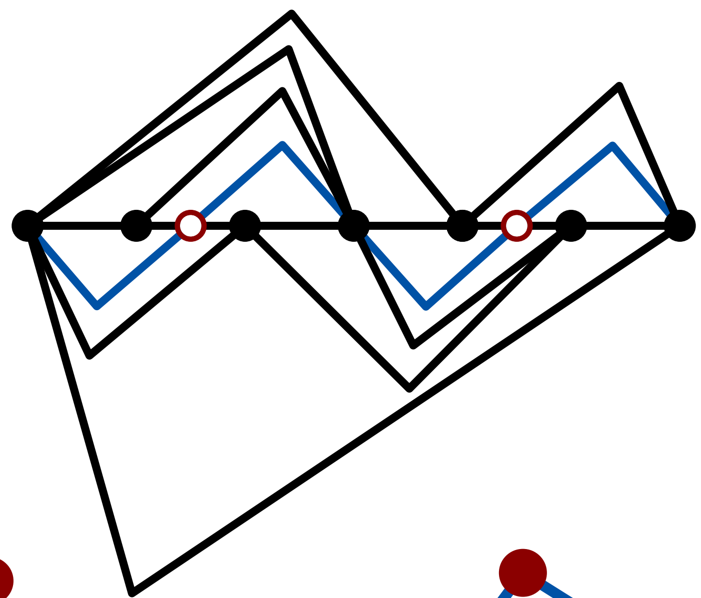

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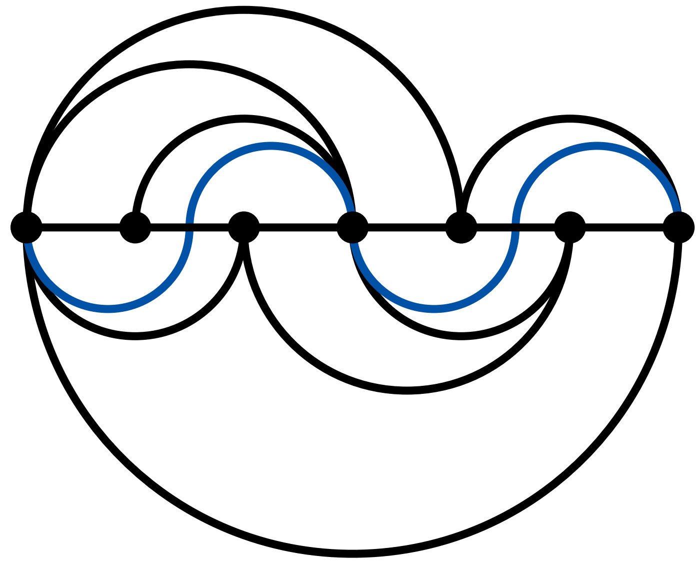

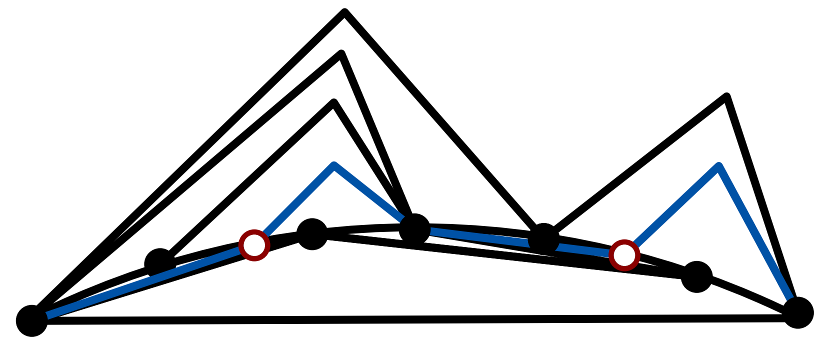

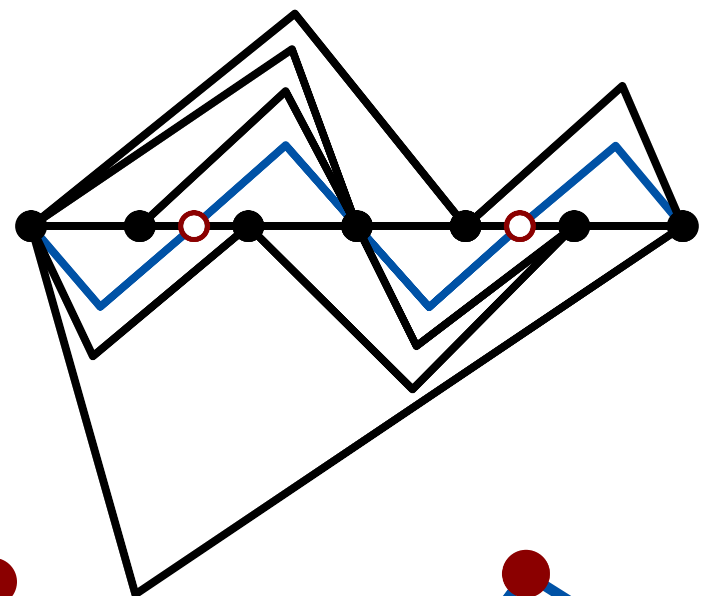

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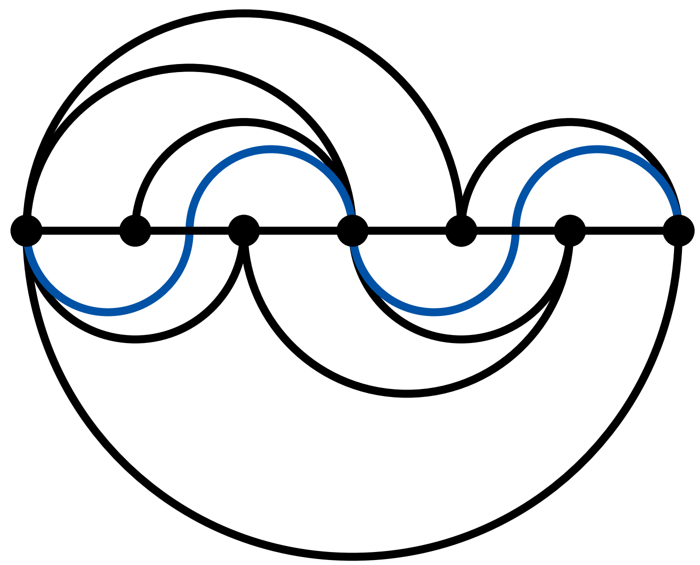

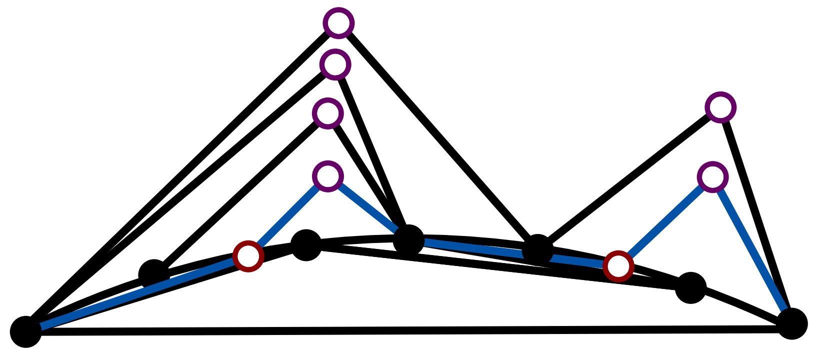

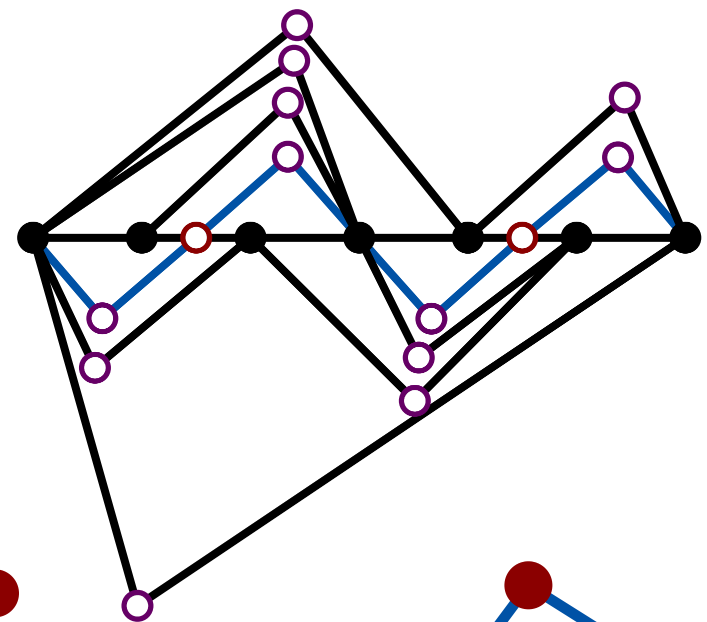

Cardinal et al (2015): Every n-vertex planar graph admits a biarc diagram with at most $n-4$ biarcs for $n \geq 4$.
$\Rightarrow$ W.l.o.g. at least $n-1$ edges are below the spine.

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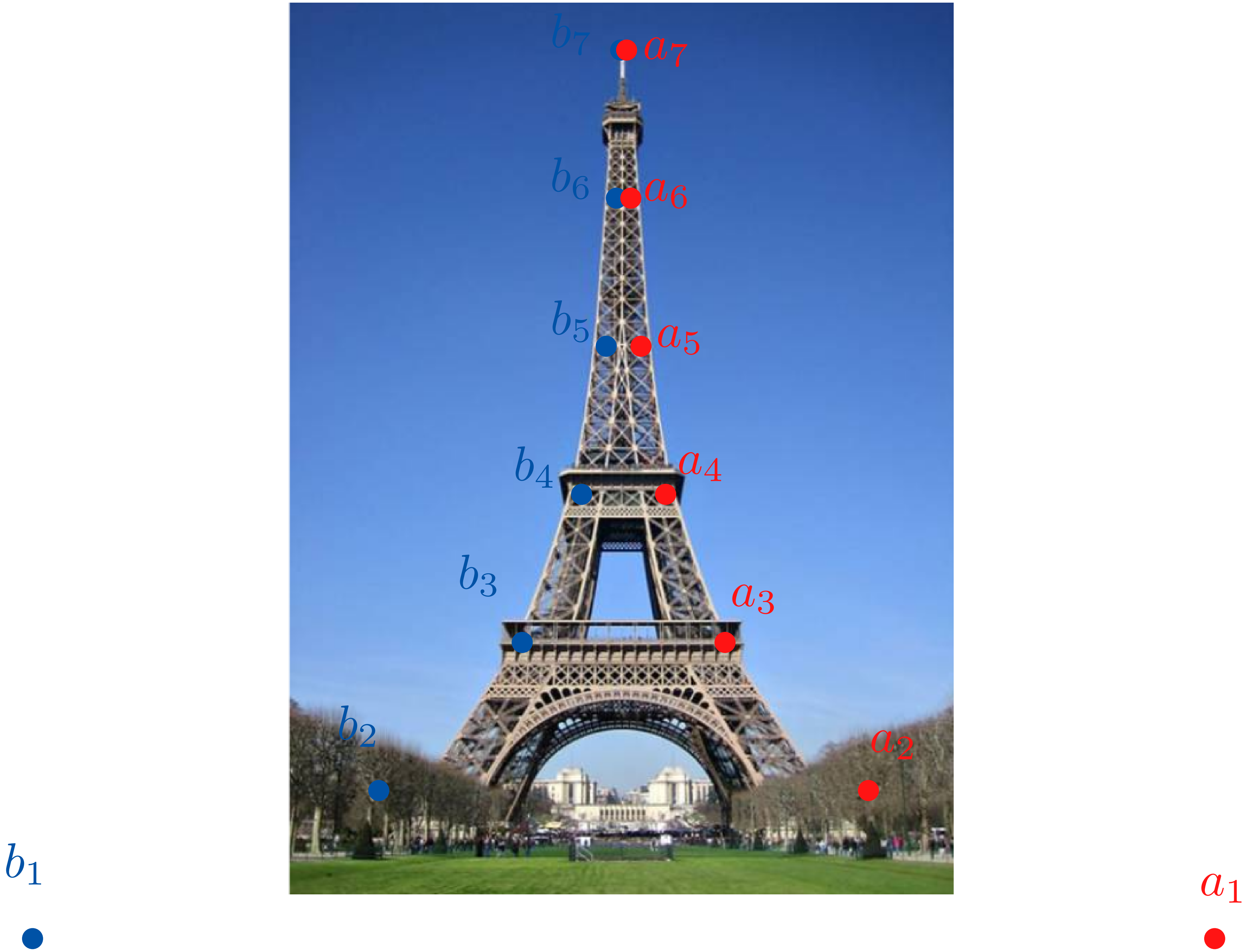

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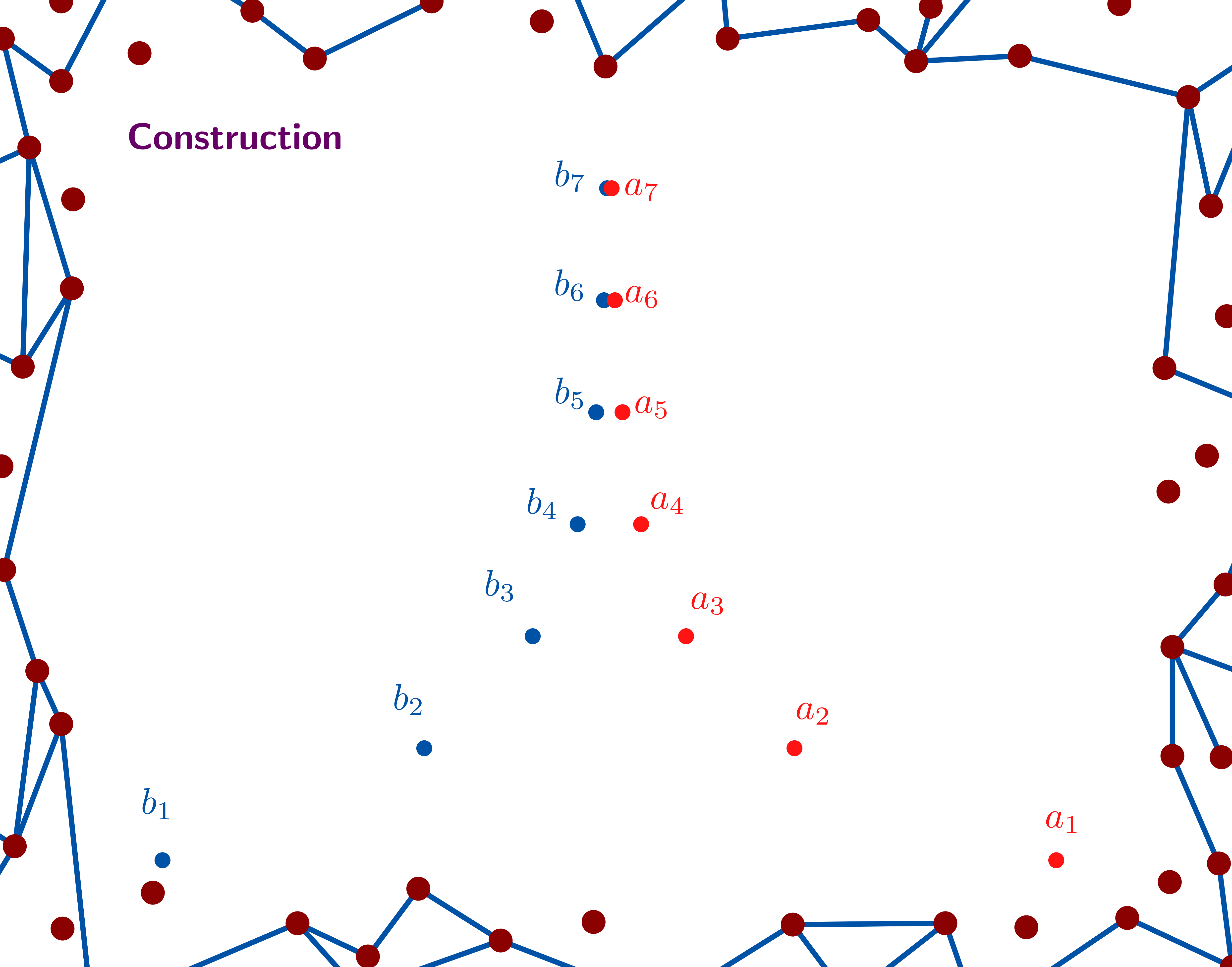

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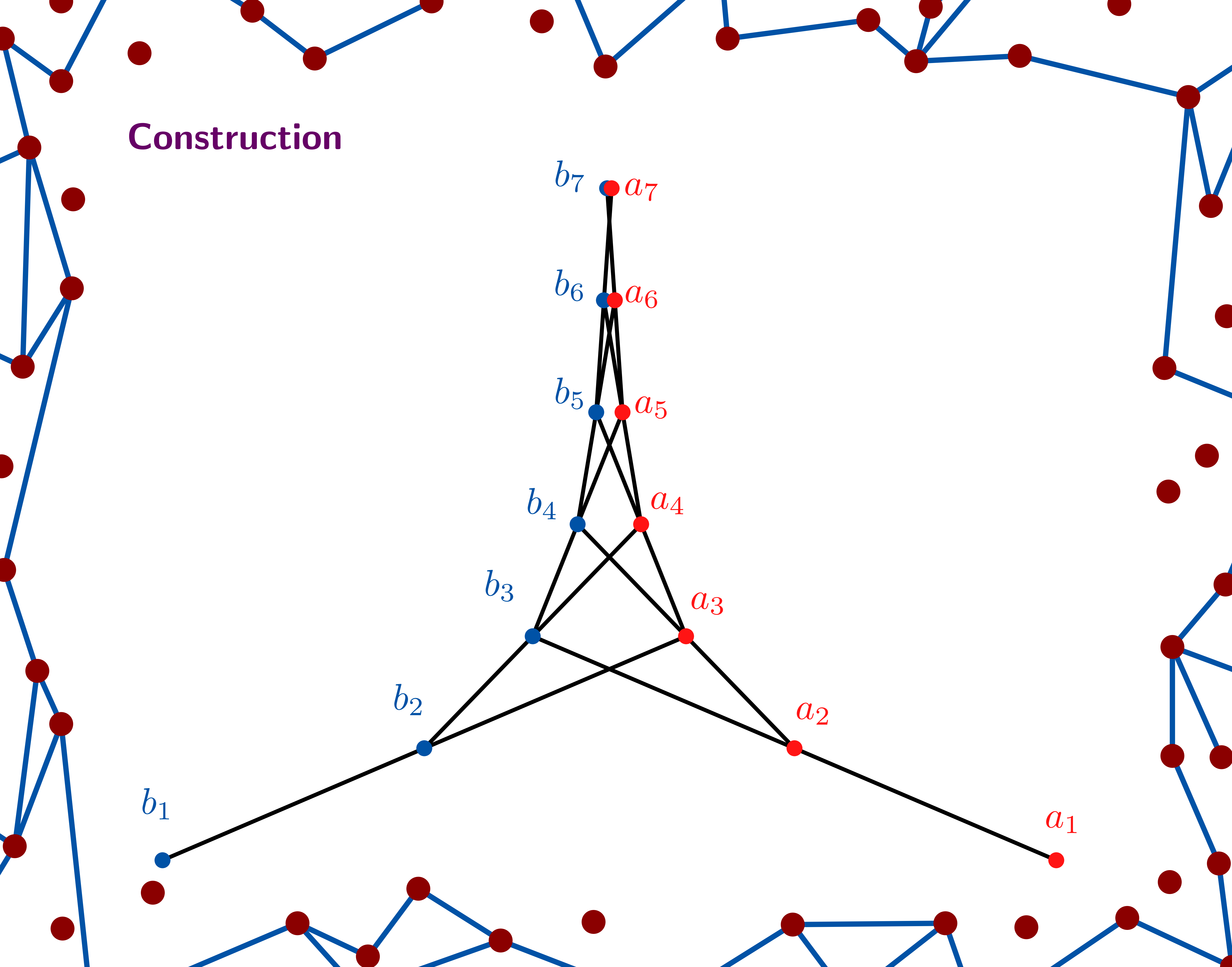

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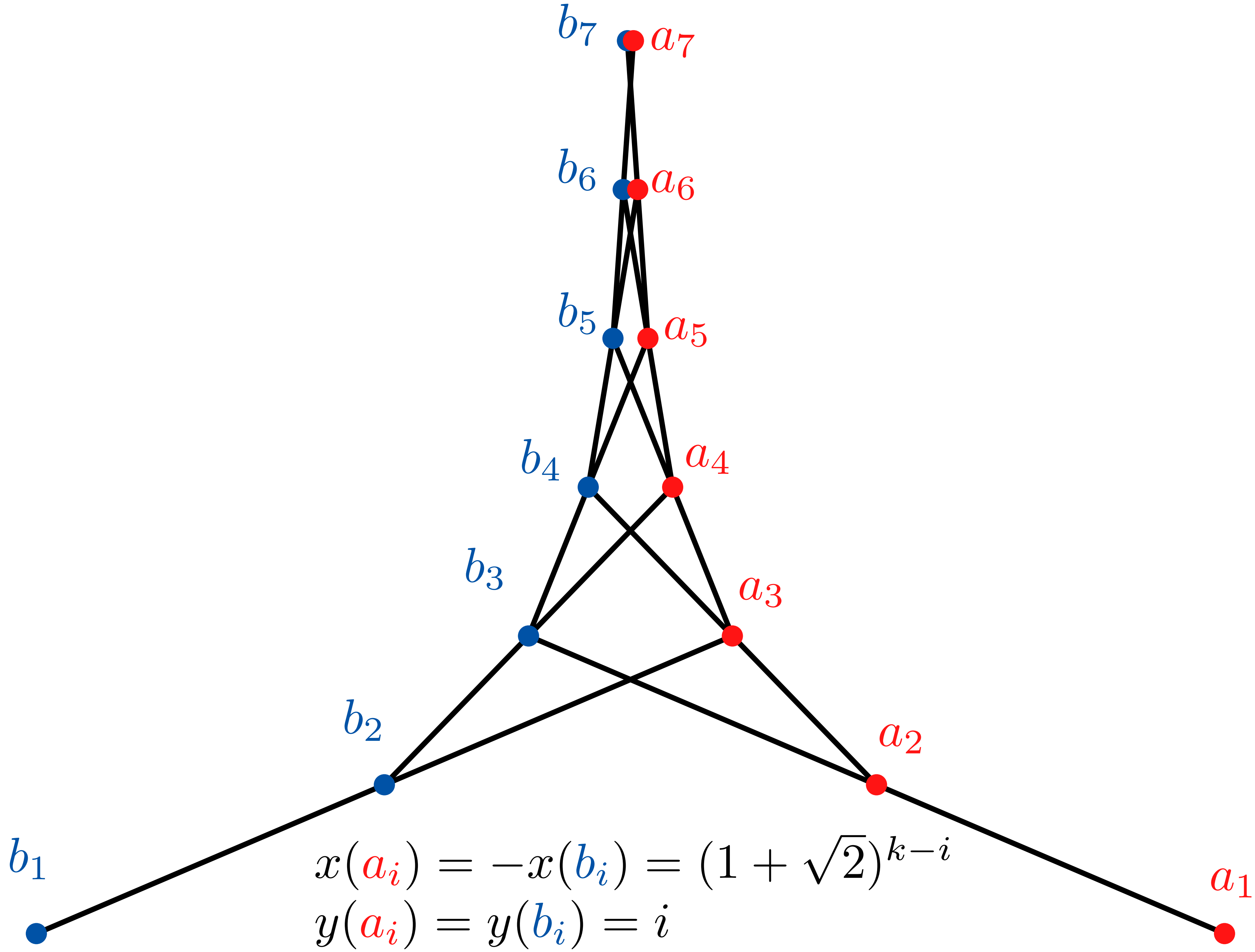

$$x(a_i) = -x(b_i) = (1+\sqrt{2})^{k-i}$$
$$y(a_i) = y(b_i) = i$$

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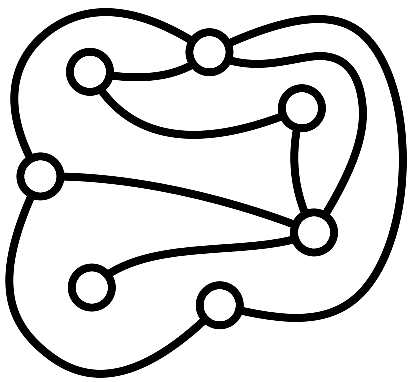

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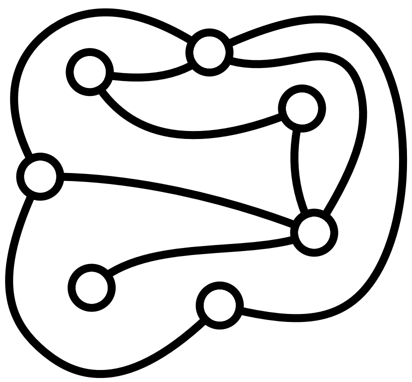

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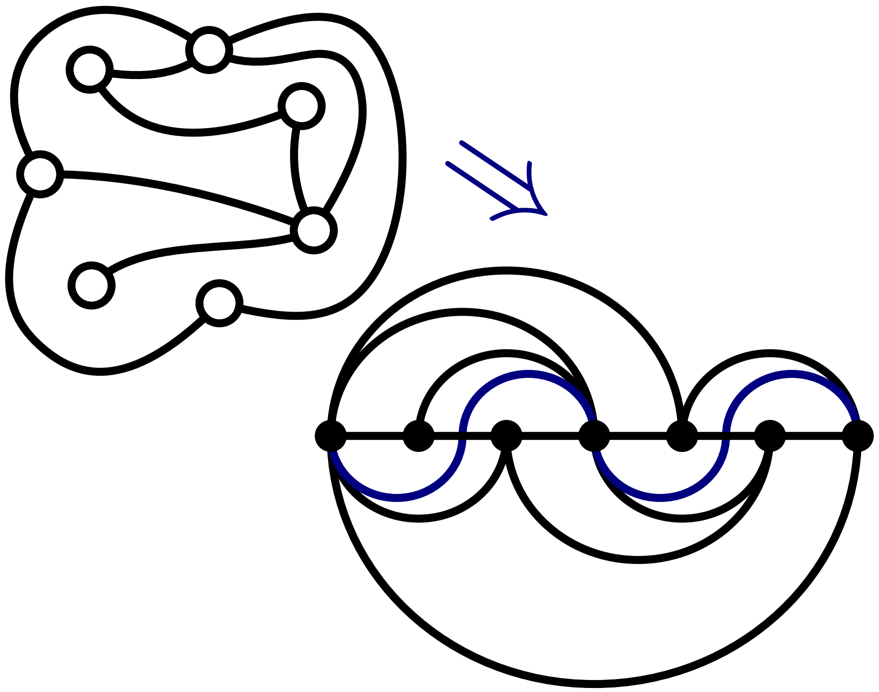

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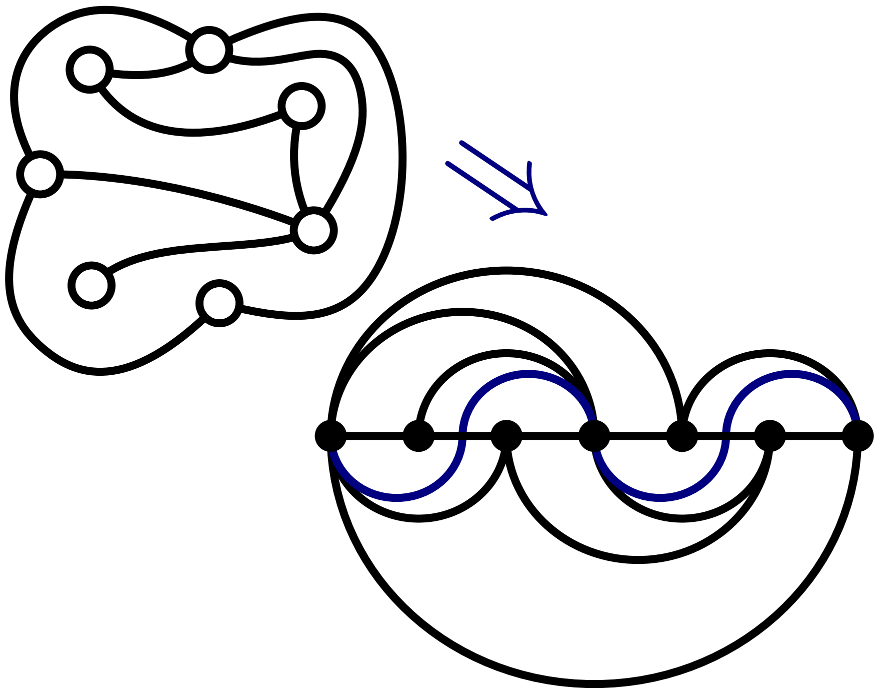

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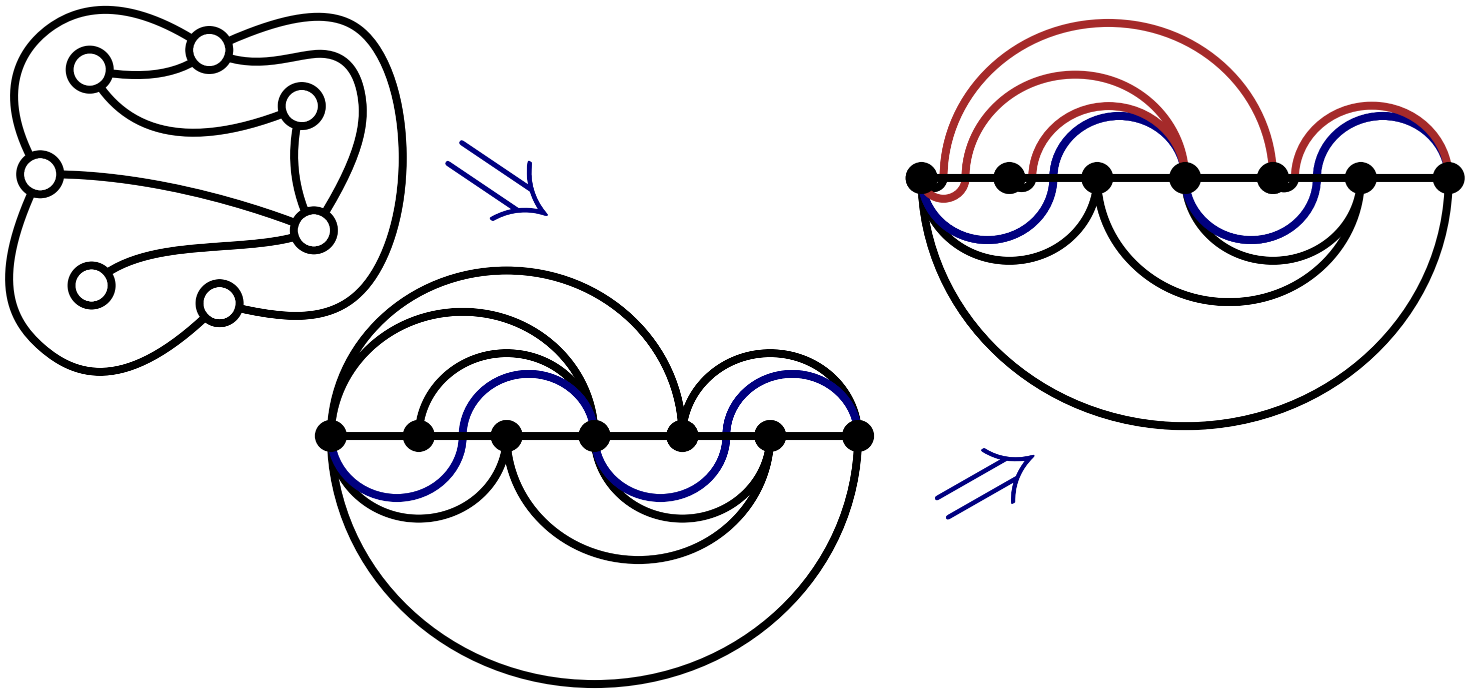

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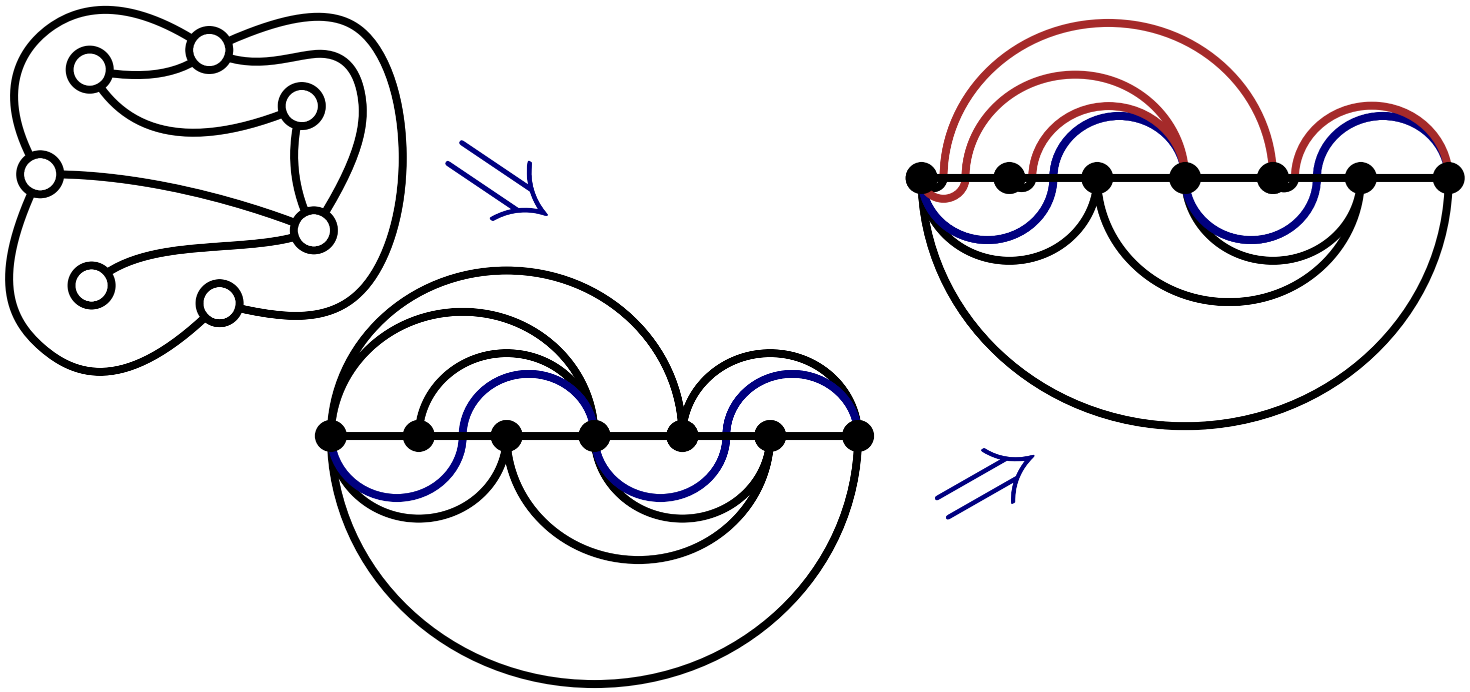

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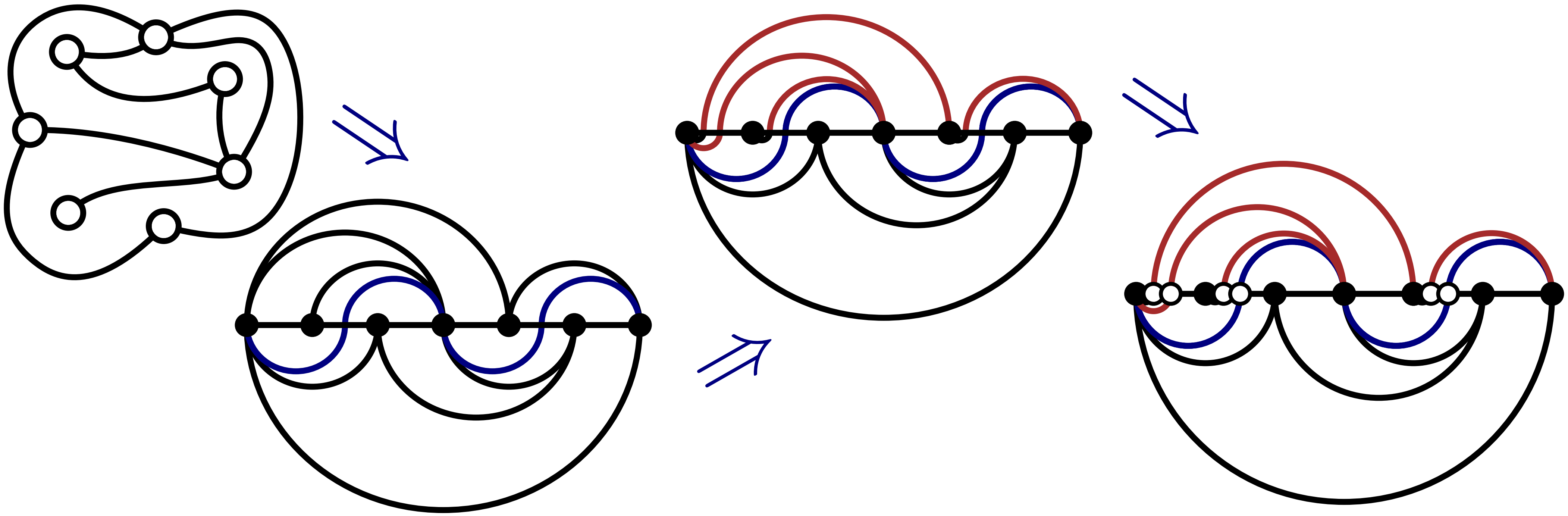

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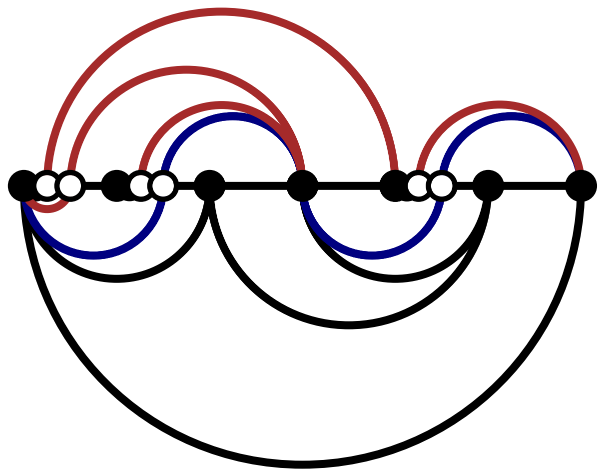

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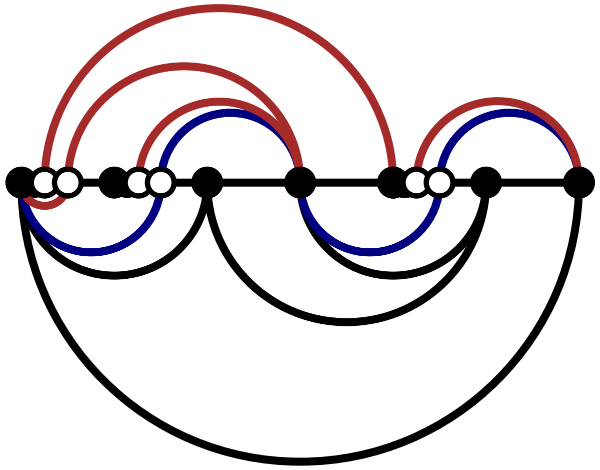

Label the vertices along the spine by $p_1, \ldots, p_m$, where $m \leq 3n - 5$.

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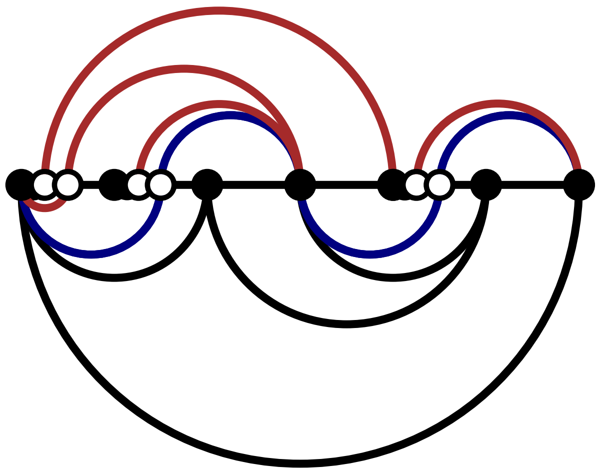

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Embed each proper vertex p_i at point a_i; and each bend point p_j at b_j.

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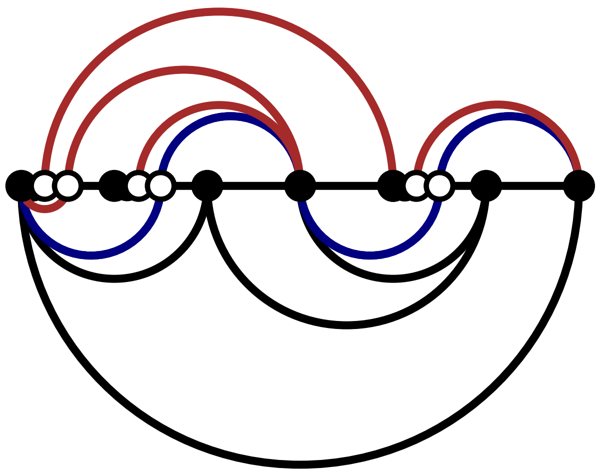

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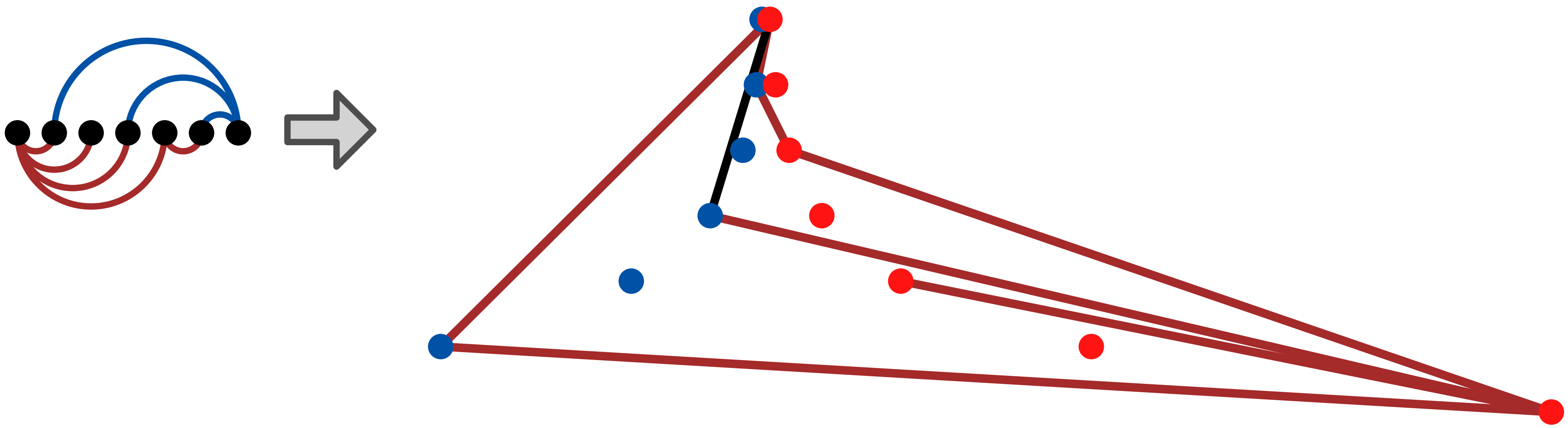

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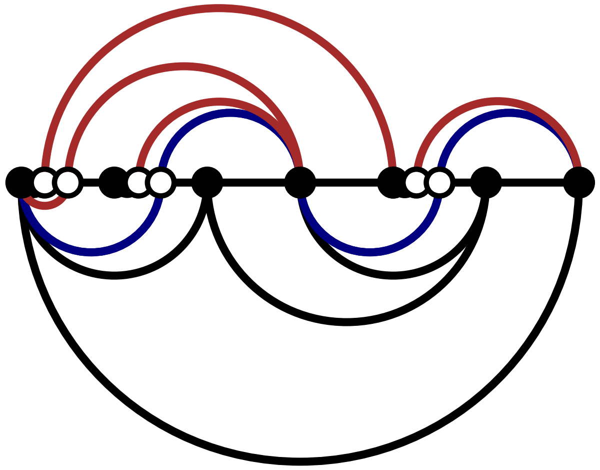

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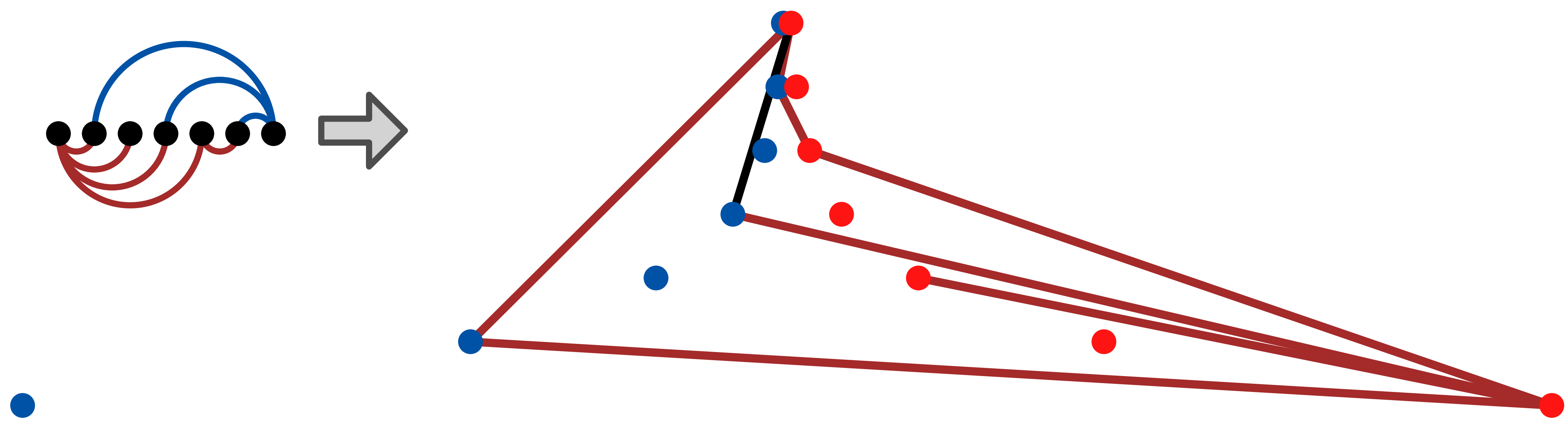

Proper arcs below the spine become straight-line edges.
Biarcs become 1-bend edges. No edge crossings.

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The point set $\{a_1, \ldots, a_{5n-4}, b_1, \ldots, b_{5n-4}\}$ in our construction is $(n, \frac{1}{3})$-bend universal, for every $n \geq 4$.
(At least $\frac{1}{3}\ |E|$ edges are proper arcs below the spine.)

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OPEN: Are there (n, ϱ)-universal point set for $\varrho > \frac{1}{3}$?

Merci!