Some algebraic geometry related to the mapping class group

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This is a summary of the talk I gave in response to the request of the organizers of this workshop to survey the algebro-geometric aspects of the mapping class group. I am grateful to them for asking me to do so, as this led me to revisit (and to rethink) a few these connections. But the reader be warned that although I have tried to tailor this survey to an audience of geometers of a non-algebraic denomination, it remained one *au goût du jour* and consisted in the end more of asking questions than of stating results.

A moduli space of curves. Let S be an oriented surface of *hyperbolic type*, in the sense that it has finite Betti numbers and each connected component has negative Euler characteristic. We consider complex structures on S compatible with the given orientation that make it in fact a nonsingular complex algebraic curve (in other words, a compact Riemann surface minus a finite subset). The group $\mathrm{Diff}^+(S)$ of orientation preserving diffeomorphisms of S acts in an evident manner on this space of complex structures. The *Teichmüller space* $\mathcal{T}(S)$ of S is the orbit space of the identity component $\mathrm{Diff}^\circ(S)$ of this group, in other words, the space of isotopy classes of such structures. It is a basic fact that $\mathcal{T}(S)$ is contractible and comes with a natural structure of a complex manifold. The action of $\mathrm{Diff}^+(S)$ on $\mathcal{T}(S)$ clearly descends to one of the *mapping class group* $\Gamma(S) := \mathrm{Diff}^+(S)/\mathrm{Diff}^\circ(S)$ of S. This action is proper and virtually free and hence the orbit space is $\mathcal{M}(S) = \Gamma(S)\backslash\mathcal{T}(S)$ is an orbifold that is a virtual Eilenberg-Mac Lane space for $\Gamma(S)$. In particular, $H^\bullet(\Gamma(S)) \cong H^\bullet(\mathcal{M}(S))$ (in this note we shall always take (co)homology with $\mathbb{Q}$-coefficients).

Deligne-Mumford compactification. A compact 1-dimensional submanifold $A \subset S$ is necessarily a disjoint union of a finitely many embedded circles. Say that A is *admissible* if $S - A$ is of hyperbolic type (this includes the case $A = \emptyset$). Then $\Gamma(S - A)$ and $\mathcal{T}(S - A)$ are defined and only depend on the isotopy class (i.e., the $\mathrm{Diff}^\circ(S)$-orbit) $[A]$ of A. We express this by sometimes writing $\Gamma(S - [A])$ and $\mathcal{T}(S - [A])$ instead. Consider the disjoint union of the $\overline{\mathcal{T}}(S)$ of the Teichmüller spaces $\mathcal{T}(S - [A])$, where $[A]$ runs over all the admissible isotopy classes. The group $\Gamma(S)$ acts in this union and the $\Gamma(S)$-stabilizer of $\mathcal{T}(S - [A])$ maps to a subgroup of $\Gamma(S - [A])$ of finite index with kernel the (free) abelian group generated by the Dehn twists along the connected components of $[A]$. There is a natural $\Gamma(S)$-invariant topology on $\overline{\mathcal{T}}(S)$ which has the property that the closure of $\mathcal{T}(S - [A])$ meets $\mathcal{T}(S - [B])$ if and only if $[A]$ is represented by a union of connected components of B. In view of the preceding, the action of $\Gamma(S)$ on $\overline{\mathcal{T}}(S)$ is not proper (unless the only admissible A is the empty set, but this happens only when S is a thrice punctured sphere). Yet the orbit space $\overline{\mathcal{M}}(S)$ has a natural complex orbifold structure extending the one on $\mathcal{M}(S)$. With this structure, $\overline{\mathcal{M}}(S)$ is even projective. In particular, it is compact and this explains the noun in its name: the *Deligne-Mumford compactification* of $\mathcal{M}(S)$. The boundary $\overline{\mathcal{M}}(S) - \mathcal{M}(S)$ is a normal crossing divisor (in the orbifold sense) whose natural partition (which counts the number of branches passing through a given point) coincides with the partition inherited from $\overline{\mathcal{T}}(S)$. The stratum $\mathcal{T}(S-[A])$ has an algebro-geometric interpretation: it parametrises complex structures on the quotient of S obtained by contracting each connected component of A to a point such that this point becomes an ordinary double point.

A virtue of this approach is that it behaves in a straightforward manner under passage of subgroups of finite index: if $\Gamma \subset \Gamma(S)$ is of finite index, then $\mathcal{M}_\Gamma := \Gamma\backslash\mathcal{T}(S)$ will be a finite cover of $\mathcal{M}(S)$ and $\overline{\mathcal{M}}_\Gamma := \Gamma\backslash\overline{\mathcal{T}}(S)$ is a projective orbifold compactification of $\mathcal{M}_\Gamma$ with a normal crossing boundary in the orbifold sense such that the evident map $\overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}(S)$ is like a ramified cover (a finite, flat and surjective morphism). In particular, $\mathcal{M}_\Gamma$ is quasi-projective, so that $H^k(\Gamma) \cong H^k(\mathcal{M}_\Gamma)$ comes with a mixed Hodge structure whose weights are $\geq k$ and $\leq 2k$.

Purity of stable classes. We now take S connected. Precisely, given nonnegative integers g, n with $2g-2+n>0$, we fix a compact connected oriented surface S_g of genus g and pairwise distinct points $x_1, \dots, x_n$ on S_g. Then $S_{g,n} := S_g - \{x_1, \dots, x_n\}$ is hyperbolic in the sense above. The connected component group of the group of orientation preserving diffeomorphisms of S_g which fix each x_i can be regarded as a normal subgroup of $\Gamma(S_{g,n})$ with factor group the permutation group of degree n. We denote this group $\Gamma_{g,n}$ and write $\mathcal{M}_{g,n}$ for $\mathcal{M}_{\Gamma_{g,n}}$. If we choose $x = x_{n+1} \in S_{g,n}$, then we have a forgetful map $\mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n}$ (fill x back in) which may in some sense be thought of as a universal family of punctured Riemann surfaces. This naturally extends as a morphism $\bar\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ between their Deligne-Mumford compactifications whose restriction over $\mathcal{M}_{g,n}$, $\pi : \bar\pi^{-1}\mathcal{M}_{g,n} \to \mathcal{M}_{g,n}$ may, again in some sense, be understood as a universal family of *pointed* Riemann surfaces: the difference $\bar\pi^{-1}\mathcal{M}_{g,n} - \mathcal{M}_{g,n+1}$ consists of the images of pairwise disjoint sections $s_1, \dots, s_n$ of π. Each s_i extends to a section $\bar s_i$ of $\bar\pi$ that takes it values in the Deligne-Mumford boundary[1].

A complex structure on $S_{g,n}$ defines one on $T_x^* S_{g,n}$. This gives rise to a line bundle $\mathcal{L}$ on $\overline{\mathcal{M}}_{g,n+1}$, whose Chern class we denote by $\bar\psi \in H^2(\overline{\mathcal{M}}_{g,n+1})$. From this class we obtain *tautological classes* on $\overline{\mathcal{M}}_{g,n}$: $\bar\psi_i := s_i^*(\bar\psi) \in H^2(\overline{\mathcal{M}}_{g,n})$ of Hodge type $(1,1)$ $(i=1,\dots,n)$ and $\bar\kappa_r := \bar\pi_*(\bar\psi^{r+1}) \in H^{2r}(\overline{\mathcal{M}}_{g,n})$ of Hodge type (r,r) $(r=1,2,\dots)$. Here $\bar\pi_*$ is 'integration along the fibers' (when we use Poincaré duality on both source and target to identify cohomology with homology, then this is just the induced map on homology). These classes define a homomorphism of graded $\mathbb{Q}$-algebras:

$$\mathbb{Q}[K_1, K_2, \dots] \otimes \mathbb{Q}[\Psi_1, \Psi_2, \dots, \Psi_n] \to H^\bullet(\overline{\mathcal{M}}_{g,n}).$$

[1] To be precise, $\bar s_i$ is induced from a section of $\overline{\mathcal{T}}(S_{g,n+1}) \to \overline{\mathcal{T}}(S_{g,n})$ that is given by an embedded closed disk $D \subset S_g - \{x_1, \dots, \widehat{x}_i, \dots, x_n\}$ whose interior $\mathring{D}$ has been endowed with a complex structure and contains x_i and x: compose the isomorphism $\overline{\mathcal{T}}(S_{g,n}) \cong \overline{\mathcal{T}}(S_{g,n} \smallsetminus D)$ defined by the natural isotopy class of diffeomorphisms $S_{g,n} \to S_{g,n} \smallsetminus D$ with the embedding $\overline{\mathcal{T}}(S_{g,n} \smallsetminus D) \hookrightarrow \overline{\mathcal{T}}(S_{g,n+1} - \partial D) \subset \mathcal{T}(S_{g,n+1})$ defined by the complex structure on $\mathring{D}$.

The theorem of Madsen-Weiss tells us (when combined with the theorems of Harer) that its composite with the restriction map $H^\bullet(\overline{\mathcal{M}}_{g,n}) \to H^\bullet(\mathcal{M}_{g,n})$ is an isomorphism in degrees $< 2g/3$. Before this was proved, it was shown by Pikaart [5] that $H^\bullet(\overline{\mathcal{M}}_{g,n}) \to H^\bullet(\mathcal{M}_{g,n})$ is onto in this range (so that the mixed Hodge structure on this part of $H^\bullet(\mathcal{M}_{g,n})$ is 'pure'). A simple instance of Pikaart's argument appears in the following observation.

Suppose M is a nonsingular complex variety and $\overline{M}$ is a nonsingular projective compactification such that $D := \overline{M} - M$ is a normal crossing divisor. If the first Chern class of the normal bundle of $D_{\rm reg}$ in $\overline{M}$ is nonzero on every connected component of $D_{\rm reg}$, then the map $H^1(\overline{M}) \to H^1(M)$ is an isomorphism. In particular, when $H^1(M) \neq 0$, then there exists a nonzero holomorphic differential on $\overline{M}$.

As Putman noted [6], this applies to the orbifold $M = \mathcal{M}_\Gamma$ and its Deligne-Mumford compactification when $\Gamma \subset \Gamma_{g,n}$ of finite index and $g \geq 3$. This is of interest in view of the *Ivanov conjecture* which states that then $H^1(\Gamma) = 0$. So when the conjecture fails for Γ, then $\overline{\mathcal{M}}_\Gamma$ admits a nonzero holomorphic differential (as an orbifold). This leads us to ask:

Question 1. Let for $g \geq 2$, $\Delta_g \subset \Gamma_g$ be a subgroup of finite index that is 'sufficiently natural' in its dependence on g (for instance, the kernel of the Γ_g-action on $H^1(S_g; \mathbb{Z}/m)$ for a fixed m). Is then $H^k(\overline{\mathcal{M}}_{\Delta_g}) \to H^k(\mathcal{M}_{\Delta_g}) \cong H^k(\Delta_g)$ onto (or equivalently, is $H^k(\Delta_g)$ pure) when $g \gg 0$?

Representations of the mapping class group. The Birman exact sequence

$$1 \to \pi_1(S_{g,n}, x) \to \Gamma_{g,n+1} \to \Gamma_{g,n} \to 1$$

can be regarded as the fundamental group sequence of the fibration $\mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n}$. Let $K \subset \pi_1(S_{g,n}, x)$ be a subgroup of finite index that is normal in $\Gamma_{g,n+1}$ and put $G := \pi_1(S_{g,n}, x)/K$ (a finite group) and $N_K := \Gamma_{g,n+1}/K$. We have then a G-covering $S_K \to S_{g,n}$ and N_K can be understood as the group of mapping classes of S_K that lift one of $S_{g,n}$. Such lifts are unique up to a covering transformation and the exact sequence $1 \to G \to N_K \to \Gamma_{g,n} \to 1$ expresses this fact. The group N_K acts on G by group automorphisms; the kernel $N_K^\flat$ of this action is the centraliser of G in N_K and is clearly of finite index in N_K. The covering $S_K \to S_{g,n}$ extends naturally to a ramified covering $\overline{S}_K \to S_g$ (with $\overline{S}_K$ a compact oriented surface) so that $H_K := H^1(\overline{S}_K)$ defines a symplectic representation $\rho_K : N_K \to \mathrm{Sp}(H_K)$ with $\rho_K(N_K^\flat) \subset \mathrm{Sp}_G(H_K)$. This makes $\rho(N_K^\flat)$ appear as the monodromy of a family $\mathcal{C}_K \to \mathcal{M}_\Gamma$ with fiber $\overline{S}_K$, where $\Gamma \subset \Gamma_{g,n}$ is the image of $N_K^\flat$. According to Deligne such a representation is semisimple. Hence the Zariski closure $\mathcal{G}_K \subset \mathrm{Sp}(H_K)$ of $\rho(N_K)$ is also semisimple with its identity component $\mathcal{G}_K^\circ$ mapping to $\mathrm{Sp}_G(H_K)$. We pose the following questions without offering a conjectured answer.

Question 2. Is $H_K^{\mathcal{G}_K^\circ} = \{0\}$, when $g \geq 3$? This is a reformulation of a question asked by Putman-Wieland [7]; they showed that a yes answer implies that the Ivanov conjecture holds in genus ≥ 4. This question is also of interest to algebraic geometers because this property is detectable infinitesimally via the period map:

a theorem of Deligne [2] implies that the Hodge structure that we get on $H_K^{\mathcal{G}_K^\circ}$ when we give S_g a complex structure is *independent of that complex structure*. As Avila-Matheus-Yoccoz observed (personal communication), the answer is *no* for $g = 2$: an example is provided by the cyclic cover of degree 6 of the Riemann sphere which totally ramifies in 6 distinct points; such a cover factors through the degree 2 cover, which is in fact the general genus 2 curve (this example appears in the work of Deligne-Mostow).

Question 3. Do we have $\mathcal{G}_K^\circ = \mathrm{Sp}_G(H_K)$?

Question 4. Is $\rho(N_K)$ arithmetic in $\mathcal{G}_K$?

Note that a 'yes' to Q3 implies also a 'yes' to Q2 and so the Ivanov conjecture would then follow for $g \geq 4$. Some time ago [4] I proved that the answer is yes for both Q3 and Q4 when $n = 0$ and G abelian. The recent work of Grünewald-Larsen-Lubotsky-Malestein [3] should furnish many examples with $n = 0$ and G non-commutative.

Potential quantum representations. Let $\alpha \subset S_{g,n}$ be an oriented embedded circle and denote by τ_α the associated Dehn twist. Every connected component $\tilde{\alpha}$ of $p^{-1}\alpha$ has the same degree m_α over α and hence $\tilde{\tau}_\alpha := \prod_{\tilde{\alpha}/\alpha} \tau_{\tilde{\alpha}}$ is lift of $\tau_\alpha^{m_\alpha}$ which lies in $N_K^\flat$. The action of this lift on H_K is a unipotent transformation given by the Picard-Lefschetz formula. Its associated 1-parameter subgroup of $\mathcal{G}_K$ is

$$T_\alpha : \mathbb{G}_a \hookrightarrow \mathcal{G}_K, \quad T_\alpha(\lambda) : v \mapsto v + \lambda \sum_{\tilde{\alpha}/\alpha} ([\tilde{\alpha}] \cdot v)[\tilde{\alpha}].$$

The subgroup $\mathcal{D}_K \subset \mathcal{G}_K$ generated by such 1-parameter groups is a normal connected subgroup of $\mathcal{G}_K$ that is defined over $\mathbb{Q}$. Hence $\mathcal{D}_K$ is also semisimple. Marco Boggi and I [1] observed that the subspace $P_K \subset H_K$ spanned by the classes $[\tilde{\alpha}]$ that are obtainable as above (so with also α varying) is the symplectic perp of $H_K^{\mathcal{D}_K}$. By Deligne's semisimplicity theorem, P_K must then be nondegenerate for the symplectic form on H_K so that we have a symplectic decomposition $H_K = P_K \oplus H_K^{\mathcal{D}_K}$ with the second summand containing the (possibly trivial) $H_K^{\mathcal{G}_K^\circ}$.

In view of the dependence of Q3 on the Ivanov conjecture, it is more reasonable to first address the following

Conjecture. For $g \geq 3$, $\mathcal{D}_K = \mathrm{Sp}_G(P_K)$ and $\rho(N_K) \cap \mathcal{D}_K$ is arithmetic.

Presumably the results of [3] imply that for $n = 0$ this conjecture has a positive answer when asked for the part of P_K on which G acts through a quotient with at most $g - 1$ generators.

Notice that the group $\mathcal{G}_K$ acts on $H_K^{\mathcal{D}_K}$ via the semisimple $\mathbb{Q}$-group $\mathcal{G}_K/\mathcal{D}_K$. Since $\Gamma_{g,n}$ is generated Dehn twists, N_K is generated by lifts of Dehn twists and as we just saw, any lift of $\tau_\alpha \in \Gamma_{g,n}$ in N_K maps to a torsion element of $\mathcal{G}_K/\mathcal{D}_K$ of order divisible by m_α. On the other hand, the image of N_K in $\mathcal{G}_K/\mathcal{D}_K$ is also Zariski dense and so we ask:

Question 5. Are there any examples for which $H_K^{\mathcal{D}_K} \neq 0$ and if so, with $\mathcal{G}_K^\circ/\mathcal{D}_K$ acting nontrivially on it?

The cases covered by [4] and [3] do not produce such examples, for we then have $P_K = H_K$ so that $H_K^{\mathcal{D}_K} = 0$ (and Q3 is equivalent to our conjecture). I have no idea what the situation is in general[2], but I would in fact be pleased if the answer to the stronger version of Q5 were yes. For we then obtain representions of N_K (some of which could define a projective representation of $\Gamma_{g,n}$) in $H_K^{\mathcal{D}_K}$ with the property that the lifts of Dehn twists act with finite order and since the quantum representations also have this property, it is then natural to ask:

Question 6. Is any quantum representation of the mapping class group (i.e., one arising from the theory of conformal blocks) obtained as a *complex* subrepresentation of this type?

The quantum representations are also conjectured to be unitary and indeed, we would then expect this inner product to come from the intersection pairing on $H_K^{\mathcal{G}_K^\circ}$ (yielding a compact factor of $\mathcal{G}_K^\circ/\mathcal{D}_K(\mathbb{R})$).

References

[1] M. Boggi, E. Looijenga: *In preparation.* 4

[2] P. Deligne: *Théorie de Hodge, II.* Publ. Math. IHÉS **40** (1971), 5–57. 4

[3] F. Grünewald, M. Larsen, A. Lubotzky, J. Malestein: *Arithmetic quotients of the mapping class group,* `arXiv:1307.2593v2`. 4, 5

[4] E. Looijenga: *Prym Representations of Mapping Class Groups,* Geometriae Dedicata **64** (1997), 69–83. 4, 5

[5] M. Pikaart: *An orbifold partition of $\overline{\mathcal{M}}_g^n$,* in: The moduli space of curves (Texel Island, 1994), 467–482, Progr. Math. **129**, Birkhäuser Boston, Boston, MA, 1995. 3

[6] A. Putman: *A note on the abelianizations of finite index subgroups of the mapping class group.* Proc. Amer. Math. Soc. **138** (2010), 753–758. 3

[7] A. Putman, B. Wieland: *Abelian quotients of the mapping class group and higher Prym representations*, J. Lond. Math. Soc. **88** (2013), 79–96. 3

[2] After my talk Julien Marché referred me to his Mathoverflow posting of January 2012 (`http://mathoverflow.net/questions/86894/`), where he asks (in the somewhat more restricted context of no ramification) whether $P_K = H_K$. Boggi and I had been wondering about that, too. Note that in view of the example mentioned above, we have to assume in our setting (where we allow ramification) that $g > 2$. It is a bit of a scandal that the answer is not known.