REMARKABLE POLYDIFFERENTIALS ON THE CONFIGURATION SPACE OF A COMPACT RIEMANN SURFACE

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ABSTRACT. We define a remarkable 'quasi-logarithmic' n-form on the configuration space of n points on a compact Riemann surface and determine its cohomology class and Hodge type. For the two-point configuration space this gives the 2-form on the complement of the diagonal of the self-product of a compact Riemann surface that was recently investigated by a number of authors.

INTRODUCTION

We describe a remarkable meromorphic 2-form on C^2 whose polar divisor is twice the diagonal and whose cohomology class is related to the Weil operator acting on $H^1(C;\mathbb{R})$ (it definitely does not represent an algebraic cycle) and find a higher dimensional generalization as a meromorphic n-form on $C^{\mathbb{Z}/n}$ whose polar divisor is the sum of the diagonal hypersurfaces defined by $z_i = z_{i+1}$, $i \in \mathbb{Z}/n$. These forms have such a classical flavor that they might well have already been known in the 19th century.

We ran into these a while ago when working on our project to relate the WZW theory to the Hodge theory of configuration spaces of curves, in particular to make the latter a receptacle of the highest weight representations a centrally extended loop algebra (also known as a Kac-Moody Lie algebra of affine type); the appearance of these forms comes then from the central extension. In the belief that this may be of independent interest, we think it worthwhile to separate this from the envisaged application as to let it be uncluttered by the somewhat involved context in which we plan to use it.

I recently became aware of some papers in which this 2-form appears, namely in the work of Colombo-Frediani-Ghigi [3], in a draft of a book by Gunning [6] and in more recent work by Biswas-Colombo-Frediani-Pirola [2] and Ghigi-Tamborini [5]. Although there is some overlap with these works when $n = 2$, the emphasis in this note is different; besides obtaining a higher dimensional generalization, we obtain a complete and direct description of the cohomology class that is represented by this form.

It is a pleasure to acknowledge some helpful discussions that I had with Ashok Raina a long time ago and some correspondence related to [1] on this material back in 2011. In hindsight, my interest in this must have the same origin as his.

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In what follows C stands for a nonsingular complex projective irreducible complex curve (equivalently, a compact connected Riemann surface) of genus g.

1. Canonical meromorphic forms on the configuration space of a curve

1.1. Review of the Künneth decomposition. What follows is classical and well-known, but since we must be careful with signs, we think it worthwhile to spell this out. We let $\mathfrak{S}_2$ act on C^2 by transposition of factors. If we take its action on the cohomology into account, then the Künneth decomposition reads as follows

$$H^2(C^2) \cong \big(H^2(C)\otimes 1 \oplus 1\otimes H^2(C)\big) \oplus \big(H^1(C) \otimes H^1(C) \otimes \operatorname{sign}_2\big),$$

where general sign_2 stands for the nontrivial character of $\mathfrak{S}_2$. Indeed, if $\mu \in H^2(C)$ is the natural generator, then the transposition exchanges $\mu \otimes 1$ and $1 \otimes \mu$, but if $\alpha, \beta \in H^1(C)$, then it exchanges $\alpha \otimes \beta$ and $-\beta \otimes \alpha$. If $\Delta : C \to C^2$ is the diagonal, then the Gysin map $\Delta_! : H^\bullet(C) \to H^{\bullet+2}(C^2)$ is $H^\bullet(C^2)$-linear. The class of the diagonal is $\Delta_!(1) \in H^2(C^2)$ is (of course) fixed under $\mathfrak{S}_2$ and is under the Künneth decomposition equal to $\mu \times 1 + 1 \times \mu + \delta$, where $\delta \in H^1(C) \otimes H^1(C)$ is defined by the intersection pairing on $H_1(C)$. To be precise, if $\alpha_{\pm 1}, \dots, \alpha_{\pm g}$ is a symplectic basis of $H^1(C)$ (i.e., if $i > 0$, then $\alpha_i \cdot \alpha_j$ is 1 when $j = -i$ and is zero otherwise), then $\delta = \sum_{i=1}^g (-\alpha_i \times \alpha_{-i} + \alpha_{-i} \times \alpha_i)$. Note that δ has the property that for $\alpha \in H^1(C)$,

$$(\alpha \times 1) \cup \delta = \mu \times \alpha, \quad (1 \times \alpha) \cup \delta = \alpha \times \mu \text{ and so}$$
$$\Delta_!(\alpha) = \mu \times \alpha + \alpha \times \mu.$$

Both μ and δ are of Hodge type $(1, 1)$.

1.2. A canonical bidifferential on C^2. Let $D \subset C^2$ stand for the diagonal. Then the biresidue defines an equivariant trivialization

$$\Omega^2_{C^2}(2D) \otimes \mathcal{O}_D \cong \mathcal{O}_D \otimes \operatorname{sign}_2.$$

According to Biswas-Raina (Prop. 2.10 of [1]) there exists a $\zeta \in H^0(C^2, \Omega^2(2D))$ whose restriction to D yields this trivialization. We may, of course, take this generator to be anti-invariant under the nontrivial element σ of $\mathfrak{S}_2$ so that $\sigma^*\zeta = -\zeta$. It is then unique up an element of $H^0(C^2, \Omega^2)^{-\sigma} \cong \operatorname{Sym}^2 H^0(C, \Omega_C)$. Note that this last space is of Hodge type $(2, 0)$. It defines an anti-invariant class in $H^2(C^2 \smallsetminus D; \mathbb{C})$. As is clear from the Künneth decomposition, this space of anti-invariant classes can be identified with the direct sum of $\operatorname{Sym}^2 H^1(C; \mathbb{C})$ and the $\mathbb{C}$-span of $-\mu \times 1 + 1 \times \mu$. We will identify an optimal choice of ζ and determine its class in terms of this decomposition.

In order to find the coefficient of $[\zeta]$ on $-\mu \times 1 + 1 \times \mu$ we proceed as follows. Choose $p \in C$ and consider $Z := -C \times \{p\} + \{p\} \times C$ as a 2-cycle. It is clear that with respect to the Künneth decomposition of $H^2(C)$, its class is annihilated by $H^1(C) \times H^1(C)$, whereas $\mu \times 1$ and $1 \times \mu$ take on it the value -1 resp. 1. So the coefficient in question is then computed by integrating ζ over any 2-cycle on $C^2 \smallsetminus D$ that is homologous to Z in $H_2(C^2)$.

We construct such a cycle by modifying Z a bit near (p,p). To this end we choose a chart $(U;z)$ centered at p, so that ζ has on U^2 the form

$$(\dagger)\qquad \zeta = (z_1 - z_2)^{-2} dz_1 \wedge dz_2 + \text{a form regular at } (p,p).$$

For any $\varepsilon > 0$ such that U contains a closed disk $D_{2\varepsilon}$ mapping onto the closed disk of radius 2ε in $\mathbb{C}$, we embed the cylinder $[-\varepsilon,\varepsilon]\times S^1$ in U^2 by

$$u_\varepsilon : [-\varepsilon,\varepsilon]\times[0,2\pi] \to U^2; \quad u^* z_1 = (\varepsilon - s)e^{\sqrt{-1}\phi}\,,\; u^* z_2 = -(\varepsilon + s)e^{\sqrt{-1}\phi}$$

Considered as a 2-chain, its boundary is $\{p\}\times\partial D_\varepsilon - \partial D_\varepsilon \times \{p\}$ (we use the complex orientation), so that if we add to this the 2-chain $-(C \smallsetminus D_\varepsilon)\times\{p\} + \{p\}\times(C\smallsetminus D_\varepsilon)$ we obtain a 2-cycle Z_ε in $C^2 \smallsetminus D$. It is clear that Z_ε is homologous to Z in $C^2\smallsetminus D$.

Lemma 1.1. *The value of the cohomology class* $[\zeta]$ *on* Z_ε *(and hence the coefficient of* $[\zeta]$ *on* $-\mu\times 1 + 1\times\mu$*) equals* $2\pi\sqrt{-1}$.

Proof. The form pull-back of ζ to Z_ε is nonzero only the cylindrical part of Z_ε and so

$$\langle[\zeta], Z_\varepsilon\rangle = \int_{Z_\varepsilon}\zeta = \int_{[-\varepsilon,\varepsilon]\times[0,2\pi]} u_\varepsilon^*\zeta$$

In order to compute the latter, we note that

$$u_\varepsilon^* dz_1 = e^{\sqrt{-1}\phi}(-ds + \sqrt{-1}(\varepsilon - s)d\phi), \quad u_\varepsilon^* dz_2 = e^{\sqrt{-1}\phi}(-ds - \sqrt{-1}(\varepsilon + s)d\phi)$$

so that $u_\varepsilon^*(dz_1\wedge dz_2) = 2\sqrt{-1}\varepsilon e^{2\sqrt{-1}\phi} ds\wedge d\phi$. On the other hand, $z_1 - z_2 = 2\varepsilon e^{2\sqrt{-1}\phi}$ and so it follows that

$$u_\varepsilon^*\zeta = \big(\frac{\sqrt{-1}}{2\varepsilon} + O(\varepsilon)\big) ds\wedge d\phi \;\text{ and hence }\; \int_{[-\varepsilon,\varepsilon]\times[0,2\pi]} u^*\zeta = 2\pi\sqrt{-1} + o(\varepsilon).$$

The left hand side of the integral is independent of ε, and so $\langle[\zeta], Z_\varepsilon\rangle = 2\pi\sqrt{-1}$. □

Any $K = \sum \alpha_i \otimes_{\mathbb{C}} \beta_i \in H^1(C)\otimes H^1(C)$ determines endomorphisms E_K and ${}_KE$ of $H^1(C)$ defined by the formulae

$$\textstyle\sum_i \alpha_i \times (\beta_i \cup x) = E_K(x)\cup\mu, \quad \sum_i (\alpha_i\cup x)\times\beta_i = \mu \times {}_KE(x).$$

Note that $E_{\sigma^* K} = -{}_KE$. In particular, if $\sigma^*K = -K$ (which means that $\sum_i \alpha_i\times\beta_i \in H^2(C^2)$ is σ-invariant), then $E_K = {}_KE$. Our main example will be the following: let $\omega_1,\dots,\omega_g$ be an orthonormal basis of $H^0(C,\Omega_C)$ in the sense that $\int_C \omega_i\wedge\overline{\omega}_j = \delta_{ij}$, so that

$$W := \omega_i\otimes\bar\omega_i + \bar\omega_i\otimes\omega_i.$$

represents the 'inverse' hermitian form on the dual of $H^0(C,\Omega_C)$. Note that this is a real $(1,1)$-form which satisfies $\sigma^*W = -W$ so that $E_W = {}_WE$. We also find that

$$E_W(\omega_j) = \sum_i \omega_i\Big(\int_C \bar\omega_i\wedge\omega_j\Big) = -\omega_j \quad ; \quad E_W(\bar\omega_j) = \sum_i \bar\omega_i\Big(\int_C \omega_i\wedge\bar\omega_j\Big) = \bar\omega_j$$

and so E_W acts as multiplication with -1 resp. 1 on $H^{1,0}(C)$ resp. $H^{0,1}(C)$. In other words $\sqrt{-1}E_W$ is the Weil operator acting in $H^1(C;\mathbb{R})$ for the convention that is used nowadays (so it has $H^{1,0}(C)$ as $-\sqrt{-1}$ eigenspace and $H^{0,1}(C)$ as $+\sqrt{-1}$ eigenspace). Thus W encodes the full Hodge decomposition of $H^1(C;\mathbb{C})$.

The 2-form ζ that appears in the proposition below was found by Colombo-Frediani-Ghigi in [3] in their study of the local geometry of the period map. We here obtain this form in a somewhat different manner and also determine its periods.

Theorem 1.2. *There exists a ζ whose class lies in $H^2(C^2 \smallsetminus D_{12}; \mathbb{R}(1))$ and is given by*

$$2\pi\sqrt{-1}(-\mu \times 1 + 1 \times \mu + W),$$

where $W = \sum_{i=1}^{g}(\omega_i \otimes \bar{\omega}_i + \bar{\omega}_i \otimes \omega_i)$ (so that $\sqrt{-1}E_W$ represents the Weil operator). It is the unique antisymmetric meromorphic 2-form on C^2 with polar divisor $-2D$ with biresidue 1 along D and whose class is of type $(1,1)$.

Proof. If we are given a $p \in C$ and a meromorphic differential η that is regular on $C \smallsetminus \{p\}$, then it defines an class $[\eta] \in H^1(C \smallsetminus \{p\}; \mathbb{C}) = H^1(C; \mathbb{C})$. Its image in $H^1(C, \mathcal{O}_C)$ is obtained as follows: write in a punctured neighborhood of p the form η as df (which is possible since the residue of η at p has to be zero) and then take the image of f in the above quotient. The theorem of Stokes combined with the Cauchy residue formula implies the (well-known) identity

$$\int_C [\eta] \wedge \omega = 2\pi\sqrt{-1}\,\mathrm{Res}_p\, f\omega.$$

We first show that for ζ be as in the lemma, $E_{[\zeta]}$ is the identity on $H^0(C, \Omega_C)$. Let p and a chart $(U; z)$ centered at p be a above so that ζ is on U^2 as above (†). It is also clear that $\zeta|U \times C$ is of the form $dz_1 \wedge \pi_2^*\eta$, where η is a meromorphic differential relative to the projection $U \times C \to U$ with $\eta|U^2 = (z_1 - z_2)^{-2}dz_2$ and η regular elsewhere. Integration of η on U^2 with respect to z_2 yields $-(z_1 - z_2)^{-1}$ plus a holomorphic function on U^2. Let ω be an abelian differential on C. So $\omega|U = f(z)dz$ for some holomorphic f. The residue pairing relative to the second coordinate (with z_1 as parameter) yields

$$\int_C [\eta] \wedge \omega = 2\pi\sqrt{-1}\,\mathrm{Res}_{z_2 \to z_1} -(z_1 - z_2)^{-1}\omega =$$
$$= 2\pi\sqrt{-1}\,\mathrm{Res}_{z_2 \to z_1}(z_2 - z_1)^{-1}f(z_2)dz_2 = 2\pi\sqrt{-1}f(z_1)$$

If we apply $dz_1 \wedge$ to both sides, we find that $E_{[\zeta]}$ takes ω to $2\pi\sqrt{-1}\omega = -2\pi\sqrt{-1}E_W(\omega)$. We now write the component of ζ in $H^1(C; \mathbb{C}) \otimes H^1(C; \mathbb{C})$ as

$$\textstyle\sum_{i,j}\big(a_{ij}\omega_i \otimes \omega_j + b_{ij}\omega_i \otimes \bar{\omega}_j + s_{ij}\bar{\omega}_i \otimes \omega_j + d_{ij}\bar{\omega}_i \otimes \bar{\omega}_j\big).$$

The assumption that $\sigma^*\zeta = -\zeta$ implies that $a_{ij} = a_{ji}$, $b_{ij} = s_{ji}$ and $d_{ij} = d_{ji}$. From what we proved, it follows that b_{ij} is $2\pi\sqrt{-1}$ times the identity matrix and that $d_{ij} = 0$. Since we have the freedom of choosing the a_{ij}'s arbitrary, the unique representative in question is then obtained by taking each $a_{ij} = 0$. If we combine this Lemma 1.1, the assertion follows. □

Remark 1.3. We prefer to think of ζ as a polydifferential. It is then σ-invariant and this makes it more canonical, as unlike the 2-form interpretation, it does not depend on how we ordered the factors. Note that for $C = \mathbb{P}^1$, ζ is simply given by $(z_1 - z_2)^{-2}dz_1dz_2$ (so that its divisor is minus twice the diagonal).

Remark 1.4. At each point $x \in C$ we can find a local holomorphic coordinate z such that ζ takes at (x,x) the simple form $(z_1-z_2)^{-2}dz_1dz_2$. Such a coordinate is not unique, but any other coordinate with that property is necessarily a fractional linear (Möbius) transform of z. In other words, ζ determines a projective structure at x. It is clearly invariant under the automorphism group of C. This group acts transitively when C has genus 0 or 1, and so this projective structure must then be the standard one. When the genus g of C is > 1, then the universal cover of C is realized by the upper half plane $\mathbb{H}$ with covering group contained in $\mathrm{PSL}_2(\mathbb{R})$. So this defines another projective structure on C. It was shown by Biswas-Colombo-Frediani-Pirola [2] that these projective structures differ in general. Since the difference between two projective structures is a quadratic differential on C, and a quadratic differential has the interpretation of a covector at the point that C defines in its Teichmüller space, it follows that thus been constructed a form of type $(1,0)$ on the moduli space of curves $\mathcal{M}_g$.

2. A HIGHER DIMENSIONAL GENERALIZATION

We will here find a higher order generalization of the 2-form that we introduced in the previous section. Consider for $n \geq 2$ on C^n the divisors

$$D_n := D_{12} + D_{23} + \cdots + D_{n-1,n}, \quad E_n := D_n + D_{n,1} = D_{12} + D_{23} + \cdots + D_{n-1,n} + D_{n,1},$$

where D_{ij} stands for the diagonal hypersurface $z_i = z_j$ (so that $E_2 = 2D_{12} = 2D$). Note that D_n is a normal crossing divisor, and that E_n is nearly so: it has this property away from the main diagonal. According to Deligne [4] this implies that $H^q(C^n, \Omega^n_{C^n}(D_n))$ can for $q \geq 0$ be identified with a term of the Hodge filtration on the cohomology of $C^n \smallsetminus D_n$, namely $F^n H^{n+q}(C^n \smallsetminus D_n)$. The second assertion of the following lemma shows that this mixed Hodge structure is in fact pure (and therefore implies formally its first assertion).

Let for $1 \leq i < j \leq n$, $\pi_{ij} : C^n \to C^2$ be given by $\pi_{ij}(z_1, \dots, z_n) = (z_i, z_j)$. We note that the map $\Delta_{ij} : C^{n-1} \to C^n$ which puts z_i also in the jth slot but otherwise preserves the order of the numbering, has image D_{ij} and has the property that $\Delta_{ij!}(1) = \pi_{ij}^*(\mu \times 1 + \delta + 1 \times \delta) = \pi_i^*\mu + \pi_{ij}^*\delta + \pi_j^*\mu$.

Lemma 2.1. *The inclusion $\Omega^n_{C^n} \subset \Omega^n_{C^n}(D_n)$ defines an isomorphism on global sections. The inclusion $C^n \smallsetminus D_n \subset C^n$ identifies $H^\bullet(C_n \smallsetminus D_n)$ with the quotient of $H^\bullet(C^n)$ by the ideal generated by the classes $\Delta_{i,i+1!}(1)$ of the irreducible components $D_{i,i+1}$ of D_n ($i = 1, \dots, n-1$) so that in the Künneth decomposition $H^\bullet(C^n) \cong H^\bullet(C)^{\otimes n}$ the subsum of the tensor products not involving $\pi_i^*(\mu)$, for $i = 2, 3, \dots, n$ maps isomorphically onto $H^\bullet(C^n \smallsetminus D_n)$.*

Proof. For the first statement it suffices to show that the inclusion in question defines an isomorphism when taking the direct image under the projection $\pi^n : C^n \to C^{n-1}$ which omits the last factor. This is clear: this defines a curve over the generic point of C^{n-1} with the divisor D_n defining a single section and the assertion thus becomes a consequence of the residue theorem.

For the second assertion, we note that for $n = 2$ this is a straightforward consequence of the fact that the Gysin sequence for the diagonal embedding splits up in short exacts sequences. So we assume $n > 2$ and, proceeding with induction, that $H^\bullet(C^{n-1}) \to$

$H^\bullet(C^{n-1} \smallsetminus D_{n-1})$ is onto with kernel generated by the classes of irreducible components of D_{n-1}. Consider the Gysin sequence associated to the closed embedding $C^{n-1}\smallsetminus D_{n-1} \hookrightarrow C^n \smallsetminus D_n$ induced by map $\Delta_{n-1,n} : C^{n-1} \to C^n$ (which repeats the last component):

$$\cdots \to H^{\bullet-2}(C^{n-1} \smallsetminus D_{n-1})(-1) \xrightarrow{\Delta_{n-1,n!}} H^\bullet((C^{n-1} \smallsetminus D_{n-1}) \times C) \to H^\bullet(C^n \smallsetminus D_n) \to \cdots$$

The coboundary in this sequence is the homomorphism of $H^\bullet(C^{n-1} \smallsetminus D_{n-1})$-modules

$$H^{\bullet-2}(C^{n-1} \smallsetminus D_{n-1})(-1) \to H^\bullet(C^{n-1} \smallsetminus D_{n-1}) \otimes H^\bullet(C)$$

which sends 1 to the image of $\Delta_{n-1,n!}(1) = \pi^*_{n-1}\mu + \pi^*_{n-1,n}\delta + \pi^*_n\mu$. Our inductive description of $H^\bullet(C^{n-1} \smallsetminus D_{n-1})$ shows that this map is injective, so that its cokernel gives $H^\bullet(C^n \smallsetminus D_n)$. This proves that $H^\bullet(C^n \smallsetminus D_n)$ is as asserted. □

The following proposition may be regarded as a generalization of the construction of our canonical bidifferential on C^2.

Theorem 2.2. *For $n \geq 2$, the space $H^0(C^n, \Omega^n_{C^n}(E_n))$ embeds in $H^n(C^n \smallsetminus E_n; \mathbb{C})$ and lands in $F^{n-1}H^n(C^n \smallsetminus E_n)$. Its subspace $H^0(C^n, \Omega^n_{C^n})$ is of codimension one and maps isomorphically onto $F^nH^n(C^n \smallsetminus E_n)$. There is a unique supplementary line of $H^0(C^n, \Omega^n_{C^n})$ in $H^0(C^n, \Omega^n_{C^n}(E_n))$ that maps to a line in $H^n(C^n \smallsetminus E_n; \mathbb{C})$ of Hodge type $(n-1, n-1)$ and that is spanned by a real class. The $\mathfrak{S}_n$-stabilizer of E_n (a dihedral group of order $2n$) acts on this line with the character of the expression $(z_1-z_n)(z_2-z_1)\cdots(z_n-z_{n-1})dz_1\wedge\cdots\wedge dz_n$. For $n \geq 3$, it has a unique generator ζ_n with the property that the residue along $\Delta_{n,n-1}$ is equal to ζ_{n-1}, where $\zeta_2 = \zeta$ is the 2-form defined earlier.*

Proof. We already established this for $n = 2$. We therefore assume $n > 2$ and the proposition verified for $n-1$. Taking the residue along $\Delta_{n,1}$ gives for $n \geq 3$ the exact sequence

$$0 \to \Omega^n_{C^n}(D_n) \to \Omega^n_{C^n}(E_n) \to \Delta_{n,1*}\Omega^{n-1}_{C^{n-1}}(D_{n-1}) \to 0$$

whose associated long exact sequence begins with

$$0 \to H^0(\Omega^n_{C^n}(D_n)) \to H^0(\Omega^n_{C^n}(E_n)) \to H^0(\Omega^{n-1}_{C^{n-1}}(E_{n-1})) \to H^1(\Omega^n_{C^n}(D_n)) \to \dots$$

This sequence is compatible with the Gysin sequence for the closed embedding $\Delta_{n,1} : C^{n-1} \smallsetminus E_{n-1} \subset C^n \smallsetminus D_n$ in the sense that we have morphism of exact sequences

$$\begin{array}{ccccccccc}
\to H^n(C^n \smallsetminus D_n;\mathbb{C}) & \to & H^n(C^n \smallsetminus E_n;\mathbb{C}) & \to & H^{n-1}(C^{n-1} \smallsetminus E_{n-1};\mathbb{C})(-1) & \to & H^{n+1}(C^n \smallsetminus D_n;\mathbb{C}) & \to \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & \\
0 \to H^0(\Omega^n_{C^n}(D_n)) & \longrightarrow & H^0(\Omega^n_{C^n}(E_n)) & \longrightarrow & H^0(\Omega^n_{C^{n-1}}(E_{n-1})) & \longrightarrow & H^1(\Omega^n_{C^n}(D_n)) & \longrightarrow
\end{array}$$

The top sequence is one of mixed Hodge structures. Since D_n is a normal crossing divisor, the first and the fourth up arrow are embeddings with image $F^nH^n(C^n \smallsetminus D_n)$ resp. $F^nH^{n+1}(C^n \smallsetminus D_n)$. According to our induction hypothesis, the third up arrow embeds $H^0(\Omega^n_{C^{n-1}}(E_{n-1}))$ in $F^{n-2}(H^{n-1}(C^{n-1}\smallsetminus E_{n-1}))(-1) = F^{n-1}(H^{n-1}(C^{n-1}\smallsetminus E_{n-1})(-1))$ and its image in $\mathrm{gr}^{n-1}_F(H^{n-1}(C^{n-1}\smallsetminus E_{n-1})(-1))$ is one dimensional. Since the (Hodge) functor F_{n-1} is exact, it follows that the second up arrow is an embedding whose image lies in

$F^{n-1}H^n(C^n \smallsetminus E_n)$ and contains $F^nH^n(C^n \smallsetminus E_n)$ as a subspace of codimension one. We also see that the residue map defines an isomorphism

$$H^0(\Omega^n_{C^n}(E_n))/H^0(\Omega^n_{C^n}) \cong H^0(\Omega^n_{C^{n-1}}(E_{n-1}))/H^0(\Omega^{n-1}_{C^{n-1}}).$$

Since the latter is of Tate type $(n-2, n-2)$, the former is of Tate type $(n-1, n-1)$. The uniqueness of the line follows from the fact that it is spanned by preimage of the real cohomology $H^n(C^n \smallsetminus E_n; \mathbb{R})$.

It remains to determine the character of the $\mathfrak{S}_n$-stabilizer of E_n on the corresponding line of polydifferentials. Near the main diagonal, any generator of that line has in terms of a local coordinate z on C the form

$$\frac{f(z_1, \dots, z_n)dz_n \wedge dz_{n-1} \cdots \wedge dz_1}{(z_1 - z_n)(z_2 - z_1) \cdots (z_n - z_{n-1})}.$$

with f holomorphic and invariant under cyclic permutation and $f(z, z, \dots, z)$ constant nonzero. We let ζ_n be the generator such that this constant is 1. Then it is clear that the residue of ζ_n along $\Delta_{n-1,n}$ is ζ_{n-1}. The fact that it transforms under the dihedral group as indicated is clear. □

Remark 2.3. In a subsequent paper we will see that Theorem 2.2 is best expressed in term of a class of polydifferentials that we have baptized *quasi-logarithmic* n-forms on the n-point configuration space of C, which as the name is intended to suggest, contains the space logarithmic n-forms. Theorem 2.2 will then imply that every quasi-logarithmic form is a sum of exterior products of logarithmic forms and forms of the type described by that theorem.

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