Hodge theory and period maps  (Fall 2020)
Literature:
R.O. Wells: Differential Analysis on Complex manifolds, Springer GTM 65
R.H. Cunefredsk, J.Harris: Principles of Algebraic Geometry, Wiley

(more will be added)
§1 Hodge theory Riemannian manifold (quick review)

(1.1) Exterior algebra of an inner product space

V real v.sp. of dimn endowed with an inner product <·,·>. Then all "natural" v.sp. associated with V inherit an inner product from <·,·>. For instance V* has one: if (e₁,...,eₘ) is an orthonormal basis for V, then the dual basis (e₁",...eₘ") is orthonormal for the inner product on V*.

Likewise {eᵢ ∧ eⱼ} is orthonormal for <·,·> on V ⊗ V and

\[ e_I = e_{i₁} \wedge ... \wedge e_{iₙ} \quad (1 ≤ i₁ < ... < iₙ ≤ n) \]

In particular, the generator e₁ ∧ ... ∧ eₘ of Λⁿ V has unit length. Assume now V oriented. Then the (e₁,...,eₘ) orthonormal and oriented.

The generator e₁ ∧ ... ∧ eₘ is then indep. of choices - denote it µ.

We can now define the * operator \( *: \Lambda^k V \longrightarrow \Lambda^{n-k} V \) characterized by \( \alpha \wedge * \beta = \langle \alpha, \beta \rangle \mu \). In coordinates \( *(e_i) = \delta_{m}^{(I)} e_I \), where where \( I^t = 1,...,m \setminus I \).

**Exercise.** Show that \( *: \Lambda^k V \to \Lambda^{n-k} V \) is mult with \( (-1)^{k(m-k)} \).

(1.2) De Rham cohomology

M an m-manifold, \( \mathcal{E}(M) \) the space of \( C^\infty \) k-forms on M,

\[ d: \mathcal{E}^k(M) \rightarrow \mathcal{E}^{k+1}(M) \text{ exterior derivative}. \]

Have dd=0 and get the \( C^\infty \)-De Rham complex \((\mathcal{E}^\infty(M),d)\).

The \( C^\infty \)-forms with compact support define a subcomplex \((\mathcal{E}^\infty_c(M),d)\).

**Note:** the wedge product makes \( \mathcal{E}^\infty(M) \) an algebra. It is in fact a DGA (differential graded algebra): if \( \alpha \in \mathcal{E}^k(M), \beta \in \mathcal{E}^l(M), \) then

\[ d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{k} \alpha \wedge d\beta \]

Moreover \( \mathcal{E}^\infty_c(M) \) is not just a subalgebra, it is also a \( \mathcal{E}^\infty_c(M) \)-module: if \( \alpha \in \mathcal{E}^\infty_c(M), \beta \in \mathcal{E}^\infty_c(M) \), then \( \alpha \wedge \beta \in \mathcal{E}^\infty_c(M) \)

**Def (De Rham cohomology)**

\[ H^p_{DR}(M) := H^p(\mathcal{E}^\infty_c(M),d) \]

\[ H^p_{DR,c}(M) := H^p(\mathcal{E}^\infty_c(M),d) \]

\( H^p_{DR}(M) \) is a graded IR-algebra.

\( H^p_{DR,c}(M) \) is a graded \( H^p_{DR}(M) \)-module.

(Also a graded IR-algebra.)
If $M$ oriented, then we know how to integrate $m$-forms with compact support:

$$\alpha \in \mathcal{E}^m_c(M) \mapsto \int_M \alpha$$

By Stokes' theorem, if $\alpha$ exact: $\alpha = d\beta$ for some $\beta \in \mathcal{E}^m_c(M)$, then

$$\int_M \alpha = \int_M d\beta = 0$$

and so $\int_M$ induces $H^*_\text{DR,}c(M) \to \mathbb{R}$.

This enables us to define the pairing

$$H^k_{\text{DR}}(M) \times H^{m-k}_{\text{DR,}c}(M) \to \mathbb{R}, \quad ([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta$$

From now assume $M$ is of finite type, meaning $M$ different from the interior of a compact manifold with boundary.

**Theorem (De Rham)** Then $H^*_{\text{DR}}(M)$ and $H^*_{\text{DR,}c}(M)$ are finite dimensional and the duality pairing is perfect; the associated map $H^{m-k}_{\text{DR,}c}(M) \to H^k_{\text{DR}}(M)^*$ is an isomorphism.

When $M$ compact, $H^{m-k}_{\text{DR,}c}(M) = H^{m-k}_{\text{DR}}(M)$ and then this is Poincaré duality.

If $f: M \to N$ is a $C^\infty$ map of manifolds, then $f$ induces

$$f^*: \mathcal{E}^*(N) \to \mathcal{E}^*(M)$$

(morphism of DGA's) and hence $H^*_f: H^*_\text{DR}(M) \to H^*_\text{DR}(N)$ (homomorphism of graded algebras). If $f$ is an open embedding, then also

$$f_*: \mathcal{E}^*_c(M) \to \mathcal{E}^*_c(N)$$

and $H^*_{\text{DR,}c}(f): H^*_{\text{DR,}c}(M) \to H^*_{\text{DR,}c}(N)$.

**1.3 Harmonic representation.**

Assume $M$ oriented and endowed with a Riemann metric $g$.

Then there $*: \mathcal{E}^k_c(M) \to \mathcal{E}^k_c(M)$, for every $\alpha \in \mathcal{E}^k_c(M)$,

$$\alpha \mapsto \mathcal{L} \alpha$$

and so $\int_M \alpha \wedge \alpha > 0$ with equality if and only if

This makes $\mathcal{E}^k_c(M)$ a real Hilbert space with inner product

$$\langle \alpha, \beta \rangle_M = \int_M \alpha \wedge \beta$$

Remark: Its Hilbert completion, sometimes denoted $L^2_c(M)$ can be understood as the space of square integrable forms on $M$; there are $k$-forms which...
In local coordinates have \( L_z \)-functions as coefficients.

Let \( a \in E^k_c(M), b \in E^k_c(M) \). So \( d(a \wedge b) = da \wedge b + (-1)^{k+1}a \wedge db \).

\[
\langle d a, b \rangle_M = \int_M a \wedge b = \int_M a \wedge b - (-1)^{k+1} \int_M a \wedge db
\]

(we used exercise)

If \( M \) compact, then at least formally (these are not Hilbert spaces), the adjoint of \( d : E^k_c(M) \rightarrow E^{k-1}_c(M) \) is given by

\[
-(-1)^{k+1} \ast d : E^k_c(M) \rightarrow E^{k-1}_c(M).
\]

(Note that when even this is just \( -\ast d \ast \). This is why \( \ast \) is called the formal adjoint of \( d \); we denote it \( d^\ast \), even if \( M \) is noncompact. Note that \( d^\ast d^\ast = 0 \).

**Lemma 1.** Suppose \( E', \varphi, E, \chi, E' \) are continuous maps of Hilbert spaces with \( \varphi \chi = 0 \). If \( E \overset{\varphi}{\leftarrow} E', \chi \overset{\chi}{\leftarrow} E'' \) are their adjoints

\[
(*) \quad \text{Ker}(\varphi^\ast \varphi + \chi \chi^\ast) = \text{Ker}(\varphi) \cap \varphi(\text{Im}(\chi))
\]

in particular \( \text{Ker}(\varphi^\ast \varphi + \chi \chi^\ast) \) maps onto \( \text{Ker}(\varphi)/\text{Im}(\chi) \).

**Proof.** First observe that \( \text{Ker}(\varphi) = \text{Im}(\varphi)^\perp \) (Ku follows from

\[
\langle \varphi(a), \chi(b) \rangle = \langle a, (\chi^\ast \varphi)(b) \rangle. 
\]

Let \( a, b \in E \). Then

\[
\langle (\varphi^\ast \varphi + \chi \chi^\ast)(a), b \rangle_E = \langle \varphi^\ast \varphi(a), b \rangle_E + \langle \chi \chi^\ast(a), b \rangle_E
\]

\[
= \langle \varphi(a), \varphi(b) \rangle_E + \langle \varphi^\ast(a), \chi(b) \rangle_E.
\]

So if \( a \in \text{Ker}(\varphi), a \perp \text{Im}(\varphi) \perp \text{Ker}(\varphi^\ast) \), then \( \text{Ker}(\varphi) \cap \text{Im}(\chi) = \text{Ker}(\varphi^\ast) \).

Conversely, if \( (\varphi^\ast \varphi + \chi \chi^\ast)(a) = 0 \), then take \( b = a \) and get

\[
0 = \langle \varphi(a), \varphi(a) \rangle_E + \langle \varphi^\ast(a), \chi(a) \rangle_E;
\]

and hence \( \varphi(a) = 0, \varphi^\ast(a) = 0 (\Rightarrow a \perp \text{Im}(\varphi)) \).

**Remark:** If \( \varphi, \chi \) are just linear maps between inner product spaces (pre-Hilbert spaces) which have adjoints, then \( (*) \) still holds, but \( \text{Ker}(\varphi^\ast \varphi + \chi \chi^\ast) \rightarrow \text{Ker}(\varphi)/\text{Im}(\varphi) \) will be only an injection.
This suggests that we introduce the operator
\[ \Delta := dd^* + d^*d : \mathcal{E}^*(M) \to \mathcal{E}^*(M) \]
It is called the Laplacian and preserves degree.

**Exercise.** Take \( M = \mathbb{R}^m \) with standard inner product regarded as a Riemann metric. Give \( \Delta \) in coordinates.

**Theorem (De Rham, Hodge).** Assume \( M \) compact (and oriented as before). Then
\[ H^k(M) := \ker (\Delta : \mathcal{E}^k(M) \to \mathcal{E}^k(M)) \]
is the orthogonal complement of the space of exact \( p \)-forms in the space of closed \( p \)-forms and maps isomorphically onto \( H^0_{DR}(M) \). We call \( H^k(M) \) the space of harmonic \( p \)-forms on \( M \).

**Rk:** By the remark following Lemma 1, the first part of the claim and the injectivity of \( H^k(M) \subseteq H^0_{DR}(M) \) is essentially linear algebra. What is nontrivial is surjectivity.

**Rk.** A wedge product of harmonic forms need not be harmonic: \( H^k(M) \) is not in general a subalgebra of \( H^0_{DR}(M) \).

Note that the group of isometries of \((M,g)\) will preserve \( H^k(M) \).

We exploit this in the following example.

**Example.** Let \( V \) be a (real) inner product space of dimension \( n \) and regard it as a Riemann manifold with trivial, invariant metric. Let \( \mathcal{L}(V) \) be lattice of rank \( n \). Then \( T := V/\mathcal{L} \) is both a group and a manifold. Since \((V,\mathcal{L}) \cong (\mathbb{R}^n,\mathbb{Z}^n)\), \( T \cong (\mathbb{R}/\mathbb{Z})^n \) as a group manifold (an \( n \)-torus). The metric on \( V \) determines one on \( T \) which is invariant under the translation maps \( x \in T \mapsto x + x_0 \in T \) (\( \forall x_0 \in T \)).

Hence \( H^k(T) \) consists of \( k \)-forms on \( T \) invariant under the translation maps. This means that they are given on \( V \) as \( k \)-forms with constant coefficients: \( H^k(T) \subseteq \Lambda^k V^* \). On the other hand \( d \) kills \( \Lambda^k V^* \) and \( * \) takes \( \Lambda^k V^* \) to \( \Lambda^{n-k} V^* \) and so we have equality: \( H^k(T) = \Lambda^k V^* \).

In this special case, \( H^k(T) = \Lambda^k V^* \) is closed under \( \wedge \).
Let \( T \) be a real vector space of even dimension \( n = 2n \).

A complex structure on \( T \) is a linear map \( J : T \to T \) with \( J^2 = -I_T \). Then we can turn \( T \) into a complex vector space of dimension \( 2n \) by letting \( V^i \) act as \( J^i \).

(Conversely a complex vector space defines a complex structure on the underlying real \( V \).)

Its complexification \( T_{\mathbb{C}} : T_{\mathbb{C}} = T \otimes \mathbb{C} \) has eigenvalues \( \pm V^i \).

In fact for \( v \in T \)

\[
J(v \pm V^i Jv) = Jv \pm V^i J^2 v = \mp V^i v + Jv = \mp V^i (v \mp V^i Jv)
\]

and so the \( V^i \)-eigenpairs of \( J : T_{\mathbb{C}} \) are \( \{ v \pm V^i Jv : v \in T \} \) so that

\[
T_{\mathbb{C}} = T^{1,0} \oplus T^{0,1} \text{ is the eigenspace decomposition of } J \text{ and } T^{0,1} = \overline{T^{1,0}}.
\]

Note that \( T_{\mathbb{C}} \) is a real isomorphism which takes \( J \) to \( J \).

This isomorphism of \( T \) is a real isomorphism of \( T_{\mathbb{C}} \) and \( T \) is \( \mathbb{C} \)-anti-linear:

- it takes \( J \) to \( J \).

We can always find \( e_1, \ldots, e_n \in T \) such that \( e_i, Je_i, \ldots, e_n, J^n \) is a basis of \( T \).

The orientation of \( T \) is defined by this basis only depends on \( J \); we call it the canonical orientation of \( T \).

Note that

\[
(e^v_i = e_v^i - V^i J e_v)^n \text{ basis of } T^{1,0}, \\
(e^v_i = e_v^i + V^i J e_v)^n \text{ basis of } T^{0,1}.
\]

We do the same for the dual of \( T \): we take as complex structure on \( T^* \) the operator \( J^* \) (we could also have taken \( (J^*)^* = -J^* \), but this is better).

So we have

\[
T^*_{\mathbb{C}} = T^{1,0} \oplus T^{0,1}
\]

Via the isomorphism \( T^*_{\mathbb{C}} = \text{Hom}_{\mathbb{C}}(T, \mathbb{C}) \), we have

\[
T^{1,0} = \{ z \in \text{Hom}_{\mathbb{C}}(T, \mathbb{C}) : z J(v) = V^i z(v) \quad \forall v \in T \} \text{ complex linear maps} \\
T^{0,1} = \{ z \in \text{anti-linear maps} \} \\
T^{1,0} \otimes \mathbb{C} = \text{anti-linear maps}
\]
If $e_1, \ldots, e_n$ is as above, then the basis dual to $e_1, \ldots, e_n$ is $e_1^*, \ldots, e_n^*, (J e_1)^*, \ldots, (J e_n)^*$. We have $(J e_i)^* = -J^* e_i^*$ and hence a $C$-basis for $T^{*1,0}$ is $\{J e_i - \bar{J} J^* e_i = e_i^* + \bar{J} (J e_i)^* \}_{i=1}^n$. The latter is dual to the $C$-basis $\{e_i = J e_i - \bar{J}^* (J e_i)^*\}_{i=1}^n$ of $T^{1,0}$.

The exterior algebra of $T$, $\Lambda^* T$ also inherits a complex structure by $J (\vee \lambda_1 \wedge \cdots \lambda_r) = J \lambda_1 \wedge \cdots J \lambda_r$.

Since $T_c = T^{1,0} \otimes T^{0,1}$ we have

$$\left(\Lambda^* T\right)_c = \Lambda^* T_c = \left(\Lambda^c (T^{1,0} \otimes T^{0,1}) \right) \cong \left(\Lambda^c T^{1,0}\right) \otimes \left(\Lambda^c T^{0,1}\right)$$

Then for $I = 1 \leq i_1 < \cdots < i_p \leq n$ $K = 1 \leq k_1 < \cdots < k_q \leq n$

$$\Sigma_{I,K} := e_{i_1}^* \wedge \cdots \wedge e_{i_p}^* \wedge \bar{e}_{k_1} \wedge \cdots \wedge \bar{e}_{k_q}$$

is in $T^{p,q}$ and such elts make up a $C$-basis for $T^{p,q}$ (so that $T^{p,q} = \binom{n}{q}(\mathbb{C})$).

Same story for $T^*_c$:

$$\left(\Lambda^* T^*\right)_c = \Lambda^* T^*_c = \bigotimes_{p+q=1} T^{p,q}$$

We can regard $T^{p,q}$ as the $C$-dual of $T^{p,q}$, or equivalently as the space of $\mathbb{R}$-multi-linear maps

$$f : T^\times T^\times \cdots \times T^\times T^\times \rightarrow \mathbb{C}$$

which are alternating and $C$-linear in the first $p$ and alternating and $C$-anti-linear in the last $q$ variables.

(2.2) Complex manifolds.

Let $M$ be a $2n$-manifold. An almost complex structure on $M$ is a $C^\infty$-endomorphism $J$ of its tangent bundle with $J^2 = -1$. This also defines a complex structure $J^*$ on its cotangent bundle; we use this to orient the cotangent bundle of $M$ and thereby $M$.

The preceding shows that $\mathcal{E}_c^k(M) = \mathcal{E}_c^k(M) + \sqrt{-1} \mathcal{E}_c^k(M)$ decomposes as
\[ E^k_c(M) = \bigoplus_{pq=k} E^p_c(M) \]

The (complexified) differential \( d : E^k_c(M) \to E^{k+1}_c(M) \) will not in general respect the bigrading, so a priori

\[ d : E^1(M) \to E^2(M) \]

but we will see that on a complex manifold the last component is zero.

**Example:** \( \mathcal{U} \) open part of \( \mathbb{C}^n \). Then \( T_x \mathcal{U} \cong \mathbb{C}^n \) so has obvious complex structure. If \( (z_1, \ldots, z_n) = (x + iy) \), the coordinates for \( \mathbb{C}^n \), then \( \mathcal{T}_x \mathcal{U} \) has the basis

\[ \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\} \]

so \( J^* dx = dx \), \( J^* dy = -dy \)

We put \( \frac{\partial}{\partial x} := \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \) ; \( \frac{\partial}{\partial y} := \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \)

These make up a basis of \( T_x^\mathbb{C} \mathcal{U} \). Hence

\[ dx_1 - V^1 J^* dx = dx_1 + i dy = d(x + iy) = d\zeta \]

rep.

\[ dx_1 + i J^* dy = dx_1 - i dy = d(x - iy) = d\bar{\zeta} \]

is a basis of \( E^1(\mathcal{U}) \) resp. \( E^0(\mathcal{U}) \) as \( E^0(\mathcal{U}) \)-module, dual to these bases, which consists of smooth forms. This yields for every \( (p,q) \) the basis

\[ \{d\zeta^I \wedge d\bar{\zeta}^J : I, J \subseteq \{1, \ldots, n\}, |I|=p, |J|=q \} \]

of \( E^{p,q}(\mathcal{U}) \) which also consists of smooth forms. The orientation is characterized by the fact that an \( n \)-form \( d\zeta^1 \wedge \cdots \wedge d\zeta^n \) not differentiable is positive. If \( f : \mathcal{U} \to \mathbb{C} \), then

\[ df = \sum_{\nu=1}^n \frac{\partial f}{\partial \zeta^\nu} d\zeta^\nu + \sum_{\nu=1}^n \frac{\partial f}{\partial \bar{\zeta}^\nu} d\bar{\zeta}^\nu \]

The identity \( df = 0 \) amounts to \( \frac{\partial f}{\partial \zeta^\nu} = 0 \) \((\nu=1, \ldots, n)\) and this just means that \( f \) obeys the Cauchy–Riemann eqns. So \( df = 0 \Leftrightarrow f \) holomorphic.

Back to \( (M, J) \). We define \( \Theta : E^0_c(M) \to E^1_c(M) \), \( \bar{\Theta} : E^0_c(M) \to E^0_c(M) \) by

\[ d = (\partial, \bar{\partial}) : E^0_c(M) \to E^1_c(M) = E^1_c(M) \oplus E^0_c(M) \]

The above observation suggests to say that \( M \stackrel{f}{\to} \mathbb{C} \) is holomorphic if \( df = 0 \). We write \( \Theta(U) = \{ \text{holomorphic functions on } U \} \) (a \( \mathbb{C} \)-algebra).
Def. We say that a chart \( M \times \mathbb{C}^n \) is holomorphic if each \( \overline{z} \) is holom. If \( M \) admits an atlas of holomorphic charts, we say that \( M \) is a \textbf{complex manifold} (and that \( J \) is \textbf{integrable}).

A \( C^\infty \)-map \( f : M \to N \) of complex manifolds is called \textbf{holomorphic} if composition with \( f \) takes a holom. function on an open \( \mathcal{V} \subset N \) to a holom. function on \( f^{-1}\mathcal{V} \). This defines the \textbf{holomorphic category}.

If \( M \) is a complex manifold, then a \textbf{chart} \( (M \times \mathbb{C}^n) \) defines a \textbf{bigraded basis} \( \{\overline{z}_I, dz_I \} \) of exact forms of \( \mathcal{E}^* \) as a \( \mathcal{O}(\mathcal{U}) \)-module. Since \( d(\overline{z}_I \wedge dz_I) = (\overline{z}_I + \overline{z}_I) \wedge dz_I \), we find that \( d \) maps \( \mathcal{E}^{0,0}(M) \) to \( \mathcal{E}^{0,1}(M) \oplus \mathcal{E}^{1,0}(M) \). We denote the first component \( \partial \) and the second component \( \bar{\partial} \).

Remark: According to Neulander-Nirenberg, a converse also holds: if \( d \) takes \( \mathcal{E}^{0,0}(M) \) to \( \mathcal{E}^{0,1}(M) \oplus \mathcal{E}^{1,0}(M) \), then \( J \) is integrable (and makes \( M \) a complex manifold).

\textbf{Corollary}. If \( M \) is a complex manifold, then \( d = \partial + \bar{\partial} \) with \( \partial \) resp. \( \bar{\partial} \) \textbf{bidegree} \((1,0)\), \((0,1)\) and we have \( \partial^2 = 0 \), \( \bar{\partial}^2 = 0 \), \( \partial \bar{\partial} + \bar{\partial} \partial = 0 \). Moreover \( \bar{\partial} \) is linear over the holomorphic functions.

\textbf{Proof.} \( d^2 = (\partial + \bar{\partial})^2 = \partial^2 + \bar{\partial}^2 + \partial \bar{\partial} + \bar{\partial} \partial \) and so

\[ \mathbb{E}^2 = (\partial + \bar{\partial})^2 = \partial^2 + \bar{\partial}^2 + \partial \bar{\partial} + \bar{\partial} \partial \]

\( \partial^2 = 0 \), \( \bar{\partial}^2 = 0 \), \( \bar{\partial} \partial + \partial \bar{\partial} = 0 \). If \( f \) holom. and \( \alpha \in \mathcal{E}^\infty \), then

\[ \bar{\partial}(f\alpha) = \bar{\partial} f \wedge \alpha + f \bar{\partial} \alpha = f \bar{\partial} \alpha. \]

\textbf{Remark.} The bar is appropriate: \( \bar{\partial} \alpha = \bar{\partial} \alpha \). It is clear that \( d \) is a \textbf{real operator}: \( \partial \bar{\partial} = 0 \). Another linear combination of \( \partial \) and \( \bar{\partial} \) that is \textbf{real} is

\[ \partial^e := \nabla^{-1}(\partial - \bar{\partial}) = \frac{\partial - \bar{\partial}}{\sqrt{2}} \]

For \( \partial^e \partial^e = -\nabla^{-1} (\partial - \bar{\partial}) = \partial^2 \). It is a \textbf{differential}:

\[ \partial^e \partial^e = (\partial - \bar{\partial})^2 = \partial^2 - (\partial \bar{\partial} + \bar{\partial} \partial) + \partial^2 = 0 \]

It also gives rise to a \textbf{2nd order differential operator}:

\[ \partial^e \partial^e = (\partial - \bar{\partial}) \cdot \nabla^{-1} (\partial - \bar{\partial}) = 2\nabla^{-1} \partial \bar{\partial} = -\partial^2. \]
In terms of a diagram chart \( \mathcal{M}^2 \to \mathbb{C}^n \), we have for \( \phi \in \mathcal{E}_c(\mathbb{C}) \):

\[
\begin{align*}
\dd \phi &= 2 \imath \partial \bar{\partial} \phi = 2 \imath \partial \sum_{\mu, \nu} \left( \frac{\partial^2 \phi}{\partial z^\mu \partial \bar{z}^\nu} - \frac{\partial^2 \phi}{\partial z^\nu \partial \bar{z}^\mu} \right) dz^\mu \wedge d\bar{z}^\nu \\
&= 2 \imath \partial \sum_{\mu, \nu} \left( \frac{\partial^2 \phi}{\partial z^\mu \partial \bar{z}^\nu} - \frac{\partial^2 \phi}{\partial z^\nu \partial \bar{z}^\mu} \right) dz^\mu \wedge d\bar{z}^\nu \\
&= 2 \imath \partial \left( \frac{\partial^2 \phi}{\partial z^\mu \partial \bar{z}^\nu} + \frac{\partial^2 \phi}{\partial z^\nu \partial \bar{z}^\mu} \right) dz^\mu \wedge d\bar{z}^\nu.
\end{align*}
\]

The classical Poincaré lemma asserts that the De Rham cohomology of

of the open hypercube \((0,1)^m\) is concentrated in degree 0 and in that degree equal
to \(k + 1\) so if \( \alpha \in \mathbb{E}^k((0,1)^m) \), \(k > 0\) is closed, then \( \alpha \) is exact.

It's proof is elementary and uses partial integration. A variation
of this argument (which is less elementary, but not too difficult) gives a
similar result for \( \partial \) and \( \bar{\partial} \): Let \( \Delta = \{ z \in \mathbb{C} : |z| < 1 \} \)

Theorem (Delhanty-Thirring Lemma). The cohomology of the complex

\[
\begin{align*}
0 &\to \mathbb{E}_c^0(\Delta^n) \xrightarrow{\bar{\partial}} \mathbb{E}_c^1(\Delta^n) \xrightarrow{\partial} \cdots \xrightarrow{\bar{\partial}} \mathbb{E}_c^k(\Delta^n) \xrightarrow{\partial} 0
\end{align*}
\]

is concentrated in degree zero and in that degree equal to the space \( \mathcal{O}(\Delta^n) \) of
holomorphic functions on \( \Delta^n \).

We omit the proof (see for instance Griffiths-Harris, p.25)

Note that \((\ast)\) is a complex of \( \mathcal{O}(\Delta^n) \)-modules (each \( \bar{\partial} \) is \( \mathcal{O}(\Delta^n) \)-linear).
Complex conjugation gives a corresponding result for \( \partial \) (we get \( \overline{\mathcal{O}(\Delta^n)} \) in degree zero and
the complex is one of \( \overline{\mathcal{O}(\Delta^n)} \)-modules)

(2.3) Holomorphic vector bundles. The \( \bar{\partial} \)-operator also manifests itself on vector bundles

Recall that a complex vector bundle of rank \( r \) on the \( \mathbb{C}^n \)-category is a sht map

\[
\begin{align*}
\xi &\colon E \to \mathbb{C}^n \text{ of } \mathbb{C}^n \text{-manifolds whose fibers are endowed with the structure of } r \text{-dim complex vector spaces admitting local trivializations: } E|_U \cong U \times \mathbb{C}^r \\
\end{align*}
\]

For \( U \subset \mathbb{C}^n \) open, the space of \( \mathbb{C}^\infty \)-sections of \( \xi|_U \) is a \( \mathbb{E}_c^0(U) \)-module that we denote by \( \mathbb{E}_c(U, \xi) \).

This has a holom. counterpart: let \( \xi \colon \mathbb{C}^r \to M \) be a complex vector bundle
over the complex manifold \( M \). A hol. structure on \( \xi \) is given by a
complex structure on \( E \) for which the projection is holom. and holom.
local trivial exist. We then write \( \Omega(U, \mathcal{E}) \subset \mathcal{E}(U, \mathcal{E}) \) for the set of holomorphic sections over \( U \) (this is \( \mathcal{O}(U) \)-module). The standard constructions \( \mathfrak{e}, \mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3 \) take holomorphic vector bundles to holomorphic vector bundles, but the passage to the complex conjugate bundle (replaces \( \mathfrak{e}_1 \) by \( \mathfrak{e}_1^* \) by \( \mathfrak{e}_1^* \)) does not.

**Example.** For the complex manifold \( M \) its holomorphic tangent bundle \( T^h_M = \mathcal{T}^h_M \) is a holomorphic vector bundle. Hence so are its dual \( T^*_hM \) and its exterior powers \( \bigwedge^p \mathcal{T}^h_M = \bigwedge^p T^*_hM \).

In terms of a holomorphic chart \( (U, z^1, \ldots, z^n) \) we have the \( \mathcal{O}(U) \)-bases
\[
\left\{ \partial / \partial z^j \right\}_{1 \leq j \leq n} \text{ and these transform holomorphically under a transition.}
\]
We often write \( \Omega^p(U) \) for \( \mathcal{O}(U, \bigwedge^p \mathcal{T}^h_M) \).

Suppose \( \xi : E \to \mathcal{T}^h_M \) a holomorphic vector bundle. Let \( U \subset M \) be such that \( \xi|_U \) is trivial

as a holomorphic bundle. So we have a basis \( \xi_1, \ldots, \xi_r \) of \( \xi|_U \).

This is then also a basis of \( \mathcal{E}(U, \xi) \) as a \( \mathcal{O}(U, \mathcal{T}^h_M) \)-module. So

\[
\mathcal{E}(U, \xi) \cong \mathcal{E}^0(U) \otimes \mathcal{O}(U, \xi).
\]

This generalizes: holomorphic vector bundle \( \mathcal{T}^h_M \) is not holomorphic.

\[
\mathcal{E}^{p,q}(U, \xi) = \mathcal{E}^{p,q}(U, \bigwedge^p \mathcal{T}^h_M \otimes \bigwedge^q \mathcal{T}^h_M)
\]

called the space of \( (p,q) \)-forms on \( U \) with values in \( \xi \); we have

\[
\mathcal{E}^{p,q}(U, \xi) \subset \mathcal{E}^{p,q}(U) \otimes \mathcal{O}(U, \xi)
\]

Since \( \overline{\delta} \) is \( \mathcal{O}(U) \)-linear, this allows us to define

\[
\overline{\delta} : \mathcal{E}^{p,q}(U, \xi) \longrightarrow \mathcal{E}^{p,q+1}(U, \xi)
\]

locally:

\[
\delta : \mathcal{E}^{p,q}(U, \xi) \otimes \mathcal{O}(U, \xi) \longrightarrow \mathcal{E}^{p,q+1}(U, \xi) \otimes \mathcal{O}(U, \xi)
\]

Corollary 5. We have \( \overline{\delta} \circ \delta = 0 \) so that we have complex

\[
\mathcal{O} \longrightarrow \mathcal{E}^{0,0}(U, \xi) \overline{\delta} \mathcal{E}^{0,1}(U, \xi) \overline{\delta} \cdots \overline{\delta} \mathcal{E}^{0,n}(U, \xi) \longrightarrow 0
\]

Its cohomology zero is \( \mathcal{O}(U, \xi \otimes \bigwedge^p \mathcal{T}^h_M) \) and if \( U = U \mathcal{M} \), then the complex is exact in degrees \( > 0 \).
We define the Dolbeault cohomology of $\xi$

$$H^q(M,\xi) := H^q\left( \mathcal{E}^{p,q}(M,\xi), \overline{\partial} \right)$$

and write

$$H^{p,q}(M,\xi) \quad \text{for} \quad H^q(M,\xi \otimes \Lambda^p T^* M) = H^q\left( \mathcal{E}^{p,q}(M,\xi), \overline{\partial} \right)$$

$$H^{p,q}(M) \quad \text{for} \quad H^q(M,\Lambda^p T^* M) = H^q\left( \mathcal{E}^{p,q}(M), \overline{\partial} \right)$$

This is a sort of analogue of De Rham cohomology.

Theorem 6 For $M$ compact, $H^{q}(M,\xi)$ is a finite dimensional complex vector space. We omit the proof (see e.g. Wells), but will say more about it later.

(2.4) Hermitian forms

Let $T$ be a finite dimensional real v.s. of dimen 2n endowed with a complex structure. Suppose we are given a bilinear form $S: T \times T \to \mathbb{K}$. Put $A(v,v') := S(v,Jv')$ (so that $S(v,v') = -A(v,Jv')$ and $H(v,v) := S(v,v) + V^{-1} A(v,v)$. The following lemma is an exercise.

Lemma 7 The following are equivalent

(i) $S$ is symmetric and $J$-invariant: $S(Jv,Jv') = S(v,v')$

(ii) $A$ is antisymmetric and $J$-invariant:

(iii) $A$ is antisymmetric and is of type $(1,1)$ when considered in $\Lambda^2 T^*$,

(iv) $H$ is a hermitian form rel. to $J$: $H(Jv,v') = \overline{H(v,v')}$ and $H(v,v) = H(v,v')$ (so that $H(v,Jv) = -\overline{H(v,v')}$).

In terms of a coordinate basis $(z_1, \ldots, z_n)$ of $T$: $H(v,v') = \sum_{\mu,\nu} h_{\mu\nu} z_\mu v(\nu)$

$\overline{h}_{\mu\nu} = h_{\nu\mu}$. (We then write $H = \sum_{\mu,\nu} h_{\mu\nu} z_\mu z_\nu$) In that case

$$A(v,v') = \text{Im} H(v,v) = \frac{i}{2^{n-1}} \sum_{\mu,\nu} \left( h_{\mu\nu} z_\mu (v) \overline{z}_\nu (w) - \overline{h}_{\mu\nu} \overline{v}_\mu z_\nu (w) \right)$$

$$= \frac{i}{2^{n-1}} \sum_{\mu,\nu} \left( h_{\mu\nu} z_\mu (w) \overline{z}_\nu (w) - h_{\nu\mu} \overline{z}_\mu (w) z_\nu (w) \right) = \frac{i}{2^{n-1}} \sum_{\mu,\nu} h_{\mu\nu} z_\mu \overline{z}_\nu (v\circ w-v\circ w).$$

The $(1,1)$-form associated to this is then $\frac{i}{2^{n-1}} \sum_{\mu,\nu} h_{\mu\nu} z_\mu \overline{z}_\nu$ for $\overline{z}_\mu z_\nu$ or $\overline{z}_\mu \overline{z}_\nu$, depending on convention; we choose the former: $\Lambda^2 T^*$ is the quotient of $T^* \otimes T^*$ by the subspace spanned by the squares $x \otimes x$ ($x \in T^*$) i.e., the annihilator in $\mathbb{K} T^*$ of this subspace is the span of $v \circ w$ ($v,w \in T^*$), i.e., the space of antisymmetric tensors. So $\Lambda^2 T^*$ is then the space of linear forms on that subspace. In any case, for
for what follows minus this form is more relevant; we denote it by
\[ \Omega = \frac{1}{2} \sum_{\mu, \nu} h_{\mu \nu} \partial \bar{z}_\mu \wedge \partial \bar{z}_\nu. \]

Via this formalism every real (1,1)-form \( \omega \) on the complex manifold \( M \) determines a hermitian form \( h_\omega \) on the tangent bundle of \( M \) for which \(-\text{Im}(h_\omega)\) defines \( \omega \). In particular:

**Lemma.** With a \( C^\infty \) function \( f : M \to \mathbb{R} \) is associated a hermitian form \( h_f \in \mathcal{T}^2(M) \), the so-called Levi form of \( f \) which is characterized by the fact that the (1,1)-form associated to \(-\text{Im}(h_f)\) is \( \frac{i}{2} \partial \bar{\partial} f \). In terms of local coordinates \((z_1, \ldots, z_n)\):

\[ h_f = \sum_{\mu, \nu} \frac{\partial f}{\partial z_\mu} \partial \bar{z}_\mu \wedge \partial \bar{z}_\nu \]

(so here \( h_f = h_\omega \), where \( \omega = \frac{i}{2} \partial \bar{\partial} f \)).

If \( f : N \to M \) is a holomorphic map of complex manifolds, then \( h_f \circ f = f^* h_f \).

**Proof.** Only the formula for \( h_f \) needs to be checked. Indeed:

\[ \text{Im}(h_f) \equiv \frac{1}{2n-1} \sum_{\mu, \nu} \frac{\partial^2 f}{\partial z_\mu \partial \bar{z}_\nu} \partial z_\mu \wedge \partial \bar{z}_\nu \]

\[ = - \frac{2n-1}{4} \sum_{\mu, \nu} \frac{\partial^2 f}{\partial z_\mu \partial \bar{z}_\nu} \partial z_\mu \wedge \partial \bar{z}_\nu = -\frac{i}{4} \partial \bar{\partial} f. \]

We say that \( f \) is **strictly pseudo-convex** if its Levi-form \( h_f \) is positive definite.

In restriction to any complex submanifold of \( M \) it then also has this property.

**Example:** \( \varphi = \sum_{\mu} |z_\mu|^2 \) is strictly ps. convex for \( h_\varphi = \sum \partial z_\mu \wedge \partial \bar{z}_\mu \) is positive definite.

**Remark.** Complex manifolds \( M \) which admit a proper function \( \varphi : M \to \mathbb{R}_{>0} \) that is strictly pseudo convex are called **Stein manifolds**. These behave like smooth affine varieties. It follows from the above that every closed submanifold of \( \mathbb{C}^n \) is a Stein manifold.

**Exercise:** Prove that a Levi pseudo-convex function which is also a Morse function has at every critical point Morse index \( \leq n \).
It is clear that a \((1,1)\)-form which is locally \(dd^c\) of a function will be \(d\)-closed. The converse is also true:

**Lemma 10.** Let \(W\) be \((1,1)\)-form on \(\Delta^n\) which is \(d\)-closed. Then there exists a \(C^\infty\)-function \(\psi : \Delta^n \to \mathbb{C}\) such that \(W = dd^c \psi\) and \(\psi\) is unique up to the sum of a holom. and an antiholom. function. If \(W\) is real, then we can take \(\psi\) real and \(\psi\) is then unique up to the real part of a holom. function.

Proof. Since \(\partial W = d\bar{\omega} + \bar{\partial} \omega\), it follows that \(\partial \omega = \bar{\partial} \omega = 0\) and so by Dolbeaull’s Lemma \(\omega = \bar{\Theta} \alpha\) for some \((1,0)\)-form \(\alpha\) on \(\Delta^n\). Then \(\Theta \alpha\) is a \((2,0)\)-form and \(\bar{\Theta} \Theta \alpha = -\bar{\Theta} \Theta \alpha = -\bar{\partial} \omega = 0\), so that \(\Theta \alpha\) is a holom. 2-form. It is also closed and so by the (holom) Poincare lemma (obtained by partial integration), \(\Theta \alpha = \Theta \psi\) for a holol \((1,0)\)-form \(\psi\). Now replace \(\alpha\) by \(\alpha - \bar{\Theta} \Theta \alpha\). We then find that still \(W = \Theta \alpha\), but now also \(\Theta \alpha = 0\). The Dolbeaull lemma implies that \(\alpha = \Theta \psi\) for some \(\psi \in \mathcal{E}^0(\Delta^n)\). So \(\Theta \Theta \psi = -\bar{\Theta} \partial \psi - \bar{\Theta} \alpha = -\bar{\omega}\). Hence \(\omega = dd^c \psi\), \(\psi = \frac{-d\bar{\omega}}{2\pi i}\).

As to uniqueness: assume \(\Theta \Theta \psi = 0\). Then \(\Theta \partial \psi = 0\) and so \(\Theta \psi\) is holom. and \(d\)-closed. By the (holom) Poincare lemma, \(\psi = \Theta \psi\) for some \(\psi \in \mathcal{O}(\Delta^n)\). Since \(\Theta(\psi - \psi) = 0\), it follows from the Dolbeaull lemma that \(\psi - \psi\) is antiholom. So \(\psi = \psi + (\psi - \psi)\) by \(\psi - \psi = 0\).

If \(W\) is real and \(W = dd^c \psi\), then this identity is still valid if we replace \(\psi\) by its real part (i.e., \(dd^c\) is real). The ambiguity is then of the form \(\psi + \bar{\psi}\) with \(\psi\) holomorphic. □

**Lemma 10.** Let \(f\) be strictly pseudo-convex in a nbhd of \(x\in M\). Then there exists a coordinate system \((z_1, \ldots, z_n)\) at \(x\) in which \(f\) takes the form

\[
    f = \sum_{\gamma \geq 4} |z_\mu|^2 + \text{hol} + \text{terms of order } \geq 4.
\]

Proof: The matrix \(\frac{\partial f}{\partial z_\mu}(x)\) represents a pos. def Hermitian form. So after a lin change of coordinates we can arrange this to be the identity matrix. Then \(f\) takes the form

\[
    \sum_{\gamma \geq 4} |z_\mu|^2 + \text{hol} + \text{terms of order } \geq 4.
\]

Now pass to \(z_\mu = \bar{z}_\mu + \text{hol} + \text{terms of order } \geq 4\).
(2.5) Kahler metrics

Let \( g \) be a Riemann metric on \( G \). We say that \( g \) is hermitian if \( J \) is orthogonal for \( g \): \( g(v, v') = g(Jv, Jv') \). Such a metric always exists: if \( g \) is any Riemann metric, then take \( \tilde{g}(v, v') := g(v, v) + g(Jv, Jv') \). A hermitian metric determines a hermitian inner product \( h \) on \( TM \) by \( h(v, v') := g(v, v) + g(Jv, Jv') \).

Def: We say that \( h \) is a Kahler metric if its imaginary part is d-closed when considered as a 2-form. We write \( \omega_h \) for the closed \((1,1)\)-form defined by \(-\text{Im}(h)\).

In view of Lemma 8.2 this is equivalent to the metric being locally given by \(-\frac{i}{2}dd^c\) (strictly ps. convex \( \phi \)). Lemma 9 even allows to take this at a given \( x \in M \) of the form \( \sum_{r=1}^n |z_r|^2 + \sum_{r=1}^n \frac{1}{2}d|z_r|^2 + \sum_{r,s} \mu_{rs} d\overline{z}_r d\overline{z}_s \) with each \( \mu_{rs} = \overline{\phi_{rs}} \) and its partial derivatives vanishing at \( x \). In other words, \( h \) is the standard flat euclidean metric up to 2nd order. As a consequence we have the Euclidean reduction principle: every identity for hermitian metrics which only involves its components and their first partial derivatives is valid whenever \( h \) is valid for the flat standard metric.

Examples

1) Let \( V \) be a complex vector space of dim \( n \), \( L \subset V \) a full lattice, i.e., the \( \mathbb{Z} \)-span of an \( \mathbb{R} \)-basis of \( V \). Then \( T := V/L \) is a torus with a translation invariant complex structure (briefly, a complex torus). Any hermitian inner product \( H : V \times V \to \mathbb{C} \) can be regarded as a translation invariant hermitian inner product \( h \) on the tangent bundle of \( V \). This determines a hermitian inner product on the tangent bundle on \( T \) (also transl. invar.). Both are Kahler metrics.

In this example we can recover the 2-form associated to \( h \) from the function \( v \in V \to H(v, v) \) by applying to \(-\frac{i}{4}d\phi_{rs} \) to it: choose a basis of \( V \) on which \( H \) takes the standard form: \( H(z, \overline{z}'') = \sum_{r=1}^n z_r \overline{z}_r \). Then
Here is a way to produce a Kähler metric on IPV; it is called the Fubini–Study metric (the $\pi$ in the formula is not important here).

Prop. 11 Let $V$ and $H$ be as above. Then the 2-form on $V_{f_0}/\tilde{g}$ defined by

$$-rac{1}{2} \log H(z, \bar{z}) = \frac{1}{2V-1} \Theta \bar{\Theta} \log H(z, \bar{z})$$

is the pull-back under $\pi: V_{f_0}/\tilde{g} \to IPV (= V_{f_0}/\mathbb{C}^x)$ of a symplectic form $\omega$ on IPV for which $h_{\omega}$ is a Kähler metric.

Proof. We check this at a point $p \in IPV$. Choose a coordinate basis $(z_1, \ldots, z_n)$ of $V$ that is orthonormal for $H$ and for which $p$ is given by $[0: \ldots: 0: 1]$. So $H = \overline{z}_i z_i$ and $(\overline{z}_i = \frac{z_i}{2})_{i=1}^n$ is a coordinate system for an affine neighborhood at $p \in IPV$. Now

$$\log H(z, \bar{z}) = \log (\overline{z}_1^{2n} |z_1|^2) = -\log (1 + \sum \frac{|z_i|^2}{2}) + \log |z_1|^2.$$  

Applying $\Theta \bar{\Theta}$ to this kills the last term and hence produces a 2-form that is expressible in terms of the $\overline{z}_i = \frac{z_i}{2}$. We get that $\eta = \omega$ is given by

$$\Theta \bar{\Theta} \log (1 + \sum \frac{|z_i|^2}{2}) = \Theta \left( \frac{\sum \overline{z}_i z_i \bar{d}z_i}{1 + \sum |z_i|^2} \right)$$

$$= \frac{\sum_{i=1}^n z_i \bar{d}z_i \bar{d}\overline{z}_i}{1 + \sum |z_i|^2} - \frac{\sum_{i=1}^n \overline{z}_i \bar{d}z_i \bar{d}\overline{z}_i}{1 + \sum |z_i|^2}.$$  

If we evaluate this at $p$ (defined by $\overline{z}_i = 0, i = 1, \ldots, n$) we get $\sum_{i=1}^n \overline{z}_i \bar{d}z_i \bar{d}\overline{z}_i$ and so associated to $\omega$ is a hamiltonian form $h_0$, which is possessions p. □

We get plenty other examples if we combine this with the observation that if $f: N \to M$ is a holomorphic map of complex manifolds whose derivative is injective everywhere (a "holomorphic immersion"), then a Kähler metric $h$ on $M$ pulls back to a Kähler metric $f^* h$ on $N$. Indeed, $d (\omega_{f^* h}) = f^* d \omega h = 0.$
It follows that every complex submanifold of a complex proj. space admits a Kähler metric. In particular a smooth complex projective variety admits one (by a theorem of Chow, every closed complex submanifold of $\mathbb{P}^n$ has in fact the structure of a complex proj. variety and that structure is unique).

In order to appreciate the power of the Kähler condition we first need to do a bit of linear algebra.

(2.6). The exterior algebra of a hermitian vector space

Here $T$ is a real vector space of dim $2n$ endowed with a complex structure $J: T \rightarrow T$ and a compatible hermitian inner product $<, >: T \times T \rightarrow \mathbb{C}$ (we want to use the letter $H$ for something else). We choose a basis $z_1, \ldots, z_n$ of $(T^*)^n$ on which $<, >$ takes the form $\sum_{v=1}^n z_v \bar{z}_v$. So $\{ z_I \wedge \bar{z}_K : |I|=p, |K|=q \}$ is then a basis of $(T^*)^p q$. If we extend $<, >$ in a natural manner to $(T^*)^p q$, then this basis is orthogonal; it is not orthonormal for $<z_v, z_v> = <x_v, x_v> + <y_v, y_v> = 2$ and so $<z_I \wedge \bar{z}_K, z_I \wedge \bar{z}_K > = 2$. It is logical to extend the $\ast$-operator in an anti-linear manner:

$*: \bigwedge^r_T \rightarrow \bigwedge^{r+1}_T, \quad \alpha^1 + i \alpha^2 \mapsto \ast \alpha^1 - \sqrt{-1} \ast \alpha^2$

for then for $\alpha, \beta \in \bigwedge^r_T$, we have

$\alpha \wedge \ast \beta = <\alpha, \beta > \mu$.

Example: For $n=1$, we have $\ast z = \ast (x + i y) = y - i x = \sqrt{-1} (x - i y) = \sqrt{-1} z$ and likewise $\ast \bar{z} = -\sqrt{-1} \bar{z}$.

This shows that $\ast$ takes $z_I \wedge \bar{z}_K$ to $(\sqrt{-1})^{1|I| - 1|K|} z_I \wedge \bar{z}_K$.

In particular, $\ast$ gives an antilinear isomorphism of $(T^*)^p q$ onto $(T^*)^{n-p, n-q}$.
We denote by $\Omega$ the 2-form of type $(1,1)$ associated to $-\text{Im}(\langle \cdot, \cdot \rangle)$.

So in terms of the basis $(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ of $T^{*1,0}$

$$\Omega = \frac{1}{2^n-1} \sum_{\alpha=0}^{n-1} \mathbf{z}_\alpha \wedge \mathbf{\bar{z}}_\beta = \sum_{\alpha=1}^{r} \mathbf{x}_\alpha \wedge y_\alpha.$$ 

We put

$$L : \wedge^r T^*_C \longrightarrow \wedge^r T^*_C \quad \alpha \longmapsto \Omega \wedge \alpha$$

This is a real operator of bi-degree $(1,1)$ (for $\Omega$ is). Its adjoint

$L^*$, characterized by $\langle L \alpha, \beta \rangle = \langle \alpha, L^* \beta \rangle$, or what amounts to the same, $\Omega \wedge \alpha \wedge \beta = \alpha \wedge (*^r(\beta))$, is given by $L^* = \ast^{-1} L^*$. It is a real operator of bi-degree $(-1,-1)$. So $[L, L^*]$ is a real operator of bi-degree $(0,0)$.

**Lemma 12.** $[L, L^*]$ acts on $\wedge^r T^*_C$ as mult. with $k-n$.

Proof. This is almost an exercise. First do the case $n=1$, then $\wedge^r T^*_C$ has the basis $1, \pi, \nu, \kappa, \lambda$ whose images under $\ast$ are resp. $\nu, \nu, \kappa, -\lambda$, and the lemma is then trivial: $L(1) = \nu, L^*(\nu) = 1$ and $L$ resp. $L^*$ is zero on the other basis elements. We proceed with induction on $n$: for $n \geq 2$, we decompose $T$ orthogonally as

$$T = T_1 \oplus T_2,$$

with $T_1$ spanned by $e_1, j_1$ and $T_2$ spanned by $\{e_v, j_v \mid v = 2\}$. Then $\wedge^r T^* = \wedge^r T_1^* \otimes \wedge^r T_2^*$ and in terms of a tensor product, $L = L_1 \otimes 1 + 1 \otimes L_2$ and on $\wedge^r T_1^* \otimes \wedge^r T_2^*$ we have $\ast = (-)^{st} \ast_1 \otimes \ast_2$. This implies that $L^* = L_1^* \otimes 1 + 1 \otimes L_2^*$ and hence

$$[L, L^*] = [L_1, L_1^*] \otimes 1 + 1 \otimes [L_2, L_2^*].$$

Our induction hypothesis implies that the last operator acts on

$\wedge^r T_1^* \otimes \wedge^r T_2^*$ as mult with $(6-1) + (4-6) = 3 + t - n$.\[\square]
Let us now write \( H \) for \([L, L^*] \); so \( H \) is mult with \( r - n \) in degree \( r \). Since \( L \) resp. \( L^* \) is of degree \( 2 \) resp. \(-2 \), it follows that \([H, L] = 2L \) and \([H, L^*] = -2L^* \). This is telling us that we have defined a representation of the Lie algebra \( sl(2) \) by

\[
X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rightarrow L \\
X_- = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow L^* \\
H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow H
\]

This has interesting consequences. Since \( sl(2) \) is a simple Lie algebra, we can decompose this representation into irreducible subrepresentations; the most canonical way to do this is by isogeny type.

(2.7) Review representation theory of \( sl(2) \): A representation of a Lie algebra \( g \) on a vector space \( V \) is called \textit{irreducible} if \( g \) does not hit any subspace \( \neq 0, V \). So this is the case when \( \text{dim} \, V = 1 \), even when this is the trivial representation in the sense that each \( X \in g \) gives the zero map in \( V \). We say that \( g \) is \textit{semi-simple} if every finite dim repr. of \( g \) can be decomposed into irreducible ones. We will see that \( sl(2) \) is semi-simple.

A representation of \( g \) on vector spaces \( V, W \) gives representations on all the natural vector spaces associated with them, like \( V^*, V \oplus W, V \otimes W, \wedge^k V, \text{Sym}^k V \), the representation of \( g \) on \( V^* \) being given by \( X: \varphi \in V^* \rightarrow -X^* \varphi \) and on \( V \otimes W \) being \( X: v \otimes w \rightarrow Xv \otimes w + v \otimes Xw \) (this makes the natural pairing \( V^* \otimes V \rightarrow \text{base field of equivariant} \)). We say that \( V \) is \textit{orthogonal} resp. \textit{symplectic} if \( g \) leaves invariant a nondegenerate symmetric resp. antisymmetric bilinear form \( B: V \times V \rightarrow \mathbb{C} \) in the sense that for all \( v, v' \in V \) \( B(Xv, v') + B(v, Xv') = 0 \). In that case the representation \( V \) is self-dual for the associated map.
$\mathbf{V}_n^{\mathbf{B}^*}, V_n \rightarrow \mathbf{B}(V_n)$ is then isomorphism of representations. This also shows that when $V$ is irreducible and a basis is nondegenerate when nonzero (for $K_{\mathbf{B}}(V^{\mathbf{B}^*}) = \mathbf{B}(V^*)$ is a $\mathbf{B}$-invariant subspace) and that $\mathbf{B}$ is unique up to scalar (for $\mathbf{B}'$ is another such then $\mathbf{B} + \mathbf{B}'$ is either nondegenerate or zero for all $t$ and this can only happen if $\mathbf{B}'$ is a scalar multiple of $\mathbf{B}$).

We develop the representation theory for $\mathbf{sl}(2, \mathbb{R})$, the one for $\mathbf{sl}(2, \mathbb{C})$ is essentially the same and is obtained by "complexification".

Denote the basis elements of $1\mathbb{R}^2$ by $e_+, e_-$ and endow it with the symplectic form defined by $\mathbf{B}(e_+, e_-) = 1$. It is convenient to denote this copy of $1\mathbb{R}^2$ by $V_1$. We think of $\mathbf{sl}(2, \mathbb{R})$ as the Lie algebra of endomorphisms of $V_1$ of trace 0; this is also the Lie algebra which preserves $\mathbf{B}_1$. Write $V_k$ for $\text{Sym}^k V_1$, as a rep. of $\mathbf{sl}(2, \mathbb{R})$ ($k=0, 1, 2, \ldots$; for $k=0$ this gives the trivial rep. of $\text{dim } V$).

Note that $V_k$ is of $\text{dim } k+1$ with basis $e^k_+, e^{k-1}_+, \ldots, e^k_-$. From the def. we see if $p, q, k$,

\[ H(\mathbf{e}_+^k \mathbf{e}_-^q) = (p-q) \mathbf{e}_+^p \mathbf{e}_-^q \]

or

\[ \mathbf{e}_+^p \mathbf{e}_-^q \]

So:

\[
\begin{array}{c}
\mathbf{X}_- & \mathbf{X}_-^{1/2} & \mathbf{X}_-^{1/2} & \mathbf{X}_-^{3/2} & \cdots & \mathbf{X}_-^{k-1/2} & \mathbf{X}_-^{k} \\
0 & e_-^{k-1} & e_-^{k} & e_-^{k+1} & \cdots & e_-^{k+1} & 0 \\
\mathbf{H} & \cup & \mathbf{X}_+^{1/2} & \cup & \mathbf{X}_+^{1/2} & \cup & \mathbf{X}_+^{1} & \cup \\
-\mathbf{H} & \cup & -\mathbf{X}_+^{1/2} & \cup & -(\mathbf{H}-\mathbf{2}) & \cup & -(\mathbf{H}-\mathbf{1}) & \cup \\
-k & (H-2) & k & k & k & k & k & k \\
\end{array}
\]

These formulas show that if $w$ is $V_{k-2}$, then after applying some power of $X_+^{1/2}$ to it we get a nonzero multiple of $e^k_+$ or $e^k_-$. This implies that every $\mathbf{sl}(2)$-invariant subspace of $V_k$ which contains $w$ must be all of $V_k$.

In other words $V_k$ is irreducible. We will see that every irreducible rep. of $\mathbf{sl}(2)$ of dim $k+1$ is isomorphic to $V_k$. Note that $\mathbf{B}_k$ gives rise to a form

\[
\mathbf{B}_k : V_k \otimes V_k \rightarrow \mathbf{C}, \mathbf{B}_k(v_i v_j) := \sum_{s \in S_k} \frac{1}{n!} \mathbf{B}(v_i, v_j)
\]

In coordinates:

\[
\mathbf{B}_k(\mathbf{e}_+^p \mathbf{e}_-^q, \mathbf{e}_+^p \mathbf{e}_-^q) = \begin{cases}
\delta(p, q', (p', q')) & \text{if } (p', q') = (q, p) \\
(-1)^q p! q! & \text{if } (p', q') = (q, p) 
\end{cases}
\]
So $B_k$ is nondegenerate, $(1)^k$-symmetric. It is also $SL(2)$-invariant, so $V_k$ is self-dual.

The diagram also shows that for $p = 0, \ldots, k$, $X^p X^p(e^k) = c_p e^k$ for some $c_p \neq 0$.

Let now $W$ be a finite-dimensional rep. of $sl(2)$. It can be shown that $\mathfrak{h}$ will always act semisimply on $W$, but we make this an assumption as this will be evidently satisfied in the cases we encounter. This means that $W = \bigoplus \lambda \mathcal{W}(\lambda)$, where $\mathcal{W}(\lambda)$ is the $\lambda$-eigenspace of $H$ (also called the weight $\lambda$ subspace of $W$).

The formula $[H, X^\pm] = \pm 2X^\pm$ implies that $X^\pm$ takes $W(\lambda)$ to $W(\lambda \pm 2\lambda)$.

Def. We say that $w \in W$ is primitive of weight $\lambda$ if we $\mathcal{W}(\lambda)$ and $X^- w = 0$. We denote the subspace of such elements by $P^+(w)$.

If we $P^+(w) \neq \{0\}$, then consider the sequence $\{X^k w \in W(\lambda + 2\lambda) \}_{k=0}^{\infty}$. It is clear that for some integer $k \geq 0, X^k w = 0$. A straightforward check (with induction on $r$) shows that

$$X^- X^k w = -(r-1)(\lambda + r) X^r w.$$

So if we take $r = k$, we get $C = (\lambda + r)(\lambda + r) X^k w$. Since $X^- w = 0$, we find $\lambda = -k$, so $\lambda$ is an integer $\leq 0$. We then see that $w$ generates a copy of the representation $V_k$: we obtain an $sl(2)$-equivariant embedding $V_k \hookrightarrow W$ which takes $e^k$ to $w$.

Its image has the basis $w, X^- w, \ldots, X^{-k} w$ (we have $X^- w = 0$).

This also shows that $X^{-k} \lambda = c_p w$ with $c_p \neq 0$ ($p = 0, \ldots, k$).

If we let $w$ run over $P^+(w)$ we obtain a natural $sl(2)$-equivariant embedding

$$V_k \otimes P^+(w) \rightarrow W \quad (k = 0, 1, \ldots)$$

whose image can be identified with $C[X^-] / (X^{-k}) \otimes_\mathbb{C} P^+(w)$.

Prop. 13. The resulting map $\otimes_\mathbb{C} V_k \otimes P^+(w) \rightarrow W$ is an iso of $sl(2)$-representations.

Moreover, $P^+(w) = \{ w \in W(\lambda) : X^k w = 0 \}$ and thus identifies $W$ with

$$\otimes_\mathbb{C} C[X^-] / (X^{-k}) \otimes_\mathbb{C} P^+(w)$$

as a $C[X^-]$-module.

Proof. Let $w \in W(\lambda) \neq \{0\}$ (we rec. primitive). Let $p \geq 0$ be such that $X^p w = 0$. We prove with induction on $p$ that $w \in \bigoplus_{k=0}^p V_k \otimes P^+(w)$. When $p = 0$ this is clear, for then $w \in P^+(w)$. Assume the result for $p > 0$.

Then $w = X^p v$ is primitive of weight $\lambda - 2p$ and so $\lambda - 2p$ is of the form $-k$ with $k$ an integer $\geq 0$. Since $X^p X^w = c_p w$ for some constant $c_p \neq 0$. Let $v = X^p v$, we get $\lambda - 2p = \lambda - 2k$ and $X^p v = 0$. So $v \in \bigoplus_{k=0}^p V_k \otimes P^+(w)$ by our
induction hypothesis. It follows that
\[ v = \chi v^I, \quad \chi = \sum \chi_\alpha \in \mathbb{P}^w(\Omega). \]

It is clear that the sum of the \( \mathbb{P}^w(\Omega) \) is valid. The last assertion of the claim follows from the first. \( \Box \)

Remark. \( \mathbb{P}(2) \) has an automorphism \( \sigma \) defined by \( \sigma x = x^{-1}, \quad \sigma I = -I. \) This has the effect on \( \mathbb{P}(2) \) of exchanging the roles of \( e_+ \) and \( e_- \). For a \( \mathbb{P} \) as above, it means that the primitive part of weight \( -k \) for this \( \mathbb{P} \) is of the form \( \mathbb{P}^w(\Omega) \). Let \( \Omega = \sum \chi \in \mathbb{P}^w(\Omega) \).

2.8 Exterior algebra of a Hermitian vector space (continued)

Let us apply this to the \( \mathbb{P}(2) \)-rep. \( \Omega = \Lambda_+^* \).

Since \( \Lambda \) acts on \( \Lambda_+^* \) as with \( \mathbb{P} \), we have \( \Lambda_+^* = (\Lambda^*)^* \). So
\[ p^k(h_+) = \{ \chi \in \Lambda_+^*: \Omega \wedge \chi = 0 \} \]
and
\[ \Lambda_+ = \Omega^k \wedge \text{maps this space isom. onto a subspace of } \Lambda_+^k \text{.} \]

We have a primitive decomposition
\[ \Lambda_+^* = \bigoplus_{k=0} \Lambda^*_{\Omega k} \mathbb{P}^k(\Lambda_+^*) \Omega \end{equation} \)

Let us make \( \mathbb{P}^k(\Lambda_+^*) \) for \( \mathbb{P}^k(\Lambda_+^*) \) (so \( \Omega \), the kernel of \( \Lambda_+^k \big/ \Lambda_+^k \)). Since \( \Omega \) is real of bidegree \( (1,1) \) it follows that \( \mathbb{P}^k(\Omega) \) is composed \( \mathbb{P}^k(\Omega) = \bigoplus_{\Omega, \Omega = (\Omega^k \wedge \chi)} \mathbb{P}^k(\Omega) \Omega \)

Example. \( \chi = \lambda \Theta \cdots \lambda \chi \) is primitive (of weight \( k \)) for \( \mathbb{P}^k = (\sum \chi \lambda \lambda \cdots \lambda \chi) \mathbb{P}^k \).

By the preceding this implies that \( L^k \chi = \Omega^k \wedge \chi \) is nonzero. Note that \( L^k \chi \) has the same degree (namely \( 2n - k \)) as \( \chi \). These elements are related in a

Simple way: First note that
\[ \chi = (\chi_1 \cdots \chi_k) \cdots (\chi_k \cdots \chi_1) \]
where the sign comes from the permutation \( (\chi_1 \cdots \chi_k \cdots \chi_1) \cdots (\chi_k \cdots \chi_1) \).

Now \( \chi_1 \cdots \chi_k \circ \chi \) and so \( \chi_1 \cdots \chi_k \circ \chi = (\chi_1 \cdots \chi_k) \circ \chi \). Hence
\[ \chi = (\chi_1 \cdots \chi_k) \cdots (\chi_k \cdots \chi_1) \]

Since \( \Omega^k = \sum \chi \wedge (\chi_1 \cdots \chi_k) \cdots (\chi_k \cdots \chi_1) \), it follows that
\[
\ast \alpha = (\varepsilon_1)^{k(\alpha n)/2} J^* \alpha \wedge \bigwedge^{n-k} \frac{L^{n-k}}{(n-k)!} (J^* \alpha).
\]

This is in fact true for all \( \alpha \in P^{-k}(T^*) \). You may verify this by computation, but a better argument is based on the following fact.

\[\text{Remark. Denote by } U(T) \text{ the group unitary transformations of } T \text{ (with respect to the complex structure } J \text{ and the Riemann form } < \cdot, \cdot >). \text{ This group acts on } \wedge^* T^*_C \text{ in a way as to commute with } \ast \text{ and } L. \text{ Hence it commutes with the } \mathfrak{sl}(2) \text{-action. In particular, } U(T) \text{ preserves each } P^k(T^*). \text{ The representations } P^k(T^*), k=0,\ldots, n, \text{ are irreducible; they are the so-called fundamental representations of } U(T).
\]

The maps \( P^k(T^*) \rightarrow \wedge^{2n-k}(T^*) \) defined by \( \ast \) and \( (\varepsilon_1)^{k(kn)/2} \frac{L^{n-k}}{(n-k)!} J^* \) are \( \mathfrak{sl}(2) \)-equivariant. Since they take the same value on \( \alpha \) and \( P^k(T^*) \) is spanned by the \( U(T) \)-orbit of \( \alpha \), they coincide. Since \( \ast \) is the antilinear extension we conclude:

**Corollary 1a.** For \( \alpha \in P^k(T^*) \), we have:

\[\ast \alpha = (\varepsilon_1)^{k(kn)/2} \frac{L^{n-k}}{(n-k)!} J^* \alpha\]

**Remark.** Since \( X_+ \rightarrow L, X_- \rightarrow L^\ast \) it follows that the \( \varepsilon_1 \)-twist of this representation of \( \mathfrak{sl}(2) \) on \( \wedge^* T^* \) is accomplished by the \( \ast \)-operator. This explains more in a priori manner why \( \ast \) preserves the primitive decomposition.

The preceding corollary suggests that we define for \( k=1, \ldots, n \)

\[Q^k : P^k(T^*) \times P^k(T^*) \rightarrow \mathbb{R}\]

characterized by the property that

\[Q^k(\alpha, \beta) \mu = (\varepsilon_1)^{k(kn)/2} \frac{\mu^{n-k}}{\mu^{n-k}} \wedge \alpha \Lambda \beta\]

so that \( Q^k(\alpha, J^* \beta) = \langle \alpha, \beta \rangle \). We denote by
\[ Q^k_c : P^1(T^*E) \times P^1(T^*_C) \rightarrow C \]

This bilinear complexification gives us the following basic properties:

(i) \( Q^k_c \) is an \( P^1(T^*E) \times P^1(T^*_C) \)-symmetric

(ii) \( Q^k_c \) has bidegree \((-k,-k)\) in the sense that \( Q^k_c \mid_\{p^1,q^1\}(T^*E) \times P^1(T^*_C) \)

is zero unless \((p+p',q+q')= (k,k)\)

(iii) \((\alpha, \beta) \in P^1(T^*E) \times P^1(T^*_C) \rightarrow Q^k_c(\alpha, T^*_C(\beta)) \in \langle \alpha, \beta \rangle\) is an inner product

Not that (i) and (ii) imply that \( Q^k_c \mid_\{p^1,q^1\}(T^*E) \times P^1(T^*_C) \) is zero unless \((p,q')= (q,p)\)

in which case this pairing is perfect.

2.9. Local identities on a Kähler manifold

If \( h \) is a hermitian metric on the complex manifold \( M \), then we found that the formal adjoint of

\[ d : \mathcal{E}^{k,1}_C(M) \rightarrow \mathcal{E}^{k}_C(M) \]

is defined by \( d^* = -d^* \). The same is true for \( \bar{d} \) and \( \bar{d}^* \):

\[ \bar{d}^* \alpha = \bar{d} \bar{d}^* \alpha \]

for \( \alpha \) of type \((n,0)\)

\[ \bar{d}^* \alpha = -\bar{d} \alpha \]

is of type \((0,1)\). Similarly, the formal adjoint of \( \bar{d} \) is \( -\bar{d}^* \).

We have corresponding Laplace operators of bidegree \((0,0)\):

\[ \Delta \alpha = \bar{d} \bar{d}^* + \bar{d}^* \bar{d} \]

In general, \( \Delta \) depends on \( h \), does not preserve the grading (it is not sum of operators of bidegree \((i,-i)\) for \( i = 0 \ldots \infty \)), but we will see that this is the case when \( h \) is Kähler.

From now on \( h \) is a Kähler metric. So then \(-\Im(h)\) defines a real closed 2-form \( \omega_h \) of type \((1,1)\). We write \( L \) for the operator \( \omega_h \).

Since \( \omega_h \) is both \( \bar{d} \)-closed and \( \bar{d}^* \)-closed, \( L \) commutes with \( \bar{d} \) and \( \bar{d}^* \) and hence also with \( d \) and \( d^* \).
The usual argument shows that for \( L = \alpha \), we have
\[
\langle L \alpha, \beta \rangle_M = \langle \alpha, *^{-1} L^* \beta \rangle_M = \langle \alpha, L^* \beta \rangle_M
\]
So this does not involve any differentiation; in the exterior algebra of a given cotangent space - it is the operator discussed before.

It is clear that the adjoint \( L^* \) of \( L \) will commute with the adjoint \( \delta^* \) resp. \( \delta \) resp. \( \delta^* \) and hence also with \( \delta^* \) and \( \delta \).

Let us consider another type of commutator \([L, \delta^*]\). It is clear that its bidegree is \((1, 1) + (-1, 0) = (0, 1)\). This is also the bidegree of \( \delta \).

These operators are in fact proportional:

\[
\text{Lemma 15} \quad [L, \delta^*] = \sqrt{-1} \delta
\]

Proof: It is clear that this identity regards a 1st order differential operator and by the Euclidean reduction principle it then suffices to check this for the flat standard metric on an open subset \( U \subset \mathbb{R}^n \). We first do the case \( n=1 \) and then proceed with induction of \( n \) as in Lemma 12.

Since left and right hand side have degree \((0, 1)\) this identity is only nontrivial for forms of bidegree \((0, 1)\) and \((1, 0)\).

Recall that then \(*d\alpha = \sqrt{-1} d_{\alpha} \), \(*d\delta = \sqrt{-1} d_{\delta} \), \(*d = dx \wedge dy = \partial \zeta \partial \zeta\) -2V^{-1}

and \(*T_{\zeta} = \delta \) resp. \(*(-2V^{-1} dx \wedge dy) = 2V^{-1} \). Since \([L, \delta^*]\) and \( \delta \) are of

So if \( f: \mathbb{R}^n \to \mathbb{C} \), then

\[
[L, \delta^*] f = L \delta^* f - \delta^* Lf = \ast \delta \astLf.
\]

We have \( Lf = \int f d\zeta d\bar{\zeta} \), hence \( \ast Lf = \bar{f} \) and so

\[
\ast \delta \ast Lf = \ast \delta \bar{f} = \ast \frac{\partial \bar{f}}{\partial \zeta} d\zeta = \frac{\partial f}{\partial \bar{\zeta}} \sqrt{-1} d\zeta = \sqrt{-1} \bar{\delta} f
\]

Next consider \( f dz \). Then

\[
[L, \delta^*] f dz = L \delta^* (f dz) - \delta^* (L f dz) = \ast \delta \ast (f dz).
\]

We have \( \ast f dz = \sqrt{-1} \bar{f} d\bar{\zeta} \) and so \( \delta \ast f dz = \sqrt{-1} \frac{\partial f}{\partial \bar{\zeta}} d\bar{\zeta} \). Hence
\[
- L \ast \Theta \ast (P(z)) = -L \ast (\nabla_\Theta \frac{\partial P}{\partial z} \wedge dz) = -\frac{\partial P}{\partial z} \ast L \ast (\nabla_1 dz \wedge d\overline{z})
\]
\[
= -\frac{\partial P}{\partial z} \cdot \nabla_1 dz \wedge d\overline{z} = \nabla_1 \Theta \ast (P(z)).
\]
For \( n > 1 \), we decompose the forms according to \( \mathcal{E}(V) = \bigoplus \mathcal{E}(s,t)(V) \) with \( \mathcal{E}(s,t)(V) = \mathcal{E}(V) \otimes (\wedge^s T \otimes \wedge^t \overline{T}) \).

The relevant operators decompose accordingly:
\[
L = \Theta+1 \otimes L > 1, \quad \ast = (\ast)^{s,t} \otimes (\ast)^{s,t}, \quad \text{and the same for } \overline{\ast}.
\]
It follows that
\[
\Theta^\ast = \ast^{-1} \Theta \ast = \Theta^\ast \ast_1 + (\ast)^{s,t} \ast_2^t, \quad \text{and hence on } \mathcal{E}(s,t)(V):
\]
\[
[L, \Theta^\ast] = \left[ L, \Theta^\ast \right] + (\ast)^{s,t} \otimes \left[ L > 1, \ast_2^t \right] = \nabla_1 \overline{\Theta}.
\]
\[
\text{by induction}
\]

We get another such identity by applying complex conjugation:
\[
[L, \overline{\Theta}^\ast] = -\nabla_1 \Theta
\]

It also follows that
\[
[L, d^\ast] = \left[ L, \delta^\ast \right] + \left[ L, \overline{\Theta}^\ast \right] = \nabla_1 \overline{\Theta} - \nabla_1 \Theta = d^\ast
\]
\[
[L, d^\ast] = \nabla_1 \left( \left[ L, \overline{\Theta}^\ast \right] - \left[ L, \delta^\ast \right] \right) = \nabla_1 ( -\nabla_1 \Theta - \nabla_1 \overline{\Theta} ) = d
\]

Remark: By taking complex conjugates of formal adjoints (using the fact that \( (AB)^\ast = B^\ast A^\ast \)), we also find the identities:
\[
[L^\ast, \Theta] = -\nabla_1 \overline{\Theta}^\ast; \quad [L^\ast, d] = [\delta^\ast, L]^\ast = -d^\ast;
\]
\[
[L^\ast, d^\ast] = -\nabla_1 \overline{\Theta}^\ast; \quad [L^\ast, d^\ast] = [d^\ast, L]^\ast = -d^\ast.
\]

Corollary 6. We have \( \Delta = \Delta_\overline{\Theta} = \frac{1}{2} \Delta_d \). In particular, \( \Delta_d \) preserves bidegree (hence commutes with \( \Delta^* \) which acts on \((p,q)\)-forms as mult with \((d_{p,q})^p_q\)). Moreover \( L \) and \( L^* \) commute with \( \Delta_d \).
Proof. We first show that $\Delta_3$ is real and hence equal to $\overline{\Delta_3} = \Delta_3$.

We have

$$\Delta_3 = \partial^* \partial + \partial \partial^* = \nabla^2 [\Delta, \overline{\Delta}] - \nabla^2 [\Delta, \overline{\Delta}].$$

Each of these terms is real (use that $L$ is real and $\partial \partial^* + \partial^* \partial = 0$).

We next prove that $\overline{\partial}$ and $\partial^*$ anticommute:

$$\partial \overline{\partial} + \overline{\partial} \partial = \nabla^2 [\Delta, \overline{\Delta}] - \nabla^2 [\Delta, \overline{\Delta}] = 0.$$

Their complex conjugates $\overline{\partial}$ and $\partial^*$ then also anticommute. So

$$\Delta_3 = \partial \overline{\partial} + \overline{\partial} \partial = \Delta_3 + \Delta_3 = \Delta_3 + \Delta_3.$$

For the last assertion it suffices to show that $\Delta_3$ and $L$ commute:

It then follows that their formal adjoints $\Delta_3^* = \Delta_3$ and $L^*$ also commute.

We compute

$$[L, \Delta_3] = [L, \overline{\partial} \partial^*] + [L, \partial \overline{\partial}^*] = [L, \overline{\partial} \partial^*] + [L, \partial \overline{\partial}^*].$$

For every $x \in M$, the harmonic form $\Delta_3$ on $T^* M$ makes $\Lambda^* T^* M$ a representation of $\text{sl}(2)$. In particular we have a primitive decomposition

$$\Lambda^* T^* M = \bigoplus L^r \mathcal{P}^k (T^* M),$$

then

Its complexification is bigraded.
\[ p^k(M)_x = \bigoplus_{p+q=k} p^{p,q}(M)_x \quad \text{and} \quad \overline{p^{p,q}(M)}_x = p^{q,p}(M)_x. \]

Since \( p^k(M)_x \) varies in a \( C^0 \)-manner with \( x \), we obtain a subbundle \( \pi_t \) of \( \Lambda^t \omega^*_M \). It can be shown that \( \pi_t \) is graded \( \pi^{p,q}_t = \bigoplus_{p+q=n} \pi^{p,q} \) and \( \overline{\pi^{p,q}} = \pi^{q,p} \). Similarly, for the spaces of \( C^\infty \)-sections \( \pi_t \) with compact support \( \pi_t \) that are \( C^\infty \), we get a \( C^\infty \)-section in \( \pi_t \) of degree \( k \), so \( \pi_t \) is an \( \infty \)-dimensional representation of \( \mathfrak{sl}(2) \). We can now also define

\[ \mathcal{Q}_M : \bigoplus_{p} \pi^p_c(M) \times \bigoplus_{p} \pi^p_c(M) \to \mathbb{K} \]

by the condition that for \( \alpha, \beta \in \pi^p_c(M) \)

\[ \mathcal{Q}_M(\alpha, \beta)_t = \int_x (\alpha_{i,j})_t \frac{c^{p,q}}{(q-1)!} x_i y_j \]

This has all the properties of the \( \mathcal{Q} \) defined earlier: it is \((\cdot, \cdot)\)-symmetric and \((\alpha, \beta) \to \mathcal{Q}_M(\alpha, \beta^t)\) defines an inner product, and has bi-degree \((-k, -k)\)

2.9 Main Theorem of Kählerian Hodge Theory: We assume \( M \) compact connected complex manifold endowed with a Kähler metric. We represent

\( H^k(M, \mathbb{R}) \) by the space of harmonic forms \( H^k(M) \) (Thm. 2). Since \( \Delta \) preserves the bigrading, it commutes with \( J \). It also commutes with \( \star \) (for \( \Delta^2 = 0 \) and \( L \) and hence also with \( L \). The following is now immediate:

Theorem 17. The space of harmonic forms \( H^k(M) \) acquires after complexification a bigrading: \( H^k(M) = \bigoplus_{p+q=k} H^{p,q}(M) \) and \( \overline{H^{p,q}(M)} = H^{q,p}(M) \) and \( H^{p,q}(M) \cong H^q(M, \Omega^p_M) \).

Moreover, \( H^k(M) \) is preserved by \( L \) and \( \star \) and thus becomes a \( \mathfrak{sl}(2) \)-representation. So \( p^k(M) = H^k(M) \cap p^k(M) = \ker(L^{n+1} - H^k(M)) \), then we have
a primitive decomposition

$$H' (M) = \bigoplus_{r+k \leq n} L^r P^k (M),$$

$$P^k (M)_{c} = \bigoplus_{p+q = k} P^p q^q (M)$$ with $P^p q^q (M) = P^p q^q (M)$ and $Q^k$ restricted to $P^k (M)$
is $(1)$ symmetric, of bidegree $(-h, k)$ and $(\alpha, \beta) \mapsto Q^k (\alpha, J^h \beta)$
is positive.

Since $H' (M)$ represents $H^\bullet_{DR} (M)$, we want to express this in cohomological terms as much as possible. We first show that the bidegree of $H^\bullet_{DR} (M)$ inherits from $H' (M)$ is independent of $h$.

Let us first note that $J^r p^q (M)$ (with $p+q = k$) is the intersection of $\bigoplus_{r'}^{r} J^{r'} p^q (M)$ and $\bigoplus_{r'}^{r} J^{r'} p^q (M) = \bigoplus_{s} J^s p^q (M)$

So it suffices to show that the image of $\bigoplus_{r'}^{r} J^r p^q (H) \in H^h_{DR} (M; C)$
is independent of $h$.

To this end we make some observations regarding an arbitrary compact complex manifold (so not necessarily Kähler). We define the Hodge filtration

on the De Rham complex by

$$F^r E^k (M) = \bigoplus_{r' \geq r} E^{r', k-r} (M).$$

This is indeed a subcomplex of $E^k (M)$, both for $d$ and $\bar{\partial}$. It is clear that $F^r E^k (M) \cap F^p E^l (M) \subset F^{p+r} E^{k+l} (M)$ and that we have a (decreasing) filtration

$$E^k (M) = F^0 E^k (M) \subset F^1 E^k (M) \subset \ldots \subset F^{r} E^k (M) \subset F^{r+1} E^k (M) = 0,$$

so that we get a chain of maps:

$$H^q_{DR} (M) \subseteq H^q (F^r E^k (M), d) \subseteq \ldots \subseteq H^q (F^{r+1} E^k (M), d) \subseteq 0.$$

On the quotient complex

$$F^r E^k (M)/F^{r+1} E^k (M) = E^{r, 0-p} (M)$$

d and $\bar{\partial}$ induce the same differential; we recognize this as the Dolbeault complex so that in cohomology in degree $q$ is $H^q (M, E^0 \ell^p M)$. Hence the short
exact sequence of complexes

\[ 0 \rightarrow F^{p-1}E^* (M) \rightarrow F^p E^* (M) \rightarrow E^p (M) \rightarrow 0 \]

gives rise to a long exact cohomology sequence

\[ \cdots \rightarrow H^{q-1} (M, \Omega^p_M) \rightarrow H^q (F^{p-1} E^* (M), \delta) \rightarrow H^q (F^p E^* (M), \delta) \rightarrow H^q (M, \Omega^p_M) \rightarrow \cdots \]

Define \textit{Hodge filtration} on \( H^k_{DR} (M) \) to be

\[ F^p H^k_{DR} (M) = \text{Im} \left( H^k (F^p E^* (M), \delta) \rightarrow H^k_{DR} (M) \right), \quad p = 0, 1, 2, \ldots \]

It is clear that \( F^p H^k_{DR} \cup F^{p+1} H^k_{DR} \subseteq F^{p+1} H^k_{DR} \). It is also clear that this is functional: if \( f: M' \rightarrow M \) is a holomorphic map of complex manifolds, then \( f^* \) takes \( F^p E^* (M) \) to \( F^p E^* (M') \) and hence determines a map from the diagram \( (f)^* \) to the diagram \( (f)^* \), which produces a commutative ladder.

**Theorem 18** Assume \( M \) endowed with a Kähler metric. Then the natural maps

\[ \bigoplus_{p=0} \cdots H_{DR}^{p+1} (M) \rightarrow H^r (F^p E^* (M), \delta) \rightarrow F^p H^r_{DR} (M) \]

are isomorphisms (so that the maps in \( (i) \) are injective). In particular, the long exact sequence \( (i) \) breaks up in short exact sequences

\[ 0 \rightarrow F^{p-1} H^r_{DR} (M) \rightarrow E^p H^r_{DR} (M) \rightarrow H^r (M, \Omega^p_M) \rightarrow 0, \]

and

\[ H^q (M, \Omega^p_M) \cong H^{p+q}_{DR} (M) \cong H^q (M, \Omega^p_M) \]

and resulting decomposition \( H^r (M, \Omega^p_M) \) is compatible with the cup product.

**Proof.** The composite map is injective and the last map is surjective by definition. So it suffices to prove that the first map is also surjective. We do this with downward induction on \( p \) starting with \( p=r \).

If \( \alpha \in E^{r,0} (M) \) is \( \delta \)-closed, then \( \delta \alpha = 0 \). Since \( \delta \) is a differential operator, it also follows that \( \delta^* \delta \alpha = 0 \), and hence \( \delta^* \delta \alpha = - \delta^* \delta \alpha = 0 \).

This implies that \( \Delta \delta \alpha = 0 \), so that \( \alpha \in E^{r,0} (M) \).
Now assume $p<r$ and the singularity proved for larger values of $p$. Let $\alpha = \alpha_p + \alpha_{p+1} + \ldots + \alpha_r$ with $\alpha_p \in \mathbb{E}^{p,r-p}(M)$ and assume $d\alpha = 0$. Then $\tilde{\delta} \alpha_p = (d\alpha)^{p+1}_{p+1-p} = 0$ and so $\alpha_p$ has a harmonic representative $u \in \mathcal{H}^{p-r-p}$ for Dolbeault cohomology meaning that
\[ \alpha_p = u_p + \tilde{\partial} \beta \quad \text{for some } \beta \in \mathbb{E}^{p-r-1-r}(M) \]

Then
\[ \alpha - u - d\beta = (\alpha_{p+1} - \tilde{\partial} \beta) + \alpha_{p+2} + \ldots + \alpha_r \]

The left hand side is still $d$-closed and the right hand side is in $\mathcal{F}^{p+1}(M)$. By our induction hypothesis, this can be written as $u_{p+1} + \ldots + u_r + d\beta'$ with $u_i \in \mathcal{H}^{i,r-c}(M)$ and $\beta' \in \mathcal{F}^{p+1}(M)$. It follows that $\alpha$ can be written similarly:
\[ \alpha = u_p + u_{p+1} + \ldots + u_r + d(\beta + \beta') \quad \text{with } \beta + \beta' \in \mathcal{F}^{p+1}(M). \]

We proved almost at the same time:

**Prop. 19.** In the situation of Thm. 19, we have $\mathcal{H}^{p,0}(M) = S^p(M)$ for $p=0, \ldots, n$. In particular, every holomorphic $p$-form on $M$ is closed, and it is not exact unless equal to zero.

**Proof.** Recall that any $\omega \in \mathcal{E}^{0,0}(M)$ is $\overline{\partial}$-harmonic if and only if $\overline{\partial} \omega = 0$ and $-\ast \overline{\partial} \ast \omega = 0$. Since $\omega$ is of type $(n-p,n)$, the last condition is automatic. The condition $\overline{\partial} \omega = 0$ is equivalent to $\omega$ being holomorphic. \(\square\)

For a compact complex manifold $M$, we write $\mathcal{H}^{p,q}(M)$ for the Hodge number $\dim H^q(M, \Omega^p_M)$. In the same spirit, we write $\mathcal{H}^q(M)$ for $\dim H^q_{\overline{\partial}}(M)$.
We have proved that if $M$ admits a Kähler metric, then $h^{p,q}(M) = h^{q,p}(M)$ and $\Sigma_{p+q = k} h^{p,q}(M) = h^k(M)$. In particular, $h^k(M)$ is even when $k$ is odd!

Example (Hopf manifolds). Let $\lambda \in \mathbb{C}$ be such that $|\lambda| > 1$. Then scalar multiplication with $\lambda$ acts on $\mathbb{C}^n \setminus \{0\}$ generates an action of $\mathbb{Z}$ on $\mathbb{C}^n \setminus \{0\}$ which properly discontinuous with fundamental domain $\{z \in \mathbb{C}^n : 1 \leq |\lambda^i| \leq |\lambda|^n\}$ (this is $\mathbb{C}^n$-diffeomorphic to $S^{2n-1} \times [0,1]$). The orbit space is a flag manifold (diffeomorphic to $S^{2n-1} \times S^1$). Note that for $n \geq 2$, $H^1_{\text{DR}}(\mathbb{C}^n \setminus \{0\}) = H^1_{\text{DR}}(S^1) = \mathbb{R}$ and so $h^1(M) = 1$ is odd. Therefore, $M$ does not possess a Kähler metric.

The $\mathfrak{sl}(2)$-representation and the associated primitive decomposition can be expressed in terms of the cohomology class $u$ of $\omega_h$. We first note that $u := [\omega_h] \in H^1_{\text{DR}}(M)$ is of type (i).

Then $L(v) = uvv$ and so for $h = 0, \ldots, n$,

$$P^k(M) = \{ v \in H^k(M) : \omega_{n-k+1}v = 0 \},$$

and we have the primitive decomposition

$$\bigoplus_{k=0}^n P^k(M) \sim [u]/(\omega^{n-k+1}) \overset{\sim}{\rightarrow} H^1_{\text{DR}}(M).$$

The $\mathfrak{sl}(2)$-representation is completely determined by the data:

- For every $v \in P^{n-k}(M) \sim \mathfrak{sl}(2)$, $L^k$ acts on $v, uvv, u^2v, \ldots, u^k v$ as $X = e^{-\frac{2\pi i k}{n}}$, $X e^j, X e^{-j}, \ldots, X e^j e^{-j}$.

$Q^k_M$ is also defined in terms of $u$: if $v_1, v_2 \in P^k(M)$, then

$$Q^k_M(v_1, v_2) := (-1)^{k(n-k)/2} \langle \frac{\omega^{n-k}}{(n-k)!} v_1 v_2 \mid [M] \rangle$$
where we are evaluating \( \frac{n-k}{(n-k)!} \) on the fundamental class \([M] \in H_{2n}(M)\) (on which the orientation, defined by the complex structure, takes the value 1).

It is worthwhile to formalize the structure that we found on the de Rham cohomology of a compact Kählerian manifold. Here we should remember that by a fundamental result (due to de Rham itself) this is canonically isomorphic to the singular cohomology tensored with \( \mathbb{R} \):

\[
H^i(M) \to H^i(\text{Hom}(H_i(M), \mathbb{R})) = H^i(M; \mathbb{R})
\]

In particular, the image of \( H^i(M) \) in \( H^i_{\text{DR}}(M) \) is a lattice of full rank \( H^i(M) \).
Here \( H^i(M) \) denotes the \( i \)-th Betti number of \( M \). The kernel of this map is finite.

**Lemma 20.** Let \( H \) be a \( d \)-finite dimensional \( \mathbb{R} \)-vector space, and let \( d \in \mathbb{Z} \). Then it is equivalent to give

(i) a decomposition \( H_C = \bigoplus_{p \geq 0} H^{p,q} \) with \( H^{p,q} = H^{p,-q} \)

(ii) a descending filtration \( H^*_C \subset F^p \subset \cdots \) with \( F^p = H^*_C \) for \( p < 0 \)

and \( F^p = 0 \) for \( p > 0 \) such that \( F^{d+p} \) is a supplement for \( F^p \) for all \( p \).

the bijection being given by \( F^p = \bigoplus_{p \geq 0} H^{p,d-p} \) and \( H^{p,d-p} = F^p \cap F^{d-p} \).

**Proof:** Exercise.

**Def.** Let \( R \) be an subring of \( \mathbb{R} \) (e.g. \( \mathbb{Z} \) or \( \mathbb{Q} \)) and let \( V \) be a f.g. \( R \)-module.

Then an \( R \)-Hodge structure of weight \( d \) on \( V \) is a structure on \( V_R := \text{Reg}_R V \) as described by Lemma 20 above. (When \( R = \mathbb{Z} \) we simply say: Hodge structure of weight \( d \)). If the Hodge numbers \( h^{p,q}(V) := \dim V^{p,q} \) have the property that \( h^{p,q}(V) = 0 \) when \( p < 0 \) or \( q < 0 \), then we will say the Hodge structure is effective.

An \( R \)-polarization on such a Hodge structure is an \( R \)-bilinear map

\[ Q : V \times V \to R \]

which is \( C^{1,1} \)-symmetric, whose complexification \( \bar{Q} \) of bidegree \((-d,d)\) and for
which the form \((v,v') \in V \times V \longrightarrow Q (v,Jv')\) is positive definite, where \(J\) is the endomorphism of \(V\) which on \(V^{\otimes q}\) is mult. \(m(\xi_i)^{q^g}.\)

Remark. It is not hard to show that an IR-Hodge structure of weight \(d\) admits an IR-polarization. We will see that a \(Q\)-Hodge structure of weight \(d\) does not always admit a \(Q\)-polarization. If that is the case, we will say that this Hodge structure is polarizable.

Example. If \(M\) is a compact Kahlerian manifold, then \(H^d(M)\) comes with an effective Hodge structure of weight \(d\). If a Kahler metric is also given, so that we have a primitive decomposition of \(H^d(M; \mathbb{R})\), then \(K\) is a decomposition into IR-Hodge summands. In particular, \(P^d(M) (d = n, m)\) is a Hodge substructure of \(H^d(M; \mathbb{R})\) and our bilinear \(Q^{(k)} : P^d(M) \times P^d(M) \rightarrow \mathbb{R}\) defines an IR-polarization of \(K\).

Example. Let \(C\) be a compact connected Riemann surface. We have seen that \(C\) admits a hermitian metric. Let \(h\) be one. Then the associated 2-form \(\omega^h\) (of type \((1,1))\) is closed (as there are no 3-forms on \(C\)) and hence \(h\) is automatically a Kahler metric. The form \(h\) is in terms of a local coordinate \(z = x + iy\) of the form \((p(x,y))dx^\wedge dy\) with \(p > 0\) and hence defines the complex orientation of \(C\). Its class \(\nu := [\omega^h] \in H^2_{\text{DR}}(C)\) is nonzero. After multiplying \(h\) with a positive real number, we may assume that \(\nu\) gives an integral generator of \(H^2(C)\). If \(g\) is the genus of \(C\), then \(H^2(C) \cong \mathbb{Z}^{2g}\) and 
\[H^1(C; \mathbb{C}) = \overline{H^{10}(C)} \oplus H^{0,1}(C)\] with \(H^{10}(C) = \Omega^1(C)\) and hence \(H^{0,1}(C) = \overline{\Omega^1(C)}\). This Hodge structure comes with a \(\mathbb{Z}\)-polarization defined by the intersection pairing \(Q' : H^1(C) \times H^1(C) \longrightarrow \mathbb{Z}, (v,v') \mapsto \langle v, v'\rangle_{[C]}\).

The properties of \(Q'\) can also be stated by saying that \(H^{10}_{\text{DR}}(C) = \Omega^1(C)\) is isotropic for the intersection pairing \((\alpha, \alpha') \in H^1_{\text{DR}}(C) \times H^1_{\text{DR}}(C) \mapsto \int_C \alpha \wedge \alpha'\) and \((\alpha, \alpha') \in \Omega^1(C) \times \Omega^1(C) \mapsto \int_C \alpha \wedge \alpha'\) is positive definite. These are
classically known as the Riemann bilinear relations.

Example. Let us first consider $\mathbb{P}^N$ endowed with its Fubini-Study metric $h$. It is known that $H^2(\mathbb{P}^N) \cong \mathbb{Z}$ and so after multiplying $h$ with a real number $>0$ we may assume that $u = [w_0] \in H^2(\mathbb{P}^N, \mathbb{R})$ is in fact an integral generator (actually our choice of $h$ had this property already). So if $M$ is a closed complex submanifold of $\mathbb{P}^N$, and $M$ is endowed with the induced Kähler metric, then the class of the associated 2-form comes from an integral class, namely the image of $u$ under the restriction map $H^2(\mathbb{P}^N) \to H^2(M)$. So then the primitive decomposition is defined over $\mathbb{Q}$ and so will be the polarization that we found on $\mathfrak{m}$'s primitive cohomology.

§3 Albanese and jacobian manifolds

(3.1) Effective Hodge structures of weight 1. Let $V$ be a finitely generated abelian group endowed with an effective Hodge structure of weight 1. The latter amounts to $V_0 = V^{1,0} \oplus V^{0,1}$ with $V^{1,0} \cong V^{0,1}$. This is equivalent to having a complex structure $J$ on $V_0^\mathbb{R}$ for which $V^{1,0}$ is the $-1$-eigen space (and hence $V^{0,1}$ the $+1$-eigen space). We claim that this is in fact $H^1$ of a complex torus $A(V)$ with its Hodge structure. It is defined as follows. First we observe that the complexification map defines an $\mathbb{R}$-linear iso.

$$\text{Hom}(V, \mathbb{R}) \cong \text{Hom}(V, \mathbb{C}) \cong (V^{1,0})^*$$

and that under this $\mathbb{R}$-isomorphism the operator $J^*$ on the left corresponds to multiplication with $-1$ on the right. We then put

$$A(V) = \text{Coker}(V^\mathbb{C} \to (V^{1,0})^*)$$

and via the above identification we see that $A \cong \text{Coker}(V^\mathbb{C} \to \text{Hom}(V, \mathbb{R})) \cong \text{Hom}(V, \mathbb{R}/\mathbb{Z})$. This shows that $A$ has the structure of a Lie group and is such that $H^1(A) \cong V^{1,0}$ canonically and so $H^1(A) \cong V$. Its Hodge structure
is indeed the given one. We call \( A(V) \) the Albanese torus associated to \( V \).

We can also associate a torus \( J \) to \( V \) without dualizing. This is based on the observation that the composite map

\[
V_R \subset V_\mathbb{C} \longrightarrow V_\mathbb{C} / V^{\perp 0}
\]

is an IR-isomorphism under which complex mul. \( V^\dagger \) in \( V_\mathbb{C} / V^{\perp 0} \) corresponds to \( V_R \) to the operator \( -J = J^{-1} \). The Jacobian torus associated to \( V \) is

\[
J(V) := V_\mathbb{C} / V^{\perp 0} + V
\]

which we may think of as \( V_R \cap V \) endowed with a complex structure. The two constructions are essentially dual to each other: if \( V' = \text{Herm}(V, \mathbb{Z}) \) is endowed with the effective weight 1 Hodge structure for which \( (V')^{\perp 0} \) is the annihilator of \( V^{\perp 0} \), then \( A(V') \cong J(V) \) and \( J(V') \subset A(V) \) (this is left as an exercise).

Remark. If in the above discussion \( V' \subset V \) is a submodule of finite index, then we natural homomorphisms a complex tori \( A(V) \rightarrow A(V') \) and \( J(V') \rightarrow J(V) \) that are surjective and with finite kernel. We say that two complex tori \( T, T' \) are isogenous if there exists a surjective holomorphic homomorphism \( T \rightarrow T' \) with finite kernel. You may check that these maps also endow a finite holomorphic homomorphism \( T' \rightarrow T \) so that this is an equivalence relation. To give an equivalence class is to give an isomorphism class of effective \( \mathbb{Q} \)-Hodge structures of weight 1. If such a Hodge structure is polarizable, then the associated Albanese and Jacob tori are isogenous.

(3.2) Polarizability as a restrictive condition. We make precise the claim that most \( \mathbb{Q} \)-Hodge structures are not \( \mathbb{Q} \)-polarizable. Let \( V \) be a \( \mathbb{Q} \)-vector space of even dimension \( 2g \) and let \( \alpha : V \times V \rightarrow \mathbb{Q} \) be a nondegenerate alternating form.

An effective Hodge structure of weight 1 on \( V \) polarized by \( \alpha \) is given by a \( g \)-dim
complex subspace $F \subset V_c$ (our $V^{(0)}$) with the property that

1. $V_c = F \oplus \overline{F}$
2. $Q_c |_{F \times F} = 0$
3. $(z, z') \in F \mapsto Q_c(z, z')$ is pos. definite.

These properties define a subset of the Grassmannian $\text{Arg}_g(V_c)$ of $g$-dim complex subspaces of $V_c$. Conditions (1) and (2) define an open subset (so this will be of co-dim $g^2$), whereas (1) defines a closed subset. To understand this condition suppose that $F$ satisfies these $3$ properties. Then a neighborhood of $F$ in $\text{Arg}_g(V_c)$ is identified with $\text{Hom}_c(F, \overline{F})$ via

$$\text{Hom}_c(F, \overline{F}) \to \text{Arg}_g(V_c), \quad \varphi \mapsto \text{graph} \varphi = : F_{\varphi}.$$ 

Since

$$Q_c(z, z') + Q_c(q(z'), z) = Q_c(z, z') + Q_c(q(z'), z) + Q_c(z, q(z')) + Q_c(q(z'), q(z'))$$

we see that $Q_c|_{F_{\varphi} \times F_{\varphi}} = 0$ if and only if the alternating part of the bilinear form $(z, z') \in F \times F \mapsto Q_c(z, q(z'))$ is identically zero (so that $t$ is a symmetric form). In other words, condition (1) defines a linear subspace of $\text{Hom}_c(F, \overline{F})$ of complex codim $(g^2) = g(g-1)/2$ (and hence of dim $g(g+1)/2$).

As there are only countably many choices for $\varphi$, we see that the $\varphi$-polarizable effective Hodge structures on $V$ of weight $1$ are parameterized by a countable union of complex submanifolds of $\text{Arg}_g(V_c)$ of codimension $g(g-1)/2$. When $g \geq 2$, the complement of this subset in $\text{Arg}_g(V_c)$, being a countable intersection of open dense subsets is dense. So most effective $\varphi$-Hodge structures of weight $1$ and of codim $2g \geq 4$ are not $\varphi$-polarizable.

(3.5) Albanese manifold and the Abel–Jacobi map. In what follows $M$ is a compact connected kählerian manifold of complex dim $n$. We define its Albanese manifold as the Albanese torus associated to the Hodge structure on $H^1(M)$, in other words by
$\text{Alb} (M) := \text{Coh} \left( H^1_*(M), \Omega^1_*(M)^* \right).$

(Use that $H^1_*(M) = \text{Hom}_\mathbb{Z} (H_1(M), \mathbb{Z})$). This is functorial: if $f: M' \to M$ is a holomorphic map of compact connected kählerian manifolds, then we have a commutative diagram

\[
\begin{array}{ccc}
H^1_*(M) & \to & \Omega^1_*(M)^* \\
\downarrow f_* & & \downarrow (f^*)_* \\
H^1_*(M') & \to & \Omega^1_*(M')^*
\end{array}
\]

With any pair $(x,y) \in M \times M$ we associate an element of $\text{Alb}(M)$ as follows: let $\gamma: [0,1] \to M$ be a $C^\infty$-path with $\gamma(0) = x, \gamma(1) = y$. Integration over $\gamma$ defines a linear map $\int_{\gamma} : \Omega^1_*(M) \to \mathbb{R}$ and hence an element of $\text{Alb}(M)$. The latter only depends on the pair $(x,y)$, for if $\gamma'$ is another such path, then $\gamma$ followed by $\gamma'$ traversed in the opposite direction defines a 1-cycle on $M$ and hence an element of $H_1(M)$.

As we divided out by the image of $H_1(M)$ in $\Omega^1_*(M)^*$ it follows that $\gamma'$ gives the same image in $\text{Alb}(M)$. The map so obtained

\[
\text{AJ}_M: M \times M \to \text{Alb}(M) \quad (x,y) \mapsto \int_{\gamma}
\]

is called the **Abel-Jacobi map**.

**Prop. 21.** The Abel-Jacobi map has the following properties

(i) $\text{AJ}_M$ is holomorphic.

(ii) It is functorial in the sense that if $f: M' \to M$ is a holomorphic map of compact connected kählerian manifolds then $\text{AJ}_M: M \times M \to \text{Alb}(M)$ commutes.

(iii) If $T$ is a complex torus, then $y \in T \mapsto \text{AJ}_T (0,y) \in \text{Alb}(T)$ is an isomorphism of complex tori.

(iv) Given $x_0 \in M$, then the holomorphic map $\text{AJ}_{M,x_0}: M \to \text{Alb}(M)$,

\[
x \in M \mapsto \text{Alb}_M (x_0, x)
\]

is universal among the holomorphic maps $f: M \to T$.
to a complex torus in the sense that $f$ factors uniquely as $AJ_M, \phi$, followed
by a homomorphism of complex tori $\text{Alb}(M) \rightarrow T$ followed by a
translation on $T(\phi)$.

Proof. (i) Let $(\mathbf{x}, \mathbf{y}) \in M \times M$ and choose disk-like neighborhoods $U_\varepsilon \ni \mathbf{x}_i$ ($i=0,1$).
Any $\mathbf{x} \in \Omega^1(M)$ is exact on $U_\varepsilon$ and hence is uniquely written as $d\mathbf{p}_{\mathbf{x},i}$ with $\mathbf{p}_{\mathbf{x},i} : U_\varepsilon \rightarrow \mathbb{C}$ holomorphic and $\mathbf{p}_{\mathbf{x},i}(\mathbf{x}_i) = 0$. If $\gamma$ is a path from $\mathbf{x}_0$ to $\mathbf{x}_1$, then the map $U_0 \times U_1 \rightarrow \Omega^1(M)^*$ defined by
\[
(y_0, y_1) \mapsto (\mathbf{x} \mapsto \int_0^1 \mathbf{p}_{\mathbf{x},i}(y) - \mathbf{p}_{\mathbf{x}}(y_0))
\]
is holomorphic and it composite with $\Omega^1(M)^* \rightarrow \text{Alb}(M)$ is $AJ \vert_{U_0 \times U_1}$. Hence $AJ$ is holomorphic.

The proofs of (ii) and (iii) are left as an exercise.

(iv) The commutative diagram
\[
\begin{array}{ccc}
\text{M} & \xrightarrow{AJ_M, \phi} & \text{Alb}(M) \\
\downarrow & & \downarrow \\
T & \xrightarrow{\phi} & \text{Alb}(T)
\end{array}
\]
gives the factorization
$AJ \vert_{T(\phi)}$

Corollary 22: Every holomorphic map $f : T \rightarrow T$ of complex tori is the composite of a holomorphic homomorphism $T' \rightarrow T$ and a translation over $f(0)$. $\square$

The three abelian group generated by the points of $M$ is the group of zero cycles
denoted $Z_0(M)$. The natural map $Z_0(M) \rightarrow H_0(M) \in \mathbb{Z}$ is the degree map
which assigns to every point of $M$ the value 1. Its kernel is the subgroup of
exact zero cycles, denoted $B_0(M)$. Any $z \in B_0(M)$ can be written as
$\sum_{i=0}^s (y_i - x_i)$ for certain $x_i, y_i \in M$. So if $\gamma_i$ is a $C^1$ path from $x_i$ to $y_i$ and is considered as a 1-chain on $M$, then $W = \frac{2\pi}{i} \gamma_i$ is a 1-chain with $\partial W = 2\pi i$ and $\int_W = \sum_{i=0}^s \int_{\gamma_i} \in \Omega^1(M)^*$. Its image in $\text{Alb}(M)$ is independent of
the choice of $W$ and yields an element $z_i \in \text{Alb}(M)$ that we shall denote
by $AJ_M(z)$. It is of course equal to $\sum_{i=0}^s AJ_M(x_i, y_i)$. We thus obtain a map
$AJ_M : B_0(M) \rightarrow \text{Alb}(M)$

which extends our earlier defined map. It is clearly a homomorphism of groups.
Prop. 23. The group homomorphism $AJ^*_M : B_0(M) \to \text{Alb}(M)$ is surjective.

Proof. Put $g = \text{dim}_{\mathbb{C}} \Omega^1(M)$ and choose $x_0 \in M$. We define

$$f : M^g \to \text{Alb}(M), \quad (y_{1}, \ldots, y_{g}) \mapsto \frac{g}{c=1} A^*_M(x_c, y_c).$$

We prove that $f$ is a submersion at some $(x_{1}, \ldots, x_{g}) \in M^g$. This will suffice, for then it follows that the image of $AJ^*_M$ contains a non-empty open subset of $\text{Alb}(M)$ and it is easy to see that a subgroup of a complex torus which contains a non-empty open subset of that torus is the whole torus.

Assume $g > 0$ and let $x_{i} \in \Omega^1(M) \setminus \{0\}$. Choose a tangent vector $v_i \in T_{x_{i}} M$ such that $x_{i}(v_i) \neq 0$. Evaluation in $v_i$ then defines a nonzero linear form $ev(v_i) : \Omega^1(M) \to \mathbb{C}$. If $g > 1$, choose a nonzero $v_2$ in the kernel of $ev(v_1)$ and let $v_2 \in T_{x_{1}} M$ be a tangent vector with $x_{2}(v_2) = 0$. With induction we find a sequence $(x_i \in \Omega^1(M), v_i \in T_{x_i} M)_{i=1}^{g}$ with $x_i(v_i) = 0$ and $x_i(v_j) = 0$ when $j \neq i$. So the matrix $(x_i(v_j))_{i,j=1}^{g}$ is nonsingular. But it is easily seen that $x_i(v_j)$ is just the derivative of $f$ at $(x_{1}, \ldots, x_{g})$ precomposed with the embedding $(x_{1}, \ldots, x_{g}) \in \mathbb{C}^{g} \to (x_{1}, y_{1}, \ldots, x_{g}, y_{g}) \in \prod_{i=1}^{g} T_{x_{i}} M = T_{(x_{1}, \ldots, x_{g})} M^g$. Hence $f$ is a submersion at $(x_{1}, \ldots, x_{g})$. $\square$

Remark. We can write $B_0(M)$ as a monothone union of copies of $M^{2g}$ by assigning to $(x_{1}, y_{1}, \ldots, x_{g}, y_{g})$ the element $\left(\sum_{i=1}^{g}, (y_{i} - x_{i}) \right)$ in $\text{Alb}(M)$ the inclusion $M^{2g} \subset M^{2g+2}$ being given by taking $x_{i+1}$ and $y_{i+1}$ equal to some fixed $x_0 \in M$. The map $AJ$ is then still holomorphic in the sense that its restriction to each $M^{2g}$ is.

Remark. If we take $M = \mathbb{C}$, a compact connected Riemann surface, then the kernel of the map $B_0(M) \to \text{Alb}(M)$ is just the group of principal divisors, i.e., the elements of $B_0(M)$ which are either zero or of the form $f^*(0) - f^*(\infty)$ for some nonconstant holomorphic map $f : \mathbb{C} \to M$. We can see that such an element lies in the kernel by observing that we have a holomorphic
map $\text{Pic}^1 \rightarrow \text{Ab}(C)$ which assigns to $\mathbb{P}^1$ $A^\circ_j(f^*(\ell) - f^*(\ell))$. Such a map must be constant because $\text{Ab}(\mathbb{P}^1) = \{0\}$. (That this precisely the kernel comes down to the assertion that $A^\circ_j$ defines an isomorphism of the identity component of the Picard group of $C$ onto $\text{Ab}(C)$.)

(3.4) Intermediate Jacobians: We generalize the previous construction to effective Hodge structures of arbitrary odd weight. So let $V$ be a finite, generated abelian group endowed with an effective Hodge structure of odd weight $2d-1$ ($d \geq 1$):

$$V_c = \frac{V^{d-1,0} \oplus \cdots \oplus V^{1,d-1} \oplus V^{d-1,0} \oplus \cdots \oplus V^{1,d-1}}{F^d V}$$

We then note that $V_c = F^d V \oplus F^d V$ so that this is defined a complex structure on $V_c$. (It is also a coarsening of the given Hodge structure). We can associate with $V$ two complex tori, namely

$$J(V) := \text{CH}_0(V \rightarrow V_c/F^{d}V), \quad A(V) := \text{CH}_0(V \rightarrow (F^{d}V)^*)$$

These are essentially dual to each other in the following sense: if $V'$ is an effective Hodge structure of weight $2d'-1$ on a lattice and there exists a $V'_{\text{tor}} \times V'_{\text{tor}} \rightarrow \mathbb{Z}$ whose complexification is of bidegree $(d+d',d+d')$ then $F^d V'$ is the annihilator of $F^d V$ and this identifies $J(V')$ with $A(V)$ and $A(V')$ with $J(V)$. We will therefore focus on $J(V)$.

For $M$ as above (compact, connected, kählerian of complex dimension $n$) we define its $d^{th}$ intermediate Jacobian $J^d_M$ by

$$J^d(M) := J(H^{2d-1}_M) \quad (d = 1, \ldots, n).$$

We could also define $A_d(M)$ as $A(H^{2d-1}_M)$, but Poincaré duality derives a perfect pairing between $H^{2d-1}_M$ (borel) and $H^{2n+1-2d}_M$ (borel) of bidegree $(n,n)$ and so $A_d(M)$ is identified with $J^{n-1-d}_M$. In particular $\text{Ab}(M) = A_1(M)$ is identified with $J^1_M$. We want to define an Abel-Jacobi map for $J^d_M$.

The construction for $\text{Ab}(M)$ suggests that this should involve analytic cycles of codimension $d$.

A subset $Z \subset M$ is said to be analytic if it is locally given as the common
zero set of a system of holomorphic functions. It is said to be irreducible if its
zero set cannot be written as $Z_0 \cup Z_1$, with $Z_i$ an closed analytic subset closed in $Z$.

An irreducible $Z$ contains a connected complex submanifold that is open-dense in $Z$. The dimension of this manifold can be taken as the
definition of the complex dimension of $Z$.

Suppose $Z$ is a closed, irreducible analytic subset of $M$ of complex codimension $d$.

Then $Z$ can be triangulated in a $C^\infty$-manner so that the $(2n-2d)$-simplices have
their interior in the smooth part of $Z$. If we give them their natural orientation
then their sum defines a fundamental cycle $[Z] \in H_{2n-2d}(M)$ (a priori this yields
an element of $H_{2n-2d}(M,\mathbb{Z})$, but since $\mathbb{Z}$ has homology in degree $2n-2d-2$, this is
uniquely representable as an element of $H_{2n-2d}(M)$). Integration over $Z$ defines $\mathbb{E}^{2n-2d}(M) \to \mathbb{C}$.
The restriction of an element of $\mathbb{E}^{2n-2d}(M)$ to the manifold part of $Z$ is
clearly zero. Its restriction to the closed forms is the complexification of the map
$H_{2n-2d}(M) \to \mathbb{C}$ given by evaluation on $[Z]$. The latter can also be represented
by the Poincaré dual of $[Z]$, $PD[Z] \in H^d(\mathbb{C})$. Since the annihilator of
$H_{2n-2d}(M) \cap H^d(\mathbb{C})$ under Poincaré duality is $F^{d}(H^d(\mathbb{C}))$, it follows
that $PD[Z] \in F^{d}(H^d(\mathbb{C}))$. This element is also called the class of $Z$.

We now define $Z^d(M)$ as the free abelian group generated by the closed
irreducible subsets of $M$ of complex codimension $d$. We have an obvious map
$$\partial^d: Z^d(M) \to H^d(M)$$
and we just proved that its image (after complexification) lies in $F^{d}(H^d(\mathbb{C}))$. Since
the image is defined over $\mathbb{Q}$, we see that we have a map
$$Z^d(M)_{\mathbb{Q}} \to F^{d}(H^d(\mathbb{C}) \cap H^d(M;\mathbb{Q})) = H^d(\mathbb{C}) \cap H^d(M;\mathbb{Q})$$
The famous and still open Hodge conjecture states that in case $M$ is a projective
manifold this map is onto. If true, then its consequences are enormous.

We note $F^{d}(M)$ for the kernel of $\partial^d$. We claim that we have an
Abel–Jacobi map \( \text{AJ}^d : B^d(M) \rightarrow J^d(M) \). For this let \( Z \in B^d(M) \) so that \( \text{im} Z \) is in fact the boundary of a \( C^\infty \)-chain \( W \) of dimension \( 2n-2d+1 \). Integration over \( W \) then defines a map:

\[
\int_W : \mathbb{E}^{2n-2d+1}(M) \rightarrow C
\]

We restrict \( W \) to the closed forms in \( F^{n-d+1} \mathbb{E}^{2n-2d+1}(M) \). If \( \alpha \in \mathbb{E}^{n-d+1}(M) \)

then by Stokes \( \int_W \alpha = \int_Z \alpha = 0 \) and so \( W \) defines a linear form on \( \mathbb{E}^{n-d+1} \mathbb{H}^{2n-2d+1}(M) \). If \( W' \) is another choice for \( W \), then \( W-W' \) is a \((2n-2d+1)\)-cycle on \( M \) so that it defines an element of \( \mathbb{H}^{2n-2d+1}(M) \). It follows that \( Z \) determines an element of

\[
\text{Coker} \left( \mathbb{H}^{2n-2d+1}(M) \rightarrow F^{n-d+1} \mathbb{H}^{2n-2d+1}(M)^* \right) = A_{n-d+1}(M) \cong J^d(M).
\]

This defines \( \text{AJ}^d(Z) \).

Prop. 24. This Abel–Jacobi map is holomorphic in the following sense: if \( S \) is a connected complex manifold and \( Z \in B^d(S \times M) \) such that \( Z \) is flat over \( S \) (this implies that for every \( s \in S \), \( Z \) restricts to an element \( Z_s \in B^d(M) \)), then the map \( S \rightarrow \text{AJ}(Z_s) \in J^d(M) \)

is a holomorphic. The proof is similar as for the case \( d = n \).

Remark. It is not clear whether \( \text{AJ}^d \) is always surjective (as in the case for \( d = n \)).

Remark 2. The forms \( J(V) \) arises by coarsening an effective Hodge structure of weight \( 2d-1 \). This coarsened Hodge structure appears as a Hodge structure of weight \( 1 \) on \( V \). If the Hodge structure on \( V \) is polarizable, then the latter need not be. This applies in particular to \( J^d(M) \) with \( M \) a projective manifold:

for \( 1 < d < n \) it may well happen that \( J^d(M) \) is not a projective manifold.

Remark 3. There exists a natural exact sequence

\[
0 \rightarrow J^d(M) \rightarrow \hat{F}^{2d}(M) \rightarrow F^d \mathbb{H}^{2d}(M; \mathbb{Z}) \rightarrow 0,
\]

where \( F^d \mathbb{H}^{2d}(M; \mathbb{Z}) \) stands for the intersection in \( \mathbb{H}^{2d}(M; \mathbb{Z}) \) of the images of \( F^d \mathbb{H}^{2d}(M) \) and \( H^d(M; \mathbb{Z}) \). It has the property that the map \( \text{AJ}^d : B^d(M) \rightarrow J^d(M) \)

extends naturally to a map \( Z^d(M) \rightarrow H^{2d}_D(M) \) such that in the
diagram below the red arrow is the one we constructed.

\[
\begin{array}{c}
0 \rightarrow B^d(M) \rightarrow Z^d(M) \\
\downarrow \\
0 \rightarrow J^d(M) \rightarrow \hat{H}^d(M) \rightarrow H^d(M, \mathbb{Z}) \rightarrow 0
\end{array}
\]

We will come back to this construction (and then find that it is better to make a "Tate twist").

5.5 The Hodge structure of Tate type

From the point of view of algebraic geometry, the Hodge structure on nonsingular projective variety "reconciles" two cohomology theories of a different flavor: the algebraically defined De Rham theory (more on this later) and singular cohomology. The latter is, when taken with integral coefficients, not of an algebraic character and makes only sense when the base field is \( \overline{\mathbb{Q}} \). However for an arbitrary base field \( k \) there exists an algebraic counterpart, namely Deligne cohomology, (we take our coefficients in \( Z(k) \) for which \( 1/k \) exists in \( k \). One can then take the coefficients in \( \hat{Z}(k) = \lim_{\rightarrow} Z(k) \). In particular, we can take for \( k \) a prime number so that our coefficient ring is a discrete valuation ring whose field of fractions \( \mathbb{Q}_k \) is of characteristic zero. The algebraic character implies that if \( k^s/k \) is a separable closure, then the Galois group \( \text{Aut}(k^s/k) \) will act on the cohomology of a variety (scheme) \( X \) over \( k \). For example if \( X \) is a variety over a finite field \( k \), then \( F_k^s \) is an algebraic closure of \( F_k \) and \( \text{Aut}(F_k^s/F_k) \) is generated by the Frobenius \( \text{Fr} = a \rightarrow a^q \in F_k^s \). A central theorem due to Deligne shows that \( \text{Fr} \) acts on \( H^q(X; \mathbb{Q}_k) \) with eigenvalues of absolute value \( q^{q/2} \), i.e., \( 0, \ldots, 2q^q \). (after we have embedded \( \mathbb{Q}_k \) in \( \overline{\mathbb{Q}} \)) and that \( q \geq 1 \) if \( X \) is smooth resp. projective. This inspired Deligne also to develop his mixed Hodge theory.

To illustrate what we mean by reconciling two cohomology theories, consider the ample case of \( \mathbb{P}^1 \). Over \( \mathbb{Q} \) this is the union of two copies on \( \mathbb{C} \) namely \( \mathbb{P}^1 \setminus \{0\} \) and \( \mathbb{P}^1 \setminus \{\infty\} \) and these intersect in a copy of \( \mathbb{C}^X \). This makes even sense over \( \mathbb{Q} \): we have a scheme \( \mathbb{P}^1_\mathbb{Q} \) that is obtained by glueing two copies of \( \text{Spec} \mathbb{Z} \) along a
diagram below the red arrow is the one we constructed.

\[
\begin{align*}
0 & \longrightarrow B^d(M) \longrightarrow Z^d(M) \longrightarrow \{\text{alg. cycle classes}\} \longrightarrow 0 \\
\downarrow & \quad \downarrow \quad \downarrow \\
0 & \longrightarrow J^d(M) \longrightarrow H^d_D(M) \longrightarrow \text{F}^dH^d(M,\mathbb{Z}) \longrightarrow 0
\end{align*}
\]

We will come back to this construction and then find that it is better to make a "Tate twist."

3.5 The Hodge structure of Tate type

From the point of view of modern algebraic geometry, the Hodge structure on a complex projective manifold "reconciles" two cohomology theories of a different flavor, namely the algebraically definable De Rham cohomology and singular cohomology which a priori is tied to the usual Hausdorff topology and not to the Zariski topology. There exists however, for schemes of finite type over a field \( k \), an algebraic counterpart of singular cohomology, namely Tate cohomology. This does not allow for integral coefficients but demands that we take coefficients in \( \mathbb{Z}/l \) for any integer \( l > 1 \) whose image in \( k \) is invertible. We can then also take our coefficients in \( \mathbb{Z}_l = \lim_{\rightarrow} \mathbb{Z}/l^n \). If we choose \( l \) prime to \( \text{char}(k) \), then \( \mathbb{Z}_l \) is a discrete valuation ring (whose field of fractions is \( \mathbb{Q}_l \), which is a field of characteristic zero). The algebraic character implies that if \( k^s/k \) is a separable closure, then the Galois group \( \text{Aut}(k^s/k) \) will act on the \( \mathbb{Q}_l \)-adic cohomology of any \( k \)-scheme \( X \) of finite type. For example, if \( k = F_q \), then \( \text{Aut}(F_q^s/F_q) \) is generated by the Frobenius \( a \mapsto a^q \in F_q^s \). A fundamental theorem due to Deligne states that \( \mathbb{Q}_l \) acts on \( H^d_{et}(X;\mathbb{Q}_l) \) with eigenvalues of absolute value \( q^{d/2} \) with \( i=0,1, \ldots, 2d \) (this requires an embedding of \( \mathbb{Q}_l \) in \( \mathbb{C} \), but the property in question is independent of the embedding) and that we have a finitely many. If \( X \) is smooth resp. projective, this inspired Deligne also to develop his mixed Hodge theory.
To illustrate what we mean by reconciling De Rham and singular cohomology consider the case of \( \mathbb{P}^1 \). Over \( \mathbb{C} \) this is the union of two copies of \( \mathbb{C} \) namely, \( \mathbb{P}^1 \setminus \{0\} \) and \( \mathbb{P}^1 \setminus \{\infty\} \) and these intersect in a copy of \( \mathbb{C}^\times \). This makes even sense over \( \mathbb{Z} \): we have a scheme \( \mathbb{P}^1_{\mathbb{Z}} \) that is obtained by gluing two copies of the affine line over \( \mathbb{Z} \), \( \mathbb{A}^1 = \text{Spec } \mathbb{Z}[t] \), along \( \mathbb{P}_m := \text{Spec } \mathbb{Z}[t, 1/t] \). Then \( \mathbb{P}^1_{\mathbb{Z}}(\mathbb{C}) \) is the Riemann sphere but with the Zariski (= finite) topology. The Mayer-Vietoris sequence for the Riemann sphere gives

\[
H^1(\mathbb{P}^1_{\mathbb{Z}}(\mathbb{C}^\times)) \to H^1(\mathbb{P}^1_{\mathbb{Z}}(\mathbb{C}^\times)) \to \cdots \to H^2(\mathbb{P}_m) \to H^2(\mathbb{P}_m) \to H^2(\mathbb{P}^1_{\mathbb{Z}}(\mathbb{C}^\times)) \to H^3(\mathbb{P}_m) \to \cdots
\]

This has a De Rham counterpart that is algebraically defined. Here \( H^1(\mathbb{P}^1_{\mathbb{Z}}(\mathbb{C}^\times)) \) has a natural generator roughly \( \frac{dz}{z} \) (this makes even sense for \( \mathbb{P}^1_{\mathbb{Z}} \setminus \{0, \infty\} \approx \text{Spec } \mathbb{Z}[t, 1/t] \)). But \( \frac{dz}{z} \) does not produce a generator of \( H^1(\mathbb{P}^1_{\mathbb{Z}}(\mathbb{C}^\times)) \).

Indeed, integration of \( \frac{dz}{z} \) over the integral generator of \( H_1(\mathbb{P}^1_{\mathbb{Z}}(\mathbb{C}^\times)) \) defined by the circle \( |z|=1 \), counterclockwise oriented gives the value \( 2\pi i \).

Note that for this identification it was necessary to choose a square root of \(-1\) (the notion counterclockwise is dependent on it).

A somewhat different approach leads to the same conclusion. The map

\[
\mathbb{C} \to \mathbb{C}^\times, \quad z \mapsto e^{z}
\]

is a universal covering with covering group \( 2\pi i \mathbb{Z} \) (acting as translations in \( \mathbb{C} \)). It is well-known that \( H_1 \) of a reasonable path connected space \( X \) is the abelianization of its fundamental group or equivalently, of the group of Deck transformations of a universal covering of \( X \). So in the present case this identifies \( H_1(\mathbb{C}^\times) \) with \( 2\pi i \mathbb{Z} \) and hence its dual \( H^1(\mathbb{C}^\times) \) with \( \mathbb{Z} \).

This motivates:

**Def.** The **Tate Hodge structure** \( \mathbb{Z}(1) \) is the Hodge structure of bidegree \((-1, -1))\) on

\[
2\pi i \mathbb{Z} = \ker ( \mathbb{C} \to \mathbb{C}^\times, \quad z \mapsto e^{z})
\]
Note that this definition doesn’t require us to choose a square root of \(-1\); the group \(2\pi\mathbb{V}1\mathbb{Z}\) is the same, it is only the generator that is ambiguous.

We denote by \(\mathbb{Z}(-1)\) the dual Hodge structure and more generally \(\mathbb{Z}(d)\) for \(d \in \mathbb{Z}\) as

\[
\mathbb{Z}(-1) \oplus \mathbb{Z}(-1), \mathbb{Z}(-1) \oplus \mathbb{Z}(-1) (d < 0) \quad \text{and} \quad \mathbb{Z}(d) = \mathbb{Z}.
\]

In other words, \(\mathbb{Z}(d)\) is the Hodge structure of bidegree \((-d, -d)\) on the lattice \((2\pi\mathbb{V}1)\mathbb{Z}^d\). For a subring \(R \subset \mathbb{C}\), we write \(R(d)\) for \(R \oplus \mathbb{Z}(d)\) (that is, an \(R\)-Hodge structure) and more generally if \(H\) is a \(R\)-Hodge structure, we write \(H(d)\) for \(H \oplus \mathbb{Z}(d)\).

From now on we adopt this algebraic point of view and so regard \(H^2(\mathbb{P}^1)\) as a copy of \(\mathbb{Z}(-1)\). Its unique generator is \(\frac{\partial z}{\partial \bar{z}}\) is. Equivalently: the canonical algebraic generator is represented by \(\frac{\partial z}{\partial \bar{z}}\) and this is an element of \(H^2(\mathbb{P}^1) = H^2(\mathbb{P}^1, \mathbb{Z}(1))\) of bidegree \((0,0)\). We can think of \(H^2(\mathbb{P}^1)\) (or equivalently \(H_1(\mathbb{C}^*)\)) as the geometric incarnation of \(\mathbb{Z}(1)\).

For a connected compact complex manifold \(M\) of dimension \(n\) this then leads to regarding \(H^{2n}(M)\) as a copy of \(\mathbb{Z}(n)\). To see this, let \(x \in M\) and choose a polyhedral nbhd \(U\) of \(x\) in \(M\) (so \(U \cong \mathbb{C}^n\)). Then

\[
H^{2n}(M) \cong H^{2n}(M, M, \pm \Delta) \cong H^{2n}(U, U, \pm \Delta) \cong H^2(\Delta, \Delta, \pm \Delta) \cong H^2(\Delta, \Delta, \pm \Delta) \cong H^2(\mathbb{P}^1, \mathbb{P}^1, \pm \Delta) \cong H^2(\mathbb{P}^1, \mathbb{P}^1) \cong \mathbb{Z}(n).
\]

Thus we have a natural generator of \(H^{2n}(M)\) (which is of bidegree \((0,0)\)). In order to get a generator of \(H^{2n}(M)\) we then need to pick a square root of \(-1\). In other words, this notation takes automatically care of orientation issues that arise from such a choice.

The exterior product is now a bilinear map \(H^d(M) \times H^{n-d}(M) \to \mathbb{Z}(n)\), which is still birational for the Hodge bidegree. This leads us to modify some our previous conventions: a polarization of an \(R\)-Hodge structure \(H\) of weight \(d\) is a birational map

\[
\Psi: H \times H \to R(1-d)\]
which is $GL$-symmetric and has the property that
$$(e_2H^i)^*V(v, j)^*$$ is positive definite.

The class map attaches to an irreducible subvariety $Z$ of codimension $d$ the Poincaré dual of
its fundamental class $[Z]$. This is an element of $FH^d(M, Z(d))$. This makes
$$\text{cl}: B^d(M) \longrightarrow F^dH^d(M, Z(d))$$
bihomogeneous of degree $(0, 0)$ if we give $B^d(M)$ the trivial Hodge structure.

The inclusion of $P^1 \subset P^n$ induces
$$H^0(P^n, Z(d)) \cong H^0(P^1, Z(d)) \otimes \mathbb{Z}.$$ So the
$M \subset P^n$ is an embedding when the canonical generator of $H^0(P^n, Z(d))$ defines
an element $v \in H^0(M, Z(d))$ (an element of bidegree $(0, 0)$). The $L$-operator is
now capping with $v$ and hence defines a bihomogeneous map.

$$L: H^d(M) \longrightarrow H^{d+2}(M, Z(d))$$

3.6 The category of Hodge structures. We want to define a category of Hodge
structures which allows to take direct sums of different weight. This is provided by
the following notion: if $R$ is a subring of $K$ with quotient field $K$ (a CKCR), then
a Hodge structure on $V$ is a decomposition of $V_\kappa = \bigoplus_{d \in \mathbb{Z}} V^d_\kappa$, called the weight
decomposition and a Hodge structure of weight $d$ on each summand $V^d_\kappa$. Equivalently,
we are given a decomposition $V = \bigoplus_{d \in \mathbb{Z}} V^d_\kappa$ such that $V^d_\kappa = V^d_{\kappa'}$ and
$$\bigoplus_{d \in \mathbb{Z}} V^d_{\kappa'}$$ is defined over $K$ for all $d \in \mathbb{Z}$.

The $R$-Hodge structures are indeed the objects of a category: a morphism
$V^d_\kappa \to V^d_\kappa$ of $R$-Hodge structures is an $R$-homomorphism whose canonicalization takes
$V^d_\kappa$ to $V^d_\kappa$. Notice that $\ker(q)$ resp. $\im(q)$ is a sub-Hodge structure of $V^d_\kappa$ resp. $V^d_\kappa$. This category has also direct sums:

$$V = \bigoplus_{d \in \mathbb{Z}} V^d_\kappa$$

This direct sum serves both as a categorical sum and as a categorical product:
both the inclusion $V \subseteq V \otimes V'$ and the projection $V \otimes V' \to V$ is a morphism of
$R$-Hodge structures. This makes the category of $R$-Hodge structures abelian. On top of that
the dual $Hom(V, R)$ and the tensor product $V \otimes V'$ have Hodge structures.

We can now say that cohomology defines a contravariant functor from the
category of compact kählerian manifolds to the abelian category of \( R \)-Hodge structures. When \( R = \mathbb{Q} \) this takes the product to the tensor product (via the Kummer formula).

The Deligne torus. The bi-grading on \( V_\mathbb{C} \) can also be given by a representation of the multiplicative group \( \mathbb{C}^\times \): we let \( \lambda \in \mathbb{C}^\times \) act on \( V^{p,q} \) as multiplication with \( \lambda^p \lambda^{-q} \) (the reason for choosing minus signs will become somewhat clear later). Note that this action is real in the sense that it preserves \( H^0 \) this is because \( z \) acts on the real subspace

\[
\langle z, \mathcal{I} \rangle \in H^{0,0} \oplus H^{0,0} \in H^{0,0}
\]

as a rotation over the angle \( -\arg(\lambda) \) and multiplication by \( \frac{1}{|\lambda|} \). In other words it is an action of the (real) Lie group underlying \( \mathbb{C}^\times \) given by the real 2x2 matrices \((a, -b, b, a)\) with \( a^2 + b^2 = 1 \), via \((a, -b, b, a) \mapsto \text{mult with } a + b\mathcal{I} \). In the theory of algebraic groups this is expressed as follows: consider \( \mathbb{C}^\times \) as the group of real points of an algebraic group \( S \) defined over \( \mathbb{R} \). (This is an example of Weil restriction of scalars: we have \( S = R_{\mathbb{C}^0}(G_{\mathbb{R}}) \).) The identification \( S(\mathbb{R}) = \mathbb{C}^\times \) can now be understood as defining a character on \( S \).

We denote this character by \( z \), so that \( z(a, -b) = a + b\mathcal{I} \).

The Hodge structure on \( V \) is then the decomposition of \( V_\mathbb{C} \) into \( S \)-eigenspaces, \( V^{p,q} \) being the eigenspace for the character \( z^{-p} \mathcal{I}^{-q} \). So \( Z(1) \) is defined by character \( z \mathcal{I} \).

The characters \( z, \mathcal{I} \) define an isomorphism

\[
S(\mathbb{C}) \xrightarrow{(z, \mathcal{I})} \mathbb{C}^\times \times \mathbb{C}^\times
\]

So \( S \) is an algebraic torus of dimension 2 with a somewhat unusual structure over \( \mathbb{R} \): complex conjugation takes \((a, -b)\) to \((-\overline{a}, \overline{b})\).

The inclusion \( \mathbb{R}^\times \subset \mathbb{C}^\times \subset S(\mathbb{R}) \) is one of real algebraic groups (it is a cocharacter). We put \( w = h^{-1} : G_m \rightarrow S \) (in matrix notation: \( w(t) = (\overline{t}, 0, t^{-1}) \)). In the above example \( w(t) \) acts on \( V^{p,q} \) as mult with \( t^p \mathcal{I}^{-q} \) and hence the restriction of the representation of \( S \) to \( G_m \) gives the weight decomposition of \( V_\mathbb{R} \). We call \( S \) the Deligne torus.
§4 Sheaves and their cohomology

4.1 Abelian sheaves. We fix for now a topological space \( X \).

An abelian presheaf on \( X \) is a covariant functor from the category of open subsets of \( X \) (morphisms are the inclusions) to the category of abelian groups. So this assigns to an open \( U \subset X \) an abelian group \( F(U) \) and to every inclusion \( U' \subset U \) a group homomorphism \( F(U) \to F(U') \) (denoted \( s \mapsto s|_{U'} \)) such that for \( U'' \subset U' \subset U \) this is the identity map and for \( U' \subset U' \subset U'' \), \( (s|_{U'})|_{U''} = s|_{U''} \). An element \( s \in F(U) \) is usually called a section of \( F \) over \( U \) and \( s|_{U'} \) is called the restriction of \( s \) to \( U' \).

An abelian presheaf \( F \) on \( X \) is called an abelian sheaf if for every collection \( \{ U_i \}_{i \in A} \) of open subsets of \( X \), the following diagram is exact:

\[
\begin{array}{ccc}
\{ s_i \}_{i \in A} & \xrightarrow{(s_i|_{U_i \cap U_j})_{i < j}} & (F(U_i \cap U_j))_{i < j} \\
\xrightarrow{\prod_{i \in A}} & \xrightarrow{\prod_{i < j}} & (F(U_i \cap U_j))_{i < j} \\
& \xrightarrow{(s_i|_{U_i \cap U_j})_{i < j}} & \prod_{i \in A} F(U_i \cap U_j) \\
\end{array}
\]

(\( s_i \) is \( s \) restricted to \( U_i \)) is exact. In other words, a collection \( \{ s_i \}_{i \in A} \) is obtained as a collection of restrictions of a section \( s \) of \( F \) over \( \bigcup_{i \in A} U_i \) if (and only if) for each pair \( i, j \in A \) \( s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \) and this \( s \) is then unique. Here we note that one can show that this property formally implies that \( F(\emptyset) \) is the initial group \( \{0\} \), so that in the above diagram we may restrict our indexes to the \( \{ i \in A \mid \text{rep } (s_i, s_j) \in A^2 \text{ with } U_i \neq \emptyset, \text{rep } U_i \cap U_j \neq \emptyset \} \).

We can extend \( F \) to the collection of all subsets of \( X \), by giving it an \( \mathcal{C}_X \) the value \( F(U) : = \lim \{ F(V) : V \text{ open neighborhood of } Y \} \). In other words, an element of \( F(Y) \) is represented by a pair \( (s, s|_{F(U)}) \) with \( Y \subset U \subset X \) and another pair \( (s', s'|_{F(U)}) \) represents the same element of \( F(Y') \) and only \( F \) there exists an \( x \in U \cap V \) and that \( s'|_{F(U)} = s|_{F(U)} \). In case \( Y \) is a single point \( \{y\} \), we call this the stalk of \( F \) at \( y \), denoted \( F_y \). So given an open \( U \subset X \), then any \( s \in F(U) \) determines \( \{ s|_{F_x} \}_{x \in U} \).

More generally, if instead of an inclusion, \( Y \) is any topological space and \( f : Y \to X \) a continuous map, then we have sheaf \( f^{-1}F \) on \( Y \) defined by \( (f^{-1}F)(V) := F(f(V)) \). It is called the preimage of \( F \) along \( f \). It is clear that for \( y \in Y \), \( (f^{-1}F)_y = F_x \).
Sheafification. A construction called sheafification turns an abelian presheaf \( \mathcal{F} \) into an abelian sheaf \( \hat{\mathcal{F}} \) with the same stalks: if \( U \subseteq X \) is open, then an element of \( \hat{\mathcal{F}}(U) \) is given by a collection \((s_x \in \mathcal{F}(U))_{x \in U}\) with the property that there exists an open covering \( \bigcup_{a} U_{a} = U \) of \( U \) and \( s_x \in \mathcal{F}(U_{a}) \) such that \( s_x = s_{x,a} \) for all \( x \in U_{a} \) (\( x \in A \)). It is not hard to check that \( \hat{\mathcal{F}} \) is an abelian sheaf and that we have \( \hat{\mathcal{F}} = \hat{\hat{\mathcal{F}}} \) in case \( \mathcal{F} \) was already an abelian sheaf.

4.2 Sheaves of rings etc. The notion of a (pre)sheaf of rings (or of \( R \)-algebras) for some fixed ring \( R \) is similar; restriction maps must then of course be ring (or \( R \)-algebra) homomorphisms. We can take this a step further: if we are given a sheaf \( \mathcal{O} \) of rings on \( X \) (in which case we call the pair \( (X, \mathcal{O}) \) a ringed space), then we have a notion of an \( \mathcal{O} \)-module: this is an abelian sheaf \( \mathcal{M} \) on \( X \) such that each \( \mathcal{M}(U) \) has the structure of a \( \mathcal{O}(U) \)-module in such a manner that when \( U \subseteq U' \subseteq X \) the section \( s \in \mathcal{M}(U) \) maps \( s|_{U'} \in \mathcal{M}(U') \) is an \( \mathcal{O}(U) \)-homomorphism: for \( r \in \mathcal{O}(U) \) we have \( r \cdot s|_{U'} = r|_{U'} \cdot s|_{U'} \). Here are some examples.

(i) \( \mathcal{C} \): \( \mathcal{O}^{\text{c}}_{x} \mathcal{O}_{U} \rightarrow \mathcal{O}(U, \mathbb{R}) = \{ \text{continuous functions } U \rightarrow \mathbb{R} \} \). This is in fact a sheaf of \( \mathbb{R} \)-algebras. We can replace \( \mathbb{R} \) by a topological ring or by a topological abelian group (we then get an abelian sheaf). For example, if \( A \) is an arbitrary abelian group, then by endowing it with the discrete topology we obtain the sheaf which assigns to every open \( U \subseteq X \) the group of locally constant functions \( U \rightarrow A \). This is also the sheaf that we get from sheafifying the constant presheaf \( U \rightarrow A \) and that is why we call this the constant sheaf (with stalks \( A \) ), denoted \( A_{X} \). If we take \( A = \mathbb{Z} \), then we get a sheaf of rings \( \mathbb{Z}_{X} \) on \( X \). So this turns \( X \) into a ringed space \( (X, \mathbb{Z}_{X}) \) and makes every abelian sheaf on \( X \) a module over \( \mathbb{Z}_{X} \).

(ii) Let \( M \) be a differentiable manifold. Then \( \mathcal{M}^{\text{dn}}_{\text{m}} \mathcal{O}_{U} \rightarrow \mathcal{C}(U) = \{ \text{smooth functions } U \rightarrow \mathbb{R} \} \) defines a sheaf of \( \mathbb{R} \)-algebras \( \mathcal{E}_{M} \). This makes \( M \) a ringed space. (The sheaf \( \mathcal{E}_{M} \) actually remembers the manifold structure of \( M \); a manifold of dimension \( m \) could
be defined as a ringed space \((M, \mathcal{C}_M)\) that is Hausdorff and locally like \((\mathbb{R}^m, \mathcal{C}_{\mathbb{R}^m})\).

If \(E^1_M\) is a real vector bundle of rank \(r\), then
\[
M \xrightarrow{\cdot^*} \mathcal{E}^M(U) = \{C^\infty\text{-sections} \text{ of } E^1_M\}
\]
defines a sheaf of \(\mathcal{E}_U\)-modules (locally free of rank \(r\)). In particular
\[
M \xrightarrow{\cdot^*} \mathcal{E}^M(U) = \{C^\infty\text{-k-forms on } U\}
\]
defines a sheaf \(\mathcal{E}_M^1\) (the sheaf of \(C^\infty\text{-k-forms}).

(i) For a complex manifold \(M\) of dim \(n\),
\[
M \xrightarrow{\cdot^*} \mathcal{O}(U) = \{\text{holomorphic functions } U \to \mathbb{C}\}
\]
defines a sheaf \(\mathcal{O}_M\) of \(\mathbb{C}\)-algebras. It is called the structure sheaf of \(M\). As in the \(C^\infty\)-setting, we could define a complex manifold of dim \(m\) as a ringed space \((M, \mathcal{O}_M)\) that is locally like \((\mathbb{C}^m, \mathcal{O}_{\mathbb{C}^m})\). Note that the stalk \(\mathcal{O}_{\mathbb{C}^m}\) is nothing but \(\mathbb{C}[[z_1, \ldots, z_m]]\), the \(\mathbb{C}\)-algebra of (absolutely) convergent power series.

A holomorphic vector bundle \(E^1_M\) of rank \(r\) determines a sheaf \(\mathcal{O}_M(E)\),
\[
M \xrightarrow{\cdot^*} \mathcal{E}^M(U) = \{\text{holomorphic sections of } E^1_M\}.
\]

If \(E\) is a sheaf of \(\mathcal{O}_M\)-modules, locally free of rank \(r\). In particular, we have
\[
\mathcal{O}^1_M : M \xrightarrow{\cdot^*} \text{local \(k\)-forms on } U^3.
\]

(ii) A ring \(R\) defines a ringed space \((\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)})\). A point of \(\text{Spec}(R)\) is given by a prime ideal \(p \subset R\) and the stalk of \(\mathcal{O}_{\text{Spec}(R)}\) at this point is the localization \(R_p\). A scheme is a ringed space that is locally \(\mathbb{C}\). If we restrict the rings \(R\) to \(k\)-algebras (\(k\ a\ \text{ring}\) we get the notion of a scheme over \(k\).

4.5 The category of modules over a ringed space. We now assume that we are also given a sheaf of rings \(\mathcal{O}\) on \(X\). We focus our attention on \(\mathcal{O}\)-modules and make these the objects of category \(\text{Mod}_X\) (as we have seen, this gives for \(\mathcal{O} = \mathbb{Z}\) all abelian sheaves on \(X\); we shall then denote this category \(\text{Ab}_X\).) The morphisms of this category are called \(\mathcal{O}\)-homomorphisms and are defined as follows. Given \(\mathcal{O}\)-modules \(F, G\), then an \(\mathcal{O}\)-homomorphism \(\alpha : F \to G\).
assigns to every \( U \subseteq X \) an \( \mathcal{O}(U) \)-homomorphism \( \varphi(U) : F(U) \to G(U) \) such that if \( U' \subseteq U \) and \( s \in F(U) \), then \( \varphi(U')(s|_{U'}) = \varphi(U)(s)|_{U'} \).

Hence we get for every \( x \in U \) a \( \mathcal{O}_x \)-homomorphism \( \varphi_x : F_x \to G_x \). We define \( \mathcal{O} \)-modules \( \ker(\varphi), \text{im}(\varphi), \text{Coker}(\varphi) \) whose stalk at \( x \) is \( \ker(\varphi_x), \text{im}(\varphi_x), \text{Coker}(\varphi_x) \). For \( \ker(\varphi) \) we simply take \( M \rightrightarrows \ker(\varphi(U)) \), this is indeed an \( \mathcal{O} \)-module. However \( U \to \text{im}(\varphi(U)) \) and \( U \to \text{Coker}(\varphi(U)) \) are in general only abelian presheaves and so we define \( \text{Im}(\varphi) \) and \( \text{Coker}(\varphi) \) as their sheafifications.

So if we have a complex of \( \mathcal{O} \)-modules

\[
\cdots \to E^i \xrightarrow{\delta^i} E^{i+1} \xrightarrow{\delta^{i+1}} E^{i+2} \to \cdots
\]

(meaning that each \( \delta^i \) is an \( \mathcal{O} \)-homomorphism and \( \delta^i \delta^{i-1} = 0 \)), then we can form the cohomology module \( H^i(E^\bullet) = \ker(\delta^i)/\text{im}(\delta^{i-1}) \). We say that the complex is exact at \( E^i \) if \( H^i(E^\bullet) = 0 \), i.e., \( \text{im}(\delta^{i-1}) = \ker(\delta^i) \).

We also have an obvious notion of direct sum \( F \oplus G \) which takes the role of both categorical product (we can project onto a summand) and sum (we have an inclusion of each summand): \( (F \oplus G)(U) = F(U) \oplus G(U) \).

These properties essentially say that the category \( \mathbf{Mod}_\mathcal{O} \) is an abelian category. This means that it is very much like the category of modules over a fixed ring \( R \) (which is in fact a special case: take \( X = \{x\} \) and \( \mathcal{O}(X) = R \)). For example, if \( F \) and \( G \) are objects of an abelian category, \( \text{Hom}_\mathcal{O}(F,G) \) is naturally an abelian group. (*)

There is more similarity; for example for two \( \mathcal{O} \)-modules \( F \) and \( G \), we have defined \( F \otimes \mathcal{O}_X G \) and \( \text{Hom}_\mathcal{O}(F,G) \) as the sheafifications of the presheaves

\[
\mathcal{O}(U) \to F(U) \otimes G(U) \text{ and } \text{Hom}_\mathcal{O}(F(U), G(U)).
\]

Here are some important examples.

1) Let \( M \) be a \( C^0 \)-manifold of dim. \( n \). A homomorphism of real \( C^0 \)-vectorbundles \( E \to F \) over \( M \) can also be regarded as a homomorphism of \( C^0 \)-modules \( \mathcal{E}_M(E) \to \mathcal{E}_M(F) \).

(*) For the formal definition of an abelian category, see ...
The exterior derivative determines a homomorphism of IR-algebras $E^* -\rightarrow E^*$. The Poincaré lemma amounts to the assertion that the complex below is exact:

$$0 \rightarrow E^* -\rightarrow E^* -\rightarrow \cdots -\rightarrow E^* \rightarrow 0$$

2) Let $M$ be a complex manifold of (complex) dimension $n$. A homomorphism of holomorphic vector bundles $\xi -\rightarrow \eta$ over $M$ can be regarded as a homomorphism of $\mathcal{O}_M$-modules. The Dolbeault complex

$$0 \rightarrow \mathcal{O}_M(\xi) -\rightarrow \mathcal{O}_M(\xi) -\rightarrow \cdots -\rightarrow \mathcal{O}_M(\xi) \rightarrow 0$$

is an exact complex of $\mathcal{O}_M$-modules (and $\bar{\partial}$ is $\mathcal{O}_M$-linear). The holomorphic Poincaré lemma asserts that the complex

$$0 \rightarrow \mathcal{O}_M -\rightarrow \mathcal{O}_M -\rightarrow \cdots -\rightarrow \mathcal{O}_M \rightarrow 0$$

is an exact complex of $\mathcal{O}_M$-modules.

The standard notions and tools of homological algebra (as used in the category of modules over a fixed ring) extend with little or no change to any abelian category $\mathcal{A}$ if that category has enough injectives. This means the following. We say that an object $I$ of $\mathcal{A}$ is injective if $\text{Hom}_{\mathcal{A}}(\cdot, I)$ is exact, i.e. if for every object $F$ of $\mathcal{A}$, an $\mathcal{A}$-homomorphism from a subobject of $F$ to $I$ always extends to $F$. We say that $\mathcal{A}$ has enough injectives if every object of $\mathcal{A}$ embeds in an injective object. The category of modules over a fixed ring $R$ has this property.

Lemma 25. The abelian category $\text{Mod}_\mathcal{O}$ has enough injectives.

Proof. Let $F$ be an $\mathcal{O}$-module. Choose for every $x \in X$, an $\mathcal{O}$-embedding of $F_x$ in an injective $\mathcal{O}$-module $J_x$ (we assume known that the category of modules over a given ring $R$ has enough injectives). Then define an $\mathcal{O}$-module $J$ by \[ J(U) = \prod_{x \in X} J_x. \] We then have an obvious embedding $F \hookrightarrow J$. It is straightforward to check that $J$ is an injective $\mathcal{O}$-module.
4.3 Homological algebra for abelian categories. Injective objects serve to define cohomological invariants (see below) but are rarely used to compute them. We list some properties that any abelian category $A$ with enough injectives possesses (but think of $A = \text{Mod } R$), the proofs are standard.

(i) The complexes $\mathbb{C}^i = (C^i \to C^{i-1})$ with $C^i = 0$ for $i < 0$ form a category $C_+(A)$: a morphism $\varphi^i : C^i \to D^i$ in this category is a chain map in $A$; i.e., a commutative ladder

\[
\begin{array}{cccc}
C^i & \to & C^{i-1} \\
\downarrow \varphi^i & & \downarrow \varphi^{i-1} \\
D^i & \to & D^{i-1}
\end{array}
\]

We then have induced maps of cohomology objects $H^i(C^i) \to H^i(D^i)$ (see $\S$). If these are all isomorphisms, then we say that $\varphi^i$ is a quasi-isomorphism.

We embed $A$ in $C_+(A)$ by identifying the object $F$ with the complex $\cdots \to 0 \to F \to 0 \to \cdots$ where $F$ is put in degree zero.

(ii) A (right) resolution of an object $F$ of $A$ is an exact sequence in $A$ of the form

\[
0 \to F \to C^\cdot \to C^2 \to \cdots
\]

which we prefer to regard as a quasi-isomorphism $F \to C^\cdot$. The resolution is said to be injective if each $C^i$ is. More generally, an injective representative of a complex $C^\cdot$ in $C_+(A)$ is a quasi-isomorphism $C^\cdot \to I^\cdot$ in $C_+(A)$ with each $I^i$ injective. Such an injective representative always exists.

Examples. For a $C^0$-manifold $M$ the De Rham complex

\[
0 \to \mathcal{E}_M \to \mathcal{E}_M \to \mathcal{E}_M \to \cdots
\]

is a resolution of $\mathbb{R}_M$.

For a complex manifold $M$ the holomorphic De Rham complex

\[
0 \to \Omega_M^0 \to \Omega_M^1 \to \Omega_M^2 \to \cdots
\]

is a resolution of $\mathbb{C}_M$. If $\mathcal{E}_M^1$ is a holomorphic vector bundle, then the Dolbeault complex

\[
0 \to \mathcal{E}_M^0 \to \mathcal{E}_M^1 \to \mathcal{E}_M^2 \to \cdots
\]
is a resolution of $\mathcal{O}(\xi)$.

Except for some trivial cases, none of these resolutions is injective.

(5) Two morphisms $\varphi, \psi : C \to D$ in $\mathcal{C}_+(\mathcal{A})$ are called chain homotopic if there exist

$$\begin{align*}
\delta^i & : C^i \to C^{i+1} \\
\delta^i & : D^i \to D^{i+1}
\end{align*}$$

such that

$$\begin{align*}
\delta^i \varphi^i + \varphi^{i+1} \delta^i & = \psi^i - \psi^{i+1} \\
\varphi^i - \psi^i & = \eta^i \text{ for all } i. \text{ Then } \varphi^i \text{ and } \psi^i \text{ induce the same maps on cohomology: } H^i(\varphi) = H^i(\psi)
\end{align*}$$

for all $i$. If conversely, $H^i(\varphi^i) = H^i(\psi^i)$ for all $i$ and each $D^i$ is injective, then $\varphi^i$ and $\psi^i$ are chain homotopic.

We also note that being chain homotopic is an equivalence relation.

(6) We say that a chain map $\varphi : C \to D$ in $\mathcal{C}_+(\mathcal{A})$ is a chain homotopy equivalence if there exists a chain map $\psi : D \to C$ such that $\varphi \circ \psi$ resp. $\psi \circ \varphi$ is chain homotopic to the identity map of $C$ resp. $D$. This is an equivalence relation. Any two injective representatives of an object $C$ of $\mathcal{C}_+(\mathcal{A})$ are chain homotopic:

$$\begin{align*}
\text{left map} : C & \cong I_o \\
\text{right map} : C & \cong I_1
\end{align*}$$

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let $T : \mathcal{A} \to \mathcal{B}$ be a functor. Such a functor automatically takes $\mathcal{F} \otimes \mathcal{G}$ to $T(\mathcal{F}) \otimes T(\mathcal{G})$ and is additive in the sense that if $\varphi_0, \varphi_1 : F \to G$ are $\mathcal{A}$-morphisms, then $T(\varphi_0 + \varphi_1) = T(\varphi_0) + T(\varphi_1)$. We say that $T$ is left-exact if $T$ takes kernels to kernels. In other words, if $0 \to C \to C' \to C''$ is exact in $\mathcal{A}$, then $0 \to T(C) \to T(C') \to T(C'')$ is exact in $\mathcal{B}$. We have a similar definition for right-exact (must preserve cokernels). If $T$ is both it is called exact.

Example. Let $(X, \theta)$ be a ringed space. Then the global section functor

$$\Gamma(X, -) : \text{Mod} \mathcal{O} \to \text{Mod} \theta(X) \quad (\text{the category of } \mathcal{O}(X)\text{-modules})$$

is left exact. Indeed, if $\varphi : F \to G$ is a $\mathcal{O}$-homomorphism, then $(\text{Ker} \varphi)(X) = \text{Ker}(\varphi(X) : F(X) \to G(X))$ by definition. On the other hand,
Cohom (p(X) : F(X) → G(X)) ⊆ Cohom (p) (X) and in general this is not an equality (we had to sheafify the Cohom presheaf to get the Cohom sheaf). In other words, R(p(X)) will in general not be right exact.

An important variation on this example is the direct image functor. Here we assume given a continuous map f : X → Y. If F is an abelian sheaf on X, then

\[ Y \rightsquigarrow V \rightarrow \mathcal{F}(f^* V) \]

defines an abelian sheaf on Y, called the direct image of \( \mathcal{F} \) and denoted \( f_* \mathcal{F} \). This defines a functor \( f_* : \text{Ab}_X \rightarrow \text{Ab}_Y \). This functor is still right exact (for the same reason as before). If we restrict to \( \mathbb{O} \)-modules then \( f_* \) determines a functor, still denoted

\[ f_* : \text{Mod}_\mathbb{O} \rightarrow \text{Mod}_{f_* \mathbb{O}} \]

Note that in case \( X \neq \emptyset \) and \( Y = f(X) \) (so \( f \) maps all of \( X \) to a single point) then \( f_* \) is just the global section functor.

On the other hand, if \( f : Y \rightarrow X \), then \( f^* \) defines a functor \( \text{Mod}_\mathbb{O} \rightarrow \text{Mod}_{f^* \mathbb{O}} \) and this functor is exact: indeed, if \( 0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0 \) is an exact sequence then \( 0 \rightarrow f^* \mathcal{F} \rightarrow f^* \mathcal{G} \rightarrow f^* \mathcal{H} \rightarrow 0 \) is still exact (just look at the stalks).

(5) Let \( T : A \rightarrow B \) be a functor between abelian categories and assume that \( A \) is as above, i.e., that it has enough injectives. Given \( C \in C_+(A) \), choose an injective representative \( C \rightarrow I' \). Then \( T(I') \in C_+(B) \). Clearly \( T \) takes a chain homotopy to a chain homotopy. So it follows from the preceding that the chain homotopy class of \( T(I') \) only depends on \( C' \). In particular, \( H^i(T(I')) \) only depends on \( C' \). This defines a functor \( C_+(A) \rightarrow B \). Indeed, a chain map \( C' \rightarrow D' \) in \( C_+(A) \) extends to injective representatives:

\[ C' \rightarrow D' \]

with the extension \( \widetilde{C}' \) being unique up to chain homotopy so that \( H^i(T(\widetilde{C}')) : H^i(T(I')) \rightarrow H^i(T(J')) \) is independent of the choice of \( \widetilde{C}' \).
We call $H^q(T(I^\bullet))$ the $q$th hypercohomology object of $C$ and denote it by $R^qT(I^\bullet)$.

Note that for a single object $C$ of $\mathcal{A}$ (viewed as a complex by placing it in degree zero) we will have $R^qT(I) = 0$ when $q < 0$. It is clear from the definition that when $I$ is injective (it is its own resolution) then $R^qT(I) = 0$ for all $q \neq 0$, and $R^0T(I) = I$.

**Prop.** Let $C$ be an object of $\mathcal{A}$. If $T$ is left exact, then $R^0T(C) \cong T(C)$. If $T$ is exact, then in addition $R^qT(C) = 0$ for $q \neq 0$.

**Proof.** Let $0 \to C \to I^\bullet \to I^\bullet \to \ldots$ be an injective resolution and consider the sequence $0 \to T(C) \to T(I^\bullet) \to T(I^\bullet) \to \ldots$. If $T$ is left exact, then $T(A) = \ker(T(I) \to T(I^\bullet)) = R^0T(C) \cong C$. If $T$ is exact, then this sequence is exact and hence $R^qT(C) = 0$ for $q > 0$.

(6) Assume now $T$ left exact. Then $R^1T$ viewed as a functor $\mathcal{A} \to \mathcal{B}$ is called the $q$th right derived functor of $T$. This is because a short exact sequence in $\mathcal{C}^\oplus(\mathcal{A})$,

$$0 \to C \to D \to E \to 0 \quad \text{(meaning: } 0 \to C \to D \to E \to 0 \text{ exact for all } i)$$

gives rise to a long exact sequence in $\mathcal{B}$ (it begins with zero, so it lies in $\mathcal{C}^\oplus(\mathcal{B})$)

$$\cdots \to R^qT(E) \xrightarrow{\delta^q} R^qT(D) \to R^qT(C) \to R^{q+1}T(E) \xrightarrow{\delta^{q+1}} R^{q+1}T(D) \to \cdots$$

and this is functorial in the sense that a morphism $0 \to C' \to D' \to E' \to 0$ of short exact sequences (a in the opposite diagram)

$$0 \to C' \to D' \to E' \to 0$$

gives rise to a commutative ladder (a morphism in $\mathcal{C}^\oplus(\mathcal{B})$).

If the short exact sequence lies in $\mathcal{A}$ rather than $\mathcal{C}^\oplus(\mathcal{A})$: $0 \to C \to D \to E \to 0$,

then the above sequence begins as

$$0 \to T(C) \to T(D) \to T(E) \xrightarrow{\delta^0} R^0T(C) \to R^1T(D) \to \ldots$$

This shows that the $T$-image of this sequence is exact if and only if $\delta^0 = 0$.

Here is one of our main examples.

**Def.** Let $(X, \mathcal{O})$ be a ringed space. Then the $q$th right derived functor of the global section functor $\mathcal{M} \xrightarrow{\operatorname{eMod}_\mathcal{O}} \mathcal{M}(X) \xrightarrow{\operatorname{cMod}_{\mathcal{O}(X)}} \mathcal{M}$ is called the $q$th cohomology module, denoted $H^q(X; \mathcal{M})$ (so this is a $\mathcal{O}(X)$-module).
If $A$ is an abelian group (and $\mathcal{O} = \mathbb{Z}_A$), then $H^q(X, A)$ is written $H^q(X; A)$.

We will shortly make the link with singular cohomology and de Rham cohomology.

It follows from the above that an exact sequence of $\mathcal{O}$-modules

$$0 \to F \to G \to H \to 0$$

gives rise to a long exact sequence of $\mathcal{O}(X)$-modules

$$0 \to F(X) \to G(X) \to H(X) \xrightarrow{g^*} H^1(X, F) \to H^1(X, G) \to H^1(X, H) \xrightarrow{g^*} H^2(X, F) \to \cdots$$

For a continuous map $f : X \to Y$ this produces such a sequence of $f^*_\mathcal{O}$-modules:

$$0 \to f^*F \to f^*G \to f^*H \xrightarrow{g^*} R^1f^*_\mathcal{O}F \to \cdots$$

On the other hand, as we noticed earlier, for a continuous map $g : Z \to X$,

$$g^{-1} : \mathcal{M}od g \to \mathcal{M}od g^* \text{ is exact}. \text{ It follows that } R^qg^{-1} = 0 \text{ for } q > 0.$$ 

In particular, if $F \to J'$ is an injective resolution, then $g^{-1}F \to g^{-1}J'$ is still a resolution.

It need not be injective, but we can always choose an injective resolution $g^{-1}J \to J'$, so that $g^{-1}F \to g^{-1}J \to J'$ is an injective resolution of $g^{-1}F$.

We then have chain maps:

$$J'(X) \xrightarrow{g^{-1}} g^{-1}J'(X) \to J'(Z)$$

which on cohomology define

$$H^q(X, F) \to H^q(Z, g^{-1}F)$$

This is independent of choices; it gives us the contravariant character of cohomology.

In case $Z$ is a subset of $X$, and $g$ is just the inclusion, we can arrange that each $g^{-1}F$ is injective: just follow the construction of Lemma 25: choose but for every $x \in X$ an embedding of $F_x$ in an injective $\mathcal{O}_x$-module $J_x$ and put $J(x) = \bigcup x J_x$.

We are now able to better understand $R^qf^*_\mathcal{O}F$ as a sheaf on $Y$. If $F \to J$ is an injective resolution with this property, then for every open $V \subseteq Y$, the restriction of $J$ to $f^{-1}V$ is an injective resolution of $f^{-1}V$. Its cohomology is therefore equal to $H^q(f^{-1}V, F)$. It follows that $R^qf^*_\mathcal{O}F$ is the sheafification of the presheaf $V \to H^q(f^{-1}V, F)$. In particular,
\[(R^qf_\# F)_y = \lim_{V \to V} H^q(f^{-1}V, F)\]

(7). We have an abstract De Rham theorem, which allows us to relax the condition that a resolution be injective. Let \( T: \mathcal{A} \to \mathcal{B} \) be as before (a left exact functor of abelian categories and \( \mathcal{A} \) has enough injectives). Let us say that an object \( C \) of \( \mathcal{A} \) is \( T \)-acyclic if \( R^i T(C) = 0 \) for all \( i > 0 \). So any injective object is \( T \)-acyclic.

Theorem 27 (abstract De Rham theorem) Let \( F \) be an object of \( \mathcal{A} \) and \( F \to C \) a resolution of \( F \) with \( C^q = 0 \) for \( q < 0 \). Then we have natural maps

\[ H^q(T(C)) \longrightarrow R^q T(F) \]

and these are isomorphisms when each \( C^q \) is \( T \)-acyclic.

Proof. Let \( Z^q = \ker(C^q \to C^{q+1}) \) so that we have short exact sequences

\[ 0 \to F \to C^q \to Z^q \to 0 \quad \text{and} \quad 0 \to Z^q \to C^q \to Z^{q+1} \to 0 \quad (q \geq 1) \]

The associated long exact sequences for \( T \) have boundary maps

\[ \cdots \to R^i T(C^q) \to R^i T(Z^{q+1}) \to R^{i+1} T(Z^q) \to R^{i+1} T(C^q) \to \cdots \]

which are isomorphisms if \( R^iT(C^q) = R^iT(Z^q) = 0 \). We can compose them to form

\[ R^q T(F) \overset{\delta}{\to} R^q T(Z^q) \overset{\delta}{\to} R^{q+1} T(Z^q) \overset{\delta}{\to} \cdots \overset{\delta}{\to} R^{q+1} T(Z^{q+1}) \overset{\delta}{\to} R^{q+1} T(C^q) \]

Since \( T \) is left exact, \( T(Z^q) = \ker(T(C^q) \to T(C^{q+1})) \) and so this produces a map

\[ H^q(T(C^q)) \longrightarrow R^q T(F). \]

This will be an isomorphism if all the boundary maps are isomorphisms and this is the case when each \( C^q \) is \( T \)-acyclic. \( \Box \)

4.4 Soft and flabby sheaves In this section we assume the spaces to be paracompact Hausdorff.

Definition. Let \( X \) be a space. We say that an abelian sheaf \( F \) is soft if for every closed subset \( C \subseteq X \) the restriction \( F(X) \to F(C) \) is onto.

Proposition 28. A soft abelian sheaf \( F \) on \( X \) is flabby for \( T(X, -) \) and more generally for \( f_* \), where \( f: X \to Y \) is continuous.
We need two lemmas.

**Lemma 29.** An injective sheaf $\mathcal{I}$ on $X$ is soft.

**Proof.** Let $\mathcal{I}_f$ be the sheaf of discontinuous sections of $\mathcal{I}$: $\mathcal{I}_f(U) = \prod_{x \in U} \mathcal{I}_f(x)$. We have an obvious embedding $\mathcal{I} \subseteq \mathcal{I}_f$. Since $\mathcal{I}$ is injective, every identity $\mathcal{I} = \mathcal{I}_f$ embeds $\mathcal{I}$ into $\mathcal{I}_f$. If $\mathcal{C}(X)$ is closed, then any $\mathcal{C} \in \mathcal{C}(X)$ extends to $\mathcal{S} \in \mathcal{I}_f(X)$ by putting $\mathcal{S}(x) = 0$ for $x \in X \setminus C$, and then $\mathcal{S} \in \mathcal{I}(X)$ restricts to $\mathcal{S}$. 

**Lemma 30.** Let $\alpha : F \to G$ be an exact sequence of abelian sheaves on $X$ with $F$ soft. Then $\alpha : F(X) \to G(X)$ is exact. If moreover $G$ is soft, then so is $F$.

**Proof.** We must show that any $\mathcal{G} \in \mathcal{C}(X)$ is the image of some $\mathcal{F} \in \mathcal{C}(X)$. We can cover $X$ by open subsets $\mathcal{U}$ such that $\mathcal{U}^\subset$ is the image of some element of $\mathcal{G}(\mathcal{U})$. Since $X$ is paracompact, this covering admits a locally finite subcovering by closed subsets $\mathcal{C}_x \in \Delta$. So we have $\mathcal{G}(x)$ for every $x \in A$, $x \in \mathcal{C}(x)$ whose image in $\mathcal{H}(\mathcal{C})$ is $\mathcal{G}(\mathcal{C})$.

For every subset $B \subseteq C \subseteq A$, $C_B = \bigcup_{x \in C} C_x$ is a locally finite union of closed subsets of $X$ and is dense closed. Suppose first for some $B \subseteq C$, we succeed in finding an $\mathcal{S}_y \in \mathcal{C}(C_B)$ which maps onto $\mathcal{S}|C_B$. If $B \subseteq C$ we are done, of course. Otherwise let $x \in B \setminus C$. We then construct an extension $\mathcal{S}_x \in \mathcal{C}(x)$ which maps to $\mathcal{S}|\mathcal{C}_x$ as follows. It is clear that the restrictions of $\mathcal{S}_y$ and $\mathcal{S}_x$ to $\mathcal{C}_x \cap C_B$ differ by an element of $\mathcal{F}(\mathcal{C}_x \cap C_B)$. Since $\mathcal{F}$ is soft, this element is the restriction of some $\mathcal{S} \in \mathcal{F}(X)$. Now replace $\mathcal{S}_y$ by $\mathcal{S}_x$, the image of $\mathcal{S}|\mathcal{C}_x$ in $\mathcal{F}(\mathcal{C}_x)$. Then $\mathcal{S}_y$ still maps to $\mathcal{S}|C_B$, but has now the additional property that it has the same restriction to $\mathcal{C}_x \cap C_B$ as $\mathcal{S}_x$.

So together they define an $\mathcal{S}|\mathcal{C}_x \in \mathcal{C}(\mathcal{C}_x)$ as asserted.

By taking the limit (basepoints if $A$ uncountable) we obtain an $\mathcal{S} \in \mathcal{G}(X)$ mapping to $\mathcal{S}$. Suppose now that $\mathcal{G}$ is also soft. If $\mathcal{C}(X)$ is closed and $\mathcal{S} \in \mathcal{H}(C)$, then by the above $\mathcal{S}$ is the image of some $\mathcal{S} \in \mathcal{G}(C)$. Since $\mathcal{G}$ is soft, $\mathcal{S}$ is the restriction of some $\mathcal{S} \in \mathcal{G}(X)$. Then the image of $\mathcal{S}$ in $\mathcal{H}(X)$ extends $\mathcal{S}$.
Proof of Prop. 28. Let $F - F^9 \to F^{q-1} \to \cdots$ be an injective resolution and put $L^q = \text{Ker}(d^q : F^q \to F^{q+1})$. So $L^0 = F$ and we have exact sequences

$$0 \to L^q \to F^q \xrightarrow{d^q} F^{q-1} \to L^{q-1} \to 0 \quad (q \geq 1)$$

By assumption $L^0 = F$ is soft. Since $F^{q+1}$ is injective, it is soft by Lemma 29. If $L^q$ is soft, then so is $L^{q-1}$ (by Lemma 30). So it follows with induction that each $L^q$ is soft. Hence by Lemma 30 the sequence

$$0 \to L^{q-1}(x) \to F^{q-1}(x) \xrightarrow{d^{q-1}} F^q(x) \to L^q(x) \to 0 \quad (q \geq 1)$$

is exact. In other words $L^q(x) = \text{Ker}(d^q(x) : F^q(x) \to F^{q+1}(x))$ equals the image of $F^{q-1}(x)$ under $d^{q-1}. This proves that $H^q(x; F) = 0$.

The restriction of $F$ to a closed subset will still be soft. So if $D \subset X$ is closed, $H^q(f^{-1}D; F) = 0$ for $q \geq 0$. Since every point of $X$ has a neighborhood basis of closed neighborhoods it follows that every stalk of $R^q f_* F$ ($q \geq 0$) is zero.

Example (Link with singular cohomology). Recall that a singular $q$-simplex of $X$ is a continuous map $\Delta^q \to X$. A $q$-cochain of $X$ assigns to each such a number $a$ an integer. These make up the group of singular $q$-cochains

$$C^q(X) = \mathbb{Z}^{\text{cont. maps } \Delta^q \to X}$$

A boundary operator $d : C^q(X) \to C^{q+1}(X)$ comes from the map which assigns to a singular $(q+1)$-simplex a singular $q$-chain, so that we have the cochain counit $C^q(X) \to C^q(X)$, and the singular cohomology $H^q(X)$ of $X$ is by definition the cohomology of this complex. Then $X \subset U \subset C^q(U)$ is an abelian presheaf.

Denote by $C^q_X$ its sheafification. So an element of $C^q_X(U)$ is represented by an open covering $U$ of $X$ and a map which assigns to each $q$-simplex $a : \Delta^q \to X$ whose image lies in a member of $U$ an integer. Thus, $C^q_X(U) \subset C^q(X(U))$. With the help of a “barycentric subdivision operator” you can show that this is in fact a chain homology equivalence. So $H^q(C^q_X(U)) = H^q(X)$. Each $C^q_X$ is soft if
$\mathcal{C}^q_x$ is closed and $s \in \mathcal{C}^q_x(Y)$, then extend $s$ to all of $\mathcal{C}^q_x(X)$ by letting it be zero on each singular $q$-simplex whose image is not in $Y$.

**Corollary 3:** Suppose that each point of $X$ has a basis of acyclic neighborhoods $U \subseteq X$ (meaning that $H^q(U_x) = \{Z \in \mathbb{Z} \mid q = 0\}$), then $\mathcal{C}^q_x$ is a soft resolution of the constant sheaf $\mathbb{Z}_x$ and hence $H^q(X, \mathbb{Z}_x) = H^q(X)$. More generally, if $f : X \to Y$ is continuous, then $(R^q f_* \mathbb{Z}_x)_y = \lim_{V \ni y} H^q(f^{-1}(V))$.

**Proof.** It is clear that $H \subseteq C^q_x$ is acyclic, then since $\mathcal{C}^q_x(U) \subseteq C^q_x(U)$ is a quasi-iso, we have $H^q(\mathcal{C}^q_x(U)) = H^q(U) = \{Z \in \mathbb{Z} \mid q = 0\}$ and so $\mathcal{C}^q_x$ is a resolution of $\mathbb{Z}_x$. Each $\mathcal{C}^q_x$ is soft and hence $H^q(\mathcal{C}^q_x(Y)) = H^q(X, \mathbb{Z}_x)$. We already noticed that $H^q(\mathcal{C}^q_x(X)) = H^q(X)$. The second assertion follows from this. $\Box$

**Remark:** The assumption of local acyclicity holds for every manifold (clear), for every simplicial complex (also clear) and in fact for any analytic variety (less clear).

For a manifold $M$, one may consider the sheafification $\mathcal{E}^q_M$ of the presheaf which assigns to $M \cup V$, $\mathbb{Z}$-maps $\Delta^q \to V$. One can show that $\mathcal{E}^q_M \to \mathcal{C}^q_M$ is a chain hom. equivalence.

§5 Return to De Rham complexes)

5.1 The holomorphic De Rham complex. We find soft resolutions with the help of:

**Prop. 32.** Let $F$ be a sheaf on $X$ with the property that for every open covering $U$ of $X$ $F$ admits a partition of 1 subordinate to $U$, in the sense for every $U \subseteq X$ we are given a sheaf homomorphism $\eta_U : F \to F$ which is zero on $X \setminus U$ and is such that the sum $\sum_{U \subseteq X} \eta_U$ makes sense (locally only finitely many terms are $\neq 0$) and equals the identity $F = F$. Then $F$ is soft.

**Proof.** Let $C \subseteq X$ be closed, and let $s \in F(C)$. So for every $x \in C$ there exists an open $\tilde{U}_x \ni x$ and an $s^x \in F(\tilde{U}_x)$ with $s \upharpoonright \tilde{U}_x = s^x \upharpoonright \tilde{U}_x$. Then...
Choose \( U_k \) to be open such that \( \overline{U}_k \subset \overline{U}_x \) and choose a partition of \( 1 \) for \( F \) subordinate to the covering \( \{ X \subset C \} \cup \{ U_x \} \). Then for every \( x \in C \), \( \eta_{\overline{U}_x}(s^x) \in F(\overline{U}_x) \) is zero on \( \overline{U}_x \). So we have a \( \delta^x \in F(X) \) characterized by \( \delta^x \mid \overline{U}_x = s^x \) and \( \delta^x \mid X \setminus U_x = 0 \). The sum \( \sum \delta^x \) is locally finite and hence defines an element \( \delta \in F(X) \). Clearly \( \delta \mid C = s \).

Let now \( M \) be a \( C^\infty \)-manifold. It is well-known that \( M \) admits a \( C^\infty \)-partition of \( 1 \) subordinate to any open covering of \( M \). This means that the hypothesis of the previous proposition is fulfilled for \( \mathcal{E}_M \), but also for \( \mathcal{E}_M^{\prime}(\xi) \) where \( \mathcal{E}_M^{\prime} \) is any real or complex vector bundle of \( M \). Hence the De Rham resolutions and the Dolbeault resolutions are soft.

**Corollary 33.** (1) For a \( C^\infty \)-manifold \( M \), \( H^0_{\text{DR}}(M) = H^0(M, \Omega^1_M) \) (\( = H^0(M, \Omega^1_{\mathbb{R}}) \)).

(2) For a holomorphic vector bundle \( \mathcal{E} \) on a complex manifold \( M \),

\[ H^q(\mathcal{E}^\prime(\xi), \overline{\delta}) = H^q(M, \Omega^p(\xi)) \]

in particular,

\[ H^q(\mathcal{E}^\prime(\Omega^p_M), \overline{\delta}) = H^q(M, \Omega^p_M). \]

If \( M \) is a complex manifold, then the holomorphic De Rham complex

\[ \Omega^*_M = 0 \rightarrow \Omega^1_M \rightarrow \Omega^2_M \rightarrow \ldots \]

is a resolution of \( \mathcal{E}_M \) but it need not be acyclic (as we have seen, \( H^q(M, \Omega^p_M) \) can be nonzero for \( q > 0 \)). But we have a natural embedding \( \Omega^*_M \subset \mathcal{E}_M^\prime \) of resolutions of \( \mathcal{E}_M \) with \( \mathcal{E}_M^{\prime} \) soft. Notice that \( \Omega^*_M \cap F^p\mathcal{E}_M^{\prime} \) is the subcomplex

\[ F^p\Omega^*_M: 0 \rightarrow 0 \rightarrow \Omega^p_M \rightarrow \Omega^{p+1}_M \rightarrow \ldots \rightarrow \Omega^n_M \rightarrow 0 \rightarrow 0 \ldots \]

in degree \( p \)

(such a truncation is called the stupid truncation, which is the poor English translation of a of the French term "filtration bête"; it is open denoted \( \Omega^*_M \)). It is clear that \( F^p\Omega^*_M \subset \Omega^*_M \) is the sheaf of closed holomorphic \( p \)-forms, then \( \Omega^*_M \in C^p \) means \( \Omega^*_M \in C^p \) placed in degree \( p \).
a resolution and \( F^p\Omega^* \rightarrow F^p F^*_{M,E} \) is a soft resolution.

**Corollary 8.** Assume \( M \) compact kählerian. Then \( F^p\Omega^*_M \rightarrow \Omega^*_M \) induces on hypercohomology the inclusion \( F^p H^q(M, \Omega_M^*) \rightarrow H^q(M, \Omega_M^*). \) Equivalently, \( F^p H^q(M) = H^q(M, \Omega^*_M). \)

**Proof.** For \( p = 0 \) there is nothing to show. We proceed with induction on \( p. \) If \( p > 0 \) and the corollary established for \( p - 1, \) then consider the short exact sequence,

\[
0 \rightarrow \Omega^p_M \rightarrow \Omega^p_M \rightarrow Z^p_M 
\]

The long exact sequence yields,

\[
\begin{align*}
H^q(M, \Omega^p_M) &\rightarrow H^q(M, \Omega^{p-1}_M) \rightarrow H^q(M, \Omega^p_M) \rightarrow H^q+1(M, \Omega^{p-1}_M) \\
F^{p-1} H^{q+1}(M) &\rightarrow H^{q+1}(M) \\
F^p H^{q+1}(M) &\rightarrow H^{q+1}(M)
\end{align*}
\]

This is onto since \( M \) is compact kählerian and so this sequence breaks up in short exact sequences,

\[
0 \rightarrow \Omega^q(M, \Omega^p_M) \rightarrow F^{p-1} H^{q+1}(M) \rightarrow H^{q+1}(M) \rightarrow 0
\]

This shows that \( H^q(M, \Omega^p_M) = F^p H^{q+1}(M). \)

Since \( F^p\Omega^*_M \) is a resolution of \( \Omega^*_M \) it follows that \( H^q(M, F^p\Omega^*_M) = F^p H^{q+1}(M). \)

The corollary shows that from the point of view of homological algebra the holomorphic De Rham complex gives us the cohomology of \( M \) with \( \mathcal{O} \)-coefficients, but also the Hodge filtration.

If you are familiar with the notion of a spectral sequence that this can be expressed by saying that the holomorphic De Rham complex gives rise to a spectral sequence

\[
E^{r,q}_1 = H^r(M, \Omega^q_M) \Rightarrow H^{q+r}(M, \mathcal{O})
\]

which for \( M \) compact kählerian degenerates on this page and whose associated filtration is the Hodge filtration \( F^p H^{q+r}(M, \mathcal{O}). \)
5.2 Local systems. An abelian sheaf $F$ on a space $S$ is called a \textbf{local system} if it is locally constant: each $s \in S$ has a neighborhood $U_s$ in $S$ such that $F|_{U_s}$ is a constant sheaf. A local system $G$ on the unit interval $[0,1]$ is easily seen to be a constant sheaf. This implies that the restriction maps $G([0,1]) \to G_0$ and $G([0,1]) \to G_1$ are isomorphic, so that we get an isomorphism $G_0 \cong G_1$. If $F$ is a local system on $S$ and $\gamma : [0,1] \to X$ is continuous, then $\gamma^*F$ is a local system on $[0,1]$ and so we get an isomorphism of $F_{\gamma(0)} = (\gamma^*F)_{\gamma(0)}$ with $F_{\gamma(0)} (\gamma^*F)_{\gamma(0)}$. It is easy to see that this isomorphism only depends on the homotopy class of $\gamma$, when considered as a path from $\gamma(0)$ to $\gamma(1)$. If $S$ is simply connected, then there is exactly one homotopy class of paths connecting two points of $S$ and so it follows that $F$ must be in fact a constant sheaf. If $S$ is nicely path connected, then we obtain for a given base point $e \in S$ a representation of $\pi_1(S,e)$ on $F_0$ called the \textbf{monodromy} of $F$. Instead of specifying a base point we can choose a universal cover $\tilde{S} \to S$. Since $\tilde{S}$ is simply connected, $\tilde{\gamma}^*F$ is a constant sheaf. The group $\text{Gal}(S/S)$ of deck transformations of $\tilde{S}$ acts on $\tilde{\gamma}^*F(S)$ and this action is equivalent to the action of $\pi_1(S,e)$ on $F_0$. This also yields a converse construction: if $\rho : \text{Gal}(S/S) \to \text{GL}(r,\mathbb{R})$ is a representation, then take the constant sheaf $\mathbb{R}^r_S$ on $\tilde{S}$ and pass to the quotient under $\text{Gal}(S/S)$ where this group acts on $\mathbb{R}^r_S$ via $\rho$. We thus obtain a bijection between isomorphism classes of locally free $\mathbb{R}_S$-modules of rank $r$ and equivalence classes of representations of $\text{Gal}(S/S)$ on $\mathbb{R}^r$. Our main examples will be of the following type.

**Example.** Let $f : E \to S$ be locally trivial: for each $s \in S$ there exists a neighborhood $U$ in $S$ and a retraction $r : E_U \to E_s$ such that $E_U \xrightarrow{(r,s)} U \times E_s$ is a homeomorphism. Assuming that $S$ has a basis of contractible subsets (which have the homotopy of a point), then $S^\text{op} \to H^0(E_S;\mathbb{R})$ is a presheaf whose sheafication is a \textbf{local system}. If we work in a reasonable category of topological spaces, then this is simply $R^0f_*\mathbb{R}_E$.

(*) We prefer to denote local systems with letters like $F$ rather than $f$. 

We now assume that $S$ is a $C^\infty$-manifold and $F$ is a local system of $E_S$-modules of rank $r$. Then $F = E_S \otimes_S F$ is a locally free $E_S$-module of rank $r$, which is the same thing as a $C^\infty$ vector bundle of rank $r$. It comes with a $E_S$-linear map

$$
\nabla : F \rightarrow E_S \otimes F
$$

obeying the Leibniz property: for $\psi \in E_S$, and $u \in F$,

$$
\nabla (\psi u) = \psi \nabla u + d\psi \otimes u
$$

The map in question is here simply given by

$$
E_S \otimes F \rightarrow E_S \otimes E_S, \quad \omega \otimes u \mapsto d\omega \otimes u
$$

(which is indeed $E_S$-linear and satisfies the Leibniz property). Conversely, if $(U_i, \psi_i)$ is a cotriple chart of $S$ and $F|_U$ has the basis $e_n, e_r$, then $u \in F(U)$ is given as

$$
u = \sum_{i=0}^{r} \psi_i \otimes e_i \quad \text{with} \quad \psi_i \in E(U_i) \quad \text{and} \quad \nabla u = \sum_{i=0}^{r} d\psi_i \otimes e_i.
$$

It is clear from this description that $\nabla u = 0$ if each $\psi_i$ is constant $\Leftrightarrow u \in F(U)$ (or rather $u \in \Omega^{r+1}(U)$).

So we can recover $F$ from the pair $(\nabla, V)$. We have two obvious variations on this:

1. $F$ could be a locally free $E_S$-module (then $F = E_S \otimes F$ is a complex vector bundle and $\nabla$ will be $E_S$-linear) and in addition $S$ could be a complex manifold, in which case we might replace $E_S$ by $\mathcal{O}_S$ (then $F = \mathcal{O}_S \otimes E_S$ is a holomorphic vector bundle and $\nabla : F \rightarrow \Omega^{1}(E_S \otimes F)$).

2. There is a converse construction: if we are given a $C^\infty$ vector bundle $E_S \otimes F$, then a connection on $S$ is a map $\nabla : E_S(S) \rightarrow E_S(S)$ which is $E_S$-linear and satisfies the Leibniz rule: $\nabla (\psi u) = \psi \nabla u + d\psi \otimes u$. We say that $u \in E(U, S)$ is flat if $\nabla u = 0$, and we say that the connection $\nabla$ is flat if the $\nabla$-flat local sections generate $E_S(S)$ (for every $s \in S$, $E_S(S)$ should have a $E_S$-basis consisting of flat sections).

In that case the flat sections define a local system $F_S \subset E_S(S)$ of $E_S$-modules for which $E_S \otimes F_S \cong E_S(S)$.

Remark: A connection $\nabla : E_S(S) \rightarrow E_S(S)$ extends to $\nabla : E_S(S) \rightarrow E_S(S)$ by the rule

$$
\nabla (\alpha \otimes \sigma) = d\alpha \otimes \sigma + (-1)^p \alpha \wedge \nabla \sigma.
$$

You may check that this is well-defined and that $\nabla \nabla : E_S(S) \rightarrow E_S(S)$ is $E_S$-linear. In fact $\nabla \nabla (\alpha \otimes \sigma) = \alpha \wedge \nabla \nabla (\sigma)$ and
\[ \nabla : \xi \to E_\xi^2(\xi) \] defines a section of \( E_0^2(\text{End}(\xi)) \), called the curvature (this is zero on a flat \( \xi \)).

5.3 The Čech-Manin connection Suppose \( f : X \to S \) is a \( C^\infty \)-submanifold. Let \( s : = \text{dim } S \), \( m : = \text{dim } X - s \). We shall define the relative \( \mathbb{R} \)-Ehresmann complex of \( f \). We first do this for the case when \( f : \mathbb{R}^{m+3} = \mathbb{R}^m \times \mathbb{R}^3 \to \mathbb{R}^3 \) is the projection \( (x, y, z) \to y \). Then for \( U \subset \mathbb{R}^{m+3} \)

\[
\mathcal{E}_U^p = \bigoplus_{i=1}^s \mathcal{E}(U) dx_i
\]

(better) \( = \mathcal{E}_U^p / \sum_{i=1}^s \mathcal{E}^{p-1}(U) \)

(best) \( \text{Cone} \left( \mathcal{E}_U^p \to \mathcal{E}_U^p / \mathcal{E}^{p-1}(U) \to \mathcal{E}_U^p \right) \)

So these are \( \mathbb{R} \)-forms on \( U \) "along the fibers of \( f|_U \)." The exterior derivative induces \( \mathcal{E}_U^p \to \mathcal{E}_U^{p+1} \). A Poincaré lemma with parameters shows that the resulting complex of sheaves \( \mathcal{E}_U^p \) is a resolution of \( f^* \mathcal{E}_0 \) (the functions that only depend on \( y \)).

The map which assigns to \( \omega_1, \ldots, \omega_s \in \mathcal{E}^p(U) \) the sum \( \sum_{i=1}^s \omega_i \cdot dy_i \cdot \omega_i \) is not injective.

in general. The best way to express this is to say that we have a \( (l, a) \)-filtration

\[ L^l \mathcal{E}_U^p : = \mathcal{E}_U^p \wedge \bigwedge^{l+1} \mathcal{E}_{U,m}^p \] (so \( \mathcal{E}_{U,m}^p \wedge \bigwedge^{l+1} \mathcal{E}_{U,m}^p = 0 \)

with the property that

\[ \mathcal{E}_U^p / L^a \mathcal{E}_U^p = \mathcal{E}_U^p \otimes_{\mathcal{E}_U^{p-l}} \mathcal{E}_{U,m}^{p-l} \]

The generalization to a \( C^\infty \)-submanifold \( f : X \to S \) as above is now straightforward. We define the sheaf of relative \( \mathbb{R} \)-forms (with respect to \( f \)) \( \mathcal{E}_X^p / \mathcal{E}_S^p \) as the cokernel of the sheaf homomorphism

\[ f_! \mathcal{E}_S^p \otimes_{\mathcal{E}_S} \mathcal{E}_X^p \to \mathcal{E}_X^p, \quad \alpha \otimes \omega \to f_! \omega \wedge \alpha \cdot \omega. \]

This results in a complex \( \mathcal{E}_X^p / \mathcal{E}_S^p \) of \( f_! \mathcal{E}_S^p \)-modules with \( \mathcal{E}_X^p \to \mathcal{E}_S^p \)

It is a soft resolution of \( f_! \mathcal{E}_S^p \). We have a \( l, a \)-filtration

\[ L^l \mathcal{E}_X^p : = \bigwedge^l \mathcal{E}_X^p \wedge \bigwedge^{l+1} \mathcal{E}_S^p \]

of \( \mathcal{E}_X^p \). This is a resolution by subcomplexes \( L^l \mathcal{E}_X^p \) in which

\[ \mathcal{E}_X^p / \bigwedge^l \mathcal{E}_X^p = f_! \mathcal{E}_S^p \otimes_{\mathcal{E}_S} \bigwedge^{l+1} \mathcal{E}_S^p \]

In particular we have an exact sequence of complexes

\[ 0 \to f_! \mathcal{E}_S^p \otimes_{\mathcal{E}_S} \mathcal{E}_X^p \to \mathcal{E}_X^p / \bigwedge^l \mathcal{E}_X^p \to \mathcal{E}_X^p / \bigwedge^{l+1} \mathcal{E}_X^p \to 0 \]

Since \( f_! \mathcal{E}_S^p \to \mathcal{E}_X^p \) is a soft resolution we have \( H^p(f_! \mathcal{E}_X^p) \cong R^p f_! f_! \mathcal{E}_S^p \). We have \( f_! (f_! \mathcal{E}_S^p \otimes_{\mathcal{E}_S} \mathcal{E}_X^p) = \mathcal{E}_X^p \otimes_{\mathcal{E}_S} \mathcal{E}_X^p \) and so (since \( \mathcal{E}_S^p \) is a flat \( \mathcal{O}_S \)-module)
\[ H^p \left( \mathcal{E}_S^f \otimes P^* \mathcal{E}^i \right) = \mathcal{L}_S^* \mathcal{E}_S^i \otimes H^p \left( f^* \mathcal{E}^i \right) = \mathcal{E}_S^i \otimes R^p f_* f^* \mathcal{E}^i. \]

So the long exact sequence associated to \((\ast)\) has a differential defining a map
\[ \nabla : R^p f_* f^* \mathcal{E}^i \longrightarrow \mathcal{E}_S^i \otimes R^p f_* f^* \mathcal{E}^i. \]

Then \(S\) (Cauchy-Manin connection). This map is \(R_S\)-linear and satisfies the Leibniz rule.

If \(f\) is \(C^\infty\)-locally trivial (by Ehresmann's theorem this is automatic if \(f\) is proper), then \(R^p f_* \mathcal{E} \simeq \mathcal{E} \otimes \Omega^p\) is a local system on \(S\) and we have an identification
\[ \mathcal{L}_S^* \mathcal{E}_S^i \otimes R^p f_* \mathcal{E} = \mathcal{E} \otimes \Omega^p \mathcal{E} \] which makes \(R^p f_* \mathcal{E}\) the local system of \(\mathcal{E}\)-valued sections. We thus call \(\nabla\) the Cauchy-Manin connection.

\[ \text{Proof.} \] Clearly \(\nabla\) is \(R_S\)-linear. In order to verify the Leibniz rule we go through the definition of the boundary operator (which is based on the snake lemma).

Suppose \(\omega \in (R^p f_* f^* \mathcal{E}_S)_0 = H^p(f_* f^* \mathcal{E}_S)_0)\) is represented by \(\omega \in (\mathcal{E}_X^p \otimes f \mathcal{E}_X^i)_0\). With the help of a partition of unity there is no problem to lift \(\omega\) to some \(\tilde{\omega} \in (\mathcal{E}_X^p \otimes f \mathcal{E}_X^i)_0\). Since \(\omega\) is closed as a relative form,
\[ d\tilde{\omega} = \sum_{j=1}^s f^* dy_j \wedge \tilde{v}_j \quad \text{for certain } \tilde{v}_j \in (\mathcal{E}_X^p) \]

Then \(0 = dd\tilde{\omega} = -\sum_{j=1}^s f^* dy_j \wedge d\tilde{v}_j\) and this implies that at least locally
\[ d\tilde{v}_j \text{ can be written as } \sum_{k=1}^s f^* dy_k \wedge \tilde{v}_j \text{ with } \tilde{v}_j = -\tilde{v}_j. \]

So the image \(\eta_j\) of \(\tilde{v}_j\) in \((\mathcal{E}_X^p \otimes f \mathcal{E}_X^i)_0\) is closed as a relative differential, and hence represents an element \(v_j \in (R^p f_* f^* \mathcal{E}_S)_0\). Then
\[ \nabla u = \sum_{j=1}^s dy_j \otimes u_j. \]

Now let \(\psi \in \mathcal{E}_S^0\) and let us compute \(\nabla(\psi u)\). Then \(\tilde{\omega}\) is replaced by \(f^* \psi \tilde{\omega}\). Since
\[ d(f^* \psi \tilde{\omega}) = f^* \psi d\tilde{\omega} + f^* df \psi \tilde{\omega} = \sum_{j=1}^s f^* (\psi dy_j) \wedge \tilde{v}_j + f^* df \psi \]
it follows that \(\nabla(\psi u) = \psi \nabla u + df \psi \otimes du\) (Leibniz rule).

Assume now \(f\) is locally trivial. Let \(U\) be a neighborhood of \(0\) in \(S\) for which we have \(C^\infty\-manifolds\) \(r : X_U \longrightarrow X_0\). Then we have an inclusion
\[ f^* \mathcal{E}_U \otimes r^* \mathcal{E}_0 ' \longrightarrow \mathcal{E}_U \otimes \mathcal{E}_0 ' \]

Both are soft resolutions of \(f^* \mathcal{E}_U\). We have \(f_* (f^* \mathcal{E}_U \otimes r^* \mathcal{E}_0 ' \longrightarrow \mathcal{E}_U \otimes \mathcal{E}_0 ' \)
and hence \(H^p (f_* (f^* \mathcal{E}_U \otimes r^* \mathcal{E}_0 ')) = \mathcal{E}_U \otimes H^p (\mathcal{E}_0)\) by the universal
coefficient theorem. It follows that
\[ R^p f_* f^! \mathcal{E}_S = \mathcal{E}_0 \otimes H^p(X_0) \cong H^p \left( f^! \mathcal{E}_0 \right) \]
and so it remains to show that the elements of \( R^p f_* f^! \mathcal{E}_0 \) are \( \mathcal{D} \)-flat.

If we take in the preceding \( \omega \) of the form \( \omega = r^* \omega_0 \) with \( \omega_0 \) a closed \( p \)-form on \( X_0 \), then clearly \( r^* \omega_0 \) follows as \( \mathcal{D} \) and since \( r^* \omega_0 \) is closed it is immediate that the resulting section of \( H^p ( f^! \mathcal{E}_0 / \mathcal{O}_S ) \) is \( \mathcal{D} \)-flat. 

5.4 Variation of Hodge structure. Let \( f: X \to S \) be a holomorphic map of complex manifolds. For every \( \pi \in X \), \( f \) induces a homomorphism \( \mathcal{O}_S \langle f \rangle \to \mathcal{O}_X \) of local rings. This defines a homomorphism of sheaves of \( \mathcal{C} \)-algebras \( \mathcal{O}_S \to f_* \mathcal{O}_X \) and makes every \( f_* \mathcal{O}_X \)-module also a \( \mathcal{O}_S \)-module. In particular, for every \( \mathcal{O}_X \)-module \( M \), \( R^q f_* M \) is naturally a \( \mathcal{O}_S \)-module.

We now assume that \( X \to S \) a proper holomorphic submersion with complex fiber dimension \( n \). By Ehresmann's theorem \( f \) is then \( C^0 \)-locally trivial. The fibers \( X_s = f^{-1}(s) \) will be compact complex submanifolds of \( X \) of dimension \( n \), but we cannot expect \( f \) to be holomorphically locally trivial. The sheaf \( \mathcal{E}_X \) of holomorphic relative \( p \)-forms is defined in complete analogy to \( \mathcal{E}_X \). It is a locally free \( \mathcal{O}_X \)-module of rank \( (n) \). The higher direct image \( R^q f_* \mathcal{E}_X \) will be an \( \mathcal{O}_S \)-module. If \( s \in S \) and \( x \in X_s \), then we have an restriction map \( f^{-1} \mathcal{E}_X \to \mathcal{E}_X \). This induces a restriction map
\[ (R^q f_* \mathcal{E}_X)_s \to H^q(X_s, \mathcal{E}_X) \]
The exterior derivative determines a complex of \( f^{-1} \mathcal{O}_S \)-modules
\[ \cdots \to \mathcal{E}_X \to \mathcal{E}_X \to \ldots \to \]
which resolves \( f^{-1} \mathcal{O}_S \). The restriction to a resolution \( F^p \mathcal{E}_X \) of \( \mathcal{E}_X \).

Note that \( R^q f_* (\omega^p_1 \mathcal{E}_X) = R^q f_* f^! \mathcal{O}_S \). A local \( C^0 \) trivialization \( X_Y = X_0 \) (a neighborhood of \( 0 \) in \( S \)) identifies the projection \( X_Y \to X \) with the projection \( V_Y \to X_0 \). A Künneth theorem shows that the natural map \( \mathcal{O}_S \to H^q(X_0) \to R^q f_* f^! \mathcal{O}_S \)
is an isomorphism of $\mathcal{O}_V$-modules (the complex structure on the fibers is here irrelevant).

It follows that $\mathcal{O}_s \otimes R^q f_* \mathcal{E}_X \rightarrow R^q f_* f^{-1} \mathcal{O}_s$ is an isomorphism of $\mathcal{O}_s$-modules and that $\Gamma(V, R^q f_* f^{-1} \mathcal{O}_s)$ is the space of holomorphic maps $V \rightarrow H^q(M, \mathcal{E})$.

This shows at the same time that $R^q f_* f^{-1} \mathcal{O}_s$ is a locally free $\mathcal{O}_s$-module (the sheaf of holomorphic sections of a holomorphic vector bundle) with a flat connection which has $R^q f_* \mathcal{E}_X$ as its sheaf of flat sections.

We also have the truncation $F^p \Omega^i_{X|S} : \rightarrow \Omega^0_{X|S} \rightarrow \Omega^1_{X|S} \rightarrow \ldots$ which appears as a subcomplex of $\Omega^*_{X|S}$. It contains $F^p \Omega^1_{X|S}$ as a subcomplex with quotient the single term complex $\Omega^p_{X|S}$ placed in degree $p$. So we have a long exact sequence

$\cdots \rightarrow R^q f_* F^p \Omega^i_{X|S} \rightarrow R^q f_* \Omega^i_{X|S} \rightarrow R^q f_* f^{-1} \Omega^i_{S} \rightarrow \ldots$

Since $\Omega^i_{X|S} \rightarrow F^p \Omega^i_{X|S}$ is a resolution we get $\mathcal{O}_s$-module homomorphism

$R^q f_* \Omega^i_{X|S} \rightarrow R^q f_* F^p \Omega^i_{X|S} \rightarrow R^q f_* f^{-1} \Omega^i_{S}$.

**Theorem 96.** Assume that $X$ admits a holomorphic metric whose restriction to every fiber $X_s$ is Kähler (in other words the associated $2$-form $\omega \in \Omega^2(X)$ has an image $\omega_s \in \Omega^2(X_s)$ that is closed). Then the exact sequence (4) breaks up in short exact sequences of locally free $\mathcal{O}_s$-modules, so that if we put $F^s R^q f_* f^{-1} \mathcal{O}_s := R^q f_* F^p \Omega^1_{X|S}$ we get a flag of holomorphic subbundles

$$R^q f_* f^{-1} \mathcal{O}_s = F^0 R^q f_* f^{-1} \mathcal{O}_s \supseteq F^1 R^q f_* f^{-1} \mathcal{O}_s \supseteq \ldots \supseteq F^d R^q f_* f^{-1} \mathcal{O}_s = 0$$

with the property that $F^0 R^q f_* f^{-1} \mathcal{O}_s / F^1 R^q f_* f^{-1} \mathcal{O}_s \cong R^q f_* \Omega^{0,1}_{X|S}$ and hence for every $s \in S$ specializes to the Hodge filtration on $H^q(X_s, \mathcal{E})$.

We will not give the proof, but make some comments to indicate what part of these assertions are special cases of general facts. For this we need the notion of a coherent sheaf. Suppose $M$ is a complex manifold. Then a $\mathcal{O}_M$-module $\mathcal{F}$ is said to be coherent if it is locally the sheaf of sections of a holomorphic vector bundle map $\mathcal{F} \rightarrow \mathcal{E}$ in the other words, we can cover $M$ by open subsets $U$ for which there exists an exact
sequence \( O^k \to O^l \to F \to 0 \). So each shell \( F_n \) is a finitely generated \( O_M \) module. In particular \( F/M_{H_n} \) is a finite chain \( O \)-vector space. If we take the coefficient of a matrix as variables, then the rank of that matrix is semi-continuous. So \( r e X \to \text{dim}_O F(x) \) is semi-continuous. Therefore if \( x \in M \) has a neighborhood \( U_x \) such that \( \text{dim}_O F(x) \geq \text{dim}_O F(x) \) for all \( x \in U_x \). It is also not hard to see that if \( \text{dim}_O F(x) \) is locally constant, then \( F \) is locally free.

Assume now \( g : M \to N \) is a proper holomorphic map of complex manifolds, and let \( F \) be a coherent \( O_M \)-module. Then we have the following basic facts:

(i) \( R^q g_* F \) is a coherent \( O_N \)-module for all \( q \).

(ii) Assume \( g \) is a submersion and \( F \) is locally free. Then for every \( q \geq 0 \) we have a natural isomorphism \( (R^q g_* F)(y) \cong H^q(N, F/\delta^{m}_N F) \).

We apply this to \( f = X \to S \) and \( F = \Omega^{\geq 0}_X/S \) (this is locally free of finite rank, hence a coherent \( O_X \)-module). It is easy to check that for any \( s \in S \),

\[ \Omega^{\geq 0}_X/S/\delta M^g_s \Omega^{\geq 0}_X \cong \Omega^{\geq 0}_X, \]

and so by the above, we have that \( R^q f_* \Omega^{\geq 0}_X/S \) is a coherent \( O_X \)-module whose fiber at \( s \in S \) can be identified with \( H^q(X_s, \Omega^{\geq 0}_X) \).

Our assumptions imply that \( \sum_{p = 0}^q \text{dim} H^p(X_s, \Omega^{\geq 0}_X) = \text{dim} H^q(X, \Omega^{\geq 0}_X) \). The right hand side is locally constant, and each term of the left-hand side is semi-continuous. Hence each of these terms must be locally constant. It then follows that each \( R^q f_* \Omega^{\geq 0}_X/S \) is locally free.

The proof of the theorem proceeds with induction on \( p \). Since \( R^q f_* \Omega^{\geq 0}_X/S \) is locally constant of finite rank, \( \Omega^{\geq 0}_X \cong R^q f_* \Omega^{\geq 0}_X/S \) is a locally free \( O_X \)-module.

If we knew that \( F^p R^q f_* \Omega^{\geq 0}_X/S \) is a locally free \( O_X \)-module, then the exact sequence (4) provides the vector bundle maps \( F^p R^q f_* \Omega^{\geq 0}_X/S \to R^q f_* \Omega^{\geq 0}_X/S \).

In fiber on \( s \in S \) is surjection \( F^p H^q(X_s) \to H^{q-p}(X_s, \Omega^{\geq 0}_X) \) and hence this vector bundle map is surjective also. The theorem follows.
The identification \( \partial_s \otimes \mathcal{R}^d_{f,*} \mathcal{C}_x \equiv \mathcal{R}^d_{f,*} f^{-1} \partial_s \) gives us a flat connection \( \nabla \) in \( \mathcal{R}^d_{f,*} f^{-1} \partial_s \) whose local system of flat sections is \( \mathcal{R}^d_{f,*} \mathcal{C}_x \). This flat connection can be obtained as a boundary operator (as in the \( \mathcal{C}^* \)-case) if we keep in mind that \( \Omega^i \mathcal{X}_{15} \) is a resolution of \( f^{-1} \partial_s \) by \( f^{-1} \partial_s \)-modules so for any \( \mathcal{R}^d_{f,*} f^{-1} \partial_s = \mathcal{R}^d_{f,*} \Omega^i \mathcal{X}_{15} \) the exact sequence of complexes

\[
0 \rightarrow f^{-1} \Omega^1_s \otimes_{f^{-1} \partial_s} \Omega^i \mathcal{X}_{15} \rightarrow \Omega^i \mathcal{X}_{15} / f^* \Omega^i_s \rightarrow \Omega^i \mathcal{X}_{15} \rightarrow 0.
\]

gives rise to a boundary operator that is our \( \nabla \) (which is still called the Gauß-Manin connection).

\[ \mathcal{R}^d_{f,*} f^{-1} \partial_s \rightarrow \Omega^1_s \otimes_{\partial_s} \mathcal{R}^d_{f,*} f^{-1} \partial_s. \]

Theorem 57 (Gauß transversality). In the situation of Thm 56, the connection \( \nabla \) takes \( \mathcal{R}^d_{f,*} f^{-1} \partial_s \) to \( \Omega^1_s \otimes_{\partial_s} \mathcal{R}^d_{f,*} f^{-1} \partial_s. \)

Proof. Consider the "\( \mathcal{F}^p \)-part" of the sequence (1). It is still exact:

\[
0 \rightarrow f^{-1} \Omega^1_s \otimes_{f^{-1} \partial_s} \mathcal{F}^p \mathcal{X}_{15} \rightarrow \mathcal{F}^p \Omega^1_s / f^* \Omega^1_s \otimes \mathcal{F}^{p-1} \mathcal{X}_{15} \rightarrow \mathcal{F}^p \Omega^1 \mathcal{X}_{15} \rightarrow 0.
\]

The boundary operator defines a \( \mathcal{C} \)-linear map:

\[ \mathcal{R}^d_{f,*} \mathcal{F}^p \mathcal{X}_{15} \rightarrow \Omega^1_s \otimes_{\partial_s} \mathcal{R}^d_{f,*} \mathcal{F}^{p-1} \mathcal{X}_{15}. \]

The left-hand side is \( \mathcal{F}^p \mathcal{R}^d_{f,*} f^{-1} \partial_s \) and the right-hand side is \( \Omega^1_s \otimes_{\partial_s} \mathcal{R}^d_{f,*} \mathcal{F}^{p-1} f^{-1} \partial_s. \)

The functional nature of the boundary operator implies that this is just the restriction of \( \nabla \) to \( \mathcal{F}^p \mathcal{R}^d_{f,*} f^{-1} \partial_s. \)

Still in the situation of Thm 56, let us now make the stronger assumption that \( X \) is endowed with a Kähler metric. Assume that its class is represented by some \( \eta \in H^2(X;K) \) for some subring \( R \) of \( \mathbb{R} \). Then for every fiber \( X_s \), the image of \( \eta \) in \( H^2(X_s;K) \) gives rise to a primitive decomposition of its cohomology

\[ H^2(X_s;K) = \bigoplus_{d=0} \mathcal{P}^d(X_s;K)[\eta_s] / (\eta_s^{n-d+1}) \]

where \( K \) is the field of fractions of \( R \). If \( V \) is a contractible neighborhood of \( s \) in \( S \), then the inclusion \( X_s \subset X_V \) is a homotopy equivalence. Hence we then also have \( H^2(X_V;K) = \bigoplus_{d=0} \mathcal{P}^d(X_V;K)[\eta_V] / (\eta_V^{n-d+1}) \). In other words, we get a decomposition of local systems...
\[ \bigoplus_{d=0}^{n} R^{d}_{\mathcal{K}X} = \bigoplus_{d=0}^{n} Pf_{\mathcal{K}X}[\eta] / (\eta^{n-d} \cdot \cdot \cdot), \]

where
\[ Pf_{\mathcal{K}X} := \text{Ker} \left( R^{d}_{\mathcal{K}X} \xrightarrow{\eta^{n-d}} R^{2n-2d+2}_{\mathcal{K}X} \right). \]

Note however that \( Pf_{\mathcal{K}X} \) is also defined (replace in the above formula \( \mathcal{K}X \) by \( \mathcal{E}X \)). The primitive decomposition \( H^*(X; K) \) is one of \( K \)-Hodge structures and hence \( Pf_{\mathcal{E}X} \) is a family of \( \mathbb{R} \)-Hodge structures: if we put \( Pf := \mathcal{O} \otimes_{\mathcal{O}_X} Pf_{\mathcal{E}X} = \mathcal{O} \otimes_{\mathcal{O}_X} Pf_{\mathcal{E}X} \), then \( Pf \) is a holomorphic vector bundle with a flat connection (having \( Pf_{\mathcal{E}X} \) as its local system of flat sections). The Hodge filtration is of course given by
\[ F^* Pf = Pf \cap F^* Pf \otimes \mathcal{O}_S = \text{Ker} \left( F^* Pf \otimes \mathcal{O}_S \xrightarrow{\eta^{n-d}} R^{2n-2d+2}_{\mathcal{K}X} \right) \]

These are holomorphic subbundles of \( Pf \). Griffiths transversality tells us that \( \mathcal{D} \) maps \( F^* Pf \) to \( \mathcal{O} \otimes_{\mathcal{O}_S} F^{-1} Pf \).

The polarizations \( \Psi : P^d(X; \mathbb{R}) \times P^d(X; \mathbb{R}) \rightarrow \mathbb{R}(-2d) \) define a homomorphism of local systems
\[ \Psi : Pf_{\mathcal{K}X} \otimes Pf_{\mathcal{K}X} \rightarrow \mathbb{R}(-2d) \]

It has of course the same nondegeneracy and positivity properties as \( \Psi \).

Its bihomogeneity property and nondegeneracy can be expressed in "holomorphic terms".

The \( \mathcal{O}_S \)-extension \( \Psi : Pf_{\mathcal{S}X} \otimes Pf_{\mathcal{S}X} \rightarrow \mathcal{O}_S \) is zero on \( F^* Pf_{\mathcal{O}_S} \otimes F^{-1} Pf_{\mathcal{O}_S} \) so that we have induced homomorphisms
\[ F^* Pf_{\mathcal{O}_S} / F^{-1} Pf_{\mathcal{O}_S} \otimes F^{-d} Pf_{\mathcal{O}_S} / F^{-d+1} Pf_{\mathcal{O}_S} \rightarrow \mathcal{O}_S \]

that define perfect pairings of holomorphic vector bundles. But for the positivity property we need to pass to the underlying \( C^\infty \) vector bundles (i.e. tensor with \( \mathcal{E}_S \)): complex conjugation identifies the \( C^\infty \)-bundle underlying \( F^* Pf_{\mathcal{O}_S} / F^{-1} Pf_{\mathcal{O}_S} \) with the one underlying \( F^{-d} Pf_{\mathcal{O}_S} / F^{-d+1} Pf_{\mathcal{O}_S} \) and via this identification the positivity can be expressed.

The preceding suggests that if we replace the notion of a (polarized) Hodge structure to a family of those, we should take into account Griffiths transversality.
Def. Let \( R \subset \mathbb{R} \) be a subring. A variation of \( R \)-Hodge structure over a complex manifold \( S \) consists of local system \( H \) of finitely generated \( R \)-modules and a descending filtration of the underlying holomorphic vector bundle \( H := \mathcal{O}_S \otimes \mathcal{H} \) by locally free \( \mathcal{O}_S \)-submodules

\[
H \supseteq F^p_H \supseteq F^{p+1}_H \supseteq \cdots
\]

with \( F^0 = 0 \) for \( p \gg 0 \) and \( F^0 = H \) for \( p \ll 0 \) such that the underlying \( \mathcal{C}^\infty \)-bundles define a Hodge structure of weight \( m \) and the connection \( D : H \to \nabla^1_S \otimes H \) which has \( \mathcal{O}_S \otimes H \) as its local of flat sections takes \( F^p \) to \( \nabla^1_S \otimes F^{p-1} \).

An \( R \)-polarization of \( (H, F) \) is an \( \mathcal{O}_S \)-homomorphism

\[
\Psi : H \otimes_{\mathcal{O}_S} \mathcal{H} \to \mathcal{R}(m)
\]

which defines a polarization on each fiber.

This definition allows us to form up Thm. 37 and the subsequent discussion by saying that under the hypotheses given, \( R^d f_* \mathcal{R}x \) resp. \( P^d f_* \mathcal{R}x \) defines an \( R \)-variation of Hodge structure (resp. polarized \( R \)-variation of Hodge structure) of weight \( d \).

5.5 The period map. Let \( H \) be an \( R \)-vector space of dim \( r \). Let us first recall that for \( 0 \leq d \leq r \), the Grassmann manifold \( \mathcal{G}_k(H^d) \) of \( d \) subspaces of complex dim. \( k \) has at \( [F] \in \mathcal{G}_k(H^d) \) a tangent space that we may identify with

\( \text{Hom}(F, H^d/F) \) To see this: first choose a supplementary space \( G \) so that \( H = F \oplus G \). Then assigning to a map \( \alpha \in \text{Hom}(F, G) \) the graph of \( \alpha \) identifies \( \text{Hom}(F, G) \) with the set of \( k \)-dim subspaces of \( H \) that supplement \( G \), which makes up a neighborhood of \( [F] \) in \( \mathcal{G}_k(H^d) \). It identifies \( \text{Hom}(F, V/F) \cong \text{Hom}(F, G) \) with the tangent space \( T_{[F]} \mathcal{G}_k(H^d) \) and you may check that this identification is