

MODULI SPACES AND LOCALLY SYMMETRIC VARIETIES

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1. SOME CLASSICAL SYMMETRIC DOMAINS AND HODGE THEORY

Let G be a connected reductive Lie group with compact center. If G acts smoothly and transitively on manifold \mathbb{X} such that the stabilizer of a point is maximal compact in G , then we call the G -manifold \mathbb{X} a *symmetric space* for G . Since the maximal compact subgroups of G belong to a single conjugacy class, there is only one symmetric space for G up to unique isomorphism (and one may think of it as the moduli space of the maximal compact subgroups of G). If \mathbb{X} happens to be a complex manifold on which G acts by biholomorphic transformations, then \mathbb{X} can be embedded as an open subset in complex vector space and that is why \mathbb{X} is then called a *bounded symmetric domain*. These domains often appear in algebraic geometry because they parametrize polarized Hodge structures. (And the philosophy is that their arithmetically enhanced versions, the Shimura varieties, should parametrize motives.)

The *irreducible* bounded symmetric domains, i.e., those for which G is simple, were classified by E. Cartan. He listed them into six types (labeling them by roman numbers), of which the first four are (classical) series and the last two are exceptional in the sense that they are associated to groups of type G_2 and E_7 respectively.

Siegel upper half spaces. Perhaps the most familiar class of bounded symmetric domains are those of type III (in the Cartan classification), usually presented as a tube domain and then known as a Siegel upper half spaces. Such an object is naturally associated with a finite dimensional real vector space V endowed with a nondegenerate symplectic form $\alpha : V \times V \rightarrow \mathbb{R}$. So V has even dimension, $2g$ say, and if $\alpha_{\mathbb{C}} : V_{\mathbb{C}} \times V_{\mathbb{C}} \rightarrow \mathbb{C}$ denotes the complexification of the symplectic form, then the Hermitian form $h : V_{\mathbb{C}} \times V_{\mathbb{C}} \rightarrow \mathbb{C}$, $h(v, v') := \sqrt{-1}\alpha_{\mathbb{C}}(w, \bar{w}')$ has signature (g, g) . We take $G = \mathrm{Sp}(V)$ and $\mathbb{X} = \mathbb{H}_V$, the subset of the Grassmannian $\mathrm{Gr}_G(V_{\mathbb{C}})$: a g -dimensional subspaces $F \subset V_{\mathbb{C}}$ is in \mathbb{H}_V if F is totally isotropic relative to $\alpha_{\mathbb{C}}$ and positive definite relative to h . The group $\mathrm{Sp}(V)$ acts transitively on \mathbb{H}_V . The stabilizer of any $F \in \mathbb{H}_V$ restricts isomorphically to the unitary group $U(F)$; it is a compact subgroup of $\mathrm{Sp}(V)$ and maximal for that property. That makes \mathbb{H}_V a symmetric space for $\mathrm{Sp}(V)$. If $F \in \mathbb{H}_V$, then F defines a Hodge structure on V of weight 1 polarized by α by taking $V^{1,0} = F$ and $V^{0,1} = \bar{F}$. Thus \mathbb{H}_V parametrizes polarized Hodge structures on V of this type. This is why the

tautological bundle of rank g over $\text{Gr}_g(\mathbb{V}_{\mathbb{C}})$ restricted to \mathbb{H}_V is often called the *Hodge bundle*. We refer to its g -th exterior power, denoted by \mathbb{L}_V , as the *automorphic line bundle* of \mathbb{H}_V .

Domains of type I. Another class of classical bounded symmetric domains (type $I_{p,q}$ in the Cartan classification) is defined by a finite dimensional complex vector space W endowed with a nondegenerate Hermitian form $h : W \times W \rightarrow \mathbb{C}$ of signature (p, q) , say. We take $G = \text{U}(W)$ and let \mathbb{X} be the open subset $\mathbb{B}_W \subset \text{Gr}_p(W)$ of positive definite subspaces of dimension p . The action of $\text{U}(W)$ on \mathbb{B}_W is transitive and the stabilizer of $F \in \mathbb{B}_W$ is the product of compact unitary groups $\text{U}(F) \times \text{U}(F^\perp)$. This is indeed a maximal compact subgroup of $\text{U}(W)$. The tautological bundle of rank p over $\text{Gr}_p(W)$ restricts to one over \mathbb{B}_W ; the *automorphic line bundle* over \mathbb{B}_W is its p -th exterior power and denoted by \mathbb{L}_W . The case $p = 1$ is of special interest to us: then \mathbb{B}_W appears in $\text{Gr}_1(W) = \mathbb{P}(W)$ as a complex unit ball and the locus W_+ of $w \in W$ with $h(w, w) > 0$ can be identified with the complement of the zero section $\mathbb{L}_W^\times \subset \mathbb{L}_W$. For $p = q = 1$, we recover the usual upper half plane: there is a cobasis (z_1, z_2) of W such that $h(v, v') = -\sqrt{-1}(z_1(v)\bar{z}_2(v') - z_2(v)\bar{z}_1(v'))$. Then \mathbb{B}_W is defined by $\text{Im}(z_1/z_2) > 0$. In fact, the set defined by $h(v, v) = 0$ defines a circle in $\mathbb{P}(W)$ and there is a unique real form V of W for which this circle is the associated real projective line. The imaginary part of h defines a symplectic form α on V for which \mathbb{B}_W gets identified with \mathbb{H}_V .

Type I domains can arise as subdomains of type III domains as follows. With (V, α) as above, suppose we are given a semisimple $\sigma \in \text{Sp}(V)$ (in the sense that it decomposes $V_{\mathbb{C}}$ into a direct sum of its eigenspaces: $V_{\mathbb{C}} = \bigoplus_{\lambda} V_{\mathbb{C}}^\lambda$). We assume that its fixed point set \mathbb{H}_V^σ is nonempty and determine what it is like. Clearly, if $F \in \mathbb{H}_V^\sigma$, then $\sigma \in \text{U}(F) \times \text{U}(\bar{F})$ and so the eigenvalues lie of σ are 1 absolute value and occur in complex conjugate pairs. Write $F^\lambda := F \cap V_{\mathbb{C}}^\lambda$.

Now note that $V_{\mathbb{C}}^{\pm 1}$ is defined over \mathbb{R} and that α is nondegenerate on the underlying real vector space $V^{\pm 1}$. So $\mathbb{H}_{V^{\pm 1}}$ is defined $\dim F^{\pm 1}$ defines an element of $\mathbb{H}_{V^{\pm 1}}$. Suppose $\lambda \neq \{\pm 1\}$. Then $\alpha_{\mathbb{C}}(v, v') = \alpha_{\mathbb{C}}(\sigma v, \sigma v') = \lambda^2 \alpha_{\mathbb{C}}(v, v')$ and so $V_{\mathbb{C}}^\lambda$ is $\alpha_{\mathbb{C}}$ -isotropic. If $\text{Im}(\lambda) > 0$, then h is nondegenerate, say of signature (p_λ, q_λ) , and F^λ will be positive definite subspace of dimension p_λ of $V_{\mathbb{C}}^\lambda$: it is an element of \mathbb{B}_{V^λ} . Note that then $V_{\mathbb{C}}^{\bar{\lambda}} = \bar{V}_{\mathbb{C}}^{\bar{\lambda}}$ has signature (q_λ, p_λ) and that $F^{\bar{\lambda}}$ is the annihilator of F^λ relative to $\alpha_{\mathbb{C}}$. We thus obtain an isometric embedding

$$\mathbb{H}_{V^1} \times \mathbb{H}_{V^{-1}} \times \prod_{\text{Im}(\lambda) > 0} \mathbb{B}_{V^\lambda} \hookrightarrow \mathbb{H}_V$$

whose image is a totally geodesic subdomain \mathbb{H}_V^σ . The centralizer of σ in $\text{Sp}(V)$, $\text{Sp}(V)_\sigma$, will act transitively on this product via a homomorphism

$\mathrm{Sp}(V)_\sigma \rightarrow \mathrm{Sp}(V^1) \times \mathrm{Sp}(V^{-1}) \times \prod_{\mathrm{Im}(\lambda) > 0} \mathrm{U}(V_\mathbb{C}^\lambda)$ that is in fact an isomorphism and \mathbb{H}_V^σ may be regarded as its symmetric domain. The pull-back of the automorphic line bundle \mathbb{L}_V can be equivariantly identified with the exterior tensor product of the automorphic line bundles of the factors.

Thus a polarized Hodge structure of type $(1, 0) + (0, 1)$ invariant under a semisimple symmetry gives rise to point in such a product decomposition. Our main interest will be in the factors that give rise to complex balls, i.e., those of type $I_{1,q}$.

Domains of type IV. A class of classical bounded domains especially dear to algebraic geometers are those of type IV. Such a domain is given by a real vector space V of dimension $n + 2 \geq 2$ endowed with a symmetric bilinear form $s : V \times V \rightarrow \mathbb{R}$ of signature $(2, n)$. We let $s_\mathbb{C} : V_\mathbb{C} \times V_\mathbb{C} \rightarrow \mathbb{C}$ denote the complexification of s and $h : V_\mathbb{C} \times V_\mathbb{C} \rightarrow \mathbb{C}$ the Hermitian form defined by $h(v, v') = s_\mathbb{C}(v, \bar{v}')$. Let $V_+ \subset V$ be the set of $v \in V_\mathbb{C}$ with $s_\mathbb{C}(v, v) = 0$ and $h(v, v) > 0$ and denote by $\mathbb{D}_V := \mathbb{P}(V_+) \subset \mathbb{P}(V)$ the corresponding projectivization. The latter is an open subset in the nonsingular quadric in $\mathbb{P}(V_\mathbb{C})$ defined by $s_\mathbb{C}(v, v) = 0$. If we assign to $[v] \in \mathbb{D}_V$ the oriented plane $P_{[v]}$ spanned by the real part and the imaginary part of v , then we obtain an identification of \mathbb{D}_V with the Grassmannian of positive definite oriented 2-planes in V . This makes it clear that that \mathbb{D}_V has two connected components that are exchanged by complex conjugation. The orthogonal group $O(V)$ acts transitively on \mathbb{D}_V . The stabilizer of $[v] \in \mathbb{D}_V$ is $\mathrm{SO}(P_{[v]}) \times O(P_{[v]}^\perp)$. Its intersection with the identity component $O(V)^\circ$ is $\mathrm{SO}(P_{[v]}) \times \mathrm{SO}(P_{[v]}^\perp)$ and this is a maximal compact subgroup of $O(V)^\circ$. So each connected component of \mathbb{D}_V is a bounded symmetric domain for $O(V)^\circ$. The *automorphic line bundle* \mathbb{L}_V on \mathbb{D}_V is simply the restriction of the tautological bundle over $\mathbb{P}(V_\mathbb{C})$. The complement of its zero section \mathbb{L}_V^\times can be identified with V_+ .

We have already seen the cases $n = 1, 2$ in a different guise: if V_1 is of real dimension two and endowed with a real symplectic form $\alpha : V_1 \times V_1 \rightarrow \mathbb{R}$, then $\mathrm{Sym}^2 V_1$ comes with a natural nondegenerate symmetric bilinear form of signature $(1, 2)$ and assigning to $F \in \mathbb{H}_{V_1}$ the line $\mathrm{Sym}^2 F$ defines an isomorphism $\mathbb{H}_{V_1} \cong \mathbb{D}_{\mathrm{Sym}^2 V_1}$ that is compatible with an isogeny of $\mathrm{Sp}(V_1)$ onto $O(\mathrm{Sym}^2 V_1)^\circ$ and for which $\mathbb{L}_{\mathrm{Sym}^2 V_1}$ pulls back to \mathbb{L}_{V_1} . Similarly, $V_1 \otimes V_1$ comes with a natural nondegenerate symmetric bilinear form of signature $(2, 2)$ and $(F, F') \in \mathbb{H}_{V_1}^2 \mapsto F \otimes F'$ defines an isomorphism $\mathbb{H}_{V_1}^2 \cong \mathbb{D}_{V_1 \otimes V_1}$ that is compatible with an isogeny of $\mathrm{Sp}(V_1) \times \mathrm{Sp}(V_1)$ onto $O(V_1 \otimes V_1)^\circ$. Here $\mathbb{L}_{V_1 \otimes V_1}$ pulls back to the exterior product $\mathbb{L}_{V_1} \boxtimes \mathbb{L}_{V_1}$.

Any $F \in \mathbb{D}_V$ gives rise to a decomposition $V_\mathbb{C} = F \oplus \bar{F} \oplus (F \oplus \bar{F})^\perp$, which we can think of as defining a Hodge structure of weight 2 polarized by s with $F = V^{2,0}$, $V^{1,1} = (F \oplus \bar{F})^\perp$ and $\bar{F} = V^{0,2}$.

In this context ball domains may arise in a similar manner. Suppose $\sigma \in O(V)$ is semisimple and \mathbb{D}_V^σ is nonempty. Clearly, if $[v] \in \mathbb{D}_V^\sigma$, then $v \in V_\mathbb{C}^\lambda$

for some eigenvalue λ of σ . Since $\sigma \in \mathrm{SO}(P_{[V]}) \times \mathrm{SO}(P_{[V]}^\perp)$, all the eigenvalues of σ lie on the complex unit circle. When $\lambda = \pm 1$, then $V_\mathbb{C}^\lambda$ is defined over \mathbb{R} and must have signature $(2, n')$ for some integer $n' \geq 0$. This yields a totally geodesic embedding $\mathbb{D}_{V^{\pm 1}} \cong \mathbb{D}_V^\sigma \subset \mathbb{D}_V$ and $O(V)_\sigma$ acts via a homomorphism $O(V)_\sigma \rightarrow O(V^{\pm 1})$. This homomorphism is onto and has compact kernel.

For $\mathrm{Im}(\lambda) > 0$, h will have the same signature on $V_\mathbb{C}^{\bar{\lambda}}$ as on $V_\mathbb{C}^\lambda$. It follows that this signature must be of the form $(1, n')$ for some integer $n' \geq 0$. Then \mathbb{D}_V^σ is the disjoint union of $\mathbb{B}_{V_\mathbb{C}^\lambda}$ and its complex conjugate $\mathbb{B}_{V_\mathbb{C}^{\bar{\lambda}}}$. Both are totally geodesically embedded and $O(V)_\sigma$ acts via a homomorphism $O(V)_\sigma \rightarrow U(V_\mathbb{C}^\lambda)$ that is onto and has compact kernel.

Remark 1.1. All the geodesic embeddings discussed here have the property that the pull-back of the automorphic line bundle is an automorphic line bundle. It is also easy to check that in all cases the canonical bundle on the domain is equivariantly identified with a positive power of the automorphic line bundle.

The remaining irreducible Hermitian domains. These are the classical domains of type II and two domains that are the symmetric spaces of real forms of simple Lie groups of type G_2 and E_7 . A domain of type II is given by a real vector space V of finite even dimension $2g$ endowed with symmetric bilinear form $s : V \times V \rightarrow \mathbb{R}$ of signature (g, g) . If $s_\mathbb{C}$ and h are as usual, then let \mathbb{D}_V be the set of $s_\mathbb{C}$ -isotropic g -dimensional subspaces $F \subset V_\mathbb{C}$ that are h -positive. The group $O(V)$ acts transitively on \mathbb{D}_V and the stabilizer of any $F \in \mathbb{D}_V$ restricts isomorphically to the unitary group $U(F)$, a maximal compact subgroup of $O(V)$. So \mathbb{D}_V is a symmetric space for the identity component of $O(V)$. If $F \in \mathbb{D}_V$, then F defines a Hodge structure on V of weight 0 polarized by h by taking $V_{1,-1} = F$ and $V_{-1,1} = \bar{F}$. so that \mathbb{D}_V parametrizes polarized Hodge structures on V of this type. The automorphic line bundle \mathbb{L}_V is the exterior power of the tautological bundle of rank g over $\mathrm{Gr}(V_\mathbb{C})$ restricted to \mathbb{D}_V .

The exceptional domains have dimension 16 and 27 respectively.

Hermitian domains admitting complex reflections. The complex balls and the domains of type IV are the only bounded symmetric domains which admit totally geodesic subdomains of codimension one (or what turns out to be equivalently, admit complex reflections). In one direction this is easy to see: if (W, h) is a Hermitian form of hyperbolic signature $(1, n)$, then any subspace $W' \subset W$ of complex codimension one of signature $(1, n-1)$ is the fixed point set of a copy of $U(1)$ (just let $z \in U(1)$ act on $W' \oplus (W')^\perp$ as multiplication by $(1, z)$) and so $\mathbb{B}_{W'} = \mathbb{B}_W \cap \mathbb{P}(W') = \mathbb{B}_W^{U(1)}$. This implies that $\mathbb{B}_{W'}$ is geodesically embedded. A similar argument shows that if (V, s) defines a domain of type IV as above and $V' \subset V$ is of real codimension of signature $(2, n-1)$, then V' is the fixed point set of a reflection

$\sigma \in O(W)$ (let it act on -1 on $V' \oplus (V')^\perp$) as multiplication by $(1, -1)$ and so $\mathbb{D}_{V'} = \mathbb{D}_V \cap \mathbb{P}(V'_\mathbb{C}) = \mathbb{D}_V^\sigma$. So both cases, any nonempty hyperplane section has this property.

2. THE BAILY-BOREL PACKAGE

Arithmetic structures. The notion of an arithmetic subgroup of an algebraic group requires the latter to be defined over a number field. A trick (‘restriction of scalars’) shows that here is in fact little or no loss in generality in assuming that this number field is \mathbb{Q} . In the cases $\mathrm{Sp}(V)$ and $O(V)$ this is simply accomplished by assuming that (V, α) resp. (V, s) is defined over \mathbb{Q} . An arithmetic subgroup Γ of $\mathrm{Sp}(V)$ resp. $O(V)$ is then a subgroup which fixes a lattice $V_\mathbb{Z} \subset V_\mathbb{Q}$ and is of finite index in $\mathrm{Sp}(V_\mathbb{Z})$ resp. $O(V_\mathbb{Z})$. Such a group is discrete in the ambient algebraic group and hence acts properly discontinuously on the associated Hermitian domain.

Examples for the case of complex ball can be obtained as in the discussion above by taking a semisimple $\sigma \in \mathrm{Sp}(V_\mathbb{Z})$ resp. $\sigma \in O(V_\mathbb{Z})$ that has fixed points in the associated Hermitian domain. Since σ lies in a unitary group and fixes a lattice, it must have finite order, m say so that its eigenvalues are m th roots of unity. If μ_m^{pr} denotes the set of primitive roots of unity, then $\bigoplus_{\lambda \in \mu_m^{\mathrm{pr}}} V_\mathbb{C}^\lambda$ is defined over \mathbb{Q} . So the subgroup G of $\mathrm{Sp}(V)_\sigma$ resp. $O(V_\mathbb{Z})$ which acts as the identity on the remaining eigen spaces is defined over \mathbb{Q} and the subgroup $G(\mathbb{Z})$ which stabilizes the lattice $V_\mathbb{Z}$ is arithmetic in G . Suppose first we are in the symplectic case. If for all $\lambda \in \mu_m^{\mathrm{pr}}$, the form h is definite on $V_\mathbb{C}^\lambda$ except for a complex conjugate pair $(\lambda_o, \bar{\lambda}_o)$ with h of signature $(1, n)$ on $V_\mathbb{C}^{\lambda_o}$ (and hence of signature $(n, 1)$ on $V_\mathbb{C}^{\bar{\lambda}_o}$), then G acts properly on $\mathbb{B}_{V_\mathbb{C}^{\lambda_o}}$ and hence $G(\mathbb{Z})$ acts on this domain properly discretely. A similar argument applies to the orthogonal case. The situation simplifies considerably when $m = 3, 4, 6$, for then there are only two primitive m th roots of unity (namely $\zeta_m := \exp(2\pi\sqrt{-1}/m)$ and its conjugate $\bar{\zeta}_m := \exp(-2\pi\sqrt{-1}/m)$). So the above decomposition reduces to $V_\mathbb{C}^{\zeta_m} \oplus V_\mathbb{C}^{\bar{\zeta}_m}$ and G has no compact factors unless G itself is compact.

The more intrinsic approach is to interpret an arithmetic structure on (W, h) as given by a lift (W_K, h_K) over a CM-field K/\mathbb{Q} with the property that for every embedding $\iota : K \rightarrow \mathbb{C}$ the corresponding Hermitian form on $W_K \otimes_\iota \mathbb{C}$ is definite except for a complex conjugate pair, for which the signatures are $(n, 1)$ and $(1, n)$.

The Baily-Borel package for arithmetic varieties. Let (G, \mathbb{L}) be one of the examples above. To be precise: G is an algebraic group defined over \mathbb{R} and $G(\mathbb{R})$ acts properly and transitively on a complex manifold \mathbb{X} such that each connected component of \mathbb{X} is a symmetric domain for the identity component $G(\mathbb{R})^\circ$ of $G(\mathbb{R})$. This action lifts to one on an automorphic line bundle \mathbb{L} over \mathbb{X} . We denote by $\mathbb{L}^\times \subset \mathbb{L}$ the complement of the zero section.

Now suppose G is defined over \mathbb{Q} and we are given an arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$. Then Γ acts properly discretely on \mathbb{D} (and hence on \mathbb{L}^\times). The action of Γ on \mathbb{L}^\times commutes with the obvious action of \mathbb{C}^\times so that we have an action of $\Gamma \times \mathbb{C}^\times$ on \mathbb{L}^\times . The Baily-Borel package that we are going to state below can be entirely phrased in terms of this action. It begins with the observation that we can form the Γ -orbit space of the \mathbb{C}^\times -bundle $\mathbb{L}^\times \rightarrow \mathbb{X}$ in the complex-analytic category of orbifolds to produce an \mathbb{C}^\times -bundle $\mathbb{L}_\Gamma^\times \rightarrow \mathbb{X}_\Gamma$. For $d \in \mathbb{Z}$, we denote by $A_d(\mathbb{L}^\times)$ the linear space of holomorphic functions $f : \mathbb{L}^\times \rightarrow \mathbb{C}$ that are homogeneous of degree $-d$ (in the sense that $f(\lambda z) = \lambda^{-d}f(z)$) and subject to a growth condition (which in many cases is empty). This growth condition is such that $A_\bullet(\mathbb{L}^\times)$ is closed under multiplication (making it a graded \mathbb{C} -algebra) and invariant under Γ .

Finiteness: The graded \mathbb{C} -subalgebra of Γ -invariants, $A_\bullet(\mathbb{L}^\times)^\Gamma$ is finitely generated with generators in positive degree. So we have defined the weighted homogeneous (affine) cone $(\mathbb{L}_\Gamma^\times)^{\text{bb}} := \text{Spec } A_\bullet(\mathbb{L}^\times)^\Gamma$ whose base is the projective variety $\mathbb{X}_\Gamma^{\text{bb}} := \text{Proj}(A_\bullet(\mathbb{L}^\times)^\Gamma)$. If Γ acts transitively on the connected components of \mathbb{L}^\times , then $A_\bullet(\mathbb{L}^\times)^\Gamma$ is normal.

Separation: The elements of $A_\bullet(\mathbb{L}^\times)^\Gamma$ separate the Γ -orbits in \mathbb{L}^\times so that the natural maps $\mathbb{L}_\Gamma^\times \rightarrow (\mathbb{L}_\Gamma^\times)^{\text{bb}}$ and $\mathbb{X}_\Gamma \rightarrow \mathbb{X}_\Gamma^{\text{bb}}$ are injective.

Topology: The underlying analytic space of $(\mathbb{L}_\Gamma^\times)^{\text{bb}}$ is obtained as Γ -orbit space of a ringed space $((\mathbb{L}^\times)^{\text{bb}}, \mathcal{O}_{(\mathbb{L}^\times)^{\text{bb}}})$ endowed with a $\Gamma \times \mathbb{C}^\times$ -action and a $\Gamma \times \mathbb{C}^\times$ -invariant partition into complex manifolds such that $\mathcal{O}_{(\mathbb{L}^\times)^{\text{bb}}}$ is the sheaf of continuous complex valued functions that are holomorphic on each part. This partition descends to one of $(\mathbb{L}_\Gamma^\times)^{\text{bb}}$ into locally closed subvarieties which has \mathbb{L}_Γ^\times as an open-dense member. (Thus $\mathbb{X}_\Gamma^{\text{bb}}$ comes with stratification into locally closed subvarieties which has the image of \mathbb{X}_Γ as an open-dense member. In particular, \mathbb{L}_Γ^\times resp. \mathbb{X}_Γ acquires the structure of a quasi-affine resp. quasi-projective variety.)

In fact, $(\mathbb{L}^\times)^{\text{bb}}$ and $A_\bullet(\mathbb{L}^\times)$ only depend on $G(\mathbb{Q})$ and the $\Gamma \times \mathbb{C}^\times$ -action is the restriction of one of $G(\mathbb{Q}) \times \mathbb{C}^\times$, but in generalizations that we shall consider the latter group is no longer acting. We will make the topological part explicit in some of the classical cases discussed above.

Ball quotients. These are in a sense the simplest because $\mathbb{X}_\Gamma^{\text{bb}}$ adds to \mathbb{X}_Γ a finite set. Denote by \mathcal{I} the collection of isotropic subspaces of W that is left pointwise fixed by an element of $G(\mathbb{Q})$ of infinite order (one can show that we can then take this element to lie in Γ). Note that every $I \in \mathcal{I}$ is of dimension ≤ 1 . It is known that Γ has finitely many orbits in \mathcal{I} .

For every $I \in \mathcal{I}$, we have a projection $\pi_{I^\perp} : \mathbb{L}^\times \subset W \rightarrow W/I^\perp$. When I is of dimension one, the image of π_{I^\perp} is the complement of the origin. As a set $(\mathbb{L}^\times)^{\text{bb}}$, is the disjoint union of \mathbb{L}^\times and its projections $\pi_{I^\perp}(\mathbb{L}^\times)$. It comes with an evident action of $G(\mathbb{Q}) \times \mathbb{C}^\times$ and the topology we put on it makes

this action topological. It has a conical structure with vertex the stratum defined by $I = \{0\}$. So $(\mathbb{L}^\times)^{\text{bb}}$ adds to \mathbb{L}^\times a finite number of \mathbb{C}^\times -orbits, one of which is a singleton. In particular, $\mathbb{X}_\Gamma^{\text{bb}} - \mathbb{X}_\Gamma$ is finite.

Quotients of orthogonal type. Here (V, s) is a real vector space endowed with a nondegenerate quadratic form defined over \mathbb{Q} and of signature $(2, n)$. Denote by \mathcal{I} the collection of isotropic subspaces of V defined over \mathbb{Q} .

For every $I \in \mathcal{I}$, we have a projection $\pi_{I^\perp} : \mathbb{L}^\times \subset V_{\mathbb{C}} \rightarrow V_{\mathbb{C}}/I_{\mathbb{C}}^\perp$. When $\dim I = 2$, then the image consists to the elements in $V_{\mathbb{C}}/I_{\mathbb{C}}^\perp$ that under the identification $V_{\mathbb{C}}/I_{\mathbb{C}}^\perp \cong \text{Hom}_{\mathbb{R}}(I, \mathbb{C})$ have real rank 2 (this has two connected components, each being the image of a connected component of \mathbb{L}^\times). The \mathbb{C}^\times -orbits space of $\pi_{I^\perp}(\mathbb{L}^\times)$ is isomorphic to $\mathbb{C} - \mathbb{R}$ and the Γ -stabilizer of each of these half planes acts there via a group which acts nicely with the orbit space a modular curve. These modular curves are noncompact, but their missing points are supplied by the $I \in \mathcal{I}$ of dimension one.

When $\dim I = 1$ the image is the punctured line $V_{\mathbb{C}}/I_{\mathbb{C}}^\perp - \{0\}$ and for $I = \{0\}$ we get a singleton. Then $(\mathbb{L}^\times)^{\text{bb}}$ is the disjoint union of \mathbb{L}^\times with its projections $\pi_{I^\perp}(\mathbb{L}^\times)$ and endowed with a $O(V_{\mathbb{Q}}) \times \mathbb{C}^\times$ -equivariant topology. So $\mathbb{X}_\Gamma^{\text{bb}} - \mathbb{X}_\Gamma$ consists of a union of finitely many modular curves and finitely many points.

Arithmetic arrangements. We noted that the domains of type $I_{1,n}$ and IV have totally geodesic hyperplane sections. For these cases, and where G is defined over \mathbb{Q} , we define an *arithmetic arrangement* as a collection \mathcal{H} of such hyperplane sections with the property that the G -stabilizer of each member is defined over \mathbb{Q} and that there is an arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$ such that \mathcal{H} is a finite union of Γ -orbits. Then the collection $\{\mathbb{L}_{\mathcal{H}}^\times\}_{\mathcal{H} \in \mathcal{H}}$ is locally finite and for $\Gamma \subset G(\mathbb{Q})$ as above, its image in $\Gamma \backslash \mathbb{L}_{\mathcal{H}}^\times$ is a hypersurface $\Delta_{\mathcal{H}} \subset \Gamma \backslash \mathbb{L}_{\mathcal{H}}^\times$. This hypersurface locally given by a single equation (it is \mathbb{Q} -Cartier). We call such a hypersurface an arrangement divisor. Its closure in $(\Gamma \backslash \mathbb{L}_{\mathcal{H}}^\times)^{\text{bb}}$ is also a hypersurface, but need not be \mathbb{Q} -Cartier.

3. MODULAR EXAMPLES

Basics of GIT. Let G be a complex reductive algebraic group and H a finite dimensional complex representation of G . Then G acts on the algebra $\mathbb{C}[H]$ of regular functions on H . Here are two basic facts

Finiteness: The graded \mathbb{C} -algebra $\mathbb{C}[H]^G$ is normal and finitely generated with generators in positive degree. So we have defined the weighted homogeneous (affine) cone $\text{Spec}(\mathbb{C}[H]^G)$ whose base is the projective variety $\text{Proj}(\mathbb{C}[H]^G)$.

Geometric interpretation: The closure of every G -orbit in H contains a closed G -orbit and this G -orbit is unique. Any two distinct closed orbits are separated by a member of $\mathbb{C}[H]^G$ and assigning to a closed G -orbit the corresponding point of $\text{Spec} \mathbb{C}[H]^G$ sets up a bijection

between the set of closed G -orbits on H and the closed points of $\text{Spec } \mathbb{C}[H]^G$ (we therefore write $G \backslash\backslash H$ for $\text{Spec } \mathbb{C}[H]^G$).

This also leads to an interpretation of $\text{Proj } \mathbb{C}[H]^G$. The obvious map $H \rightarrow G \backslash\backslash H$ has as fiber through $0 \in H$ the set of $v \in H$ with $0 \in \overline{Gv}$. The complement, called the *semistable locus* and denoted H^{ss} , then maps to the complement of the vertex of the cone $\text{Spec } \mathbb{C}[H]^G$ and we get a bijection between the set of closed orbits in $\mathbb{P}(H^{ss})$ and $\text{Proj } \mathbb{C}[H]^G$. This is why we denote the latter sometimes by $G \backslash\backslash \mathbb{P}(H^{ss})$. Often these separated quotients contain an genuine orbit space as an open-dense subset: the *stable locus* $H^{st} \subset H^{ss}$ consists of the $v \in H - \{0\}$ whose orbit is closed and for which the stabilizer G_v is finite. Then H^{st} is Zariski open (but it need not be dense; it could even be empty while H^{ss} is nonempty). Then the morphisms $H^{st} \rightarrow G \backslash\backslash H$ and $\mathbb{P}(H^{st}) \rightarrow G \backslash\backslash \mathbb{P}(H^{ss})$ are open with image the G -orbit space of the source. In case H^{st} is dense in H^{ss} , we may regard $G \backslash\backslash \mathbb{P}(H^{ss})$ as a projective compactification of the G -orbit space of H^{st} .

Example: cyclic covers of the projective line (Deligne-Mostow). Let U be complex vector space of dimension 2. Then we have the irreducible $SL(U)$ -representation $H := \text{Sym}^{12}(U^*)$. We regard H as the space of homogeneous polynomials of degree 12 on U and $\mathbb{P}(H)$ as the linear system of degree 12 divisors on the projective line $\mathbb{P}(U)$. It is known that H^{ss} resp. H^{st} parametrizes the divisors which have no point of multiplicity > 6 resp. ≥ 6 . The only closed orbit in $H^{st} - H^{ss}$ is the one for which the divisor is given by two distinct points, each of multiplicity 6 and so this will represent the unique point of $SL(U) \backslash\backslash H^{ss} - SL(U) \backslash\backslash H^{st}$.

Denote by $H^\circ \subset H$ the locus which defines reduced divisors (12 element subsets of $\mathbb{P}(U)$). It is the complement of the discriminant $D \subset H$, a hypersurface. For $F \in H^\circ$ we have a cyclic cover of degree 6 $C_F \rightarrow \mathbb{P}(U)$ ramified over the divisor Δ_F defined by F : its homogeneous equation is $w^2 = F(u)$, where it is understood that u has degree 2 so that for every $\lambda \in \mathbb{C}^\times$, $(\lambda^2 u, \lambda w)$ and (u, w) define the same point. This only depends on the image of F in $\mathbb{P}(H)$, but this is not so for the differential ω_F we define next: fix a translation-invariant 2-form α on U . Then ω_F is a residue of the 2-form of α/w restricted to the surface defined by $w^6 = F(u)$ at infinity. To be concrete: if (u_0, u_1) is a coordinate pair for U , such that $\mu = du_0 \wedge du_1$, then an affine equation for C_F is $w'^6 = F(1, u')$ with $u' = u_1/u_0$ and $w' := w/u_0^2$. Then ω_F is in terms of these coordinates given by du'/w' . The Galois group of the cover is the group μ_6 of sixth roots of unity which acts on the C_F via the w -coordinate according to the inverse of the (tautological) character $\chi : \mu_6 \hookrightarrow \mathbb{C}^\times$. Hence ω_F is an eigenvector for the character χ . Now C_F has genus 25. The group μ_6 has no invariants in $H^1(C_F, \mathbb{C})$ and the eigenspaces of μ_6 with nontrivial characters all have dimension 10. Moreover, $H^{1,0}(C_F, \mathbb{C})^{\chi^i}$ is for $i = 1, \dots, 5$ of dimension $2i - 1$ with $H^{1,0}(C_F, \mathbb{C})^\times$ spanned by ω_F . So $H^{1,0}(C_F, \mathbb{C})^\times$ has signature $(1, 9)$.

Now fix a unimodular symplectic lattice $V_{\mathbb{Z}}$ of genus 25 endowed with an action of μ_6 such that there exists a μ_6 -equivariant isomorphism $H^1(C_F, \mathbb{Z}) \cong V_{\mathbb{Z}}$ of symplectic lattices. Such an isomorphism will be unique up to an element of $\mathrm{Sp}(V_{\mathbb{Z}})_{\mu_6}$, the centralizer of μ_6 in $\mathrm{Sp}(V_{\mathbb{Z}})$. If Γ denotes the image of $\mathrm{Sp}(V_{\mathbb{Z}})_{\mu_6}$ in $\mathrm{U}(V_{\mathbb{C}}^{\times})$, then ω_F defines an element of $(\mathbb{L}_{V_{\mathbb{C}}^{\times}}^{\times})_{\Gamma}$ so that we have defined a map $H^{\circ} \rightarrow (\mathbb{L}_{V_{\mathbb{C}}^{\times}}^{\times})_{\Gamma}$. This map is easily seen to be constant on the $\mathrm{SL}(\mathrm{U})$ -orbits in H° . The work of Deligne-Mostow shows that this map extends to an isomorphism $\mathrm{SL}(\mathrm{U}) \backslash H^{\mathrm{st}} \cong (\mathbb{L}_{V_{\mathbb{C}}^{\times}}^{\times})_{\Gamma}$. One can show that this isomorphism extends from one of $\mathrm{SL}(\mathrm{U}) \backslash H^{\mathrm{ss}}$ onto $(\mathbb{L}_{V_{\mathbb{C}}^{\times}}^{\times})_{\Gamma}^{\mathrm{bb}}$ so that the GIT compactification $\mathrm{SL}(\mathrm{U}) \backslash \mathbb{P}(H^{\mathrm{ss}})$ gets identified with the Baily-Borel compactification $(\mathbb{B}_{V_{\mathbb{C}}^{\times}})_{\Gamma}^{\mathrm{bb}}$ (both are one-point compactifications). This also identifies their algebras of regular functions: the graded algebra of automorphic forms $A_{\bullet}(\mathbb{L}_{V_{\mathbb{C}}^{\times}}^{\times})_{\Gamma}$ gets identified with the algebra of invariants $\mathbb{C}[H]^G$.

Remark 3.1. In particular, there is an automorphic form that defines the discriminant $D \subset H$. It would be interesting to see such an automorphic form written down. We expect this to have an infinite product expansion for the following reason: if two of the 12 points in $\mathbb{P}(\mathrm{U})$ coalesce, then the curve C_F is degenerate: it acquires an A_5 -singularity with local equation $w^6 = u^2$, but the differential ω_F (locally like du/w) does not degenerate in the sense that it becomes a regular differential on the normalization of C_F (this is clear if we write $u = \pm w^3$). This accounts for the integral of ω_F over a vanishing cycle to vanish. This implies that near such a point D maps to a hyperplane section of $\mathbb{L}_{V_{\mathbb{C}}^{\times}}$.

Remark 3.2. We could also consider the other eigenspaces. This amounts to passing to the intermediate Galois covers of the projective line: the μ_3 -cover and the hyperelliptic cover ramified in 12 points. The above isomorphism then leads to interesting morphisms from the 9-dimensional ball quotient to a locally symmetric variety of type $I_{3,7}$ (of dimension 21) and to an arithmetic quotient of a Siegel upper half space of genus 5 (of dimension 15).

The Deligne-Mostow theory provides many more examples, but the one discussed here is the one of highest dimension in their list. This includes for instance the μ_4 -coverings of a projective line totally ramified in 8 points. This case is related to the example that we discuss next.

Example: quartic plane curves (Kondo). Let U be complex vector space of dimension 3 and let $H := \mathrm{Sym}^4(W^*)$ (an irreducible representation of $\mathrm{SL}(\mathrm{U})$). So $\mathbb{P}(H)$ is the linear system of degree 4 divisors on the projective plane $\mathbb{P}(W)$. We denote the divisor associated with $F \in H - \{0\}$ by C_F and we let $H^{\circ} \subset H$ be the set of $F \in H$ for which C_F is a smooth quartic curve. This is the complement of the discriminant hypersurface $D_F \subset H$. It is classical fact that $F \in H^{\mathrm{st}}$ if and only if C_F is a reduced curve whose singularities are only nodes or cusps and a closed orbit a $H^{\mathrm{ss}} - H^{\mathrm{st}}$ is representable by

$(u_1u_2 - u_0^2)(su_1u_2 - tu_0^2)$ for some $(s, t) \in \mathbb{C}^2$ with $s \neq 0$ (hence defines the sum of two conics, one of which is smooth, which meet at two points with multiplicity ≥ 2). This includes as the case of a smooth conic with multiplicity 2. The orbit space $SL(U) \backslash \mathbb{P}(H^\circ)$ has a simple modular interpretation: it is the moduli space of nonhyperelliptic curves of genus 3 (for each such curve is canonically embedded in a projective plane as a quartic curve).

With every $F \in H^\circ$ we associate the μ_4 -cover $S_F \rightarrow \mathbb{P}(U)$ which totally ramifies along C_F : the smooth quartic surface in $\mathbb{P}(U \oplus \mathbb{C})$ defined by $w^4 = F(u)$; indeed, this is a polarized K3 surface of degree 4 with μ_4 -action. The same argument as in the previous example shows that if α is a translation 3-form on U , then the residue of α/w^3 defines a nowhere zero 2-form ω_F on S_F . We let μ_4 act on w with the tautological character $\chi : \mu_4 \hookrightarrow \mathbb{C}^\times$ so that $\omega_F \in H^{2,0}(S_F)^\times$. In fact, ω_F generates $H^{2,0}(S_F)$. The abelian group $H^2(S_F, \mathbb{Z})$ is free abelian of rank 22 and the intersection pairing $H^2(S_F, \mathbb{Z}) \times H^2(S_F, \mathbb{Z}) \rightarrow \mathbb{Z}$ is even unimodular of signature $(3, 19)$. The μ_4 -fixed point subgroup in $H^2(S_F, \mathbb{Z})$ is the image of $H^2(\mathbb{P}(U), \mathbb{Z}) \rightarrow H^2(S_F)$ and contains the polarization of S_F . The other character spaces in $H^2(S_F, \mathbb{C})$ all have dimension 7. Since $H^{2,0}(S_F, \mathbb{C})^\times = H^{2,0}(S_F, \mathbb{C})$ is spanned by ω_F , it follows that the orthogonal complement of $H^{2,0}(S_F, \mathbb{C})^\times$ (relative to the Hermitian form) in $H^2(S_F, \mathbb{C})^\times$ is primitive cohomology of type $(1, 1)$ and of dimension 6. In particular, $H^{2,0}(S_F, \mathbb{C})^\times$ has signature $(1, 6)$.

Fix a unimodular even lattice Λ of signature $(3, 19)$ endowed with an action of μ_4 such that there exists a μ_4 -equivariant isomorphism $\phi : H^2(S_F, \mathbb{Z}) \cong \Lambda$ of lattices. Let us write (W, h) for the Hermitian vector space $\Lambda_{\mathbb{C}}^\times$ and \mathbb{L}^\times for \mathbb{L}_W^\times , the set of $w \in W$ with $h(w, w) > 0$. The isomorphism ϕ will be unique up to an element of $O(\Lambda)_{\mu_4}$. If Γ denotes the image of this group in $U(W)$, then ω_F defines an element of \mathbb{L}_Γ^\times so that we have defined a map $H^\circ \rightarrow \mathbb{L}_\Gamma^\times$. This map is constant on the $SL(U)$ -orbits in H° . Kondo has shown that this map extends to an open embedding $SL(U) \backslash \mathbb{H}^{\text{st}} \hookrightarrow \mathbb{L}_\Gamma^\times$.

The situation is however not as nice as in the previous example: the map is not surjective and, related to this, does not extend to $SL(U) \backslash \mathbb{H}^{\text{ss}}$. In fact, its image is the complement of an arrangement divisor. This can be explained by the fact that we miss out some K3 surfaces of degree 4 with μ_4 -action. It is also related to the fact that we miss out some of the genus 3 curves, namely the hyperelliptic ones. This locus is well-represented in \mathbb{L}_Γ^\times an arrangement divisor $\Delta_{\mathcal{H}}$ so that we have an isomorphism $SL(U) \backslash \mathbb{H}^{\text{st}} \cong (\mathbb{L}_{\mathcal{H}}^\times)_\Gamma$. The divisor $\Delta_{\mathcal{H}}$ is irreducible: the group Γ acts transitively on \mathcal{H} , and its normalization, the quotient of a member $H \in \mathcal{H}$ by its Γ -stabilizer, is a copy the Deligne-Mostow ball quotient for the pair μ_4 -covers of projective line totally ramified in 8 points. This locus is not visible in $SL(W) \backslash \mathbb{P}(H^{\text{ss}})$, for $\mathbb{P}(H^{\text{ss}}) - \mathbb{P}(H^{\text{st}})$ is just a singleton (represented by the conic with multiplicity 2). Since this difference is codimension one, $\mathbb{C}[H]^{SL(U)}$ is also the algebra of regular functions on $SL(U) \backslash \mathbb{H}^{\text{st}}$. So via the isomorphism $SL(U) \backslash \mathbb{H}^{\text{st}} \cong (\mathbb{L}_{\mathcal{H}}^\times)_\Gamma$ the graded algebra $\mathbb{C}[H]^{SL(U)}$ is reproduced as the graded algebra of

Γ -invariant analytic functions on $(\mathbb{L}_{\mathcal{H}}^{\times})_{\Gamma}$ that are sums of homogeneous functions. There is no growth condition here and these functions are automatically meromorphic on \mathbb{L}^{\times} . From the ball quotient perspective, it is quite a surprise that this algebra is finitely generated and has positive degree generators!

Example: double covers of sextic curves. Let U be complex vector space of dimension 3 and let $H := \text{Sym}^6(U^*)$ be a $SL(U)$ -representation. So $\mathbb{P}(H)$ is the linear system of degree 6 divisors on the projective plane $\mathbb{P}(W)$. As above, we denote by $H^{\circ} \subset H$ be the set of $F \in H$ for which C_F is a smooth sextic curve. According to Jayant Shah, H^{st} contains H° and allows C_F to have simple singularities in the sense of Arnol'd. The closed orbits in $\mathbb{P}(H^{\text{ss}} - H^{\text{st}})$ make up a longer list and come in families, but two of them deserve special mention: the closed orbit represented by a smooth conic of multiplicity 3 and the the closed orbit represented by a coordinate triangle of multiplicity 2. We proceed as before: we associate to $F \in H^{\circ}$ the double cover $S_F \rightarrow \mathbb{P}(W)$ which totally ramifies along C_F : it has the weighted homogeneous equation $w^2 = F(u)$ (where $\deg w = 3$) and $\omega_F := \alpha/w$ defines a nowhere zero regular 2-form on S_F . The surface S_F is a K3 surface. The obvious involution of S_F acts on $H^2(S_F, \mathbb{Z})$ with the image of $H^2(\mathbb{P}(U), \mathbb{Z})$ in $H^2(S_F, \mathbb{Z})$ (a copy of \mathbb{Z}) as its fixed point set. The sublattice $H^2(S_F, \mathbb{Z})^{-} \subset H^2(S_F, \mathbb{Z})$ on which this involution acts as minus the identity is nondegenerate and of signature $(2, 19)$. Notice that its complexification contains ω_F .

We therefore take the same lattice (Λ, s) of signature $(3, 19)$ as above, but now endowed with an involution ι such that there exists an isomorphism $\phi : H^2(S_F, \mathbb{Z}) \cong \Lambda$ of lattices with involution. We denote by $V \subset \Lambda \otimes \mathbb{R}$ the subspace on which ι acts as minus the identity (it has signature $(2, 19)$ and write \mathbb{L}^{\times} for \mathbb{L}_V^{\times} (the set $v \in V_{\mathbb{C}}$ for which $s_{\mathbb{C}}(v, v) = 0$ and $s_{\mathbb{C}}(v, \bar{v}) > 0$). The isomorphism ϕ will be unique up to an element of $\Gamma := O(\Lambda)_{\iota} \subset O(V_{\mathbb{Z}})$. The above construction produces a map $SL(U) \backslash H^{\circ} \rightarrow \mathbb{L}_{\Gamma}^{\times}$. This map is constant on the $SL(U)$ -orbits in H° and J. Shah has shown that it extends to an open embedding $SL(U) \backslash H^{\text{st}} \hookrightarrow \mathbb{L}_{\Gamma}^{\times}$. But he also observed that as in the previous case its map is not onto: the image is the complement of an irreducible arrangement divisor $\Delta_{\mathcal{H}}$ and the map does not extend to $SL(U) \backslash H^{\text{ss}}$. The explanation is similar: $\Delta_{\mathcal{H}}$ parametrizes the K3 surfaces of degree 2 that we missed, namely the hyperelliptic ones. Since $SL(U) \backslash H^{\text{ss}} - SL(U) \backslash H^{\text{st}}$ is of dimension 2 (hence of codimension > 1 in $SL(U) \backslash H^{\text{ss}}$, $\mathbb{C}[H]^{SL(U)}$ can be understood as the graded algebra of Γ -invariant analytic functions on $(\mathbb{L}_{\mathcal{H}}^{\times})_{\Gamma}$ that are sums of homogeneous functions (these functions are automatically meromorphic on \mathbb{L}^{\times}). Again there seems reason for surprise.

A potential example: Allcock's 13-ball. Consider an even unimodular lattice $V_{\mathbb{Z}}$ of signature $(2, 26)$ endowed with an action of $\mu_3 \subset O(V_{\mathbb{Z}})$ which leave no nonzero vector fixed. The pair $(V_{\mathbb{Z}}, \mu_3 \subset O(V_{\mathbb{Z}}))$ is unique up to

isomorphism. We put $\Gamma := O(V_{\mathbb{Z}})_{\mu_3}$ and let $\chi : \mu_3 \subset \mathbb{C}^\times$ denote the tautological character. Then $V_{\mathbb{C}}^\chi$ has signature $(1, 13)$ and so we have a 13-dimensional ball quotient $\Gamma \backslash \mathbb{B}_{V_{\mathbb{C}}^\chi}$. Allcock makes a number of intriguing conjectures about this ball quotient and suspects that it has a modular interpretation (that is why call it a potential example).

4. A BAILY-BOREL PACKAGE FOR ARITHMETIC ARRANGEMENT COMPLEMENTS

In the preceding two examples we found that an algebra of invariants can be understood as an algebra Γ -invariant functions on an arrangement complement. These Γ -invariant functions are meromorphic, when considered as functions on \mathbb{L}^\times : they are meromorphic automorphic forms. This cannot be just a coincidence: it suggests that there should be an Baily-Borel package in that setting. This is indeed the case. Let us first focus on the case of a complex ball:

Arithmetic arrangements on complex balls. Let (W, h) be a hermitian vector space of Lorentz signature $(1, n)$ with $n \geq 1$ and $\Gamma \in U(W)$ the image of an arithmetic group acting on W (so this presupposes that we have a CM-subfield $K \subset \mathbb{C}$ and a K -form $W_K \subset W$ such that $h|_{W_K \times W_K}$ takes its values in K , and $\Gamma \subset U(W_K)$). Let be given a Γ -arrangement \mathcal{H} on W : every $H \in \mathcal{H}$ is a hyperplane of W defined over K and of signature $(1, n-1)$, and the collection \mathcal{H} is a finite union of Γ -orbits. We let $\mathbb{L}^\times = \mathbb{L}_W^\times \subset W$ be as usual, $\Delta_{\mathcal{H}} := \cup_{H \in \mathcal{H}} \mathbb{L}_H^\times$ (a locally finite union) and $\mathbb{L}_{\mathcal{H}}^\times := \mathbb{L}^\times - \Delta_{\mathcal{H}}$ the arrangement complement.

Denote by $\mathcal{J}_{\mathcal{H}}$ the collection of subspaces $J \subset W$ that are not positive for h and can be written as an intersection of members of $\mathcal{H} \cup \mathcal{I}^\perp$, where \mathcal{I}^\perp is the set of I^\perp with $I \subset W$ an isotropic line. This includes W , because that corresponds to the empty intersection, but does not include $\{0\}$, because $\{0\}$ is positive definite. Suppose first that $J \in \mathcal{J}_{\mathcal{H}}$ is nondegenerate: so J Lorentzian and J^\perp is negative definite. Then J^\perp maps isomorphically to W/J and the Γ -stabilizer of J will act on W/J through a finite group. The image of \mathbb{L}^\times in W/J is all of W/J , whereas the image of $\mathbb{L}_{\mathcal{H}}^\times$ is the complement in W/J of the union of the hyperplanes H/J with $H \in \mathcal{H}$ and $H \supset J$ (there are only finitely many such).

Next consider the case when $J \in \mathcal{J}_{\mathcal{H}}$ is degenerate. Then $I := J \cap J^\perp$ is an isotropic line and the image of \mathbb{L}^\times in W/J is the complement of the hyperplane I^\perp/J . In other words, $\pi_{I_J}(\mathbb{L}^\times)$ is a principal bundle over $W/I^\perp - \{0\}$ with structure group the vector group $\text{Hom}(W/I^\perp, I^\perp/I_J)$. The Γ -stabilizer of J_I has a subgroup of finite index which is abelian and acts on $\pi_{I_J}(\mathbb{L}^\times)$ via a lattice in $\text{Hom}(W/I^\perp, I^\perp/I_J)$ so that the orbit space is a \mathbb{C}^\times -bundle over (a torsor of) an abelian variety. We get $\pi_J(\mathbb{L}_{\mathcal{H}}^\times)$ by also removing the hyperplanes H/J , with $H \in \mathcal{H}$ and $H \supset J$. This amounts the removal of a finite arrangement in the \mathbb{C}^\times -bundle over the abelian torsor.

The disjoint union

$$(\mathbb{L}_{\mathcal{H}}^{\times})^{\text{bb}} := \mathbb{L}_{\mathcal{H}}^{\times} \sqcup \coprod_{J \in \mathcal{J}_{\mathcal{H}}} \pi_J(\mathbb{L}_{\mathcal{H}}^{\times}),$$

can be endowed with a Satake type of topology that is invariant under $\Gamma \times \mathbb{C}^{\times}$. We then find that if Γ is sufficiently small (in the sense that no eigenvalue $\neq 1$ of an element of Γ is of finite order), then the orbit space $(\mathbb{L}_{\mathcal{H}}^{\times})_{\Gamma}^{\text{bb}}$ is an extension of $\mathbb{L}_{\mathcal{H}}^{\times}$ by a finite number strata, each of which is either a linear arrangement complement or an affine arrangement complement. With $J = V$ is associated a singleton stratum. If we remove that singleton and divide by \mathbb{C}^{\times} , we get a space denoted $(\mathbb{B}_{\mathcal{H}})^{\text{bb}}$.

If we are lucky and all these strata have codimension > 1 (which means that no member of \mathcal{J} has dimension 1, then a Koecher principle applies and we have an easily stated Baily-Borel package:

Finiteness: If $A_d(\mathbb{L}_{\mathcal{H}}^{\times})$ denotes the space of holomorphic functions on $\mathbb{L}_{\mathcal{H}}^{\times}$ that are homogeneous of degree $-d$, then the graded \mathbb{C} -algebra $A_{\bullet}(\mathbb{L}_{\mathcal{H}}^{\times})^{\Gamma}$ is finitely generated with generators in positive degree. So we have defined the weighted homogeneous (affine) cone $\text{Spec } A_{\bullet}(\mathbb{L}_{\mathcal{H}}^{\times})^{\Gamma}$ whose base is the projective variety $\text{Proj}(A_{\bullet}(\mathbb{L}_{\mathcal{H}}^{\times})^{\Gamma})$.

Separation and Topology: The underlying topological spaces (for the Hausdorff topology) are naturally identified with $(\mathbb{L}_{\mathcal{H}}^{\times})_{\Gamma}^{\text{bb}}$ resp. $(\mathbb{B}_{\mathcal{H}})^{\text{bb}}$ and via these identifications, $\text{Spec } A_{\bullet}(\mathbb{L}_{\mathcal{H}}^{\times})^{\Gamma}$ and $\text{Proj}(A_{\bullet}(\mathbb{L}_{\mathcal{H}}^{\times})^{\Gamma})$ acquire a partition into locally closed subvarieties (the former invariant under \mathbb{C}^{\times}).

We think of $A_{\bullet}(\mathbb{L}_{\mathcal{H}}^{\times})^{\Gamma}$ as an algebra of meromorphic Γ -automorphic forms (we allow poles along the hyperplane sections indexed by \mathcal{H}).

Returning to the example of quartic curves, recall that Kondo's theorem asserts that $\text{SL}(\mathbb{U}) \backslash \mathbb{H}^{\text{st}}$ maps isomorphically onto an arrangement complement $(\mathbb{L}_{\mathcal{H}}^{\times})_{\Gamma}$. This assertion can now be amplified: since isomorphism gives rise to an isomorphism of \mathbb{C} -algebras $\mathbb{C}[\mathbb{U}]^{\text{H}} \cong A_{\bullet}(\mathbb{L}_{\mathcal{H}}^{\times})^{\Gamma}$, this isomorphism must extend to an isomorphism of $\text{SL}(\mathbb{U}) \backslash \mathbb{H}^{\text{ss}}$ onto $(\mathbb{L}_{\mathcal{H}}^{\times})_{\Gamma}^{\text{bb}}$.

Arithmetic arrangements for type IV domains. Let (V, s) be a nondegenerate symmetric bilinear form of signature $(2, n)$ defined over \mathbb{Q} with $n \geq 1$, $\Gamma \subset \text{O}(V)$ an arithmetic group and \mathcal{H} a Γ -arrangement in V . We write $\mathbb{L}_{\mathcal{H}}^{\times}$ for $\mathbb{L}_{\mathbb{V}}^{\times}$, the set of $v \in V_{\mathbb{C}}$ with $s_{\mathbb{C}}(v, v) = 0$ and $s_{\mathbb{C}}(v, \bar{v}) > 0$.

We are going to define an indexed collection $\{J\}_{J \in \mathcal{J}_{\mathcal{H}}}$ of nonpositive subspaces $J \subset V$ defined over \mathbb{Q} . When this collection is defined, then we define $(\mathbb{L}_{\mathcal{H}}^{\times})^{\text{bb}}$ as in the ball quotient case and the Baily-Borel package holds verbatim.

So let us then define $\mathcal{J}_{\mathcal{H}}$. A linear subspace $J \subset V$ of signature $(2, n')$ with $n' > 0$ is in $\mathcal{J}_{\mathcal{H}}$ if and only if it is an intersection of members of \mathcal{H} . A linear subspace $J \subset V$ with radical $I := J^{\perp} \cap J$ of dimension 2 is in $\mathcal{J}_{\mathcal{H}}$ if and only if it is an intersection of I^{\perp} and members of \mathcal{H} . The linear subspaces in $\mathcal{J}_{\mathcal{H}}$ whose radical I is of dimension 1 are some harder to index. First note

that such an I is defined over \mathbb{Q} . So s induces on I^\perp/I a nondegenerate symmetric bilinear s_I form of signature $(1, n-1)$ such that the pair $(I^\perp/I, s_I)$ is defined over \mathbb{Q} . Choose a connected component C of the set $v \in I^\perp/I$ with $s_I(v, v) > 0$. This is a quadratic cone. The members of \mathcal{H} which contain I define a locally finite collection of hyperplane sections of C and these decompose C into locally rational polyhedral cones. Then the members of $\mathcal{J}_{\mathcal{H}}$ containing I are indexed by these locally rational polyhedral cones: if σ is such a cone, then we assign to its linear span, or rather the preimage of that linear span in I^\perp . This means that the indexing leads to repetitions. For instance, all the open cones in C_I determine the same space I^\perp .