These notes were made using handwritten notes by E. Looijenga that appeared in the fall of 1992. They will be used during a mini-seminar on **Kähler manifolds and Hodge theory** at the University of Nijmegen in the fall of 2001. They are definitely **not** meant to fully cover the whole theory, many proofs and details have been omitted. I only wrote these to get everything clear for myself, and therefore they might be useful for other participants in the seminar.

Some references are:

1. R.O. Wells, *Differential Analysis on Complex Manifolds*, Springer 1980.
2. P. Griffiths and J. Harris, *Principles of Algebraic Geometric*, Wiley and sons 1978.
3. A. Weil, *Variétés Kähleriennes*, Hermann 1971.
4. S. Kobayashi, *Differential Geometry of Complex Vector Bundles*, Iwanami Shoten / Princeton University Press 1987.

1\. is closest to these notes, while 2. (which has a more analytic approach) can be used as a reference guide for complex algebraic geometry in general (covering far more than only the subject this seminar). 3. is in French, which makes it quite unreadable for me... I have never seen 4. actually, but others may want to use it as a reference.

1 Analytic functions in several variables

We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via $z = (z_1, \ldots, z_n) \leftrightarrow (x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ where $z_\nu = x_\nu + iy_\nu$. We give $\mathbb{C}^n$ the norm $\|z\| = \max_{\nu=1}^n |z_\nu|$ where $|z_\nu| = \sqrt{x_\nu^2 + y_\nu^2}$. If $p \in \mathbb{C}^n$, $\varepsilon > 0$ then we define $B(p, \varepsilon) = \{Z \in \mathbb{C}^n : \|z - p\| < \varepsilon\}$

Definition. Let $\Omega \subseteq \mathbb{C}^n$ be open, $p \in \Omega$, $f : \Omega \to \mathbb{C}$. We say that f is **analytic** or **holomorphic** at p if for some $\varepsilon > 0$ $B(p, \varepsilon) \subseteq \Omega$ and f is given at $B(p, \varepsilon)$ as a power series

$$f(z) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} a_{k_1,\ldots,k_n} (z_1 - p_1)^{k_1} \cdots (z_n - p_n)^{k_n}$$

such that for $0 \leq r < \varepsilon$

$$\sum_{k_1,\ldots,k_n} |a_{k_1,\ldots,k_n}| r^{k_1+\cdots+k_n} < \infty$$

These notes are available through http://www-math.sci.kun.nl/math/~grooten/kahler.shtml

Lemma 1.1 (Osgood). *Let $f : \Omega \to \mathbb{C}$ continuous and analytic in every variable. Then f itself is analytic.*

Let $\Omega \subseteq \mathbb{C}^n$ be open, $f : \Omega \to \mathbb{C}$ a C^1-function. Write $f(z) = f(x + iy)$, considered as a function from some open subset of $\mathbb{R}^{2n}$ to $\mathbb{C}$ and define for $\nu = 1, \ldots, n$:

$$\begin{aligned}\frac{\partial f}{\partial z_\nu} &= \frac{1}{2}\left(\frac{\partial f}{\partial x_\nu} - i\frac{\partial f}{\partial y_\nu}\right)\\ \frac{\partial f}{\partial \bar{z}_\nu} &= \frac{1}{2}\left(\frac{\partial f}{\partial x_\nu} + i\frac{\partial f}{\partial y_\nu}\right)\end{aligned}$$

then we have

Proposition 1.2 (Cauchy-Riemann criterion). *Let f be as above. Then f is analytic on Ω if and only if $\frac{\partial f}{\partial \bar{z}_\nu} = 0$ for $\nu = 1, \ldots, n$.*

Two other important properties of analytic functions are

Proposition 1.3 (Uniqueness of analytic continuation). *Let $\Omega \subseteq \mathbb{C}^n$ be open and connected and let $f : \Omega \to \mathbb{C}$ be analytic. If for some $p \in \Omega$ all the partial derivatives (of arbitrary order) vanish, then $f \equiv 0$ on Ω.*

Proposition 1.4 (Maximum principle). *Let $\Omega \subseteq \mathbb{C}^n$ be open and connected and let $f : \Omega \to \mathbb{C}$ be a non-constant analytic function. Then f is an open mapping and hence $|f|$ has no local maximum.*

2 Smooth and complex manifolds

Definition. For an open subset Ω of $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) we denote by $\mathcal{E}(\Omega)$ (resp. $\mathcal{O}(\Omega)$) the $\mathbb{R}$-algebra (resp. $\mathbb{C}$-algebra) of C^∞ (resp. analytic) functions $\Omega \to \mathbb{R}$ (resp. $\Omega \to \mathbb{C}$).

Definition. A **topological n-manifold** ($n = 1, 2, 3, \ldots$) is a Hausdorff-space which admits a countable basis for its topology and which is locally homeomorphic to $\mathbb{R}^n$.

We will treat the C^∞ and analytic cases at the same time. Therefore, let K stand for $\mathbb{R}$ or $\mathbb{C}$, $\mathcal{S}$ stand for $\mathcal{E}$ and $\mathcal{O}$ respectively and m stand for n resp. $\frac{1}{2}n$ (thus, in the latter case, n should be even!).

Definition. Let[1] M be a topological n-manifold. An **$\mathcal{S}$-structure** on M consists of a sheaf $\mathcal{S}_M$ of K-valued functions on M, such that the 'coordinate transformations' (which are functions between open subsets of K^m) are in $\mathcal{S}$ and furthermore for small open subsets U, $\mathcal{S}_M(U)$ is isomorphic to the space of $\mathcal{S}$-functions on the corresponding open subset of K^m. An n-manifold with an $\mathcal{S}$-structure is called a **smooth** (or **real**) resp. **complex manifold**.

Nothing deep here so far, and we can define the notions of **$\mathcal{S}$-morphism**, **isomorphisms** and **$\mathcal{S}$-submanifolds** in a straightforward way. If you have only seen real manifolds so far: the analytic case is completely analogous to the C^∞ case, but one should be aware that not every smooth $2m$-manifold can be given the structure of a complex m-manifold! For example, the Cauchy-Riemann criterion implies that complex manifolds are always oriented.

Easy examples of $\mathcal{S}$-manifolds are (open subsets of) K^n, $\mathbb{P}^n(K)$, real and complex tori, smooth submanifolds of these etc. Of course, since $\mathbb{C}^m \cong \mathbb{R}^{2m}$, any complex manifold has a natural smooth structure.

Due to the maximum principle (which is a local statement and therefore also valid for complex manifolds), the only connected compact submanifolds of $\mathbb{C}^n$ are the single point sets. However, there are several examples of compact submanifolds of $\mathbb{P}^n(\mathbb{R})$.

3 Vector bundles

Definition. A **K-vector bundle of rank r** (or **K^r-bundle**) ξ consists of a continuous mapping $\pi : E \to B$ of topological spaces such that every fiber $E_p = \pi^{-1}(p)$ had been given the structure of an r-dimensional K-vector space. Furthermore, for every $p \in B$, there exists and open neighborhood U of p and a continuous $h : E_U = \pi^{-1}(U) \to K^r$ such that

(i) for every $q \in U$ $h|_{E_q} : E_q \to K^r$ is an isomorphism of vector spaces.
(ii) the map $\tilde{h} = (h, \pi_U) : E_U \to K^r \times U$ is a homeomorphism.

A pair (U, h) as above is called a **local trivialisation** of ξ. If $U = B$ can be chosen, ξ is called **trivial**. The spaces E and B are called the **total space** resp. **base** of ξ.

If $U \subseteq B$ is open, then a **section of ξ over U** is a continuous mapping $s : U \to E$ such that $\pi \circ s = \mathbf{1}_U$. The sections clearly form a sheaf of K-algebras on B, denoted by $\mathcal{C}(\xi)$ or, with a little abuse of notation, simply by ξ.

[1]Although it works, the definition is a bit confusing: it mixes up two possible definitions. However, since I assumed most of the readers have seen this before, I use this definition as it is quite short

It follows easily that if ξ is trivial over U, then $\mathcal{C}(U,\xi) \cong \mathcal{C}_U(K)^r$ where $\mathcal{C}_U(K)$ is the sheaf on continuous functions to K. From this we see that the notions of K^r-bundle and locally free $\mathcal{C}(K)$-modules of rank r are equivalent.

Given two trvivializations (U_α, h_α) and (U_β, h_β) of ξ, we define $U_{\alpha\beta} = U_\alpha \cap U_\beta$ and then we get a map

$$K^r \times U_{\alpha\beta} \xrightarrow{\tilde{h}_\beta^{-1}} E_{U_{\alpha\beta}} \xrightarrow{\tilde{h}_\alpha} K^r \times U_{\alpha\beta}$$

which is trivial on the first coordinate and the first coordinate gives a continuous mapping $g_{\alpha\beta} : U_{\alpha\beta} \to GL_r(K)$. These so-called **transition functions** enjoy the properties:

(i) $g_{\alpha\alpha} = \mathbf{1}_r$
(ii) $g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = \mathbf{1}_r$

Conversely, given an open covering $\{U_\alpha\}$ of B and continuous functions $g_{\alpha\beta} : U_{\alpha\beta} \to GL_r(K)$ satisfying (i) and (ii), the we can glue the trivial bundles $K^r \times U_\alpha$ via the $g_{\alpha\beta}$'s to get a K^r-bundle over B.

If B has the structure of an $\mathcal{S}$-manifold, we can define **holomorphic** ($\mathcal{S} = \mathcal{O}, K = \mathbb{C}$) resp. **smooth** ($\mathcal{S} = \mathcal{E}, K = \mathbb{R}$) **vector bundles** by demanding the space E to be an $\mathcal{S}$-manifold and all the maps being $\mathcal{S}$-morphisms. Similarly as above, we get an equivalence between holomorphic resp. smooth K^r-bundles and locally free $\mathcal{S}(K)$-modules of rank r.

Since vector bundles can be seen as vector spaces varying continuously (holomorphically, smoothly) with the base, we can extend several procedures on vector spaces to vector bundles and thus get vector bundles

- $\xi \oplus \eta$
- $\xi \otimes \eta$
- $\mathrm{Hom}(\xi, \eta)$
- $\xi^* = \mathrm{Hom}(\xi, K \times B)$ (the **dual bundle** of ξ)
- $\bigwedge^k \xi$ (the ***k*-th exterior product** of ξ)
- $S_k\xi$ (the ***k*-th symmetric product** of ξ)

Now if ξ and η are vector bundles over the same base space B, a vector bundle **homomorphism** from ξ to η (simply denoted as f: $\xi \to \eta$) is continuous mapping $f : E \to F$ such that $f(E_p) \subseteq F_p$ and $f|_{E_p} : E_p \to F_p$ is K-linear for all p. If ξ and

η are vector bundles with an $\mathcal{S}$-structure, f is called an **$\mathcal{S}$-homomorphism** if the defining map $f : \xi \to \eta$ is an $\mathcal{S}$-morphism.

Given a K^r-bundle $\xi = (\pi : E \to B)$, a **sub(vector)bundle** of rank s ($0 \le s \le r$) is a subspace $E' \subseteq E$ with the property that for every $p \in B$ there exists a local trivialisation $(U, h : E_U \to K^r = K^s \times K^{r-s})$ with $p \in U$ and $E'_U = h^{-1}(K^s \times \{0\})$. It follows easily that $\xi' = (\pi|_{E'} : E' \to B)$ has the natural structure of a vector bundle. If ξ is an $\mathcal{S}$-bundle and the (U, h) can be chosen to be $\mathcal{S}$-morphisms, then ξ' is called an **$\mathcal{S}$-subbundle**.

Finally, given a vector bundle $\xi = (\pi : E \to B)$ over a space B and a continuous map $f : A \to B$ of topological spaces, we can **pull back** the bundle ξ to get a bundle $f^*\xi = (f^*\pi : f^*E \to B)$ over A where

$$\begin{array}{ccccc} f^*E & = & \{(x,a) \in E \times A : \pi(x) = f(a)\} & \ni & (x,a) \\ \downarrow{\scriptstyle f^*\pi} & & & & \downarrow \\ A & & & & a \end{array}$$

If ξ has an $\mathcal{S}$-structure and f is an $\mathcal{S}$-morphism, then $f^*\xi$ has an $\mathcal{S}$-structure.

Tangent bundles

Let's conclude this paragraph with some important examples of vector bundles that are essential for the remainder. For a real m-manifold M and $p \in M$, we define the **real tangent space at** p, $T_{\mathbb{R},p}(U)$ to be the vector space of $\mathbb{R}$-linear derivations on the ring of germs of real-valued C^∞-functions near p. Clearly, if $(x_1, \ldots, x_m)$ are coordinates on U, we have that $\{\frac{\partial}{\partial x_1}|_p, \ldots, \frac{\partial}{\partial x_m}|_p\}$ is a basis for $T_{\mathbb{R},p}(U)$.

These vector bundles glue via the chain-rule for derivations to get a vector-bundle $T_{\mathbb{R},M} = T_{\mathbb{R}}(M)$ of rank m over M. This is called the **tangent bundle** over M. To be a bit more precise: if $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$ are two sets of coordinates with $y = F \circ x$ for some diffeomorphism F, then $\frac{\partial}{\partial x_i} = \sum_{j=1}^m (JF)_{i,j} \frac{\partial}{\partial y_j}$, where (JF) is the Jacobian matrix of F.

Now if M is a complex m-manifold, we still have the $\mathbb{R}^{2m}$-bundle $T_{\mathbb{R},M}$ over M, but we can also look at $\mathbb{C}$-linear derivations on complex-valued C^∞-functions. This gives a smooth vector bundle, called the **complexified tangent bundle** over M and is denoted by $T_{\mathbb{C}}(M)$ or[2] $T_{\mathbb{C},M}$. Since $T_{\mathbb{C},M} = T_{\mathbb{R},M} \otimes_{\mathbb{R}} \mathbb{C}$, we see that if $(z_1, \ldots, z_m)$ (with $z_\nu = x_\nu + iy_\nu$) are local coordinates near $p \in M$, then we see that $T_{\mathbb{C},p}$ has $\{\frac{\partial}{\partial x_1}|_p, \frac{\partial}{\partial y_1}|_p \ldots, \frac{\partial}{\partial x_n}|_p, \frac{\partial}{\partial y_n}|_p\}$ as a basis, but also $\{\frac{\partial}{\partial z_1}|_p, \frac{\partial}{\partial \bar{z}_1}|_p \ldots, \frac{\partial}{\partial z_n}|_p, \frac{\partial}{\partial \bar{z}_n}|_p\}$.

[2] I want to be consequent in my notation and write between brackets the space on which the thing 'is defined on', while I write all other things on which it depends as subindices. However, in some cases there would be too much subindices and I have to abuse notation a bit.

Now let M be a real manifold and T_M its (real) tangent bundle. We take a look at (smooth) sections of $\bigwedge^r T_M^*$, which are called **r-forms** on M and on an open $U \subseteq M$ we write $\mathcal{E}_M^r(U) = \mathcal{E}^r(U)$ for these sections. We also have the sheaf of skew commutative $\mathcal{E}_M$-algebras $\mathcal{E}_M^\bullet(U) = \oplus_{r=0}^m \mathcal{E}^r(U)$, called the algebra of **differential forms** on M (note that $\mathcal{E}^r(U) = 0$ if $r > m$). The multiplication in $\mathcal{E}_M^\bullet(U)$ sends $(\alpha, \beta) \in \mathcal{E}^a(U) \times \mathcal{E}^b(U)$ to $\alpha \wedge \beta \in \mathcal{E}^{a+b}(U)$, where $\alpha \wedge \beta = (-1)^{ab} \beta \wedge \alpha$. Furthermore we have that if $f : M \to N$ is a smooth mapping between manifolds, then f defines a vector bundle homomorphism $f_* : T_M \to f^* T_N$, which on its way defines homomorphisms $f^* : f^* \mathcal{E}_N^r \to \mathcal{E}_M^r$.

Another important notion is that we have $\mathbb{R}$-linear mappings $\mathrm{d} : \mathcal{E}^r(U) \to \mathcal{E}^{r+1}(U)$ for all r, satisfying

(a) $\mathrm{d}^2 = 0$,
(b) $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^a \alpha \wedge \mathrm{d}\beta$ (where $\alpha \in \mathcal{E}^a(U)$),
(c) If $f : M \to N$ is a smooth mapping and $\alpha \in \mathcal{E}^a(N)$, then $\mathrm{d} f^* \alpha = f^* \mathrm{d}\alpha$,
(d) If φ is a smooth function on (an open subset of) $\mathbb{R}^m$, so $\varphi \in \mathcal{E}^0_{\mathbb{R}^m}(U)$, then $\mathrm{d}\varphi = \sum_{k=1}^m \frac{\partial \varphi}{\partial x_k} \mathrm{d}x_k$.

These properties characterize d. We say that a differential form α is **closed** if $\mathrm{d}\alpha = 0$, it is called **exact** if it is of the form $\alpha = \mathrm{d}\beta$.

4 Differential forms on a complex manifold

We will start with some linear algebra.

A real vector space W can be **complexified** to get a complex vector space $W_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{C}} W$. Elements of $W_{\mathbb{C}}$ can be uniquely written as $x + iy$, with $x, y \in W$. We have an $\mathbb{R}$-linear involution $x + iy \mapsto x - iy$ called **complex conjugation**. W can be recovered as the fixed point subspace $\{y = 0\}$.

We could also start with a complex vector space V and regard it as a real vector space of twice its dimension. Multiplication by i becomes an $\mathbb{R}$-linear transformation $J : V \to V$, with the property $J^2 = -\mathbf{1}_V$. Conversely, given a real vector-space V with an $\mathbb{R}$-linear map $J : V \to V$ such that $J^2 = -\mathbf{1}_V$, we can give V the structure of a complex vector space via $(a + ib)x = ax + bJ(x)$. We therefore say that J defines a **complex structure** on V. Also $-J$ defines a complex structure on V, called the **conjugate** of V.

If we have a $\mathbb{R}$-vector space V with a complex structure J, we can extend J to to $V_{\mathbb{C}}$ and get a $\mathbb{C}$-linear map $J_{\mathbb{C}} : V_{\mathbb{C}} \to V_{\mathbb{C}}$ by defining $J_{\mathbb{C}}(x + iy) = J(x) + iJ(y)$ and it of course still satisfies $J_{\mathbb{C}}^2 = -\mathbf{1}_V$. For example, let M be a complex manifold and $p \in M$. If we choose local coordinates $(z_1, \ldots, z_m)$ where $z_\nu = x_\nu + iy_\nu$ then $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1} \ldots, \frac{\partial}{\partial x_m}, \frac{\partial}{\partial y_m}\}$ is a basis for $T_{\mathbb{R},p}$. We have a $\mathbb{R}$-linear map J on $T_{\mathbb{R},p}$, such that $J(\frac{\partial}{\partial x_\nu}) = \frac{\partial}{\partial y_\nu}$ and $J(\frac{\partial}{\partial y_\nu}) = -\frac{\partial}{\partial x_\nu}$. This J clearly satisfies $J^2 = -\mathbf{1}_{T_{\mathbb{R},p}}$. Now if we look at the complexified tangent space at p $T_{\mathbb{C},p}$, we now have $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1} \dots, \frac{\partial}{\partial x_m}, \frac{\partial}{\partial y_m}\}$ as a $\mathbb{C}$-basis for this vector space, but besides the complex structure from $J_{\mathbb{C}}$ we now also have a complex structure coming from the multiplication by i.

Since all coordinate transformations on M were required to be holomorphic, $J_{\mathbb{C}}$ extends to an automorphism of the complexified tangent bundle of M, which is independent of the choice of coordinates (See Wells, p. 28-29 for the details). Therefore, we may apply the above local theory to tangent bundles of complex manifolds. However, we first need to go into the linear algebra some more and need a useful lemma.

Lemma 4.1. *Let V be a real vector space with complex structure J. Then*

(i) The eigenvalues of $J_{\mathbb{C}}$ are i and $-i$
(ii) If $V^{1,0} \subset V_{\mathbb{C}}$ and $V^{0,1} \subset V_{\mathbb{C}}$ denote the i resp. $-i$ eigenspaces of $J_{\mathbb{C}}$. Then $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ and complex conjugation interchanges these summands.
(iii) The maps $\alpha : V \to V_{\mathbb{C}},\ x \mapsto x - iJ(x)$ and $\bar{\alpha} : V \to V_{\mathbb{C}},\ x \mapsto x + iJ(x)$ map V isomorphically onto $V^{1,0}$ resp. $V^{0,1}$. Moreover α is $\mathbb{C}$-linear with regard to the complex structure coming from J (i.e. $\alpha(J(x)) = i\alpha(x)$) and $\bar{\alpha}$ is $\mathbb{C}$-anti-linear (i.e. $\bar{\alpha}(J(x)) = -i\bar{\alpha}(x)$)

Proof. This is an easy excercise using elementary linear algebra. □

Of course, a complex structure J on V determines a transformation J^* in $V^* = \mathrm{Hom}(V, \mathbb{R})$ still satisfying $(J^*)^2 = -\mathbf{1}_{V^*}$, so it determines a complex structure on V^*. Since $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = \mathrm{Hom}(V, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$, we can apply the lemma to get a decomposition

$$\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = (V^*)^{1,0} \oplus (V^*)^{0,1}$$

where

$$\begin{aligned} (V^*)^{1,0} &= \{f \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) : f(Jx) = if(x)\} \\ (V^*)^{0,1} &= \{f \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) : f(Jx) = -if(x)\} \end{aligned}$$

Using the well-known isomorphism $\bigwedge^r (A \oplus B) \cong \oplus_{p+q=r} \bigwedge^p A \otimes \bigwedge^q B$ we get

$$\bigwedge_{\mathbb{C}}^r \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = \bigoplus_{p+q=r} \bigwedge_{\mathbb{C}}^p (V^*)^{1,0} \otimes \bigwedge_{\mathbb{C}}^q (V^*)^{0,1} =: (V^*)^{p,q}$$

We may regard $\bigwedge_{\mathbb{C}}^r \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$ as the complexification of $\bigwedge_{\mathbb{R}}^r \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) = \bigwedge^r V^*$. We therefore have complex conjugation defined on $\bigwedge_{\mathbb{C}}^r \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$ and the elements fixed by it can be viewed as elements of $\bigwedge^r V^*$, they are said to be **real**.

It follows easily that complex conjugation interchanges $(V^*)^{p,q}$ and $(V^*)^{q,p}$, so if $p \neq q$, there are no real elements in $(V^*)^{p,q}$.

The following important theorem is easy to prove:

Theorem 4.2. *Let* $\{z_1, \ldots, z_n\}$ *be a* $\mathbb{C}$*-basis for* $(V^*)^{1,0}$*. Then*

$$\{z_{i_1} \wedge \cdots \wedge z_{i_p} \wedge \bar{z}_{j_1} \wedge \cdots \wedge \bar{z}_{j_q} : 1 \leq i_1 < \ldots < i_p \leq n, 1 \leq j_1 < \ldots < j_q \leq n\}$$

is a $\mathbb{C}$*-basis for* $(V^*)^{p,q}$

An important notion is the fact that the above discussion is functorial. If (V, J) and (W, K) are real vector spaces with a complex structure, and $f : V \to W$ is $\mathbb{R}$-linear such that $f \circ J = K \circ f$ (so that f is in fact $\mathbb{C}$-linear with the complex structures). Then the natural map $f_{\mathbb{C}}$ has the property that $f_{\mathbb{C}}(V^{1,0}) \subseteq W^{1,0}$ etcetera and therefore also $f^*_{\mathbb{C}}((W^*)^{p,q}) \subseteq (V^*)^{p,q}$.

Let M again be a complex manifold of complex dimension n. As we saw above, the local complex structures on the complexified tangent bundles glue to get a vector bundle homomorphism $J : T_{\mathbb{R},M} \to T_{\mathbb{R},M}$ with $J^2 = -\mathbf{1}_{T_M}$. We therefore get splittings $T_{\mathbb{C},M} = T^{1,0}_{\mathbb{C},M} \oplus T^{0,1}_{\mathbb{C},M}$ and $\bigwedge^r T^*_{\mathbb{C},M} = \oplus_{p+q=r}(T^*_{\mathbb{C},M})^{p,q}$. We usually denote $\bigwedge^r T^*_{\mathbb{C},M}$ by $\mathcal{E}^r_{\mathbb{C},M}$ (which is the complexification of $\mathcal{E}^r_{\mathbb{R},M}$) and $(T^*_{\mathbb{C},M})^{p,q}$ by $\mathcal{E}^{p,q}_M$. Sections[3] of this latter space are called (p,q)**-forms** or **forms of type** (p,q) on M.

If we choose local coordinates $(z_1, \ldots, z_n)$ near a point $P \in M$ with $z_\nu = x_\nu + iy_\nu$ then we get the following spaces:

$$\begin{aligned}
T_{\mathbb{R},P} &= \mathbb{R}\left\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n} \right\rangle \\
T_{\mathbb{C},P} &= \mathbb{C}\left\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n} \right\rangle \\
(T_{\mathbb{C},P})^{1,0} &= \mathbb{C}\left\langle \frac{\partial}{\partial x_1} - i\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n} - i\frac{\partial}{\partial y_n} \right\rangle = \mathbb{C}\left\langle \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n} \right\rangle \\
(T_{\mathbb{C},P})^{0,1} &= \mathbb{C}\left\langle \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n} + i\frac{\partial}{\partial y_n} \right\rangle = \mathbb{C}\left\langle \frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n} \right\rangle \\
(T^*_{\mathbb{C},P})^{1,0} &= \mathbb{C}\langle \mathrm{d}x_1 + i\mathrm{d}y_1, \ldots, \mathrm{d}x_n + i\mathrm{d}y_n \rangle = \mathbb{C}\langle \mathrm{d}z_1, \ldots, \mathrm{d}z_n \rangle \\
(T^*_{\mathbb{C},P})^{0,1} &= \mathbb{C}\langle \mathrm{d}x_1 - i\mathrm{d}y_1, \ldots, \mathrm{d}x_n - i\mathrm{d}y_n \rangle = \mathbb{C}\langle \mathrm{d}\bar{z}_1, \ldots, \mathrm{d}\bar{z}_n \rangle
\end{aligned}$$

[3]We will usually consider $\mathcal{E}^{p,q}_M$ as a locally free $\mathcal{E}_M$-sheaf of certain rank, rather than as a vector bundle of the same rank. As mentioned before, these two notions are in fact equivalent

Therefore a (p,q)-form near P can be uniquely written as

$$\omega = \sum_{i_1<\cdots<i_p, j_1<\cdots<j_q} \omega_{i_1,\cdots,i_p,j_1,\cdots,j_q} \mathrm{d}z_{i_1} \wedge \ldots \wedge \mathrm{d}z_{i_p} \wedge \mathrm{d}\bar{z}_{j_1} \wedge \mathrm{d}\bar{z}_{j_q}$$

which will usually be abbreviated to $\sum_{\underline{i},\underline{j}} \omega_{\underline{i},\underline{j}} \mathrm{d}z_{\underline{i}} \wedge \mathrm{d}\bar{z}_{\underline{j}}$. Here the $\omega_{\underline{i},\underline{j}}$ are C^∞-functions.

We can extend our operator d linearly to complex-valued r-forms. Here we write for a complex-valued function f near P $\mathrm{d}f = \partial f + \bar{\partial} f$ where $\partial f = \sum_\nu \frac{\partial f}{\partial z_\nu} \mathrm{d}z_\nu$, $\bar{\partial} f = \sum_\nu \frac{\partial f}{\partial \bar{z}_\nu} \mathrm{d}\bar{z}_\nu$. Applying this to ω above, we get

$$\mathrm{d}\omega = \underbrace{\sum_{\underline{i},\underline{j}} \partial\omega_{\underline{i},\underline{j}} \wedge \mathrm{d}z_{\underline{i}} \wedge \mathrm{d}\bar{z}_{\underline{j}}}_{=:\, \partial\omega} + \underbrace{\sum_{\underline{i},\underline{j}} \bar{\partial}\omega_{\underline{i},\underline{j}} \wedge \mathrm{d}z_{\underline{i}} \wedge \mathrm{d}\bar{z}_{\underline{j}}}_{=:\, \bar{\partial}\omega}$$

We thus may conclude:

Proposition 4.3. *On a complex manifold M, we have* $\mathrm{d}(\mathcal{E}_M^{p,q}) \subseteq \mathcal{E}_M^{p+1,q} \oplus \mathcal{E}_M^{p,q+1}$. *Furthermore we may write* $\mathrm{d} = \partial + \bar{\partial}$ *where ∂ and $\bar{\partial}$ are the projections on the first resp. second summand*

Corollary 4.4. *We have $\partial^2 = 0$, $\bar{\partial}^2 = 0$ and $\partial\bar{\partial} + \bar{\partial}\partial = 0$.*

PROOF. Using $\mathrm{d}^2 = 0$, we get for $\alpha \in \mathcal{E}_M^{p,q}(U)$ for some open $U \subseteq M$:

$$0 = \mathrm{d}^2\alpha = (\partial + \bar{\partial})^2\alpha = \underbrace{\partial^2\alpha}_{\in\, \mathcal{E}_M^{p+2,q}(U)} + \underbrace{(\partial\bar{\partial} + \bar{\partial}\partial)\alpha}_{\in\, \mathcal{E}_M^{p+1,q+1}(U)} + \underbrace{\bar{\partial}^2\alpha}_{\in\, \mathcal{E}_M^{p,q+2}(U)}$$

□

As a final remark, we notice that of the bundles of (p,q)-forms, only the $(p,0)$-forms for arbitrary p have a natural analytic structure. We will prove this for $(T^*_{\mathbb{C},M})^{1,0}$, which suffices, since $(T^*_{\mathbb{C},M})^{p,0} = \bigwedge^p (T^*_{\mathbb{C},M})^{1,0}$.

If we have a local chart $(h_1, \ldots, h_n)$ on an open subset $U \subseteq M$ then $(\mathrm{d}h_1, \ldots, \mathrm{d}h_n)$ is a local trivialisation of $(T^*_{\mathbb{C},M})^{1,0}$. If $h' : U \to \mathbb{C}^n$ is another chart, then we have

$$\mathrm{d}h'_j = \sum_{k=1}^n g_{jk} \mathrm{d}h_k, \text{ with } g_{jk} = \frac{\partial h'_j h^{-1}}{\partial z_k} \circ h$$

and thus the g_{jk} are analytic functions.

We have the following proposition which is easy to prove:

Proposition 4.5. *A $(p,0)$-form ω defines an analytic section if and only if $\bar{\partial}\omega = 0$.*

We will denote the sheaf of analytic $(p,0)$-forms on M by Ω^p_M. Those forms are called **holomorphic** of **analytic** p**-forms** and from the proposition it follows that they can be written locally as $\sum_{\underline{i}} \omega_{\underline{i}} \cdot \mathrm{d}z_{\underline{i}}$ where all the $\omega_{\underline{i}}$ are *holomorphic* functions.

Remark 4.5.1. Instead of assuming M to be a complex manifold, we could also have assumed that M is a real manifold of even dimension, with a vector bundle automorphism $J : T_{\mathbb{R},M} \to T_{\mathbb{R},M}$ satisfying $J^2 = -\mathbf{1}$. A real manifold with this property is called an **almost complex manifold**, while J is called the **almost complex structure**. Some (but not all!) of the above theory still holds for almost complex manifolds. However, one should be aware that there are examples of almost complex structures that don't arise from complex structures. See also the book of Wells.

5 Grothendieck's Lemma

An important tool in the study of manifolds is the following lemma, which is also known as the $\bar{\partial}$**-Poincaré lemma** or as **Dolbeaults theorem**:

Lemma 5.1 (Grothendieck). *Let $0 < r \leq \infty$ and consider $\Delta_r = B(0,r)$. If $q \geq 1$ and $\alpha \in \mathcal{E}^{p,q}(\Delta_r)$ satisfies $\bar{\partial}\alpha = 0$, then $\alpha = \bar{\partial}\beta$ for some $\beta \in \mathcal{E}^{p,q-1}(\Delta_r)$.*

Proof. I will not give the proof, since it is quite technical and not really interesting (the idea is to integrate α locally and then using limits). Its methods will not be used in the remainder of these notes. □

An analogue statement for smooth functions is the following

Lemma 5.2 (Ordinary Poincaré Lemma). *Let $0 < r \leq \infty$ and $B_r = \{x \in \mathbb{R}^n : |x_j| < r, j = 1, \ldots, n\}$. If $\alpha \in \mathcal{E}^q(B_r)$ for some $q \geq 1$ satisfies $\mathrm{d}\alpha$=0, then $\alpha = \mathrm{d}\beta$ for some $\beta \in \mathcal{E}^{q-1}(B_r)$.*

Since smooth (resp. complex) manifolds can be covered by open subsets difeomorphic to the unit ball in $\mathbb{R}^n$ (resp. $\mathbb{C}^n$), these lemmas imply that that sheaf homomorphisms $\mathrm{d} : \mathcal{E}^{q-1} \to \mathcal{E}^q$ and $\bar{\partial} : \mathcal{E}^{p,q-1} \to \mathcal{E}^{p,q}$ are exact. This is an important property which will be used in the following sections.

Remark 5.2.1. Although we will not use it, the equivalent of Grothendieck's Lemma with ∂ instead of $\bar{\partial}$ is also true. This follows immediately from the fact that ∂ is the conjugate of $\bar{\partial}$.

6 Resolutions

In this and the following sections we will study sheaf cohomology. Hereto we will develop the important tool of resolutions of sheaves. Therefore of course, some sheaf theory should be known. There are several good introductions to these, for example the book of Wells. We will assume all sheaves to be sheaves of a so-called **abelian category**, that is a category $\mathfrak{A}$ with the following properties

- For all objects A and B of $\mathfrak{A}$, $\mathrm{Hom}(A, B)$ has the structure of an abelian group with the composition law being linear
- Finite direct sums exists
- Every morphism has a kernel and a cokernel in $\mathfrak{A}$
- Every monomorphism is the kernel of its cokernel
- Every epimorphism is the cokernel of its kernel
- Every morphism can be factored into an epimorphism followed by a monomorphism

Such sheaves are called **abelian sheaves**.

Examples of abelian categories are the category of abelian groups and the category of vector spaces over some field k (these are in fact the only two examples we will use). Be aware that rings and k-algebras do *not* form an abelian category (the kernel of a homomorphism of rings is not a ring anymore!), at least not if we assume them to have a unit element for the multiplication. If X is some topological space and $\mathfrak{A}$ some abelian category, we write $\mathfrak{A}(X)$ for the category of $\mathfrak{A}$-sheaves on X, it is easy to see that $\mathfrak{A}(X)$ is also an abelian category.

Definition. Let X be a topological space. A sequence of sheaves and homomorphisms

$$\mathcal{C}^\bullet = (\cdots \longrightarrow \mathcal{C}^{q-1} \xrightarrow{\mathrm{d}^{q-1}} \mathcal{C}^q \xrightarrow{\mathrm{d}^q} \mathcal{C}^{q+1} \xrightarrow{\mathrm{d}^{q+1}} \cdots)$$

is called a **sheaf complex** if $\mathrm{d}^q\mathrm{d}^{q-1} = 0$ for all q. If in fact $\mathrm{Ker}(\mathrm{d}^q) = \mathrm{Im}(\mathrm{d}^{q-1})$ for all q, then we say that the sheaf complex is **exact**. We define $H^q(\mathcal{C}(X))$, the **q-th cohomology group**[4] of the sequence $\mathcal{C}$ to be $\mathrm{Ker}(\mathrm{d}^q)/\mathrm{Im}(\mathrm{d}^{q-1})(X)$, which is the global sections of the sheaf $\mathrm{Ker}(\mathrm{d}^q)/\mathrm{Im}(\mathrm{d}^{q-1})$.

If $\mathcal{C}^\bullet$ and $\mathcal{D}^\bullet$ are sheaf complexes on X, the a **homomorphism** from $\mathcal{C}^\bullet$ to $\mathcal{D}^\bullet$ is a sequence of sheaf homomorphisms $f^q : \mathcal{C}^q \to \mathcal{D}^q$ such that $\mathrm{d}^q f^q = f^{q+1}\mathrm{d}^q$ for all q. From this property it follows that the f^q induce homomorphisms of the corresponding cohomology groups.

[4]Be aware of a little abuse of notation: if the sheaves are all $\mathbb{C}$-vector spaces, which they usually will be in our case, then so are the cohomology 'groups'!

Given a sheaf $\mathcal{F}$ on X, then a **right resolution** of X is an “embedding” of $\mathcal{F}$ into an exact sheaf complex

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^0 \longrightarrow \mathcal{C}^1 \longrightarrow \mathcal{C}^2 \longrightarrow \cdots$$

If $F \to \mathcal{C}^\bullet$ and $\mathcal{F} \to \mathcal{D}^\bullet$ are both right resolutions of $\mathcal{F}$, then a **homomorphism** of the first resolution to the second is a homomorphism $\mathcal{C}^\bullet \to \mathcal{D}^\bullet$ such that

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{C}^\bullet \\ & \searrow & \downarrow \\ & & \mathcal{D}^\bullet \end{array}$$

commutes.

The reason for introducing resolutions is that they play an important role in the computation of cohomology for sheaves. However, it turns out to be useful to work with resolutions with one extra property.

Definition. A sheaf $\mathcal{F}$ is called **flabby** (resp. **soft**) is for every nonempty open (resp. closed) subset $A \subseteq X$ the restriction map $\mathcal{F}(X) \to \mathcal{F}(A)$ is surjective. We have similar notions of flabby and soft sheaf complexes and resolutions $\mathcal{F} \to \mathcal{C}^\bullet$, with the important notion that all the $\mathcal{C}^q$'s need to be flabby (resp. soft), not necessarily $\mathcal{F}$!

Remark 6.0.2. Since for a closed subset $A \subset X$, $\mathcal{F}(A)$ is defined as

$$\mathcal{F}(A) = \varinjlim_{U \supseteq A} \mathcal{F}(U)$$

where the U is taken over the open subsets, we have that flabby implies soft.

On smooth manifolds, we have many soft sheaves:

Lemma 6.1. *Let $\xi = (\pi : E \to M)$ be a smooth vector bundle over a smooth manifold. Then its sheaf of smooth sections $\mathcal{E}(\xi)$ is soft.*

PROOF. Let $A \subseteq M$ be a closed subset and $s \in \mathcal{E}(\xi)(A)$, which means that s can be represented by a smooth section of ξ over some open neighborhood U of A. Using so-called ‘bump-functions’[5] together with a partition of the unity, it is easy to construct another representative of s, defined on the whole M, which is 0 outside a smaller open neighborhood of A. □

[5] that is, a C^∞-function that is constant 1 on some closed subset and constant 0 outside a neighborhood of that closed subset

Remark 6.1.1. This lemma can easily be generalized as follows: Let $\mathcal{F}$ be some sheaf of $\mathcal{E}_M$-modules. Then $\mathcal{F}$ is a soft sheaf.

Remark 6.1.2. From the maximum principle (proposition 1.4) it follows easily that holomorphic bump-functions don't exist, so the analogue statement for holomorphic vector bundles is not true! For example, take $M = \mathbb{P}^1(\mathbb{C})$ (which has no non-constant holomorphic functions), A some small closed subset and $\mathcal{F}$ the sheaf $\mathcal{O}_M$ of holomorphic functions.

We already have many important examples of (soft) resolutions:

(i) (**the De Rham complex**) Let M be a smooth manifold of dimension m and $\mathbb{R}_M$ be the constant sheaf $\mathbb{R}$ on X, then we have a soft resolution of $\mathbb{R}_M$:

$$0 \longrightarrow \mathbb{R}_M \longrightarrow \mathcal{E}^0_M \xrightarrow{\mathrm{d}} \mathcal{E}^1_M \xrightarrow{\mathrm{d}} \mathcal{E}^2_M \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \mathcal{E}^m_M \xrightarrow{\mathrm{d}} 0$$

The exactness at $\mathcal{E}^q_M$ ($q \geq 1$) follows from the ordinary Poincaré lemma, the exactness at $\mathcal{E}^0_M$ follows from the fact that for some C^∞-function f we have $\mathrm{d}f = 0 \iff f$ locally constant.

(ii) (**the Dolbeault complex**) We have a complex

$$0 \longrightarrow \Omega^p_M \longrightarrow \mathcal{E}^{p,0}_M \xrightarrow{\bar\partial} \mathcal{E}^{p,1}_M \xrightarrow{\bar\partial} \mathcal{E}^{p,2}_M \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{E}^{p,n}_M \xrightarrow{\bar\partial} 0$$

By Grothendieck's lemma, this complex is exact in degrees ≥ 1 and by an easy exercise it also is exact in degree 0.

(iii) (**the Holomorphic De Rham complex**) Let M be a complex manifold of complex dimension n, then using a holomorphic version of the Poincaré lemma, we have a resolution

$$0 \longrightarrow \mathbb{C}_M \longrightarrow \Omega^0_M \xrightarrow{\mathrm{d}} \Omega^1_M \xrightarrow{\mathrm{d}} \Omega^2_M \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^n_M \xrightarrow{\mathrm{d}} 0$$

However, it is easy to see that the sheaves Ω^p_M are not soft, so this is not a soft resolution of $\mathbb{C}_M$.

(iv) We can combine the situations of (ii) and (iii). Hereto we mention that $(-1)^p\partial$ defines a homomorphism of complexes $\mathcal{E}^{p,\bullet}_M \to \mathcal{E}^{p+1,\bullet}_M$, since $\partial\bar\partial = -\bar\partial\partial$. This homomorphism is compatible with $\mathrm{d} : \Omega^p_M \to \Omega^{p+1}_M$, i.e.

$$\begin{array}{ccc} \mathcal{E}^{p,\bullet}_M & \longrightarrow & \mathcal{E}^{p+1,\bullet}_M \\ \uparrow & & \uparrow \\ \Omega^p_M & \xrightarrow{\mathrm{d}} & \Omega^{p+1}_M \end{array}$$

commutes. We therefore obtain a so-called **double complex**

$$\begin{array}{ccccccccccc}
 & & & & \vdots & & \vdots & & \vdots & & \\
 & & & & \uparrow{\scriptstyle \bar\partial} & & \uparrow{\scriptstyle \bar\partial} & & \uparrow{\scriptstyle \bar\partial} & & \\
 & & & & \mathcal{E}_M^{0,2} & \xrightarrow{\partial} & \mathcal{E}_M^{1,2} & \xrightarrow{\partial} & \mathcal{E}_M^{2,2} & \xrightarrow{\partial} & \cdots \\
 & & & & \uparrow{\scriptstyle \bar\partial} & & \uparrow{\scriptstyle \bar\partial} & & \uparrow{\scriptstyle \bar\partial} & & \\
 & & & & \mathcal{E}_M^{0,1} & \xrightarrow{-\partial} & \mathcal{E}_M^{1,1} & \xrightarrow{-\partial} & \mathcal{E}_M^{2,1} & \xrightarrow{-\partial} & \cdots \\
 & & & & \uparrow{\scriptstyle \bar\partial} & & \uparrow{\scriptstyle \bar\partial} & & \uparrow{\scriptstyle \bar\partial} & & \\
 & & & & \mathcal{E}_M^{0,0} & \xrightarrow{\partial} & \mathcal{E}_M^{1,0} & \xrightarrow{\partial} & \mathcal{E}_M^{2,0} & \xrightarrow{\partial} & \cdots \\
 & & & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbb{C}_M & \longrightarrow & \Omega_M^0 & \xrightarrow{\mathrm{d}} & \Omega_M^1 & \xrightarrow{\mathrm{d}} & \Omega_M^2 & \xrightarrow{\mathrm{d}} & \cdots \\
 & & & & \uparrow & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & & 0 & &
\end{array}$$

where all horizontal and vertical complexes are exact. This is an example of a soft resolution (namely $\mathcal{E}_M^{\bullet,\bullet}$) of a resolution (namely $\Omega_M^\bullet$). Note that if we take down-right-diagonal direct sums, we get the complexified De Rham Complex ($\mathcal{E}^r_{\mathbb{C},M} = \oplus_{p+q=r}\mathcal{E}^{p,q}, \bar\partial + \partial = \mathrm{d}$).

(v) The above example can be generalized as follows: Let η be a holomorphic vector bundle over M of rank r. Let $\mathcal{E}^{p,q}(\eta)$ be the sheaf of smooth sections of $(T_M^*)^{p,q} \otimes \eta$. If $U \subseteq M$ is open and $\{s_1, \dots, s_r\}$ is a basis of holomorphic sections of $\eta(U)$ (this means that for all $x \in U$ $\{s_1(x), \dots, s_n(x)\}$ is a basis for the vector space η_x), then every $s \in \mathcal{E}^{p,q}(\eta)(U)$ can be written uniquely as

$$s = \sum_{\nu=1}^{r} \omega_\nu \otimes s_\nu \text{ with } \omega_\nu \in \mathcal{E}^{p,q}(U)$$

Now we with to define $\bar\partial s \in \mathcal{E}^{p,q+1}(\eta)(U)$ by $\sum_{\nu=1}^r \bar\partial\omega_\nu \otimes s_\nu$, but we should first check that this is independent of the chosen basis. This is indeed the case and follows from the easy fact that the natural homomorphisms $\mathcal{E}^{p,q} \otimes \eta \to \mathcal{E}^{p,q}(\eta)$ are in fact isomorphisms. Now we can act as above and we see that the sequence

$$0 \longrightarrow \eta \longrightarrow \mathcal{E}^{0,0}(\eta) \xrightarrow{\bar\partial} \mathcal{E}^{0,1}(\eta) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{E}^{0,n}(\eta) \longrightarrow 0$$

is a resolution for η, since η is locally just the direct sum of r copies of the structure sheaf $\mathcal{O}_M$ and the exactness may be checked locally. For $\eta = (T_M^*)^{p,0}$ we get the Dolbeault resolution of Ω_M^p.

Remark 6.1.3. The reason why this last example worked (and why it doesn't work with ∂ instead of $\bar\partial$) is that $\bar\partial$ is $\mathcal{O}_M$-linear. This means that if f is a holomorpic function on some open subset of X and ω is a smooth section of some $\mathcal{O}_M$-module (e.g. $\omega \in \mathcal{E}^{p,q}(U)$), then by the chain-rule: $\bar\partial(f\omega) = \bar\partial(f)\omega + f\bar\partial\omega = f\bar\partial\omega$. Of course this does not hold with ∂ instead of $\bar\partial$.

7 Cohomology of sheaves

To start this section we will repeat some of the main properties of sheaf cohomology. There are several introductions to cohomology theory in the context of algebraic geometry. The famous book of Hartshorne[6] contains a precise introduction to the subject, but it is quite abstract, without much explaining why things are done. Better for our purpose are the books of Wells and Griffiths-Harris.

We will repeat the main properties of sheaf cohomology:

(I) For $q \geq 0$ $H^q(X, \cdot)$ is a covariant functor from $\mathfrak{A}(X)$ to $\mathfrak{A}$, where again $\mathfrak{A}$ is an abelian category. This means that for every abelian sheaf $\mathcal{F}$ on X we have defined objects $H^q(X, \mathcal{F})$ $(q \geq 0)$ and for each homomorphism $f : \mathcal{F} \to \mathcal{G}$ a morphism $H^q(X, f) : H^q(X, \mathcal{F}) \to H^q(X, \mathcal{G})$ such that $H^q(X, \mathbf{1}_{\mathcal{F}}) = \mathbf{1}_{H^q(X,\mathcal{F})}$ and $H^q(X, f \circ g) = H^q(X, f) \circ H^q(X, g)$.
The object $H^q(X, \mathcal{F})$ is called the **q-th cohomology group** of $\mathcal{F}$, again with a little abuse of notation.

(II) $H^0(X, \mathcal{F})$ is equal to the object of global sections of $\mathcal{F}$ with $H^0(X, f : \mathcal{F} \to \mathcal{G})$ being the natural morphism induced by f.

(III) A short exact sequence of abelian sheaves

$$0 \longrightarrow \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H} \longrightarrow 0$$

gives rise to a long exact sequence of cohomology groups

$$0 \longrightarrow H^0(X, \mathcal{F}) \xrightarrow{H^0(X,f)} H^0(X, \mathcal{G}) \xrightarrow{H^0(X,g)} H^0(X, \mathcal{H}) \xrightarrow{\delta^0}$$

$$\longrightarrow H^1(X, \mathcal{F}) \xrightarrow{H^1(X,f)} H^1(X, \mathcal{G}) \xrightarrow{H^1(X,g)} H^1(X, \mathcal{H}) \xrightarrow{\delta^1} \cdots$$

[6] R. Hartshorne, *Algebraic Geometry*, Springer 1977.

where the homomorphisms $\delta^q : H^q(X, \mathcal{H}) \to H^{q+1}(X, \mathcal{F})$ are functorial in the sense that homomorphisms of short exact sequences induce homomorphisms of the corresponding long exact sequence.

(IV) For a flabby sheaf $\mathcal{F}$ on X we have that $H^q(X, \mathcal{F}) = 0$ for all $q \geq 1$.

Remark 7.0.4. One possible way to define a cohomology theory for sheaves on some topological space X is the following method, due to Godement:

Let for a sheaf $\mathcal{F}$ on X $C^0(\mathcal{F})$ be the sheaf which asserts to each open $U \subseteq X$ the product $\coprod_{x\in U} \mathcal{F}_x$ (where $\mathcal{F}_x$ is the stalk of $\mathcal{F}$ at x). Furthermore, let for $n \geq 1$ $C^n(\mathcal{F}) = C^0(C^{n-1}(\mathcal{F}))$. We then get a sheaf complex

$$0 \longrightarrow \mathcal{F} \longrightarrow C^0(\mathcal{F}) \longrightarrow C^1(\mathcal{F}) \longrightarrow \cdots$$

and we define $H^n(X, \mathcal{F}) := H^n(C^\bullet(\mathcal{F})(X))$.

A sheaf $\mathcal{F}$ with the property $H^q(X, \mathcal{F}) = 0$ for all $q \geq 0$ is called **acyclic**. Acyclic sheaves might help us to compute cohomology of general sheaves, for example using the following propositions:

Proposition 7.1 ('Abstract De Rham Theorem'). *Let $0 \to \mathcal{F} \to \mathcal{C}^\bullet$ be a resolution of an abelian sheaf. Then there are natural homomorphisms*

$$\gamma^q : H^q(\mathcal{C}(X)) \to H^q(X, \mathcal{F}) \quad (q \geq 0)$$

which are isomorphisms if $\mathcal{C}^\bullet$ consists of acyclic sheaves.

Proof. For $q = 0$ this is easily verified.

Let $\mathcal{G} := \mathrm{Ker}(\mathcal{C}^1 \to \mathcal{C}^2)$, then we have a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^0 \longrightarrow \mathcal{G} \longrightarrow 0$$

which gives rise to a long exact sequence

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{C}^0(X) \longrightarrow \mathcal{G}(X) \longrightarrow H^1(X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{C}^0) \longrightarrow \cdots$$

Now use $\mathcal{G}(X) = \mathrm{Ker}[\mathcal{C}^1(X) \to \mathcal{C}^2(X)]$ so that we get an exact sequence

$$0 \longrightarrow H^1(\mathcal{C}(X)) \xrightarrow{\gamma^1} H^1(X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{C}^0) \longrightarrow \cdots$$

which settles the case $q = 1$.

For $q \geq 2$ we proceed with induction: write $\tilde{\mathcal{C}}^\bullet$ for the complex $\mathcal{C}^2 \to \mathcal{C}^3 \to \cdots$, which gives rise to a resolution $\mathcal{G} \to \mathcal{C}^2 \to \mathcal{C}^3 \to \cdots$ of $\mathcal{G}$. By the induction hypothesis we have a homomorphism

$$\tilde{\gamma}^{q-1} : H^{q-1}(\tilde{\mathcal{C}}) \to H^{q-1}(X, \mathcal{G})$$

which is an isomorphism if $\tilde{\mathcal{C}}^\bullet$ is acyclic. Notice that $H^{q-1}(\tilde{\mathcal{C}}) = H^q(\mathcal{C}(X))$. From the first exact sequence above we get a homomorphism $\delta^{q-1} : H^{q-1}(X, \mathcal{G}) \to H^q(X, \mathcal{F})$, which is an isomorphism if $\mathcal{C}^0$ is acyclic. Now take $\gamma^q := \delta^{q-1} \circ \tilde{\gamma}^{q-1}$. □

Proposition 7.2. *Let X be a paracompact*[7] *Hausdorff space, then every soft abelian sheaf on X is acyclic.*

PROOF. See Wells, p. 55. The idea behind the proof is that the global sections functor is exact for soft sheaves, while the H^q measure the 'non-exactness' of this functor. □

Using these propositions and the examples of the previous section, we have some concrete descriptions of certain cohomology groups:

(i) Let M be a smooth manifold of dimension m. Then the De Rham complex $\mathcal{E}^\bullet_M$ is a soft hence acyclic resolution of the constant sheaf $\mathbb{R}_M$ on M. We therefore get

$$H^q(M, \mathbb{R}_M) = H^q(\mathcal{E}^\bullet_M(M), \mathrm{d}) = \frac{\text{global closed forms}}{\text{global exact forms}} \text{ in degree } q$$

In particular: $H^q(M, \mathbb{R}_M) = 0$ for $q > m$.

(ii) Let M be a complex manifold of complex dimension n. Let η be a holomorphic vector bundle over M. We saw that $\mathcal{E}^{0,\bullet}(\eta)$ is a soft resolution of η, hence

$$H^q(M, \eta) = H^q(\mathcal{E}^{0,\bullet}(\eta)(M), \bar{\partial})$$

and in particular $H^q(M, \eta) = 0$ if $q > n$.

Remark 7.2.1. The cohomology groups $H^q(M, \mathbb{R}_M)$ are sometimes called the **De Rham cohomology groups**. If we want to distinguish them from several other possible cohomology groups, we will denote them by $H^q_{\mathrm{dR}}(M, \mathbb{R})$. The cohomology groups $H^q(M, \eta)$ are called the **Dolbeault cohomology groups** of η.

8 Hodge and De Rham theory on compact manifolds

Let us return to some linear algebra. Let V be a real m-dimensional vector space. A nondegenerate positive definite inner product $S : V \times V \to \mathbb{R}$ on V determines

[7] Every open covering of X has a locally finite subcovering. It is easy to show that topological manifolds are paracompact

one on its dual V^* and on all the exterior algebras of V and V^*. They can be characterized as follows:

Let $\{e_1, \dots, e_m\}$ be an orthonormal basis for V (which fully determines S), then the dual basis $\{x_1, \dots, x_m\}$ is orthonormal for the induced form on V^* and $\{e_{\underline{i}}\}_{\underline{i}}$ resp. $\{x_{\underline{i}}\}_{\underline{i}}$ are orthonormal bases for the forms on the exterior algebras. The two elements of the form $x_1 \wedge \cdots \wedge x_m$ of $\bigwedge^m V^*$ that arise in this manner are the two elements of the one-dimensional vector space $\bigwedge^m V^*$. Picking out one of them (which will call μ) is the same as choosing an orientation of V. μ gives rise to an isomorphism $\det : \bigwedge^m V^* \xrightarrow{\sim} \mathbb{R}$ sending μ to 1.

The inner product S and the orientation on V enable us to define the so-called **star operator**

$$* : \bigwedge^q V^* \to \bigwedge^{m-q} V^*$$

characterized by the property that $S(\alpha, \beta)\mu = \alpha \wedge *\beta$. In terms of an oriented orthonormal basis $\{x_1, \dots, x_m\}$ of V^* we get:

$$*x_{\underline{i}} = \pm x_{\underline{i}'}$$

where $\underline{i}'$ is the complementary set of $\underline{i}$ in $\{1, \dots, n\}$ and the sign is the sign of the permutation $(1, \dots, m) \mapsto (\underline{i}, \underline{i}')$. From this it follows easily that $**$ is multiplication by $(-1)^{q(m-q)}$ in $\bigwedge^q V^*$. In particular, $*$ is an isomorphism of $\bigwedge^q V^*$ onto $\bigwedge^{q-m} V^*$.

Now suppose that V has a complex structure J (so that $m = 2n$ is even!). This determines an orientation of V since we can choose a complex basis $\{e_1, \dots, e_n\}$ of V and then $\{e_1, Je_1, \dots, e_n, Je_n\}$ is an oriented basis. This construction is independent of the chosen complex basis.

A **Hermitian form** on V is an $\mathbb{R}$-linear mapping $H : V \times V \to \mathbb{C}$ such that $H(Ja, b) = iH(a, b)$ ($\mathbb{C}$-linearity in the first variable) and $H(b, a) = \overline{H(a, b)}$ (hence $\mathbb{C}$-anti-linearity in the second variable). We can write $H = S + iA$, with $S, A : V \times V \to \mathbb{R}$ both $\mathbb{R}$-linear and then these properties become:

(i) $A(a, b) = -S(Ja, b)$
(ii) $S(a, b) = S(b, a)$
(iii) $A(a, b) = -A(b, a)$

It follows that both S and A are J-invariant ($S(Ja, Jb) = S(a, b)$, $A(Ja, Jb) = A(a, b)$). Conversely, given a J-invariant symmetric form $S : V \times V \to \mathbb{R}$, we can define a Hermitian form H on V by $H(a, b) := S(a, b) - iS(Ja, b)$ and similarly, a J-invariant anti-symmetric form A defines a Hermitian form on V via $H(a, b) := A(Ja, b) + iA(a, b)$.

We may regard A as an element of $\bigwedge^2 V^*$ via the projection $V^* \otimes V^* \to V^* \wedge V^*$ $(\alpha \otimes \beta \mapsto \alpha \wedge \beta)$ [8], and $\bigwedge^2 V^*$ as a subspace of $\bigwedge^2_{\mathbb{C}} V^*_{\mathbb{C}}$. The J-invariance of A then means that $A \in (V^*)^{1,1}$. We see that it is equivalent to give

(i) a Hermitian form on V
(ii) a J-invariant symmetric form on V
(iii) a J-invariant anti-symmetric form on V
(iv) an element of $\bigwedge^2 V^*$ of type (1,1) (i.e. an element of $(V^*)^{1,1} \cap \bigwedge^2 V^*$)

Now suppose the H is positive definite, so $H(a,a) > 0$ if $a \neq 0$ (this is equivalent to S being positive definite). Then there exists an orthonormal basis (for S) of V of the form $\{e_1, Je_1, \ldots, e_n, Je_n\}$. Let $\{x_1, y_1, \ldots, x_n, y_n\}$ be the corresponding dual basis and let $\{z_\nu = x_\nu + iy_\nu\}_{\nu=1,\ldots,n}$ be the associated basis of $(V^*)^{1,0}$. It is easy to see that H is given by

$$H(a,b) = \sum_{k=1}^{n} z_k(a)\overline{z_k(b)}$$

and hence

$$A(a,b) = \frac{1}{2i}\sum_{k=1}^{n} z_k(a)\overline{z_k(b)} - \overline{z_k(a)}z_k(b) = \frac{1}{i}\sum_{k=1}^{n} z_k \wedge \bar{z}_k(a,b)$$

since the images of $z_k \otimes \bar{z}_k$ and $-\bar{z}_k \otimes z_k$ in $\bigwedge^2_{\mathbb{C}} V^*_{\mathbb{C}}$ are the same. This shows once more that A is of type $(1,1)$. In terms of the x's and the y's, this 2-form becomes

$$A = \frac{1}{i}\sum_{k=1}^{n}(x_k + iy_k) \wedge (x_k - iy_k) = -2\sum_{k=1}^{n} x_k \wedge y_k$$

We will later consider $\Omega := -\frac{1}{2}A = \sum_{k=1}^{n} x_k \wedge y_k$.

We transport the Hermitian form to $V^*_{\mathbb{C}}$ as follows. The real part S of H is a positive definite inner product on V which is invariant under J and hence defines a positive definite inner product on V^*, which is again invariant under J. We extend this inner product to a Hermitian form on $V^*_{\mathbb{C}}$ via

$$\langle \alpha + i\beta, \gamma + i\delta \rangle = (S(\alpha,\gamma) + S(\beta,\delta)) + i(S(\beta,\gamma) - S(\alpha,\delta))$$

(Note that this is the only possible choice!). The orthonormal basis $\{x_1, y_1, \ldots, x_n, y_n\}$ of V^* for S is also orthonormal for $\langle\ ,\ \rangle$. Another orthonormal basis for $\langle\ ,\ \rangle$ is

$$\left\{ \tfrac{1}{\sqrt{2}} z_1, \ldots, \tfrac{1}{\sqrt{2}} z_n, \tfrac{1}{\sqrt{2}} \bar{z}_1, \ldots, \tfrac{1}{\sqrt{2}} \bar{z}_n \right\}$$

[8] This gives in fact an isomorphism between $\bigwedge^2 V^*$ and the space of anti-symmetric forms on V

so we see that $(V^*)^{1,0}$ and $(V^*)^{0,1}$ are perpendicular. We can apply the same construction to $\bigwedge^\bullet V^*$ and define a Hermitian form $\langle\ ,\ \rangle$ on $\bigwedge^\bullet V_{\mathbb{C}}^*$ which has as orthonormal basis

$$\left\{\tfrac{1}{\sqrt{2}}z_{i_1}\wedge\cdots\wedge\tfrac{1}{\sqrt{2}}z_{i_p}\wedge\tfrac{1}{\sqrt{2}}\bar{z}_{j_1}\wedge\cdots\wedge\tfrac{1}{\sqrt{2}}\bar{z}_{j_q}\right\}_{\underline{i},\underline{j}}$$

and we see that all the summands $(V^*)^{p,q}$ of $\bigwedge^\bullet V_{\mathbb{C}}^*$ are mutually perpendicular with respect to $\langle\ ,\ \rangle$.

We can also extend our star operator $* : \bigwedge^r V^* \to \bigwedge^{2n-r} V^*$, which is defined by S, to $\bigwedge^\bullet V_{\mathbb{C}}^*$ in an anti-linear[9] way:

$$*(\alpha+i\beta) = *\alpha - i*\beta$$

We want to know the effect of $*$ on the above basis. Hereto we first consider the one-dimensional case: we have $\{x,y\}$ as an oriented orthonormal basis, and $*x = y$, $*y = -x$ so we get

$$\begin{aligned}
*1 &= x\wedge y = \tfrac{1}{-2i}z\wedge\bar{z}\\
*z &= *x - i*y = y + ix = i(x-iy) = i\bar{z}\\
*\bar{z} &= *x + i*y = y - ix = -i(x+iy) = -iz\\
*(\tfrac{1}{2i}z\wedge\bar{z}) &= *(x\wedge y) = -1
\end{aligned}$$

In order to generalize this, we first adapt some notation:

$$\omega_\nu^{0,0} := 1,\quad \omega_\nu^{1,0} := z_\nu,\quad \omega_\nu^{0,1} := \bar{z}_\nu,\quad \omega_\nu^{1,1} := \tfrac{1}{2i}z_\nu\wedge\bar{z}_\nu\ (= x_\nu\wedge y_\nu)$$

Then every element of the above basis can be uniquely written, up to a sign and multiplication by some constant, as

$$\omega_1^{p_1,q_1}\wedge\cdots\wedge\omega_n^{p_n,q_n}$$

If we denote by $*_\nu$ the 1-dimensional star operator in the ν-th coordinate, then an easy computation shows that

$$*(\omega_1^{p_1,q_1}\wedge\cdots\wedge\omega_n^{p_n,q_n}) = \pm(*_1\omega_1^{p_1,q_1})\wedge\cdots\wedge(*_n\omega_n^{p_n,q_n})$$

where the sign is in fact

$$(-1)^{\sum_{j<k}(p_j+q_j)(p_k+q_k)}$$

We therefore see

[9] some authors extend $*$ in a linear way, but this appears to be more natural.

Proposition 8.1. *$*$ defines an isomorphism of $\bigwedge^{p,q} V_{\mathbb{C}}^*$ onto $\bigwedge^{n-q,n-p} V_{\mathbb{C}}^*$.*

Let us return to a smooth manifold M of dimension m, which we from now on assume to be *compact*. We suppose that M is oriented (that is, every tangent space T_pM has been oriented and this orientation is locally constant). We have a bilinear map

$$(\,|\,) : \mathcal{E}^r(M) \times \mathcal{E}^{m-r}(M) \to \mathbb{R}$$
$$(\alpha|\beta) := \int_M \alpha \wedge \beta$$

(here we already need M to be oriented!). We extend this map to $\mathcal{E}^\bullet(M) \times \mathcal{E}^\bullet(M)$ by putting it zero if the degrees do not add up to m. If $\alpha \in \mathcal{E}^r(M)$ and $\beta \in \mathcal{E}^{m-r-1}(M)$ then we have

$$\begin{aligned}
(\mathrm{d}\alpha|\beta) &= \int_M \mathrm{d}\alpha \wedge \beta \\
&= \int_M \{\mathrm{d}(\alpha\wedge\beta) - (-1)^r \alpha \wedge \mathrm{d}\beta\} \\
&= (-1)^{r+1} \int_M \alpha \wedge \mathrm{d}\beta \quad \text{(by Stokes' theorem)} \\
&= (-1)^{r+1} (\alpha|\mathrm{d}\beta)
\end{aligned}$$

From this it follows that if α and β are closed forms, then $(\alpha|\beta)$ only depends on their cohomology classes, so that $(\,|\,)$ induces a pairing

$$(\,|\,) : H^r(M,\mathbb{R}) \times H^{m-r}(M,\mathbb{R}) \to \mathbb{R}$$

This is a so-called **perfect pairing** in the sense that the involved vector spaces are finite dimensional and that

$$H^r(M,\mathbb{R}) \to H^{m-r}(M,\mathbb{R})^*, \quad \alpha \mapsto (\alpha| \cdot)$$

is an isomorphism. This is in fact a topological result (a consequence of **Poincaré duality**), but will shall give an analytic proof below.

Give M a Riemannian metric S. Since M is oriented, we can extend the local operator $*$ to get a sheaf homomorphism $* : \mathcal{E}^r_M \to \mathcal{E}^{m-r}_M$. The element $*1 \in \mathcal{E}^m(M)$ is called the **volume element** of M, it will be denoted by $\mathrm{d}\mu$ [10]. We

[10] This notation is standard but confusing: in general $\mathrm{d}\mu$ need not to be exact. In fact: if M is compact it is not, by Stokes theorem

can extend S in a natural way to $\mathcal{E}^r(M)$ and we get for $\alpha, \beta \in \mathcal{E}^r(M)$: $\alpha \wedge *\beta = S(\alpha, \beta)\mathrm{d}\mu$. We define

$$\langle \alpha, \beta \rangle := (\alpha | *\beta) = \int_M \alpha \wedge *\beta = \int_M S(\alpha, \beta)\mathrm{d}\mu$$

which is $\mathbb{R}$-bilinear. The latter equality shows that $\langle \alpha, \alpha \rangle > 0$ unless $\alpha = 0$, so that $\langle\ ,\ \rangle$ is a positive definite inner product on $\mathcal{E}^\bullet(M)$.

Lemma 8.2. *Let* $\mathrm{d}^* : \mathcal{E}^r_M \to \mathcal{E}^{r-1}_M$ *be defined by* $\mathrm{d}^* := (-1)^r *^{-1} \mathrm{d} *$. *Then for* $\alpha, \beta \in \mathcal{E}^\bullet(M)$ *we have* $\langle \mathrm{d}\alpha, \beta \rangle = \langle \alpha, \mathrm{d}^*\beta \rangle$. *(We say that* d^* *is a* **formal adjoint** *of* d.*)*

PROOF. Let $\alpha \in \mathcal{E}^r(M)$, $\beta \in \mathcal{E}^{r+1}(M)$. Then $\langle \mathrm{d}\alpha, \beta \rangle = (\mathrm{d}\alpha | *\beta) = (-1)^{r+1}(\alpha | \mathrm{d} * \beta) = (-1)^{r+1}\langle \alpha, *^{-1}\mathrm{d} * \beta \rangle = \langle \alpha, \mathrm{d}^*\beta \rangle$. ☐

Definition. We define the **Laplace operator** to be $\Delta := \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d} : \mathcal{E}^r_M \to \mathcal{E}^r_M$.

Remark 8.2.1. If we take $M = \mathbb{R}^m$ with the standard metric $(\mathrm{d}x_1)^2 + \cdots + (\mathrm{d}x_m)^2$, then even though M is not compact, we can define d^* and Δ. Now if $f : M \to \mathbb{R}$ is a smooth function, then an easy exercise shows that $\Delta(f) = -\sum_{\nu=1}^m \frac{\partial^2 f}{\partial x_\nu^2}$, which explains the name of the Laplace operator.

The following lemma gives an important property of the Laplace operator:

Lemma 8.3. *For* $\alpha \in \mathcal{E}^r(M)$*, the following are equivalent*

(i) $\mathrm{d}\alpha = 0$ *and* $\langle \alpha, \mathrm{d}\beta \rangle = 0$ *for all* $\beta \in \mathcal{E}^{r-1}(M)$.
(ii) $\mathrm{d}\alpha = \mathrm{d}^*\alpha = 0$
(iii) $\Delta\alpha = 0$

Moreover, Δ *commutes with both* d *and* $*$*, hence also with* d^*.

PROOF. (i) $\Longleftrightarrow$(ii) follows from the above discussion, while (ii) $\Longrightarrow$(iii) is trivial. If $\Delta\alpha = 0$, then

$$0 = \langle \alpha, \Delta\alpha \rangle = \langle \alpha, \mathrm{d}^*\mathrm{d}\alpha \rangle + \langle \alpha, \mathrm{d}\mathrm{d}^*\alpha \rangle = \langle \mathrm{d}\alpha, \mathrm{d}\alpha \rangle + \langle \mathrm{d}^*\alpha, \mathrm{d}^*\alpha \rangle$$

and since $\langle\ ,\ \rangle$ is positive definite, this implies $\mathrm{d}\alpha = \mathrm{d}^*\alpha = 0$. The second statement is trivial from the definition of Δ. ☐

We call a global r-form $\alpha \in \mathcal{E}^r(M)$ **harmonic** if it satisfies $\Delta\alpha = 0$. The harmonic r-forms form a linear subspace $\mathcal{H}^r(M)$ of $\mathcal{E}^r(M)$. According to the lemma, this is just the orthogonal complement relative to $\langle\ ,\ \rangle$ of the exact forms in the space of closed forms. In particular we have an injection

$$\mathcal{H}^r(M) \to H^r(\mathcal{E}^\bullet(M), \mathrm{d}) \cong H^r(M, \mathbb{R}) \ , \quad \alpha \mapsto [\alpha]$$

We have the following important but highly nontrivial theorem:

Theorem 8.4 (Hodge-Kodaira). *$\mathcal{H}^r(M)$ is a finite dimensional vector space and $\mathcal{H}^r(M) \to H^r(M,\mathbb{R})$ is an isomorphism.*

Proof. See Wells or Griffiths-Harris. □

Corollary 8.5 (Poincaré duality). *The pairing*

$$(\ |\ ) : H^r(M,\mathbb{R}) \times H^{m-r}(M,\mathbb{R}) \to \mathbb{R}$$

is perfect.

Proof. From the lemmas it follows that $*$ defines an isomorphism of $\mathcal{H}^r(M) \xrightarrow{\sim} \mathcal{H}^{m-r}(M)$, so $([\alpha]|[*\alpha]) = (\alpha|*\alpha) = \langle \alpha, \alpha\rangle \neq 0$ if $\alpha \neq 0$. Therefore, if $([\alpha]|[\beta]) = 0$ for all $[\beta] \neq 0$, then $[\alpha]$ must be 0, showing that the pairing is perfect. □

For the remainder of this section, we assume M to be a compact *complex* manifold of complex dimension n, while η is a holomorphic vector bundle over M of rank k. We will discuss the analogue of the above for the (Dolbeault) cohomology groups of η.

If we denote by η^* the holomorphic dual bundle of η, we can define a sheaf homomorphism

$$\wedge : \mathcal{E}^{p,q}(\eta) \otimes \mathcal{E}^{n-p,n-q}(\eta^*) \to \mathcal{E}^{n,n}_M\ , \quad (\alpha \otimes s) \otimes (\beta \otimes \sigma) \mapsto \sigma(s) \cdot \alpha \wedge \beta$$

Lemma 8.6. *If $\omega \in \mathcal{E}^{p,q}(\eta)(U)$ and $\zeta \in \mathcal{E}^{n-p,n-q-1}(\eta^*)(U)$ for some open $U \subseteq M$ then $\mathrm{d}(\omega \wedge \zeta) = \bar\partial\omega \wedge \zeta + (-1)^{p+q}\omega \wedge \bar\partial\zeta$.*

Proof. Write $\omega = \sum \alpha_j \otimes s_j$ and $\zeta = \sum \beta_k \otimes \sigma_k$. Then

$$\begin{aligned}
\mathrm{d}(\omega \wedge \zeta) &= \bar\partial(\omega \wedge \zeta) \quad (\text{since } \omega \wedge \zeta \in \mathcal{E}^{n,n-1}_M) \\
&= \bar\partial\left[\sum_{j,k} \sigma_k(s_j)\alpha_j \wedge \beta_k\right] \\
&= \sum_{j,k} \sigma_k(s_j)\bar\partial(\alpha_j \wedge \beta_k) \\
&= \sum_{j,k} \sigma_k(s_j)\left\{\bar\partial\alpha_j \wedge \beta_k + (-1)^{p+q}\alpha_j \wedge \bar\partial\beta_k\right\} \\
&= \bar\partial\omega \wedge \zeta + (-1)^{p+q}\omega \wedge \bar\partial\zeta
\end{aligned}$$

□

Now define

$$(\mid) : \mathcal{E}^{p,q}(\eta)(M) \times \mathcal{E}^{n-p,n-q}(\eta^*)(M) \to \mathbb{C}$$

$$(\omega|\zeta) := \int_M \omega \wedge \zeta$$

which is $\mathbb{C}$-bilinear. We can extend it to a bilinear form on $\mathcal{E}^{\bullet,\bullet}(\eta)(M) \times \mathcal{E}^{\bullet,\bullet}(\eta^*)(M)$ by putting it zero if the bidegrees do not add up to (n, n). From the lemma and Stokes' theorem, it follows that $(\mid)$ induces a pairing on the Dolbeault cohomology groups

$$(\mid) : H^{p,q}(M, \eta) \times H^{n-p,n-q}(M, \eta^*) \to \mathbb{C}$$

(where $H^{p,q}(M, \eta) = H^q(\mathcal{E}^{p,\bullet}(M, \eta), \bar\partial) \cong H^q(M, \Omega^p_M \otimes \eta))$. Just as in the smooth case, we will show that this is a perfect pairing of finite dimensional vector spaces.

Let us be given smooth Hermitian metrics H_M and H_η on the bundles T_M resp. η.[11]. The Hermitian form H_η on η defines an anti-linear homomorphism $u_H : \eta \to \eta^*$, $s \mapsto H(\cdot, s)$, which is smooth but not necessarily holomorphic. Together with the anti-linear star operator on $\mathcal{E}^{\bullet,\bullet}$, this defines an anti-linear operator

$$\mathcal{E}^{p,q}(\eta) \to \mathcal{E}^{n-p,n-q}(\eta^*)$$

$$\sum_j \alpha_j \otimes s_j \mapsto \sum_j (*\alpha_j) \otimes u_H(s_j)$$

which we will also denote by $*$.

Now define

$$\langle\ ,\ \rangle : \mathcal{E}^{\bullet,\bullet}(\eta)(M) \times \mathcal{E}^{\bullet,\bullet}(\eta)(M) \to \mathbb{C}$$

$$\langle \alpha, \beta\rangle := (\alpha| *\beta)$$

One can easily check that $\langle\ ,\ \rangle$ is $\mathbb{C}$-linear in the first variable, that $\langle \alpha, \beta\rangle = \overline{\langle \beta, \alpha\rangle}$ and that $\langle \alpha, \alpha\rangle > 0$ unless $\alpha = 0$. So we see that $\langle \alpha, \beta\rangle$ is a positive definite Hermitian form. This makes $\mathcal{E}^{\bullet,\bullet}(\eta)(M)$ a so-called **pre-Hilbert space**[12].

We now can proceed as in the real case and define $\bar\partial^* := (-1)^{p+q} *^{-1} \bar\partial * : \mathcal{E}^{p,q}(\eta) \to \mathcal{E}^{p,q-1}(\eta)$. Then we again have that $\langle \bar\partial\omega, \zeta\rangle = \langle \omega, \bar\partial^*\zeta\rangle$. We further define the **Laplace $\bar\partial$-operator** $\Delta_{\bar\partial}$ to be

$$\Delta_{\bar\partial} := \bar\partial\bar\partial^* + \bar\partial^*\bar\partial : \mathcal{E}^{p,q}(\eta) \to \mathcal{E}^{p,q}(\eta)$$

and just as above we see that the following are equivalent:

[11] A **smooth Hermitian form** on a smooth bundle ξ is simply a positive definite Hermitian form in every fiber, which varies smoothly with the base point. Using a partition of the unity, one can show that such forms always exist.

[12] that is, a complex vector space with a positive definite Hermitian form on it.

(i) $\bar{\partial}\alpha = 0$ and $\langle \alpha, \bar{\partial}\beta \rangle = 0$ for all $\beta \in \mathcal{E}^{p,q-1}(\eta)(M)$.
(ii) $\bar{\partial}\alpha = \bar{\partial}^*\alpha = 0$
(iii) $\Delta_{\bar{\partial}}\alpha$=0

The subspace of $\mathcal{E}^{p,q}(\eta)(M)$ consisting of the forms α with $\Delta_{\bar{\partial}}\alpha = 0$ is denoted by $\mathcal{H}^{p,q}(M, \eta)$. We have the following theorem which follows from the same general result as the Hodge-Kodaira theorem:

Theorem 8.7. *The space $\mathcal{H}^{p,q}(M, \eta)$ is finite-dimensional and the natural map $\mathcal{H}^{p,q}(M, \eta) \to H^{p,q}(M, \eta)$ is an isomorphism.*

Corollary 8.8 (Serre Duality). $\dim H^{p,q}(M, \eta) < \infty$ *and the pairing*

$$(\mid) : H^{p,q}(M, \eta) \times H^{n-p,n-q}(M, \eta^*) \to \mathbb{C}$$

is perfect.

Proof. Similar as the proof of 8.5. □

9 The exterior algebra of a Hermitian vector space

Just as before, we start with some linear algebra. Let V be again some real vector space of dimension $m = 2n$ with complex structure $J : V \to V$ and a positive definite Hermitian form H on it. Let $\{x_1, y_1, \ldots, x_n, y_n\}$ be an orthonormal basis of V^* as in the previous section (so we have $Jx_\nu = y_\nu$). We saw there that $H(a, b) = \sum_{\nu=1}^n z_\nu(a)\overline{z_\nu(b)}$, where $z_\nu = x_\nu + iy_\nu$ and that the imaginary part A of H corresponds to the 2-form $\sum_{\nu=1}^n z_\nu \wedge \bar{z}_\nu$. We define

$$\Omega = -\frac{1}{2}A = \sum_{\nu=1}^n x_\nu \wedge y_\nu = \sum_{\nu=1}^n \omega_\nu^{1,1}$$

and we notice that

$$\Omega^n = \underbrace{\Omega \wedge \cdots \wedge \Omega}_{n} = n! \cdot x_1 \wedge y_1 \wedge \cdots x_n \wedge y_n = n! \cdot \mu$$

We define an operator L on $\bigwedge^\bullet V_\mathbb{C}^*$ by

$$L(\omega) := \Omega \wedge \omega$$

(so L maps r-forms to $(r + 2)$-forms). Recall from the previous section that we have a natural Hermitian form $\langle\ ,\ \rangle$ on $\bigwedge^\bullet V_\mathbb{C}^*$ and an anti-linear star operator $* : \bigwedge^r V_\mathbb{C}^* \to \bigwedge^{2n-r} V_\mathbb{C}^*$ such that

$$\langle \alpha, \beta \rangle \mu = \alpha \wedge *\beta$$

Let L^* be the adjoint of L with respect to $\langle\ ,\ \rangle$, which means by definition that for all α, β

$$\langle L\alpha, \beta\rangle = \langle \alpha, L^*\beta\rangle$$

Since $\langle L\alpha, \beta\rangle\mu = \Omega\wedge\alpha\wedge *\beta = \alpha\wedge\Omega\wedge *\beta = \langle\alpha, *^{-1}L(*\beta)\rangle\mu$ it follows that in fact $L^* = *^{-1}L*$. Note that L^* maps r-forms to $(r-2)$-forms.

Proposition 9.1. *$L^*L - LL^*$ acts on $\bigwedge^r V_{\mathbb{C}}^*$ as scalar multiplication by $n-r$.*

PROOF. Let ω be a basis element of $\bigwedge^r V_{\mathbb{C}}^*$ of the form $\omega = \omega_1^{p_1,q_1}\wedge\cdots\wedge\omega_n^{p_n,q_n}$. Then

$$\begin{aligned}
L^*L(\omega) &= *^{-1}L*(\Omega\wedge\omega)\\
&= *^{-1}L*\left(\sum_{\nu:(p_\nu,q_\nu)=(0,0)} \omega_1^{p_1,q_1}\wedge\cdots\wedge\omega_\nu^{1,1}\wedge\cdots\wedge\omega_n^{p_n,q_n}\right)\\
&= *^{-1}L\left(\sum_{\nu:(p_\nu,q_\nu)=(0,0)} \pm(*_1\omega_1^{p_1,q_1})\wedge\cdots\wedge\omega_\nu^{0,0}\wedge\cdots\wedge(*_n\omega_n^{p_n,q_n})\right)\\
&= *^{-1}\left(\sum_{\nu:(p_\nu,q_\nu)=(0,0)} \pm(*_1\omega_1^{p_1,q_1})\wedge\cdots\wedge\omega_\nu^{1,1}\wedge\cdots\wedge(*_n\omega_n^{p_n,q_n})\right)\\
&= \sum_{\nu:(p_\nu,q_\nu)=(0,0)} \omega_1^{p_1,q_1}\wedge\cdots\wedge\omega_\nu^{0,0}\wedge\cdots\wedge\omega_n^{p_n,q_n} = n_{0,0}\cdot\omega
\end{aligned}$$

where $n_{0,0} = \#\{\nu : (p_\nu, q_\nu) = (0,0)\}$. Similarly, if $n_{1,1} = \#\{\nu : (p_\nu, q_\nu) = (1,1)\}$, then $LL^*(\omega) = n_{1,1}\cdot\omega$. So $(L^*L - LL^*)(\omega) = (n_{0,0} - n_{1,1})\omega$.

We finally observe that $n_{0,0} - n_{1,1} = \sum_{\nu=1}^n (1 - p_\nu - q_\nu)$ (this is far more easy than it appears to be!), so that $n_{0,0} - n_{1,1} = n -$ total degree of $\omega = n - r$. □

Let $B : \bigwedge^\bullet V_{\mathbb{C}}^* \to \bigwedge^\bullet V_{\mathbb{C}}^*$ be defined as multiplication by $n-r$ in degree r. The preceding proposition then says

$$[L^*, L] = B$$

and the fact that L increases the degree by 2 and L^* decreases it by 2 can be expressed as:

$$\begin{aligned}
[B, L] &= -2L\\
[B, L^*] &= 2L^*
\end{aligned}$$

so the $\mathbb{C}$-linear span of these three operators within $\mathrm{End}(\bigwedge^\bullet V_\mathbb{C}^*)$ is closed under $[\ ,\ ]$, which makes it a subalgebra of the Lie algebra $\mathrm{End}(\bigwedge^\bullet V_\mathbb{C}^*)$. This subalgebra is isomorphic[13] to the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ of complex 2×2 matrices with zero trace. The isomorphism is in fact given by

$$\begin{aligned} L^* &\mapsto X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ L &\mapsto Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ B &\mapsto H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

It can be checked easily that $[X,Y]=H$, $[H,X]=2X$ and $[H,Y]=-2Y$.

We will now discuss some representation theory, and since we are only interested in the case of $\mathfrak{sl}_2(\mathbb{C})$ we will restrict to this special case, of course there is a far more general theory behind this.

Definition. A **representation** of $\mathfrak{sl}_2(\mathbb{C})$ is a finite dimensional complex vector space W with a homomorphism of Lie algebras $\mathfrak{sl}_2(\mathbb{C}) \to \mathrm{End}(W)$. In other words, we have three operators X, Y and H in W satisfying

$$[X,Y]=H, [H,X]=2X, [H,Y]=-2Y$$

An example of a representation is of course the case $W=\mathbb{C}^2$ with ordinary matrix multiplication. This is called the **tautological representation**. The standard operations on vector spaces (such as $\otimes$, $\oplus$, Hom, Sym^k, dual) can be used to make new representations from existing ones. For example, if W_1 and W_2 are representations of $\mathfrak{sl}_2(\mathbb{C})$, then $W_1\otimes W_2$ is a representation defined by sending $\sigma \in \mathfrak{sl}_2(\mathbb{C})$ to the endomorphism:

$$x\otimes y \mapsto \sigma(x)\otimes 1 + 1\otimes \sigma(y)$$

We are especially interested in symmetric powers of the tautological representation. Note that we have

$$\mathrm{Sym}^k\,\mathbb{C}^2 \cong \mathbb{C}\text{-span of } \{e_1^j e_2^{k-j}\}_{j=1}^k \ (= \text{polynomials of degree } k \text{ in } e_1 \text{ and } e_2)$$

[13]isomorphic as Lie algebras, which means an isomorphism of vector spaces, leaving the Lie bracket invariant. So we will disregard multiplication that we may have: H^2 is multiplication by $(n-r)^2$ in degree r, while B^2 is the identity

We define the representation $\mathrm{Sym}_k \mathbb{C}^2$ by sending $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathfrak{sl}_2(\mathbb{C})$ to the endomorphism which sends $e_1^j e_2^{k-j}$ to

$$\tfrac{b}{j} \cdot e_1^{j-1} e_2^{k-j+1} + \tfrac{a}{j(k-j)} \cdot e_1^j e_2^{k-j} + \tfrac{c}{k-j} \cdot e_1^{j+1} e_2^{k-j-1}$$

where $e_1^{-1} = e_2^{-1} = 0$. The case $k = 0$ gives the representation which sends every $\sigma \in \mathfrak{sl}_2(\mathbb{C})$ to $0 \in \mathrm{End}(\mathbb{C})$. This is called the **trivial representation**.

It can be shown that the action of H on W is diagonalizable. If you do not want to belief this: it is trivially true in our cases, so you may assume we restrict to the cases in which it is true. If we denote by $W_\lambda = W_\lambda(H) \subseteq W$ the λ-eigenspace of H within W, then this yields $W = \bigoplus_\lambda W_\lambda$ (of course, only finitely many summands are $\neq 0$).

We need some technical but not really deep lemmas:

Lemma 9.2. *X maps W_λ to $W_{\lambda+2}$ and Y maps W_λ to $W_{\lambda-2}$.*

Proof. For $w \in W_\lambda$ we have $HX(w) = (XH + 2X)(w) = X(\lambda w) + 2X(w) = (\lambda + 2)X(w)$ and similarly $HY(w) = (\lambda - 2)Y(w)$. □

Definition. We call an eigenvector $v \in W_\lambda$ **primitive (of weight λ)** if $Xv = 0$. The space of primitive vectors of weight λ is denoted by P_λ.

Lemma 9.3. *If $v \in W_\lambda$ is nonzero then there is an integer $k \geq 0$ such that $X^k u$ is nonzero and primitive (of weight $\lambda + 2k$).*

Proof. The elements $X, Xv, X^2v, \ldots$ are eigenvectors of H with eigenvalues $\lambda, \lambda + 2, \lambda + 4, \ldots$ and hence are all different (as long as they are nonzero). Since W is finite dimensional, for some $k \geq 0$ we have $X^{k+1}v = 0$, while $X^k v \neq 0$. □

Lemma 9.4. *If $v \in P_\lambda$, then $XY^k(v) = k(\lambda + 1 - k)Y^{k-1}(v)$.*

Proof. Using induction on k. The case $k = 0$ is trivial. For $k > 0$ we have

$$\begin{aligned}
XY^k(v) &= XY \cdot Y^{k-1}(v) \\
&= (YX + H)Y^{k-1}(v) \\
&= Y \cdot XY^{k-1}(v) + HY^{k-1}(v) \\
&= Y \cdot (k-1)(\lambda + 1 - (k-1))Y^{k-2}(v) + (\lambda - 2(k-1))Y^{k-1}(v) \\
&= \{(k-1)(\lambda + 2 - k) + (\lambda - 2k + 2)\}Y^{k-1}(v) \\
&= k(\lambda + 1 - k)Y^{k-1}(v)
\end{aligned}$$

using induction and lemma 9.2. □

Corollary 9.5.

(i) *If* $W_\lambda \neq 0$, *then* $\lambda \in \mathbb{Z}$ *and if* $P_\lambda \neq 0$ *then* $\lambda \in \mathbb{N}$.
(ii) *If* $v \in P_\lambda - \{0\}$, *then* $Y^{\lambda+1}(v) = 0$ *and* $\{v, Y(v), \ldots, Y^\lambda(v)\}$ *is a basis of a subspace* $W(v) \subseteq W$ *which is invariant under the action of* $\mathfrak{sl}_2(\mathbb{C})$.
(iii) *If* $v \in P_\lambda$, *then* $\frac{X^k}{k!}\frac{Y^k}{k!}(v) = \binom{\lambda}{k} v$ *for* $0 \leq k \leq \lambda$.

Proof. If $v \in P_\lambda - \{0\}$, then $Y^l v \in W_{\lambda-2l}$ for $l \in \mathbb{N}$ by lemma 9.2. So for some $k \in \mathbb{N}$, we have $Y^k(v) \neq 0$ and $Y^{k+1}(v) = 0$. By lemma 9.4 we then have

$$0 = XY^{k+1}(v) = (k+1)(\lambda - k)Y^k(v)$$

and since both $k+1$ and $Y^k(v)$ are nonzero, we should have $\lambda = k \in \mathbb{N}$. This proves the second part of (i), while the first part then follows from this and lemma 9.3.

Now we also have proven the first part of (ii). For the second part, we mention that since $Y^l v \in W_{\lambda-2l}$, the Y^l for $l = 0, \ldots, \lambda$ are linearly independent and nonzero. From lemma 9.4 it follows that their span is invariant under X, Y and H, hence under $\mathfrak{sl}_2(\mathbb{C})$.

Finally, (iii) follows with induction from lemma 9.4. □

Proposition 9.6.

(i) W_λ *is the direct sum of the subspaces* $\{Y^k P_{\lambda+2k}\}_{k \geq \max\{0, -\frac{\lambda}{2}\}}$
(ii) $P_\lambda = \{v \in W_\lambda : Y^{\lambda+1} v = 0\}$ $(\lambda \geq 0)$.

Proof. We first show that $W_\lambda \subseteq \sum_{k\geq 0} Y^k P_{\lambda+2k}$ (the other inclusion is clear). Let $v \in W_\lambda$ and define the integer $k(v) \geq 0$ by the property $X^{k(v)}(v) \neq 0$, $X^{k(v)+1}(v) = 0$. With induction on $k(v)$ we prove that $v \in \sum_{l=0}^{k(v)} Y^l P_{\lambda+2l}$. If $k(v) = 0$ there is nothing to prove, so assume $k = k(v) \geq 1$. Then $X^k(v) \in P_{\lambda+2k}$. If we take $c := (k!)^{-2}\binom{\lambda+2k}{k}^{-1}$, then it follows from the third part of the corollary that

$$X^k(v - c \cdot Y^k X^k(v)) = (\mathbf{1} - c \cdot X^k Y^k)(X^k v) = 0$$

Since $v - c \cdot Y^k X^k(v) \in W_\lambda$ it follows by the induction hypothesis that this element is in $\sum_{l=0}^{k-1} Y^l P_{\lambda+2l}$. So $v \in Y^k P_{\lambda+2k} + \sum_{l=0}^{k-1} Y^l P_{\lambda+2l} \subseteq \sum_{l=0}^{k} Y^l P_{\lambda+2l}$. This completes the induction.

To prove that the sum is in fact a direct sum assume we have

$$0 = \sum_k Y^k v_k$$

where $v_k \in P_{\lambda+2k}$. Let l be the largest l such that $v_l \neq 0$. Then $0 = Y^{\lambda+l}0 = Y^{\lambda+2l}v_l$ by (ii) of the corollary (apply Y^k on both sides of the above equation) and applying (ii) again, it follows that $v_l = 0$.

For (ii), we remark that if $v \in P_\lambda$, then by (ii) of the corollary $Y^{\lambda+1}v = 0$. If on the other hand $v \in W_\lambda$ for some $\lambda \geq 0$, then by (i) we can write v uniquely as

$$v = v_\lambda + Y v_{\lambda+2} + Y^2 v_{\lambda+4} + \cdots \quad (v_{\lambda+2l} \in P_{\lambda+2l})$$

so if $Y^{\lambda+1}v = 0$, then $Y^{\lambda+2}v_{\lambda+2} + Y^{\lambda+3}v_{\lambda+4} + \cdots = 0$, which implies (since $Y^k|_{P_{\lambda+l}}$ is injective if $k \leq \lambda + l$) that $v_{\lambda+2} = v_{\lambda+4} = \ldots = 0$ and thus $v = v_\lambda \in P_\lambda$. □

Corollary 9.7.

(i) For $k = 0, 1, 2, \ldots$ we have that $W[k] := P_k \oplus Y P_k \oplus \cdots \oplus Y^k P_k$ is a $\mathfrak{sl}_2(\mathbb{C})$-invariant subspace of W and $W = \bigoplus_{k\geq 0} W[k]$ as $\mathfrak{sl}_2(\mathbb{C})$-representations.
(ii) Y^k maps W_k isomorphically onto W_{-k}.

PROOF. Using $W = \bigoplus_{\lambda\in\mathbb{Z}} W_\lambda$ and $W_\lambda = \bigoplus_{k\geq \max\{0,-\frac{\lambda}{2}\}} Y^k P_{\lambda+2k}$ we see by re-ordering the terms that $W[k] := P_k \oplus Y P_k \oplus \cdots \oplus Y^k P_k$. The fact that this is $\mathfrak{sl}_2(\mathbb{C})$-invariant is just an easy computation using lemma 9.4.

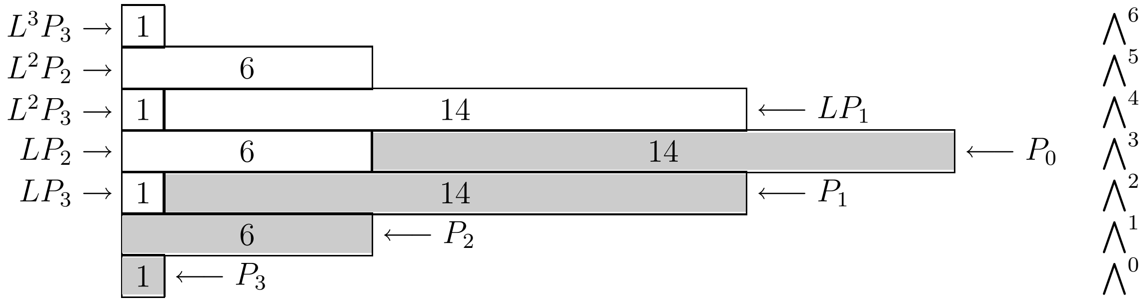
(ii) follows from the direct sum in (i) of the previous proposition: $Y^k(Y^l P_{k+2l}) = Y^{k+l} P_{-k+2(k+l)}$ and $Y^l|_{Y^k(Y^l P_{k+2l})}$ is injective as long as $l \geq 0$, which is always true. □

We now have developed enough tool for the $\mathfrak{sl}_2(\mathbb{C})$-action on $\bigwedge^\bullet V_\mathbb{C}^*$. Recall that X, Y and H correspond to L^*, L and B respectively. Using the above notation, we find (with $2n = \dim_\mathbb{R} V$):

Proposition 9.8.

- $(\bigwedge^\bullet V_\mathbb{C}^*)_\lambda = \bigwedge^{n-\lambda} V_\mathbb{C}^*$
- $P_\lambda := \{\alpha \in \bigwedge^{n-\lambda} V_\mathbb{C}^* : L^*\alpha = 0\} = \{\alpha \in \bigwedge^{n-\lambda} V_\mathbb{C}^* : L^{\lambda+1}\alpha = 0\}$
- $\bigwedge^\bullet V_\mathbb{C}^* = \bigoplus\{L^k P_\lambda;\ \lambda = 0, \ldots, n,\ 0 \leq k \leq \lambda\}$
- $L^k : \bigwedge^{n-k} V_\mathbb{C}^* \to \bigwedge^{n+k} V_\mathbb{C}^*$ *is an isomorphism.*

Below is a picture in which this is illustrated in the case $n = 3$. Here numbers indicate dimension.



Remark 9.8.1. If W is a $\mathfrak{sl}_2(\mathbb{C})$-representation, then the elements of W_λ are said to have **weight** λ. If we take $W = \bigwedge^\bullet V_\mathbb{C}^*$, then we have $W = \bigoplus_r \bigwedge^r V_\mathbb{C}^*$ and then elements of $\bigwedge^r V_\mathbb{C}^*$ are said to have **degree** r. So we see that weight λ corresponds to degree $n - \lambda$.

Since $P_\lambda \subseteq \bigwedge^{n-\lambda} V_\mathbb{C}^*$ we shall write $P^{n-\lambda}$ for this space. Note that $\bigwedge^\bullet V_\mathbb{C}^*$ is an $\mathbb{N}\times\mathbb{N}$-graded $\mathbb{C}$-algebra and that L and L^* are homogeneous operators (of bidegrees $(1,1)$ and $(-1,-1)$ respectively). This implies that if $\alpha \in P^{n-\lambda} = \mathrm{Ker}(L^*|_{\bigwedge^{n-\lambda} V_\mathbb{C}^*})$ then all the bihomogeneous components of α are also in this kernel, so that we have a decomposition

$$P^r = \bigoplus_{p+q=r} P^{p,q} \quad (0 \leq r \leq n)$$

As was mentioned before, the operator $J : V \to V$ induces an operator J^* on $\bigwedge^\bullet V_\mathbb{C}^*$ as follows:

$$\begin{aligned} J^*(z_{i_1} \wedge \cdots \wedge z_{i_p} \wedge \bar{z}_{j_1} \wedge \cdots \wedge \bar{z}_{j_q}) &= J^* z_{i_1} \wedge \cdots \wedge J^* z_{i_p} \wedge J^* \bar{z}_{j_1} \wedge \cdots \wedge J^* \bar{z}_{j_q} \\ &= i^{p-q} \cdot z_{i_1} \wedge \cdots \wedge z_{i_p} \wedge \bar{z}_{j_1} \wedge \cdots \wedge \bar{z}_{j_q} \end{aligned}$$

and we thus see that J^* is multiplication by i^{p-q} on $(V^*)^{p,q}$. This automorphism is called the **Weil operator** and therefore sometimes denoted by w. Since $(J^*)^2$ on $(V^*)^{p,q}$ is multiplication by $(-1)^{p-q} = (-1)^{p+q}$ to that $(J^*)^2 = **$.

Lemma 9.9. *If* $\alpha \in P^r (0 \leq r \leq n)$, *then* $*\frac{L^k}{k!}\alpha = (-1)^{\frac{1}{2}r(r+1)} \frac{L^{n-r-k}}{(n-r-k)!} J^* \bar{\alpha}$

Proof. The proof is just a long and boring (hence not so interesting) computation, so we refer to the book of Weil. □

If $\alpha, \beta \in P^r$, then by the lemma we have

$$\alpha \wedge *\beta = \alpha \wedge (-1)^{\frac{1}{2}r(r+1)} \frac{L^{n-r}}{(n-r!)} J^* \bar{\beta} = (-1)^{\frac{1}{2}r(r+1)} \frac{\Omega^{n-r}}{(n-r)!} \wedge \alpha \wedge J^* \bar{\beta}$$

Therefore we introduce a mapping $Q : P^r \times P^r \to \mathbb{C}$ defined by

$$Q(\alpha,\beta)\mu = (-1)^{\frac{1}{2}r(r+1)} \frac{\Omega^{n-r}}{(n-r)!} \wedge \alpha \wedge \beta$$

Note that we have

$$\langle \alpha, \beta \rangle = Q(\alpha, J^* \bar{\beta})$$

To conclude this section, let us state some properties of Q

Proposition 9.10.

(i) *Q is bilinear and $Q(\beta,\alpha) = (-1)^r Q(\alpha,\beta)$ where $\alpha,\beta \in P^r$.*
(ii) *The restriction of Q to $P^{p,q} \times P^{p',q'}$ is the zero mapping, unless $(p',q') = (q,p)$ and $Q : P^{p,q} \times P^{q,p} \to \mathbb{C}$ is a perfect pairing.*
(iii) *$(\alpha,\beta) \in P^r \times P^r \mapsto Q(\alpha, J^*\bar{\beta})$ is a positive definite Hermitian form on P^r.*

Proof. (i) follows immediately, using $(-1)^{r^2} = (-1)^r$. The first part of (ii) follows by looking at bidegrees, which should add up to (n,n). The second part of (ii) and (iii) follows by the above formulas and the properties of $\langle\ ,\ \rangle$. $\square$

10 Kähler manifolds

In this (final) section, we will apply the theory of the previous one to come to the definition of a Kähler manifold and its most important properties. So let M be complex manifold of complex dimension n, equipped with a Hermitian metric H. H defines the anti-linear star operator $* : \mathcal{E}_M^{p,q} \to \mathcal{E}_M^{n-p,n-q}$. Recall that we have a positive definite Hermitian form

$$\langle\ ,\ \rangle : \mathcal{E}_M^{\bullet,\bullet} \times \mathcal{E}_M^{\bullet,\bullet} \to \mathbb{C}$$

and we have formal adjoints

$$\begin{array}{rclcrcl} d^* &=& (-1)^r *^{-1} \mathrm{d}* : \mathcal{E}_M^r \to \mathcal{E}_M^{r-1} & \quad & \langle \mathrm{d}\alpha, \beta\rangle &=& \langle \alpha, \mathrm{d}^*\beta\rangle \\ \bar\partial^* &=& (-1)^{p+q} *^{-1} \bar\partial * : \mathcal{E}_M^{p,q} \to \mathcal{E}_M^{p,q-1} & & \langle \bar\partial\alpha, \beta\rangle &=& \langle \alpha, \bar\partial^*\beta\rangle \\ \partial^* &=& (-1)^{p+q} *^{-1} \partial * : \mathcal{E}_M^{p,q} \to \mathcal{E}_M^{p-1,q} & & \langle \partial\alpha, \beta\rangle &=& \langle \alpha, \partial^*\beta\rangle \end{array}$$

There is much analogy between these three cases and we will treat them all at once.

We have corresponding Laplacians

$$\begin{array}{rcl} \Delta_{\mathrm{d}} &=& \mathrm{dd}^* + \mathrm{d}^*\mathrm{d} \\ \Delta_{\bar\partial} &=& \bar\partial\bar\partial^* + \bar\partial^*\bar\partial \\ \Delta_{\partial} &=& \partial\partial^* + \partial^*\partial \end{array}$$

We have that $\overline{\Delta_\partial} = \Delta_{\bar\partial}$ and both Δ_∂ and $\Delta_{\bar\partial}$ respect the bidegrees (which Δ_{d} a priori does not in general).

Remark 10.0.1. Note that d, ∂ and $\bar\partial$ are defined independent of the metric H, but since the star operator is defined using H, we see that d^*, ∂^* and $\bar\partial^*$, and therefore also the corresponding Laplacians depend on this metric. It is important to be aware of this during the sequel.

Definition. Let M be as complex manifold (not nessecarily compact), equipped with a Hermitian metric on M and let Ω be the $(1,1)$-form associated to $-\frac{1}{2}\operatorname{Im}(H)$. We say that H is a **Kähler metric** if Ω is closed. M, or more precisely the pair (M,H), is called a **Kähler manifold**.

If M is a *compact* manifold with Kähler metric H, then the class $[\Omega] \in H^2(M,\mathbb{R})$ must be nontrivial: $[\Omega]^n = [\Omega\wedge\cdots\wedge\Omega] \in H^{2n}(M,\mathbb{R})$ is $n!$ times the volume element $\mathrm{d}\mu$ and since $\int_M \mathrm{d}\mu = \operatorname{vol}(M) > 0$, $\Omega\wedge\cdots\wedge\Omega$ cannot be exact by Stokes theorem. So $[\Omega]^n \neq 0$, hence $[\Omega] \neq 0$.

Remark 10.0.2. One can define a complex manifold $H_m(\lambda)$ (where $m \geq 0$, $\lambda \in \mathbb{C} - \{0\}$ $|\lambda| \neq 1$) by $\mathbb{C}^m - \{0\}/\sim$ where $z \sim z' \iff z = \lambda^k z$ for some $k \in \mathbb{Z}$. This is an example of a so-called **Hopf manifold** and it can be checked that $H_m(\lambda) \cong S^1 \times S^{2m-1}$ as real manifolds. Therefore $H^2(H_m(\lambda),\mathbb{R}) = H^2(S^1 \times S^{2m-1},\mathbb{R}) = 0$ by Künneths formula, hence $H_m(\lambda)$ does not admit a Kähler metric.

If N is a complex submanifold of a Hermitian manifold (M,H), then the restriction H_N of H to the tangent bundle of N is a Hermitian metric on N. The $(1,1)$-form defined by $-\frac{1}{2}\operatorname{Im}(H_N)$ on N is the restriction of the $(1,1)$-form defined by $-\frac{1}{2}\operatorname{Im}(H)$. So if (M,H) is a Kähler manifold, then so is (N,H_N).

Although we saw above that not every complex manifold admits a Kähler metric, we have quite some examples of Kähler manifolds:

(i) If M is a 1-dimensional complex manifold with an Hermitian metric H, then $\mathrm{d}\Omega$ is a 3-form which is therefore 0 and so H is a Kähler metric.

(ii) If V is a complex vector space (or more generally a real vector space with a complex structure), then it follows easily that the Fubini-Study metric, as is defined in proposition 10.1 below is a Kähler metric on $\mathbb{P}(V)$.

(iii) From this an the above it follows that every complex manifold that can be embedded into some complex projective space admits a Kähler metric. Furthermore, we see that Hopf manifold cannot be embedded into complex projective space.

Proposition-definition 10.1. *Let V be a real vector space with complex structure J and positive definite Hermitian form $H : V \times V \to \mathbb{C}$. Then $\mathbb{P}^V$ admits a Kähler metric, called the* **Fubini-Study** *metric.*

PROOF. Denote by $\pi : \mathbb{P}(V) - \{0\} \to V$ the natural projection. Let $U \subset \mathbb{P}(V)$ be an open set and $Z : U \to V - \{0\}$ a **lifting** of U, i.e. a holomorphic map with $\pi \circ Z = \mathbf{1}_U$. Consider the differential form

$$\omega = \frac{-1}{2\pi i}\partial\bar{\partial}\log H(Z, Z)$$

If $Z' : U \to V - \{0\}$ is another lifting, then $Z' = f \cdot Z$ for some nonzero holomorphic function f on U, so that

$$\begin{aligned}\frac{-1}{2\pi i}\partial\bar{\partial}\log H(Z', Z') &= \frac{-1}{2\pi i}\partial\bar{\partial}(\log H(Z, Z) + \log f + \log \bar{f}) \\ &= \omega + \frac{-1}{2\pi i}(\partial\bar{\partial}\log f - \bar{\partial}\partial\log\bar{f}) \\ &= \omega\end{aligned}$$

so that the definition of ω is independent of the chosen lifting. Since liftings always exist locally, ω defines a global (1,1)-form on $\mathbb{P}(V)$, which clearly satisfies $\mathrm{d}\omega = 0$. so it defines a Kähler metric. To see that ω is positive definite, note that the unitary group $U(n+1)$ acts transitively on $\mathbb{P}(V)$, leaving ω invariant, so it is positive definite everywhere if it is at one point.

Let U_0 be the open set $\{Z_0 \neq 0\}$ in $\mathbb{P}(V)$ and $\{z_j = Z_j/Z_0\}$ be coordinates on U_0. Use the lifting $Z = (1, z_1, \ldots, z_n)$ on U_0. We get

$$\begin{aligned}\omega &= \frac{-1}{2\pi i}\partial\bar{\partial}\log(1 + \sum_j z_j\bar{z}_j) \\ &= \frac{-1}{2\pi i}\partial\left(\frac{\sum_j z_j \mathrm{d}\bar{z}_j}{1 + \sum_j z_j\bar{z}_j}\right) \\ &= \frac{-1}{2\pi i}\left(\frac{\sum_j \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j}{1 + \sum_j z_j\bar{z}_j} - \frac{\left(\sum_j \bar{z}_j \mathrm{d}z_j\right) \wedge \left(\sum_j z_j \mathrm{d}\bar{z}_j\right)}{\left(1 + \sum_j z_j\bar{z}_j\right)^2}\right)\end{aligned}$$

so at the point $[1, 0, \ldots, 0]$ on $\mathbb{P}(V)$, we have

$$\omega = \frac{-1}{2\pi i}\sum_j \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$$

which is indeed positive definite. □

Now let (M, H) be a Kähler manifold. We define $L : \mathcal{E}_M^{p,q} \to \mathcal{E}_M^{p+1,q+1}$ to be $\alpha \mapsto \Omega \wedge \alpha$, and $L^* := *^{-1}L* : \mathcal{E}_M^{p,q} \to \mathcal{E}_M^{p-1,q-1}$, the formal adjoint to L with respect to $\langle\ ,\ \rangle$. Let for $0 \leq r \leq n$, $\mathcal{P}_M^{n-r} \subseteq \mathcal{E}_{\mathbb{C},M}^{n-r}$ be defined as the kernel of $L^{r+1} : \mathcal{E}_{\mathbb{C},M}^{n-r} \to \mathcal{E}_{\mathbb{C},M}^{n+r+2}$. From the results of the previous section it follows that

$$\bigoplus_{s=0}^{2n} \mathcal{E}_{\mathbb{C},M}^{s} = \bigoplus_{r=0}^{n} \bigoplus_{k=0}^{r} L^k \mathcal{P}_M^{n-r}$$

We need a rather technical proposition that has important consequences:

Proposition 10.2. *$[L, \partial^*] = i\bar{\partial}$ and $[L, \bar{\partial}^*] = -i\partial$*

Proof. It suffices to prove the first formula, as the second one is just the complex conjugate of the first one. Conjugating this formula with $*$ gives the equivalent formula

$$[L^*, \partial] = -i * \bar{\partial} * \quad (= i\bar{\partial}^*)$$

By the previous section it suffices to prove this on elements of the form $\alpha = L^k\beta$ with β primitive of degree $r \leq n$ and $k \leq n - r$. Then we have $L^{n-r+1}\beta = 0$ and so we have $0 = \partial L^{n-r+1}\beta = L^{n-r+1}\partial\beta$. Here we use that L and ∂ commute since $\mathrm{d}\Omega = 0$, hence also $\partial\Omega = 0$, and Ω has degree 2. Since $\partial\beta$ has degree $r + 1$, it follows from proposition 9.8 that $\partial\beta$ is of the form $\beta_0 + L\beta_1$, with β_0 and β_1 primitive of degrees $r + 1$ resp. $r - 1$ (use $L^{n-r+1}\partial\beta = 0$). We see

$$\begin{aligned} L^*\partial\alpha &= L^*L^k\partial\beta \\ &= L^*L^k(\beta_0 + L\beta_1) \\ &= k(n - (r + 1) - k + 1)L^{k-1}\beta_0 + (k + 1)(n - (r - 1) - (k + 1) + 1)L^k\beta_1 \\ &= k(n - r - k)L^{k-1}\beta_0 + (k + 1)(n - r - k + 1)L^k\beta_1 \end{aligned}$$

whereas

$$\begin{aligned} \partial L^*\alpha &= \partial L^*L^k\beta \\ &= \partial\left[k(n - r + 1 - k)L^{k-1}\beta\right] \\ &= k(n - r + 1 - k)L^{k-1}\partial\beta \\ &= k(n - r + 1 - k)(L^{k-1}\beta_0 + L^k\beta_1) \end{aligned}$$

using lemma 9.4 at both parts.

Combining these two results leads to

$$[L^*, \partial](\alpha) = -k \cdot L^{k-1}\beta_0 + (n - r + 1 - k)L^k\beta_1 \tag{1}$$

and it remains to show that the right hand side equals $-i * \bar{\partial} * \alpha$. Thereto we will make use of lemma 9.9.

Since L is of type $(1,1)$ and J^* equals multiplication with i^{p-q} in bidegree (p,q) we have that J^* and L commute and that $\bar\partial J^* = -iJ^*\bar\partial$. Since $*$ sends $\mathcal{E}_M^{p,q}$ to $\mathcal{E}_M^{n-p,n-q}$ we also have $J^* * = (-1)^r * J^*$ in degree r. So we get

$$\begin{aligned}
-i * \bar\partial * \alpha &= -i * \bar\partial * L^k\beta \\
&= -i * \bar\partial (-1)^{\frac{1}{2}r(r+1)} \frac{k!}{(n-r-k)!} L^{n-r-k} J^* \overline{\beta} \quad \text{(by lemma 9.9)} \\
&= (-1)^{\frac{1}{2}r(r+1)} \frac{k!}{(n-r-k)!} \cdot \left(-i * \bar\partial L^{n-r-k} J^* \overline{\beta}\right) \\
&= (-1)^{\frac{1}{2}r(r+1)} \frac{k!}{(n-r-k)!} \cdot \left((-1)^r J^* * L^{n-r-k} \overline{\partial\beta}\right) \\
&= (-1)^{\frac{1}{2}r(r+1)} \frac{k!}{(n-r-k)!} \cdot \left((-1)^r J^* \left(* L^{n-r-k}\overline{\beta_0} + * L^{n-r-k+1}\overline{\beta_1}\right)\right) \quad (2)
\end{aligned}$$

Applying lemma 9.9 again yields

$$\begin{aligned}
*L^{n-r-k}\overline{\beta_0} &= (-1)^{\frac{1}{2}(r+1)(r+2)} \cdot \frac{(n-r-k)!}{(k-1)!} L^{k-1} J^* \beta_0 \\
*L^{n-r-k+1}\overline{\beta_1} &= (-1)^{\frac{1}{2}(r-1)r} \cdot \frac{(n-r-k+1)!}{k!} L^{k} J^* \beta_1
\end{aligned}$$

and substituting these two equations into the last term of (2) gives

$$\begin{aligned}
-i * \bar\partial * \alpha &= (-1)^{r+1} k \cdot \left((-1)^r J^* L^{k-1} J^* \beta_0\right) + (-1)^r (n-r+1-k) \cdot \left((-1)^r J^* L^k J^* \beta_1\right) \\
&= -k \cdot L^{k-1} \cdot (J^*)^2 \beta_0 + (n-r+1-k) L^k \cdot (J^*)^2 \beta_1 \\
&= (-1)^{r+1} \cdot \left(-k \cdot L^{k-1}\beta_0 + (n-r+1-k) L^k \beta_1\right)
\end{aligned}$$

which indeed equals the right hand side of (1) (up to that $(-1)^{r+1}$ that I do not know how to get of...). ☐

Corollary 10.3. $\Delta_{\mathrm{d}} = 2\Delta_\partial = 2\Delta_{\bar\partial}$, *in particular* Δ_{d} *respects the bidegree.*

PROOF. We will first show that $\partial\partial^* + \partial^*\partial = 0$:

$$\begin{aligned}
\bar\partial\partial^* + \partial^*\bar\partial &= -i[L,\partial^*]\partial^* - \partial^* i[L,\partial^*] \quad \text{(by the proposition)} \\
&= -iL\partial^*\partial^* + i\partial^* L \partial^* - i\partial^* L\partial^* + i\partial^*\partial^* L \\
&= -i[L,\partial^*\partial^*] = 0
\end{aligned}$$

since $\partial^*\partial^* = \pm *^{-1} \partial * \cdot *^{-1} \partial * = \pm *^{-1} \partial^2 * = 0$.

Hence we have

$$\begin{aligned}\Delta_\partial &= \partial\partial^* + \partial^*\partial \\ &= i[L,\bar\partial^*]\partial^* + i\partial^*[L,\bar\partial^*] \quad \text{(again by the proposition)} \\ &= iL\bar\partial^*\partial^* - i\bar\partial^* L\partial^* + i\partial^* L\bar\partial^* - i\partial^*\bar\partial^* L\end{aligned}$$

Since $\bar\partial^*\partial^* = -\partial^*\bar\partial^*$, the right hand side of this equality does not change under complex conjugation (L is a real operator), so that $\Delta_\partial = \overline{\Delta_\partial} = \Delta_{\bar\partial}$.

We therefore get

$$\begin{aligned}\Delta_{\mathrm{d}} &= \mathrm{dd}^* + \mathrm{d}^*\mathrm{d} \\ &= (\partial+\bar\partial)(\partial^*+\bar\partial^*) + (\partial^*+\bar\partial^*)(\partial+\bar\partial) \\ &= \big(\partial\partial^* + \bar\partial\bar\partial^*\big) + \big(\partial^*\partial + \bar\partial\bar\partial^*\big) \\ &= \partial\partial^* + \partial^*\partial + \bar\partial\bar\partial^* + \bar\partial^*\bar\partial \\ &= \Delta_\partial + \Delta_{\bar\partial} = 2\Delta_\partial\end{aligned}$$

□

This corollary shows that, up to a constant 2, Δ_{d}, Δ_∂ and $\Delta_{\bar\partial}$ are the same operators. We will therefore write Δ instead of Δ_{d} as we already did before. Note that Δ, Δ_∂ and $\Delta_{\bar\partial}$ do depend on the Hermitian form H, so they depend on the Kähler metric.

Corollary 10.4. *L commutes with Δ_{d}*

Proof. We have

$$\begin{aligned}[L,\Delta_\partial] &= L\partial\partial^* + L\partial^*\partial - \partial\partial^* L - \partial^*\partial L \\ &= \partial L\partial^* + (\partial^* L + i\bar\partial)\partial - \partial(L\partial^* - i\bar\partial) - \partial^* L\partial \quad \text{(by the proposition)} \\ &= i(\bar\partial\partial + \partial\bar\partial) = 0\end{aligned}$$

which suffices since $2\Delta_\partial = \Delta$. □

We now have developed enough tools for some nice theory.

Suppose M is a compact Kähler manifold. Let $\mathcal{H}^r(M)$ denote the space of harmonic r-forms (that is the r-forms α satisfying $\Delta_\partial(\alpha) = 0$). By the Hodge-Kodaira theorem (8.4) we have an isomorphism $\mathcal{H}^r(M) \xrightarrow{\sim} H^r(M,\mathbb{R})$. Its complexification $\mathcal{H}^r(M)_{\mathbb{C}}$ is the kernel of Δ acting on $\oplus_{p+q=r}\mathcal{E}_M^{p,q}$ and by corollary 10.3 it is also the kernel of $\Delta_{\bar\partial}$ acting on this space. But $\Delta_{\bar\partial}$ sends $\mathcal{E}_M^{p,q}$ to itself and its kernel, which we will denote by $\mathcal{H}^{p,q}(M)$, maps isomorphically to $H^{p,q}(M) = H^q(M,\Omega_M^p)$. In particular we see that $\mathcal{H}^r(M)_{\mathbb{C}} = \oplus_{p+q=r}\mathcal{H}^{p,q}(M)$

According to corollary 10.4 Δ commutes with L and since Δ clearly commutes with $*$, we see that Δ commutes with $L^* = *^{-1}L*$. This implies that $H^\bullet(M,\mathbb{C}) \cong \mathcal{H}^\bullet(M)_\mathbb{C}$ is a representation of $\mathfrak{sl}_2(\mathbb{C})$ (via $X \mapsto L$, $Y \mapsto L^*$, $H \mapsto [L^*, L] =: B$). The primitive part in degree $n-r$ is just

$$P^{n-r}_\mathbb{C} = \operatorname{Ker}\left[L^{r+1} : \mathcal{H}^{n-r}(M)_\mathbb{C} \to \mathcal{H}^{n+r+2}(M)_\mathbb{C}\right]$$

Note that if $\alpha \in P^{n-r}$, then its value at every point x of M is an element of the primitive part $\bigwedge^\bullet T^*_{x,\mathbb{C}}M$, since L is defined 'pointwise'. We therefore get the main theorem for Kähler manifolds:

Theorem 10.5. *Let M be a compact Kähler manifold of complex dimension n and let $[\Omega] \in H^2(M,\mathbb{R})$ be the class of $-\frac{1}{2}$ times the imaginary part of the Kähler metric. Denote by $L : H^r(M,\mathbb{R}) \to H^{r+2}(M,\mathbb{R})$ taking the wedge product with $[\Omega]$ (so $L[\alpha] = [\Omega \wedge \alpha]$). Then the following hold*

(i) (Hodge decomposition) We have

$$H^k(M,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(M)$$

with $\overline{H^{p,q}(M)} = H^{q,p}(M)$ and the pairing

$$(\ |\ ) : H^{p,q}(M) \times H^{p',q'}(M)$$

being perfect if $(p',q') = (n-p, n-q)$ and zero otherwise.

(ii) (Lefschetz decomposition) Let for $(0 \le r \le n)$

$$P^{n-r} := \operatorname{Ker}[L^{r+1} : H^{n-r}(M,\mathbb{R}) \to H^{n+r+2}(M,\mathbb{R})]$$

The $\oplus_{k\ge 0} L^k$ maps $\oplus_{k \ge \max\{0, \frac{r-n}{2}\}} P^{r-2k}$ isomorphically onto $H^r(M,\mathbb{R})$.

(iii) (Hodge-Riemann bilinear relations) The decomposition in (i) induces one of $P^k_\mathbb{C}$: $P^k_\mathbb{C} = \sum_{p+q=k} P^{p,q}$ where $P^{p,q} = P^{p+q} \cap H^{p,q}(M)$ and if we define

$$Q : P^k \times P^k \to \mathbb{C}, \quad Q(\alpha,\beta) = (-1)^{\frac{1}{2}k(k+1)} \left(\frac{L^{n-k}}{(n-k)!}\alpha \middle| \beta \right)$$

then Q is bilinear and

(a) $Q(\alpha,\beta) = (-1)^k Q(\beta,\alpha)$ if $\alpha,\beta \in P^k$.

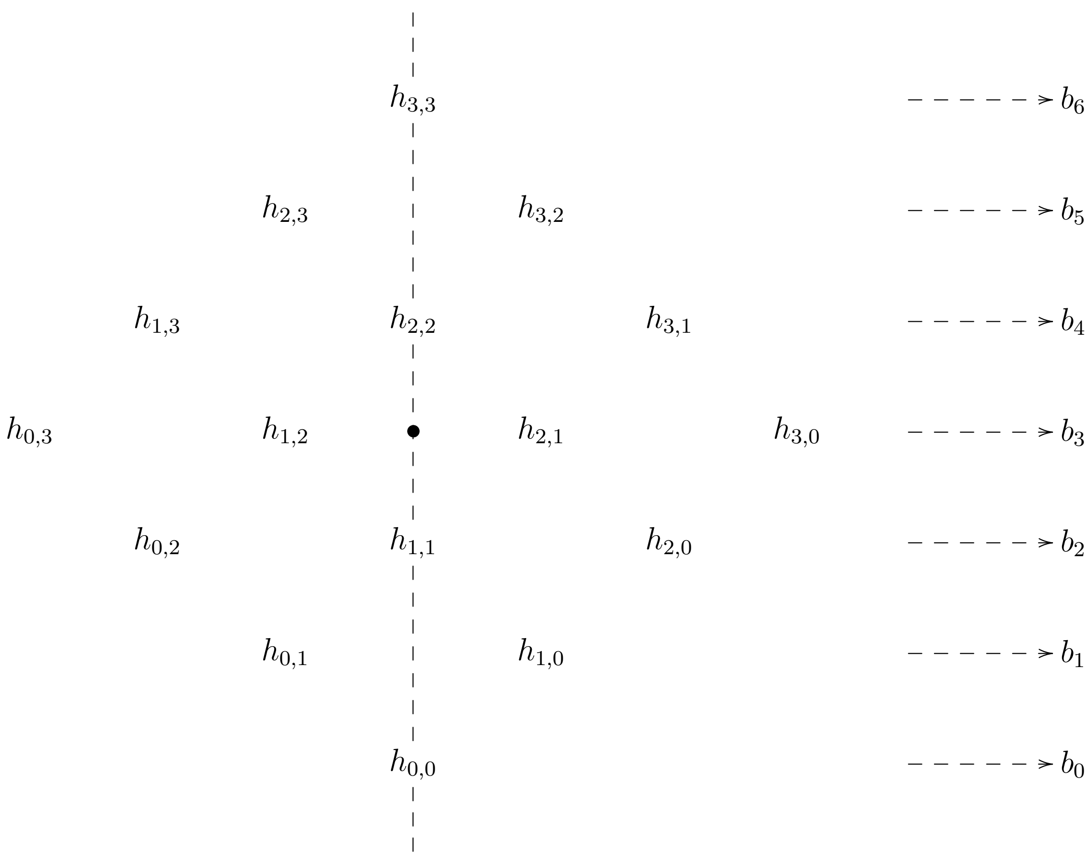
(b) $Q_\mathbb{C} : P^{p,q} \times P^{p',q'} \to \mathbb{C}$ is perfect if $(p',q') = (q,p)$ and zero otherwise.

(c) $P^{p,q} \times P^{p,q} \ni (\alpha,\beta) \mapsto Q(\alpha, J^*\overline{\beta})$ is a positive definite Hermitian form on $P^{p,q}$.

PROOF. All of this follows from the above discussion and the previous section. Most of (i) has been done already in a previous section. The fact that $\overline{H^{p,q}(M)} = H^{q,p}(M)$ follows from the fact that Δ is a real operator so it commutes with complex conjugation. Since complex conjugation interchanges $\mathcal{E}_M^{p,q}(M)$ and $\mathcal{E}_M^{q,p}(M)$ it therefore also interchanges $\mathcal{H}^{p,q}(M)$ and $\mathcal{H}^{q,p}(M)$.

(ii) is in fact a reformulation of the theory of the previous section, since $H^\bullet(M, \mathbb{C})$ is a representation of $\mathfrak{sl}_2(\mathbb{C})$. (iii) finally is just proposition 9.10. □

Remark 10.5.1. To see how powerful the Hodge decomposition is, look at the below diagram (for the case $n = 3$), which is called the **Hodge diamond**. Here $H_{p,q} = \dim_{\mathbb{C}} H^{p,q}(M)$ and $b_r = \dim_{\mathbb{C}} H^r(M, \mathbb{C})$, the **$r$-th Betti number of** M. We see that this diagram is symmetric in the central vertical line, and point-symmetric in the center. Furthermore the horizontal lines add up to the Betti numbers.



Note that a priori both the Lefschetz and the Hodge decomposition seem to depend on the Kähler metric. However, we will show that the Hodge decomposition is independent of the chosen metric. We have to do some work first.

Proposition 10.6. *Let $F^pH^k(M,\mathbb{C})$ denote the space of classes in $H^k(M,\mathbb{C})$ that admit a representative in $\oplus_{l\geq p}\mathcal{E}^{l,k-l}(M)$. Then $F^pH^k(M,\mathbb{C}) = \oplus_{l\geq p}H^{l,k-l}(M)$.*

PROOF. Since $H^{l,k-l}(M,\mathbb{C})$ is just the kernel of Δ acting on $\mathcal{E}^{l,k-l}(M)$, the inclusion $\supseteq$ is clear. So assume $\alpha \in \oplus_{l\geq p}\mathcal{E}^{l,k-l}(M)$ is closed. We will prove using descending induction on p that there exists an $\alpha' \in \oplus_{l\geq p}\mathcal{H}^{l,k-l}(M)$ with $\alpha - \alpha'$ exact. If $p = k$, then $\alpha \in \mathcal{E}^{k,0}(M)$, so clearly $\bar{\partial}^*\alpha = 0$. Since $\mathrm{d}\alpha = 0$ and thus $\bar{\partial}\alpha = 0$, we have $\Delta\alpha = 0$, so we may take $\alpha' = \alpha$.

For arbitrary p, write

$$\alpha = \alpha_p + \cdots + \alpha_k \quad \text{with } \alpha_l \in \mathcal{E}^{l,k-l}(M)$$

Since $\bar{\partial}\alpha_p$ is the $(p, k+1-p)$-part of $\mathrm{d}\alpha$ (which we assumed to be 0), it follows that $\bar{\partial}\alpha_p = 0$. Hence using theorem 8.7 we can write $\alpha_p = \alpha'_p + \bar{\partial}\beta$ with $\alpha'_p \in \mathcal{H}^{p,k-p}(M)$ and $\beta \in \mathcal{E}^{p,k-p-1}(M)$. It follows that

$$\alpha - \alpha'_p - \mathrm{d}\beta = -\partial\beta + \alpha_{p+1} + \alpha_{p+2} + \cdots + \alpha_k$$

and since $\partial\beta \in \mathcal{E}^{p+1,k-p-1}(M)$ we can apply our induction hypothesis on $\alpha - \alpha'_p - \mathrm{d}\beta$ and we find that this element is modulo an exact form in $\oplus_{l\geq p+1}\mathcal{H}^{l,k-l}(M)$. This completes the induction step. □

Corollary 10.7. *The Hodge decomposition is independent of the Kähler metric.*

PROOF. Using Hodge decomposition and the proposition we see that

$$H^{p,q}(M) = F^pH^{p+q}(M,\mathbb{C}) \cap \overline{F^qH^{p+q}(M,\mathbb{C})}$$

here the complex conjugation is with respect to the real subspace $H^{p+q}(M,\mathbb{R})$. Now use that the $F^pH^k(M,\mathbb{C})$ are defined independent of the metric. □

A remarkable consequence the above discussion is the following

Proposition 10.8. *Let M be a compact complex Kähler manifold. Then every holomorphic differential form on M is closed, but never exact, unless it is the zero form.*

PROOF. A differential-form ω is holomorphic if and only if it is of type $(r, 0)$ for some r and $\bar{\partial}\omega = 0$. Since it is of type $(r, 0)$ also $\bar{\partial}^*\omega = 0$, hence $\Delta\omega = 0$, so that ω is harmonic and therefore closed. If it would be exact, say $\omega = \mathrm{d}\eta$, then η should also be holomorphic and therefore closed, implying that $\omega = 0$. □

For $1 \leq r \leq n$, we consider $H^{2r-1}(M, \mathbb{R})$. By Hodge-decomposition we see that its complexification is the direct sum $H^{2r-1,0}(M) \oplus \cdots \oplus H^{0,2r-1}(M)$. Put $V_r := H^{r-1,r}(M) \oplus \cdots \oplus H^{0,2r-1}(M)$. Then it follows that

$$H^{2r-1}(M, \mathbb{R})_{\mathbb{C}} = V_r \oplus \overline{V_r}$$

so that this decomposition defines a complex structure I on $H^{2r-1}(M, \mathbb{R})$ characterized by the fact that V_r and $\overline{V_r}$ are the i resp. $-i$ eigenspaces of $I_{\mathbb{C}}$. In particular we see that $H^{2r-1}(M, \mathbb{R})$ has even dimension.

From the universal coefficient theorem it follows that $H^k(M, \mathbb{R})$ may be identified with $H^k(M, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$. Here $H^k(M, \mathbb{Z})$ is the k-th singular cohomology group with integer coefficients or equivalently, the k-th cohomology group of the constant sheaf $\mathbb{Z}_M$ on M. The inclusion $\mathbb{Z} \subset \mathbb{R}$ defines a homomorphism $H^k(M, \mathbb{Z}) \to H^k(M, \mathbb{R})$ and the theorem then tells us that the resulting map $H^k(M, \mathbb{Z}) \otimes \mathbb{R} \to H^k(M, \mathbb{R})$ is an isomorphism. So if $H^k(M, \mathbb{Z}) \cong$ (finite abelian group) $\oplus \mathbb{Z}^s$, then $H^k(M, \mathbb{R}) \cong \mathbb{R}^s$.

We see that the cokernel $\mathrm{Coker}[H^k(M, \mathbb{Z}) \to H^k(M, \mathbb{R})]$ is a torus $\mathbb{R}^s/\mathbb{Z}^s$ where $s = \dim H^k(M, \mathbb{R})$. We conclude that for $k = 2r - 1$ this torus has a complex structure (for $H^k(M, \mathbb{R})$ has and the action of $\mathbb{Z}^s$ is holomorphically with discrete orbits). It is called the **rth (Griffiths) intermediate Jacobian**[14] of M and is denoted by $J_r(M)$. We may view the $J_r(M)$ as 'continuous invariants' of the manifold M.

The composition

$$H^{2r-1}(M, \mathbb{R}) \hookrightarrow H^{2r-1}(M, \mathbb{C}) = \overline{V_r} \oplus V_r \xrightarrow{\text{proj}} V_r$$

is an isomorphism of real vector spaces, and the complex structure on $H^{2r-1}(M, \mathbb{C})$ is such that this is also an isomorphism of complex vector spaces. We therefore may regard $J_r(M)$ as the cokernel of the composition

$$H^{2r-1}(M, \mathbb{Z}) \longrightarrow H^{2r-1}(M, \mathbb{C}) \xrightarrow{\text{proj}} V_r$$

Two cases deserve a special attention:

If $r = 1$, we get $V_1 = H^{0,1}(M) \cong H^1(M, \mathcal{O}_M)$ and the map $H^1(M, \mathbb{Z}) \to H^1(M, \mathcal{O}_M)$ comes from the inclusion $Z_M \subset \mathcal{O}_M$.

[14] There is another intermediate Jabocian, named after Weil. Hereto one takes $W_r = H^{2r-1,0} \oplus H^{2r-3,2} \oplus \cdots \oplus H^{1,2r-2}$, so that also $H^{2r-1}(M, \mathbb{R})_{\mathbb{C}} = W_r \oplus \overline{W_r}$. The Weil intermediate Jacobians are projective, which the Griffiths intermediate Jacobians are not in general, but the advantage of the Griffiths intermediate Jacobians is that they depend holomorphically on M. However, in the two most interesting cases, namely $r = 1$ and $r = n$, both intermediate Jacobians coincide.

If $r = n$, we have $V_n = H^{n-1,n}(M)$ as all the other summands are zero. This space is complex dual to $H^{1,0}(M) \cong H^0(M, \Omega_M^1)$ via Serre duality. The integral form of Poincaré duality identifies $H^{2n-1}(M, \mathbb{Z})$ with the homology group $H_1(M, \mathbb{Z})$ and the map $H^{2n-1}(M, \mathbb{Z}) \to V_n$ corresponds to the natural map $i : H_1(M, \mathbb{Z}) \to H^0(M, \Omega_M^1)^*$ given by

$$Z \mapsto \left(\omega \mapsto \int_Z \omega \right)$$

Hence we see

$$J_n(M) \cong \operatorname{Coker}[H_1(M, \mathbb{Z}) \to H^0(M, \Omega_M^1)^*]$$

This manifold is also called the **Albanese manifold**, denoted by $\operatorname{Alb}(M)$.

Now suppose that M is connected and fix a point $p_0 \in M$. Given $p \in M$, choose a piecewise differentiable path γ_p from p_0 to p and define $\tilde{\mu}(\gamma_p) \in H^0(M, \Omega_M^1)^*$ by $\omega \mapsto \int_{\gamma_p} \omega$. If γ_p' is another such path, then γ_p followed by γ_p' in the opposite direction is a closed, piecewise differentiable path and hence a 1-cycle Z on M. This shows that $\tilde{\mu}(\gamma_p) - \tilde{\mu}(\gamma_p')$ is given by $\omega \mapsto \int_Z \omega$ and is therefore equal to $i(Z)$. We conclude that the image of $\tilde{\mu}(\gamma_p)$ in $\operatorname{Alb}(M)$ is independent of the chosen path γ_p and therefore $\tilde{\mu}$ defines a map

$$\mu = \mu_{p_0} : M \to \operatorname{Alb}(M)$$

called the **Albanese map**.

Lemma 10.9. *μ is analytic.*

Proof. Let U be a polycylindrical neighborhood of p in M. If $\omega \in \Omega^1(M)$, then $\mathrm{d}\omega = 0$ by proposition 10.8. So by the Poincaré lemma (applied to $\omega|_U$) we find that $\omega|_U = \mathrm{d}f$ for some C^∞-function $f : U \to \mathbb{C}$. Since $\bar{\partial} f = \omega^{0,1} = 0$ it follows that f is in fact holomorphic.

If $q \in U$, let γ_q be the path γ_p followed by a piecewise differentiable map δ_q from p to q in U. Then

$$\tilde{\mu}(\gamma_q)(\omega) = \tilde{\mu}(\gamma_p)(\omega) + \int_{\delta_q} \omega = \tilde{\mu}(\gamma_p)(\omega) + f(q) - f(p)$$

so that $\tilde{\mu}(\gamma_q)(\omega)$ is an analytic function of $q \in U$. If we let ω run over a basis $\{\omega_j\}$ of $H^0(M, \Omega_M^1)$, then the mapping $q \in U \mapsto (\tilde{\mu}(\gamma_q)(\omega_j))_j$ is analytic, in other words, $\tilde{\mu}(\gamma_q)$ is analytic for $q \in U$. The composition of this map with the projection $\Omega^1(M)^* \to \operatorname{Alb}(M)$ is then also analytic. □

The Albanese map plays an important role in the classification of compact analytic manifolds. We summarize some of the properties:

- If M and N are compact, connected Kähler manifolds and $f : M \to N$ is holomorpic, then $f^* : H^\bullet(N, \mathbb{C}) \to H^\bullet(M, \mathbb{C})$ sends $H^{p,q}(N)$ to $H^{p,q}(M)$. Furthermore f^* induces an analytic homomorphism of complex tori $J_r(f) : J_r(N) \to J_r(M)$.

- f_* induces an analytic homomorphism $\mathrm{Alb}(f) : \mathrm{Alb}(M) \to \mathrm{Alb}(N)$.

- If $p_0 \in M$ and $q_0 = f(p_0) \in N$, then the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow{\scriptstyle \mu_{p_0}} & & \downarrow{\scriptstyle \mu_{q_0}} \\ \mathrm{Alb}(M) & \xrightarrow[\mathrm{Alb}(f)]{} & \mathrm{Alb}(N) \end{array}$$

  commutes.

- If N is a complex torus with origin q_0, then μ_{q_0} is an isomorphism of analytic tori.

- Every analytic map $M \to N$ where N is a complex torus can be factored as

$$M \xrightarrow{\mu_{p_0}} \mathrm{Alb}(M) \longrightarrow N$$

  therefore $\mathrm{Alb}(M)$ is also called the **universal solution** of M into a complex torus.