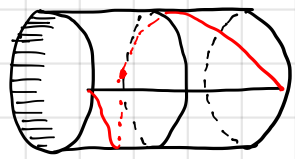
Farb-Looijenga Course Spring 2020

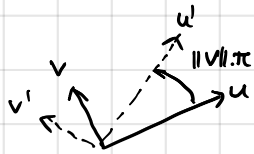
Higher dimensional Dehn twists



Usual Dehn twist: diffeo of $E = S^1 \times [0,1]$ that is the identity on ∂E, antipodal on $S^1 \times \{0\}$ and commutes with involution $(u,v) \mapsto (-u,-v)$.

A natural generalization in dim n is $E = E(S^n)$, the unit disk bundle in TS^n. The associated Dehn twist $h: E \circlearrowleft$ has the same 3 properties.

It can be defined as follows: regard S^n as the unit sphere in Euclidean $(n+1)$-space $\mathbb{R}^{n+1}$ and let $h': E \circlearrowleft$ be the geodesic flow evaluated at time π. So if $E = \{(u,v) \in S^n \times \mathbb{R}^{n+1} : u \perp v,\ \|v\| \leq 1\}$, then $h(u,v) = (u',v')$ with (u',v') as depicted. Now take $h(u,v) = h'(-u,-v)$.

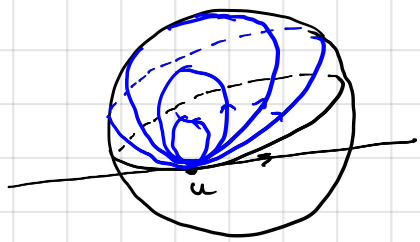


We use the metric to identify E with the unit disk bundle in T^*S^n. It then comes with a natural symplectic form. It is invariant under h: h is a symplectomorphism. The symplectic form defines an orientation of E – this is the orientation that we will use.

The isomorphisms $\tilde{H}_\cdot(E) \cong \tilde{H}_\cdot(S^n) = \tilde{H}_n(S^n)$ shows that $\tilde{H}_\cdot(E)$ is concentrated in degree n and is ∞ cyclic. Let $\delta \in \tilde{H}_n(E)$ be a generator. Since h acts on S^n as the antipodal invol^n we have $h_*(\delta) = (-1)^{n+1}\delta$. The self-intersection $\delta \cdot \delta$ is $\pm e(S^n)$:

$$\delta \cdot \delta = \begin{cases} 0 & n \text{ odd} \\ 2 & n \equiv 0\ (4) \\ -2 & n \equiv 2\ (4) \end{cases}$$

We are mostly interested in $n = 0, 1$.



<u>Prop</u> If $n=2$, then h^2 is isotopic to the identity.

<u>Proof</u> by picture. Choose an orientation of S^n, and let an oriented geodesic shrink via oriented circles on (say) the left of the geodesic to a constant map. This defines a canonical isotopy rel. ∂E of h^2 to the identity map.

<u>Rem 1</u>: A theorem of Paul Seidel (1997) implies that no nontrivial power of h is isotopic to 1 by symplectomorphisms.

<u>Rem 2</u>: We get another isotopy of h^2 to the identity by taking the shrinking circles on the right (or equivalently, by changing the orientation of S^n). The two isotopies define a loop in $\mathrm{Diff}(E, \partial E)$. Kronheimer has shown that this loop represents a nontrivial elt of $\pi_1 \mathrm{Diff}(E, \partial E)$.

Its appearance in algebraic geometry

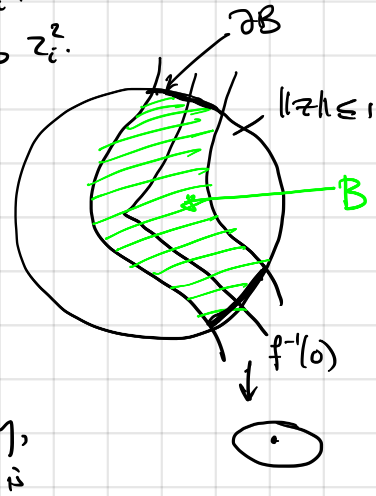
Let M be a complex man. of dim $n+1$, $f: M \to \mathbb{C}$ holom. We say that f has an ordinary double point at p if $df(p)=0$ and the hessian of f at p is nondegenerate ($\Leftrightarrow$ df has a simple zero at p).

A holom. version of the Morse lemma implies that we can then find local coord$^{\text{ns}}$ $(z_0,\dots,z_n)$ at p such that f is there given as $f(p)+\sum_{i=0}^n z_i^2$.

This suggests we investigate $f: \mathbb{C}^{n+1} \to \mathbb{C}$, $z \mapsto \sum_{i=0}^n z_i^2$.

Let $0<\eta \ll 1$ and consider

$$\begin{array}{l} B := \{z \in \mathbb{C}^{n+1} : \|z\| \le 1,\ |f(z)| < \eta\} \\ \quad f \big\downarrow \\ \Delta = \{t \in \mathbb{C} : |t| < \eta\} \end{array}$$



We let $\partial B = \{z \in B : \|z\| = 1\}$.

Then $\partial B \to \Delta$ is locally C^∞-trivial (hence trivial) and $(B, \partial B) \to \Delta$ is locally trivial over $\Delta \setminus \{0\}$. So if $0<t<\eta$, then the monodromy of $B \to \Delta$ over the circle of radius $\sqrt{t}$ is naturally an isotopy class of $(B_t, \partial B_t)$

We construct a diffeo $(B_t, \partial B_t) \to (E, \partial E)$ as follows.

A point $z = x + \sqrt{-1}\,y \in \mathbb{C}^{n+1}$ lies in B_t if and only if $\|x\|^2 + \|y\|^2 \le 1$, $\|x\|^2 - \|y\|^2 = t$ and $\langle x, y\rangle = 0$. This implies $\|x\|^2 \ge t$ and $\|y\|^2 \le 1-t$.

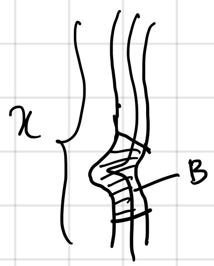
Then $z \mapsto \left(\dfrac{x}{\|x\|}, \dfrac{y}{\sqrt{1-t}}\right)$ is as desired

A more detailed analysis shows that the monodromy isotopy class of $(B_t, \partial B_t)$ is naturally represented by $h : (E, \partial E) \circlearrowleft$

A simple ordinary double point degeneration is when we have

$$\begin{array}{c} \mathcal{X} \\ f \big\downarrow \text{ proper holom} \\ \Delta \end{array}$$

such that df has a simple zero in a point $p \in X_0 = f^{-1}(0)$ and has no other zeros.



Then we can realize B as a closed nbhd of p in $\mathcal{X}$ such that $(\mathcal{X} \setminus \mathrm{int}(B), \partial B) \to \Delta$ is C^∞-trivial. The monodromy of $\mathcal{X}$ over the circle of radius $\sqrt{t}$ is then a diffeo $X_t \circlearrowleft h_t$ whose support is contained in B_t and there essentially equal to h.

The action of h_t on $H_n(X_t)$ is given by $a \mapsto a \pm \langle a, \delta\rangle \delta$ (reflection when n even, symplectic transvection when n odd) and is trivial on $H_k(X_t)$ for $k \ne n$.

(Picard-Lefschetz formula)

Atiyah flop

In the preceding situation assume $n=2$. If we blow up p in X_0 we resolve the singularity:

$$\hat{X}_0 \to X_0$$

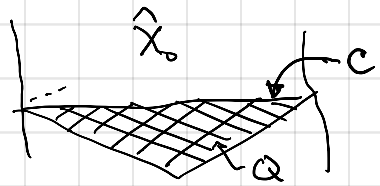
The exc. divisor C (over p) is like a conic in $\mathbb{P}^2$, so $\cong \mathbb{P}^1$. We have $C.C=-2$
The monodromy of f becomes trivial if we make a base change of order 2:

$$\begin{tikzcd} \mathcal{Y} = \mathcal{X}\times_\Delta \Delta' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \Delta' \arrow[r, "2:1"] & \Delta \\ s & t=s^2 \end{tikzcd}$$

$\mathcal{Y} = \{(z,s)\in \mathcal{X}\times\Delta' : f(z)-s^2=0\}$
this has an ordinary double pt of dim 3 at $(p,0)$. We resolve this with a single blow up:

$$\hat{\mathcal{Y}} \to \mathcal{Y}$$

The exc. divisor Q is like a nonsing quadric in $\mathbb{P}^3$, so $\cong \mathbb{P}^1\times\mathbb{P}^1$.
We have $Y_0 = X_0$ and the strict transform of Y_0 is just a copy of $\hat{X}_0$
It meets Q transversally in a plane section, $\cong$ the diagonal of $\mathbb{P}^1\times\mathbb{P}^1$.
We can retract Q onto C in two ways (corresp to the two factors $\mathbb{P}^1\times\mathbb{P}^1$). This can in fact be done holom inside $\hat{\mathcal{Y}}$ so that we get



$$\begin{tikzcd} \mathcal{Y}^+ & \mathcal{Y} \arrow[l] \arrow[r] \arrow[d] & \mathcal{Y}^- \arrow[dl, "f^-"] \\ & \Delta' \arrow[from=1-1, "f^+"'] & \end{tikzcd}$$

The maps $f^\pm$ are now without singularities (submersions). They are equal over $\Delta'\setminus\{0\}$ and they also have central fiber $\cong X_0$. Yet they are different: you pass from $\mathcal{Y}^+$ to $\mathcal{Y}^-$ by taking in $\mathcal{Y}^+_0 \cong X_0$ the copy of C, blow it up in $\mathcal{Y}^+$ and then blow down in a different way to get $\mathcal{Y}^-$. (This the Atiyah flop).
This implies a nonseparatedness property for moduli spaces of certain K3 surfaces.
This phenomenon will not occur in a projective setting.

Rem. Since $\mathcal{Y}^\pm \to \Delta'$ is C^∞-loc-trivial, we get two isotopies of the square of the monodromy to the identity. These are essentially the isotopies found earlier.

Kodaira-Spencer theory (generalities)

X: a connected complex manifold of dim n.

A deformation of X over a pointed complex manifold $S \ni 0$ is given by:

maps X isom onto X_0

$$\begin{tikzcd} X \arrow[r, hook, "i"] & \mathcal{X} \arrow[d, "\pi"] \\ 0 \in & S \end{tikzcd}$$

proper hol. submersion

This is a reasonable def. because of the Ehresmann fibration theorem which tells us that π is then locally trivial in the C^∞-category.

What only matters here is what happens near 0 and so we should actually think of this as a deformation over the germ $(S,0)$. This replaces $\mathcal{X}$ by its germ at $i(X) = X_0$.

With this in mind the deformations of X are objects of a category $\underline{\mathrm{Def}}(X)$: a morphism from $(X \overset{i'}{\hookrightarrow} \mathcal{X}' \xrightarrow{\pi'} S')$ to $(X \overset{i}{\hookrightarrow} \mathcal{X} \xrightarrow{\pi} S)$ is given by a pair $(f, \tilde{f})$ which makes the diagram commute and the square cartesian.

$$\begin{tikzcd} & X \arrow[dl, "i'"'] \arrow[dr, "i"] & \\ (\mathcal{X}', X_0') \arrow[rr, "\tilde{f}"] \arrow[d, "\pi'"] \arrow[drr, phantom, "\sqcap"] & & (\mathcal{X}, X_0) \arrow[d, "\pi"] \\ (S', 0) \arrow[rr, "f"] & & (S, 0) \end{tikzcd}$$

$\underline{\mathrm{Def}}(X)$ has an obvious initial object: take $S = \{0\}$.

A deformation is called trivial if it admits a morphism to the initial object.

A deformation is called universal if it is a final object of $\underline{\mathrm{Def}}(X)$ (need not exist!)

Remark 1. $\mathrm{Aut}(X)$ naturally acts on a universal deformation (π, i) of X: if $g \in \mathrm{Aut}(X)$, then have a unique morphism

$$\begin{tikzcd} & X \arrow[dl, "ig"'] \arrow[dr, "i"] & \\ \mathcal{X} \arrow[rr, dashed, "\tilde{f}_g"] \arrow[d, "\pi"] \arrow[drr, phantom, "\sqcap"] & & \mathcal{X} \arrow[d, "\pi"] \\ S \arrow[rr, dashed, "f_g"] & & S \end{tikzcd}$$

Applying this to g^{-1} shows that this morphism is invertible. The action is given by $g \mapsto (f_g, \tilde{f}_g)$

Remark 2. If $(X \overset{i}{\hookrightarrow} \mathcal{X} \xrightarrow{\pi} S)$ is a deformation and $0 \in T \subset S$ is singular, then we want to say that the restriction of π over T is a deformation of X over T. So it is desirable to allow S to be any analytic germ at a point. The condition for π to be a submersion then becomes: π is smooth, meaning local-analytically like the projection of a product $(\text{complex } n\text{-man}) \times (T,0) \to (T,0)$. This includes the case when $(S,0)$ is a "thickened point". For example for (π, i) as above we can take T the 1st order part of S: $T_0 S$. The restriction over this deformation is

$$\begin{tikzcd} X \arrow[r, hook] & T\mathcal{X}|_{X_0} \arrow[d, "d\pi"] \\ & T_0 S \end{tikzcd}$$

So this amounts to a hol. extension of vector bundles:

$$(*) \qquad 0 \to TX \to i^* T\mathcal{X} \to \pi^* T_0 S \to 0$$

and this deformation over $T_0 S$ is trivial if and only if this extension splits in the hol. category.

Let us write Θ_X for $\mathcal{O}(TX)$ (just as we write Ω_X for $\mathcal{O}(T^*X)$). Then $(*)$

gives the exact sequence of $\mathcal{O}_X$-modules:

$$0 \to \Theta_X \to \iota^*\Theta_{\tilde{\mathcal{X}}} \to \mathcal{O}_X \otimes_{\mathbb{C}} T_0S \to 0$$

and gives rise to the long exact sequence

$$0 \to H^0(X,\Theta_X) \to H^0(X,\iota^*\Theta_{\tilde{\mathcal{X}}}) \to T_0S \xrightarrow{\delta} H^1(X,\Theta_X) \to \cdots$$

Kodaira-Spencer map.

So the sequence (*) splits hol. $\iff \delta = 0$.

The definition of the KS-map is functorial:

$$\begin{tikzcd} T_{0'}S' \arrow[r] \arrow[dr] & T_0S \arrow[d] \\ & H^1(X,\Theta_X) \end{tikzcd}$$

As of now our category $\underline{\mathrm{Def}}(X)$ includes as objects def. over arb. analytic germs.

<u>Thm</u> (Kodaira-Spencer,...) If $H^0(X,\Theta_X) = H^2(X,\Theta_X) = 0$, then X admits a univ. def. over a complex manifold germ $(S,0)$, and its Kodaira map will be an isom of $\mathbb{C}$-vector spaces (so $\dim S = \dim H^1(X,\Theta_X)$).

Classical example: X compact Riem. surface of genus $g \geq 2$. Satisfies hyp of the above thm and so have a univ. def. with base $(S,0)$, $T_0S \cong H^1(X,\Theta_X)$ and hence $T_0^*S \cong H^0(X,\Omega_X^{\otimes 2})$ (Serre duality), which is of dim $3g-3$.

We now assume X is a K3 surface. Let $\alpha \in H^0(X,\Omega_X^2)$ be nowhere zero. Then

$$\Theta_X \to \Omega_X^1 = \Omega_X, \quad \xi \mapsto \iota_\xi(\alpha)$$

is an isom of $\mathcal{O}_X$-modules and so this induces an isom

$$H^i(X,\Theta_X) \xrightarrow{\cong} H^i(X,\Omega_X^1) \underset{\text{Hodge theory}}{=} \begin{cases} 0 & \text{if } i \neq 1 \\ H^1(X,\Omega_X^1) & \text{of dim } 20. \end{cases}$$

Hence X has a univ. def. whose base $(S,0)$ is a complex man of dim 20. The above identification $T_0S \cong H^1(X,\Theta_X) \cong H^1(X,\Omega_X)$ can be expressed w/o reference to a generator of $H^0(X,\Omega_X^2)$: we have a canonical isom

$$T_0S \xrightarrow{\cong} H^1(X,\Theta_X) \to \mathrm{Hom}_{\mathbb{C}}(\underbrace{H^0(X,\Omega_X^2)}_{= H^{2,0}(X)}, \underbrace{H^1(X,\Omega_X^1)}_{= H^{1,1}(X)})$$

We will interpret this as the derivative of a period map.

<u>Hodge structure and the period map for a K3.</u>

Let M be a 4-manifold which underlies a K3 surface X. Then $p_1(M) = 2c_2(X) = 2e(X) = 2.24.$ generator of $H^4(M)$. So M has a canonical or. $[M] \in H_4(M)$ (preserved by any self-diffeo of M). Let us write H for $H^2(M)$ equipped with its quadratic form

$$q(\alpha) := \tfrac{1}{2}\int_{[M]} \alpha \cup \alpha$$

(defines an even unimodular symm. bil. form of sign $(3,19)$). For $\alpha,\beta \in H_{\mathbb{C}}$, set

$$\langle \alpha,\beta\rangle := \int_{[M]} \alpha \cup \bar{\beta}$$

(a herm form of sign $(3,19)$)

Suppose X K3-surface with underlying man M. Then Hodge dec.

$$H_{\mathbb{C}} = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$$

(easy) standard facts

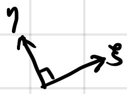
1) $H^{0,2}(X) = \overline{H^{2,0}(X)}$
2) this dec. is perpendicular wrt. $\langle\,,\rangle$
3) $H^{2,0}(X)$ is $q_{\mathbb{C}}$-isotropic, but $\langle\,,\rangle$-positive (hence same for $H^{0,2}(X)$)

It follows that $H^{0,2}(X)$ completely determines the Hodge decomposition.

Remark: A complex line in $H_{\mathbb{C}}$ (such as $H^{2,0}(X)$) which is $q_{\mathbb{C}}$-isotropic and $\langle\,,\rangle$ pos. determines a pos. oriented 2-plane in $H_{\mathbb{R}}$: choose a generator α an write this generator as $\xi + \sqrt{-1}\,\eta$ with $\xi,\eta \in H_{\mathbb{R}}$. Then

$$0 < \langle\alpha,\alpha\rangle = \langle\xi,\xi\rangle + \langle\eta,\eta\rangle$$

$$0 = q_{\mathbb{C}}(\alpha) = \tfrac{1}{2}(\langle\xi,\xi\rangle - \langle\eta,\eta\rangle) + \sqrt{-1}\,\langle\xi,\eta\rangle$$



and so ξ,η spans a pos. or. 2-plane in $H_{\mathbb{R}}$ (indep of choice of α).

Conversely, if $\Pi \subset H_{\mathbb{R}}$ is a pos or. 2-plane in $H_{\mathbb{R}}$, then for an oriented orthonormal basis (ξ,η) the line spanned by $\xi + \sqrt{-1}\,\eta$ only depends on Π and is as above.

We write Π_X for the or. 2-plane associated to $H^{2,0}(X)$.

Let $\mathcal{D} = \{[\ell] \in \mathbb{P}(H_{\mathbb{C}}) : \underbrace{\ell\ q_{\mathbb{C}}\text{-isotropic}}_{\text{defines a nons. quadric in } \mathbb{P}(H_{\mathbb{C}})},\ \underbrace{\langle\,,\rangle\text{-pos}}_{\text{open property}}\}$.

This is a connected open subset of a nons. quadric in $\mathbb{P}(H_{\mathbb{C}})$. On it acts $O(q_{\mathbb{R}})$ transitively, but not properly; $\mathcal{D} \cong O(3,19)/SO(2) \times \underbrace{O(1,19)}_{\text{not compact}}$.

The tangent space of $\mathbb{P}(H_{\mathbb{C}})$ at $[\ell]$ is

$$\mathrm{Hom}_{\mathbb{C}}(\ell, H_{\mathbb{C}}/\ell)$$

If $[\ell] \in \mathcal{D}$, then $T_{[\ell]}\mathcal{D} = T_{[\ell]}(\text{quadric defined by } q_{\mathbb{C}}) = \mathrm{Hom}_{\mathbb{C}}(\ell, \ell^{\perp}/\ell)$, where $\ell^{\perp}$ is the perp wrt to q.

So if a complex structure on M makes it a K3 surface X, then we get a point $[H^{2,0}(X)] \in \mathcal{D}$. Let

$$\begin{array}{ccc} X & \overset{\iota}{\hookrightarrow} & \mathcal{X} \\ & & \downarrow \\ & 0 \in & S \end{array}$$

now be a deformation of X with contractible base S. Then for every $s \in S$, we have an isomorphism $H^2(X_s) \xleftarrow{\cong} H^2(\mathcal{X}) \xrightarrow[\cong]{\iota^*} H^2(X) = H$. Via this isom $H^{2,0}(X_s)$ defines a point $[H^{2,0}(X_s)] \in \mathcal{D}$ and so we have a (hol) map $P: S \longrightarrow \mathcal{D}$ which we call the period map.

Prop. The derivative of $\mathcal{P}$ at o,
$$D_o\mathcal{P} : T_oS \longrightarrow T_{[H^{2,0}(X)]}\mathcal{D} \cong \mathrm{Hom}\big(H^{2,0}(X), \underbrace{H^{2,0}(X)^{\perp}/H^{2,0}(X)}_{H^{1,1}(X)}\big)$$
is the K-S map.

Proof is not so difficult

Cor. A def. $(X \hookrightarrow \mathcal{X} \xrightarrow{\pi} S)$ is universal if and only if the germ $(S,o) \xrightarrow{\mathcal{P}} (\mathcal{D}, [H^{2,0}(X)])$ is an isom of hol. germs.

Spinor orientation and Kähler cone

Here M and $H = H^2(M)$ are as above. The pos. def. 3-planes in $H_{\mathbb{R}}$ make up an open subset $Gr_3^+(H_{\mathbb{R}})$. The group $O(q_{\mathbb{R}})$ acts transitively and properly on it; it is in fact its symm. space $(\cong O(3,19)/O(3)\times O(19))$ and (hence) contractible. This implies that $\gamma^3_{H_{\mathbb{R}}} \mid Gr_3^+(H_{\mathbb{R}})$ is trivial and hence orientable. An element of $O(q_{\mathbb{R}})$ may or may not preserve its orientation. This defines a hom.
$$\sigma : O(q_{\mathbb{R}}) \longrightarrow \mu_2 \qquad \text{(the spinor norm for } -q_{\mathbb{R}})$$
Together with $\det : O(q_{\mathbb{R}}) \to \mu_2$ this defines a surjection $O(q_{\mathbb{R}}) \to \mu_2 \times \mu_2$ whose kernel is the identity component of $O(q_{\mathbb{R}})$.

A spinor orientation of $H_{\mathbb{R}}$ is an orientation of $\gamma^3_{H_{\mathbb{R}}} \mid Gr_3^+(H_{\mathbb{R}})$.
We will see that if a K3 surface X has M as underlying man., it determines a spinor orientation of $H_{\mathbb{R}}$. We use:

Thm (Yum-Tong Siu, 1983) Every K3 surface admits a Kähler metric.

Recall that to give a Kähler metric on a complex manifold Y is to give a symplectic form ω on Y which is of type $(1,1)$ and is such that $(u,v) \in T_pY \longmapsto \omega_p(J_pu, v)$ is pos. def. everywhere. So the Kähler classes $[\omega] \in H^{1,1}(Y;\mathbb{R})$ make up an open convex cone in $H^{1,1}(Y,\mathbb{R})$.

For X as above, a Kähler class $[\omega] \in H^{1,1}(X,\mathbb{R})$ has the property that $q_{\mathbb{R}}([\omega]) > 0$. So $\Pi_X + \mathbb{R}[\omega]$ is a 3-plane and nat. oriented. The spinor orientation thus obtained is independent of the choice of $[\omega]$, since these lie in a convex cone. More is true:

Thm (Donaldson 1990) This spinor orientation is even intrinsic to M: every complex structure making it a K3 surface defines the same spinor orientation. In particular $\mathrm{Diff}(M)$ preserves this spinor orientation.

Twistor construction

Thm (Yau) Let X be a K3 surface and $\kappa \in H^{1,1}(X;\mathbb{R})$ the class of a Kähler metric. Then there is a unique Kähler-Einstein metric g on X whose imaginary part represents κ (K-E metric means here: Ricci flat)

This has important consequences.

Thom construction $\quad (\mathbb{P}^3, \mathbb{P}^3 \setminus U) \simeq (U, \partial U)$