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Article

Characteristics and energy rays of equatorially trapped, zonally symmetric internal waves

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Abstract

Characteristic curves of partial differential equations (PDEs) in general differ from short wave energy rays. We give conditions for linear, two dimensional second order PDEs that guarantee an exact correspondence between characteristics and energy rays. The findings are applied to time-harmonic, zonally-symmetric small-amplitude equatorial internal waves. It is shown that for a fairly general model of inertia-gravity waves, an exact characteristic-ray correspondence holds. When characteristics of internal waves, trapped in a meridional plane, are followed over several boundary reflections, a convergence towards a limit cycle (called equatorial wave attractor) can generally be found. The results on the characteristic-ray correspondence help to interpret physically equatorial wave attractors. Recent ideas on energy accumulation by near inertial waves, trapped on wave attractors at deep ocean sites are confirmed by our results, at least in the short wave (WKB) sense.

Zusammenfassung

Im allgemeinen unterscheiden sich die charakteristischen Kurven und die Strahlen der Wellenausbreitung von partiellen Differentialgleichungen (PDEs). Wir geben hier Bedingungen für lineare, zweidimensionale PDEs zweiter Ordnung an, die eine exakte Übereinstimmung zwischen charakteristischen Kurven und Wellenstrahlen garantieren. Die Ergebnisse werden auf zeitlich harmonische, zonal symmetrische, äquatoriale interne Wellen mit kleiner Amplitude angewandt. Es wird gezeigt, dass für ein recht allgemeines Modell von Schwere-Trägheitswellen, diese exakte Beziehung zwischen charakteristischen Kurven und Wellenstrahlen gilt. Folgt man Charakteristiken von internen Schwerewellen, die in einer meridionalen Ebene gefangen sind, über eine Anzahl von Randreflexionen, so zeigt sich für ein solches Netz von Charakteristiken ein Grenzzyklus. Dieser wird äquatorialer Wellenattraktor genannt. Die gefundenen Ergebnisse der Charakteristiken-Wellenstrahlen-Korrespondenz können helfen, die äquatorialen Wellenattraktoren physikalisch zu interpretieren. Aktuelle Vorstellungen zur Rolle von Wellenattraktoren als „Energiesammelgebiete“ im tiefen Ozean werden durch unsere Analyse gestützt.

1 Introduction

A feature of equatorial dynamics common to both the atmosphere and the ocean is the phenomenon of equatorially trapped waves. The waves are trapped due to a changing Coriolis parameter, leading to an exponential decay of wave amplitude in the meridional direction (PHILANDER, 1990). In contrast to Kelvin and Rossby waves that propagate horizontally in the equatorial waveguide, internal waves can be trapped in a meridional plane due to their vertical component of propagation. An example of this kind was discussed by STERN (1963) for small amplitude zonally symmetric and time-harmonic equatorial inertial waves. Inertial waves have similar properties as internal gravity waves, but in contrast to the latter they need a stratification in angular momentum and not in density to propagate. Inertial waves can hence exist in the rotating homogeneous medium considered by STERN. In the present study, Stern's model plays a central role owing to the fact that it can be solved exactly for arbitrary boundary geometry by the characteristic web method of MAAS and LAM (1995).

Already BRETHERTON (1964) saw that for certain frequencies inertial waves may be trapped on periodic webs of characteristics (referred to as wave attractors). Such orbits were later studied by STEWARTSON (1971) and ISRAELI (1972) for the equatorial beta-plane and recently also for spherical geometry (TILGNER, 1999; RIEUTORD et al., 2001). Their physical relevance was shown in a series of laboratory experiments (MAAS et al., 1997; MAAS, 2001; MANDERS and MAAS, 2003, 2004).

By comparing exact solutions of the inviscid Stern equation (MAAS and HARLANDER, accepted; HARLANDER and MAAS, submitted) with the corresponding characteristic web, wave attractors can easily be identified as singularities of the velocity field. An obvious interpretation of this observation is that, given an energy source, kinetic energy (propagating along the characteristics) should pile up *ad infinitum* in regions of converging characteristics. Although the interpretation of

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characteristics as lines of energy propagation appears straightforward for Stern's equation, it is problematic for more general problems. The reason is that rays of energy propagation result from a short wavelength approximation to the original equation (frequently called Wenzel, Kramers, and Brillouin (WKB) or Liouville-Green approximation in literature) and such rays are in general not related to the characteristic curves of the original equation. This was already noted by ECKART (1960): 'There are two kinds of ray theory whose results are sometimes identical, sometimes very different. One is based on the theory of characteristic curves and surfaces associated with partial differential equations. Its results are always rigorous and of importance for the theory of shock waves and for certain technical purposes. The other theory is rigorous only when the differential equations have constant coefficients.' It should be added that also characteristic curves can give rigorous results only for a restricted class of equations. The other theory mentioned is the WKB theory which in fact is rigorous only for *open* constant coefficient problems. For constant coefficient boundary value problems WKB solutions are approximations (KELLER, 1985).

A robust feature of all the equatorial wave models studied here (forming certain variations of Stern's equation) is the occurrence of wave attractors. Unfortunately, only for Stern's equation we can pin down the meaning of wave attractors for the solution. The other boundary value problems are not separable in terms of characteristics and exact solutions are not available. A physical interpretation of the characteristics (e.g. as rays of energy propagation) is therefore compelling to understand the role of wave attractors for the equatorial wave motion. Even more so since there is experimental evidence that wave breaking and mixing in regions of convergence of characteristics induces a mean flow (MAAS, 2001). Equatorial wave attractors might therefore help sustain the equatorial ocean current system. Moreover, some aspects of observed deep ocean spectra are consistent with the idea of deep ocean point attractors (VAN HAREN et al., 2002; MAAS, 2001; GERKEMA and SHRIRA, 2005 a,b).

The purpose of our paper is to derive conditions for linear second order partial differential equations that lead to an exact characteristic-energy ray correspondence. After formulating the conditions we inspect different equatorial wave models for such a correspondence. Obviously, for models possessing this correspondence, wave attractors can be interpreted as regions where energy accumulates and fluid mixing can be expected.

In section 2 we discuss the mathematical theory underlying characteristic curves and short wavelength energy rays. Moreover, we derive conditions for an exact characteristic-ray correspondence. In section 3 we employ different models describing trapped equatorial internal waves. One model is a simplification of Stern's model, the other two are extensions to Stern's equation, including sphericity or stratification. We discuss briefly exact solutions of Stern's equation obtained by using the characteristic web method. The main purpose of section 3 is to verify whether the characteristic-ray correspondence, holding for Stern's model, can be carried over to the other models. In section 4 we give explicit results for the model that adds stratification to Stern's model (called stratified Stern equation). This model appears to be most important for geophysical applications since it allows for inertia-gravity waves. We discuss and illustrate the wave focusing effect which is at the end the reason for the wave attractor's energy pile up. Finally, the validity of the short wavelength assumption for the stratified Stern equation is verified which legitimates the application of WKB ray tracing. In section 5 we summarize the results and give conclusions.

2 Mathematical theory

Before we discuss concrete examples let us first develop a general theory on the correspondence of characteristics and short wave energy rays.

2.1 Characteristics and rays of energy propagation

As was mentioned in the introduction, the results of both ray theories (the theory of characteristics and the theory of energy propagation) are very different in general. However, for special cases they are identical, even for non-constant coefficients problems. A rather general model for our purpose is the second order linear partial differential equation

$$\begin{aligned} A(Y,Z,\sigma)\psi_{zz} &+ B(Y,Z,\sigma)\psi_{zy} + C(Y,Z,\sigma)\psi_{yy} \\ &+ D(Y,Z,\sigma)\psi_z + E(Y,Z,\sigma)\psi_y \\ &+ F(Y,Z,\sigma)\psi = 0, \end{aligned} \tag{2.1}$$

where the coefficients vary only weakly in meridional (Y) and vertical (Z) direction, as is indicated by their dependency on 'slow' variables $(Y,Z) = \varepsilon(y,z)$, $\varepsilon \ll 1$. Note that in general (at least one of) the coefficients depend on frequency σ also. The reason is that we have separated a time harmonic part $\exp(-i\sigma t)$ from the unknown wave field.

2.1.1 Characteristics

Characteristic coordinates $\zeta = \zeta(y,z)$, $\eta = \eta(y,z)$ are introduced such that (2.1), written in (ζ,η) coordinates, has only a mixed second order derivative term. This leads to the two conditions (MYINT-U, 1987)

$$A(\zeta,\eta,\sigma) = A\zeta_z^2 + B\zeta_z\zeta_y + C\zeta_y^2 = 0 \tag{2.2}$$

$$C(\zeta,\eta,\sigma) = A\eta_z^2 + B\eta_z\eta_y + C\eta_y^2 = 0. \tag{2.3}$$

Dividing through by ξ_y^2 (where ξ stands for ζ or η), (2.2) and (2.3) read

$$A\left(\frac{\xi_z}{\xi_y}\right)^2 + B\left(\frac{\xi_z}{\xi_y}\right) + C = 0. \tag{2.4}$$

Along the curve $\xi = const.$, $d\xi = \xi_z dz + \xi_y dy = 0$ holds, and therefore

$$\frac{dy}{dz} = -\frac{\xi_z}{\xi_y}. \tag{2.5}$$

Solving (2.4) with respect to dy/dz we find the characteristic curves of (2.1) from the ordinary differential equation

$$\frac{dy}{dz} = \frac{1}{2A}\left(B \pm (B^2 - 4AC)^{1/2}\right). \tag{2.6}$$

We note that characteristics do not depend on coefficients of zero and first order terms (D to F).

There are several examples of partial differential equations that can be solved by using the method of characteristics. Essentially, this method makes use of the knowledge of certain functions that remain invariant along the characteristics and that are prescribed at the boundaries. Solutions can be found by tracing characteristics back to the boundaries and reading off the values of these functions. The wave equation is probably the most simple example of an equation that can be solved by using characteristics (see section 3.2).

It should be noted that characteristics are not merely a mathematical tool, they also have a physical interpretation: singularities, arising due to non-regular boundary data or due to boundary conditions, propagate along the characteristics.

2.1.2 Energy rays

Short wave energy (or group velocity) rays follow from the so called Eikonal equation. To derive this equation from (2.1) we first write the equation in terms of Y and Z and insert

$$\psi \sim \psi_0(Y,Z)\exp(i\theta(Y,Z)/\varepsilon), \tag{2.7}$$

where $\psi_0(Y,Z)$ is an unknown slowly varying complex amplitude, and $\theta(Y,Z)/\varepsilon$ an unknown fast varying real phasefunction. In a next step we collect terms of order ε^0, which gives

$$-n^2A - nlB - l^2C + inD + ilE + F = 0, \tag{2.8}$$

where we have used the definitions

$$l := \frac{\partial\theta}{\partial Y}, \quad n := \frac{\partial\theta}{\partial Z}. \tag{2.9}$$

The Eikonal equation (2.8) is a nonlinear partial differential equation of first order for $\theta(Y,Z)$. By using the so called characteristic strip equations (ZWILLINGER, 1989), an implicit solution for θ can usually be found from (2.8). But θ is usually not needed explicitly since with the definitions (2.9) and the interpretation of l and n as wavenumbers, (2.8) is also a dispersion relation. Often we have quadratic dependency of the coefficients on σ and we confine our study to this case. Therefore, (2.8) can easily be solved for σ to obtain

$$\sigma = \sigma(Y,Z,l,n). \tag{2.10}$$

The characteristics of (2.8) are along the group velocity $c_{g_y} = dy/dt = \partial\sigma/\partial l$, $c_{g_z} = dz/dt = \partial\sigma/\partial n$. Hence, the short wave energy rays are given by the differential equation

$$\frac{dy}{dz} = \frac{\partial\sigma/\partial l}{\partial\sigma/\partial n}, \tag{2.11}$$

where the numerator is the meridional component of the group velocity, and the denumerator the vertical component.

Comparing (2.6) and (2.11), exact correspondence between characteristics and energy rays cannot be expected in general. Contrasting characteristic curves and energy rays, the latter have a somewhat more straightforward physical interpretation. In some applications, they are useful for a qualitative understanding of wave dispersion, even if the phase θ cannot be found explicitly from the Eikonal equation (2.8) (KAROLY, 1983; HARLANDER et al., 2000; EDWARDS and STAQUET, 2005).

2.2 Correspondence condition for the general model

A classical example for an equation with exact characteristic-ray correspondence is $\sigma^2 p_{zz} + (\sigma^2 - N^2)p_{yy} = 0$, describing internal gravity (pressure) waves in a stratified fluid with frequency σ less than the constant Brunt-Väisälä frequency N. In contrast to this hyperbolic equation, characteristic ray-correspondence cannot hold for elliptic problems for which characteristic curves are complex but energy rays are real. An example of this type is the linear beta plane Rossby wave equation $i\sigma(\psi_{xx} + \psi_{yy}) - \beta\psi_x = 0$, where β is constant. Let us now investigate the general problem (2.1), where the only restriction is the assumption of slowly varying coefficients. Equation (2.8) then reads

$$\begin{aligned}\Sigma &= -A(Y,Z,\sigma(Y,Z,n,l))n^2 \\ &- B(Y,Z,\sigma(Y,Z,n,l))ln - C(Y,Z,\sigma(Y,Z,n,l))l^2 \\ &+ D(Y,Z,\sigma(Y,Z,n,l))in + E(Y,Z,\sigma(Y,Z,n,l))il \\ &+ F(Y,Z,\sigma(Y,Z,n,l)) \\ &= 0.\end{aligned} \tag{2.12}$$

By differentiating Σ with respect to l and n we get

$$\frac{\partial \Sigma}{\partial l} = \frac{\partial \sigma}{\partial l}\gamma - 2Cl - Bn + iE = 0, \quad (2.13)$$
$$\frac{\partial \Sigma}{\partial n} = \frac{\partial \sigma}{\partial n}\gamma - 2An - Bl + iD = 0, \quad (2.14)$$

where

$$\gamma = -\frac{\partial A}{\partial \sigma}n^2 - \frac{\partial B}{\partial \sigma}nl - \frac{\partial C}{\partial \sigma}l^2 + \frac{\partial D}{\partial \sigma}in + \frac{\partial E}{\partial \sigma}il + \frac{\partial F}{\partial \sigma}. \quad (2.15)$$

From these equations we find for the energy ray

$$\frac{dy}{dz} = \frac{\partial \sigma/\partial l}{\partial \sigma/\partial n} = \frac{2Cl + Bn - iE}{2An + Bl - iD}, \quad (2.16)$$

that differs from (2.6). By assuming $D = E = F = 0$ and using (2.12) the right hand side of (2.16) can be reduced to $-n/l$. Next we divide (2.12) by l^2 and solve the equation for n/l to find

$$\frac{\partial \sigma/\partial l}{\partial \sigma/\partial n} = -\frac{n}{l} = \frac{1}{2A}\left(B \mp (B^2 - 4AC)^{1/2}\right), \quad (2.17)$$

which does correspond to (2.6). Hence we have shown that for any equation

$$A(Y,Z,\sigma)\psi_{zz} + B(Y,Z,\sigma)\psi_{zy} + C(Y,Z,\sigma)\psi_{yy} = 0, \quad (2.18)$$

with arbitrary coefficients A, B, C, characteristic curves correspond to short wave energy rays.

One might ask if the ray-characteristic correspondence of (2.18) cannot artificially be removed by introducing a new dependent variable $\psi = f(Y,Z)\phi$. The equation for ϕ has the same characteristics as (2.18), but might have different rays since D, E, F are no longer zero. However, since f is slowly varying, these terms are order ε and do not show up in the order one equation (2.12). Hence, the ray-characteristics correspondence remains intact under a transformation like $\psi = f(Y,Z)\phi$.

2.3 Ray-characteristic correspondence for equations with *y*-dependent coefficients only

Let us now study the case where all coefficients in (2.1) depend on y only. In the context of linear equatorial zonally-uniform internal waves this case is most relevant since the governing equations are of this type (STERN, 1963; STEWARTSON, 1971; MAAS and HARLANDER, accepted). Consider

$$A(y)\psi_{zz} + B(y)\psi_{yz} + C(y)\psi_{yy} + D(y)\psi_z + E(y)\psi_y + F(y)\psi = 0, \quad (2.19)$$

where ψ is a function of y and z, but the coefficients B, C, D, E, F depend on y (and possibly on σ) only. Note that as previously we assume that the coefficients depend only weakly on y, however, for convenience we introduce the slow Y and Z coordinates later on. By separating the y and z dependency and by using the transformation given by POLYANIN and ZAITSEV (1995) (see page 129)

$$\psi = \phi(y)\exp(inz)\exp\left(-\int(\alpha_1/2)dy\right), \quad \alpha_1 = (inB + E)/C, \quad (2.20)$$

where $i = \sqrt{-1}$ and n is a vertical wavenumber, (2.19) can be transformed to

$$\phi'' + q(y)\phi = 0, \quad (2.21)$$

with

$$q(y) = \alpha_0 - \alpha_1^2/4 - \alpha_1'/2, \qquad \alpha_0 = (inD + F - n^2A)/C. \quad (2.22)$$

Introducing slow space variables $(Y,Z) = \varepsilon(y,z)$, (2.21) reads

$$\varepsilon^2\phi'' + q(Y)\phi = 0, \quad (2.23)$$

where $q(Y)$ is given by (2.22), written in the slow variable. Note that up to this point we have not made any approximations. We start approximating by inserting a short wave (WKB) ansatz into (2.23)

$$\phi \sim \phi_0(Y)\exp(i\theta(Y)/\varepsilon), \quad (2.24)$$

and collecting terms with different orders in ε. Keeping only the lowest orders gives the well known WKB solution. Its validity depends on the size and variation of the coefficient $q(Y)$.

We obtain the phase θ from the order one problem (called eikonal equation, see HOLMES (1995)), where n is now a constant

$$\theta' = \pm q(Y)^{1/2}. \quad (2.25)$$

Via the transformation (2.20) we have introduced a phase function

$$\Omega(Z,Y) = nZ + i/2\int \alpha_1(Y)dY + \theta(Y), \quad (2.26)$$

that can be used to write the streamfunction as $\psi \sim \phi_0\exp(i\Omega/\varepsilon)$. Using the definition

$$(\partial\Omega/\partial Y, \partial\Omega/\partial Z) = (l,n), \quad (2.27)$$

where l (n) denotes the meridional (vertical) wavenumber, from (2.26) and (2.25) we obtain the dispersion relation

$$-\frac{\alpha_1^2}{4} - \alpha_1 il^{\pm} + l^{\pm 2} - q(Y) = 0. \quad (2.28)$$

The superscript in $l^{\pm}$ indicates that the meridional wavenumber possesses two roots, as is obvious from (2.25). Writing (2.28) in terms of the coefficients A to F by using the definitions for α_0 and α_1 from (2.20) and (2.22), we find that the equation can be reduced to the form

$$An^2 + Bl^{\pm}n + Cl^{\pm 2} = 0 \tag{2.29}$$

if the three conditions

$$D = \frac{C}{2}\left(\frac{B}{C}\right)', \tag{2.30}$$

$$E = F = 0, \tag{2.31}$$

hold. In section 2.2 we have proven that a dispersion relation of the form (2.29) (i.e., (2.12) with $D = E = F = 0$) has an exact characteristic-ray correspondence. Therefore we can say that for any second order linear partial differential equation with y dependent coefficients, characteristics correspond to short wave energy rays if first, the ansatz (2.24) can be justified, and second, the conditions (2.30) and (2.31) hold.

2.4 Separability

Finally, we want to address the question of separability of (2.19). In general there are two kinds of separability. The first one is separability of variables that means a solution can be written as $\psi = \phi(y)\chi(z)$. We have used separation of variables in (2.20). As we have shown, this kind of separability implies a characteristic-ray correspondence only if the conditions (2.30) and (2.31) hold. Moreover, only for a very restricted class of boundary value problems (i.e., (2.19) with $\psi = 0$ at a given boundary), solutions can be found by separation of variables. For example, MOON and SPENCER (1971) list eleven coordinate systems that can be used to solve Laplace- and Helmholtz-type boundary value problems where the boundaries correspond to coordinate lines. If boundaries deviate from coordinate lines, analytical solutions can be found to a certain approximation only (HARLANDER and MAAS, 2004).

The other kind of separability is the separability of the equation's operator. This means that $\mathscr{L}\psi = 0$ (where $\mathscr{L}$ is the linear second order operator of (2.19)) can be written as $\mathscr{P}_1\mathscr{P}_2\psi = 0$ with the two first order linear operators $\mathscr{P}_i = a_i(y,z)\partial/\partial y + b_i(y,z)\partial/\partial z$, $i = 1,2$. In that case we can introduce new coordinates such that $\partial_\eta = \mathscr{P}_1$ and $\partial_\zeta = \mathscr{P}_2$ (with $\eta_z\zeta_y - \eta_y\zeta_z \neq 0$). Then (2.19) can be written as

$$\psi_{\eta\zeta} = 0. \tag{2.32}$$

This equation has the structure (2.18) with $A = C = 0$. Thus, operator separability implies an exact correspondence between characteristics and short wave energy rays in the characteristic coordinate frame. Moreover, the problem (2.32) with $\psi = 0$ at the boundary can be solved for arbitrary boundary geometry by the characteristic web method of MAAS and LAM (1995).

It can be shown that the operator of (2.19) is separable if

$$\begin{aligned} &A/C = g(y)^2 - \sigma^2, \quad B/C = 2g(y), \\ &D/C = g'(y), \\ &E = 0, \quad \text{and} \quad F = 0, \end{aligned} \tag{2.33}$$

where $g(y)$ is an arbitrary function, and σ an arbitrary constant. Inspection shows that the coefficients (2.33) satisfy the conditions (2.30) and (2.31) and hence imply an exact characteristic-ray correspondence. However, these conditions do not depend on A, while operator separability does. Therefore we can expect that some equations that do not possess separable operators will still have an exact characteristic-ray correspondence.

The characteristic coordinates can be found from (2.6), that reads for the coefficients (2.33)

$$\frac{dy}{dz} = (g(y) \pm \sigma)^{-1}, \tag{2.34}$$

as curves of constant η and ζ, defined by the solutions of (2.34) as

$$\eta(y,z) = \int g(y)dy + \sigma y - z, \tag{2.35}$$

$$\zeta(y,z) = \int g(y)dy - \sigma y - z. \tag{2.36}$$

Note that (2.32) can easily be transformed to the classical wave equation

$$\psi_{\hat{Y}\hat{Y}} - \psi_{\hat{Z}\hat{Z}} = 0 \tag{2.37}$$

via the coordinates $\hat{Y} = (\eta - \zeta)/2$, $\hat{Z} = (\eta + \zeta)/2$. In that frame, the characteristics are straight lines with slope ± 1.

With respect to hyperbolic boundary value problems, separation of the operator is more powerful than separation of variables. To demonstrate this let us consider (2.37) with $\psi = 0$ along a rectangle with length L and height H. This problem is separable in terms of the variables as well as in terms of the operator. This means that the boundary value problem can be solved by the separation of variables method as well as by the characteristic web method. The solution reads simply $\psi_n = A_n \sin(n\pi\hat{Y}/L)\sin(n\pi\hat{Z}/H)$, $n \in N$. However, if one boundary is tilted with respect to the vertical, no eigenfunctions can be found by separation of variables for arbitrary tilt. Nevertheless, the operator of the non-rectangular boundary value problem is still separable; hence, the problem can still be solved by the characteristic web method (MAAS and LAM, 1995).

2.5 Summary

In summary we advise the following strategy to verify whether an equation possesses an exact characteristic-ray correspondence: first, check if the equation has the form of (2.18). If not, but the equation has y dependent coefficients only, verify whether the equation is operator separable by inspecting (2.33). If this is not the case, check the validity of (2.30) and (2.31). If the conditions fail, the equation under consideration has no characteristic-ray correspondence.

3 Governing equations for internal zonally-uniform equatorial waves

Let us now turn to the equations governing zonally symmetric equatorial waves confined between a bottom (at $z = 0$) and a surface (at $z = H$). In non-dimensional form, the linearized, zonally symmetric equations on the sphere read in Boussinesq approximation[1]

$$i\sigma u + \cos\phi w - \sin\phi v = 0, \quad (3.1)$$

$$i\sigma v + \sin\phi u = -p_y, \quad (3.2)$$

$$i\sigma w - \cos\phi u = -p_z + b, \quad (3.3)$$

$$i\sigma b + w = 0, \quad (3.4)$$

$$v_y + w_z = 0. \quad (3.5)$$

Here y stands for the meridional and z for the vertical direction, defined by the differential relations $dy = rd\phi$ and $dz = dr$, where ϕ is latitude and r radius. The zonal, meridional, and vertical velocity components are denoted by u, v, w; b is buoyancy (which is proportional to density), and p is pressure divided by a constant reference density. To keep the analysis brief, we do not discuss the scaling variants that lead to the different approximations of (3.1)-(3.5) that will be considered below; they are given e.g. by STERN (1963), GILL (1982), and MAAS and HARLANDER (accepted).

The purpose of the following four sections is to discuss different equatorial internal wave equations in the light of the findings above. It will be illustrated that short wave energy ray-characteristic correspondence is a useful concept for a better understanding of boundary value problems that lack explicit solutions.

3.1 Standard equation

The standard equatorial beta-plane equation is obtained from (3.1)–(3.5) when the so-called traditional and hydrostatic approximations are made (BRETHERTON, 1964; MATSUNO, 1966; PHILLIPS, 1990). The traditional approximation neglects the second term of (3.1) and the second term of (3.3); the hydrostatic approximation neglects the first term of (3.3). Moreover, expanding $\sin\phi$ in a Taylor series about $\phi = 0$ gives the equatorial beta-plane that replaces $\sin\phi$ by y. With these assumptions and the introduction of a meridional streamfunction $v = -\psi_z, w = \psi_y$, (3.1)-(3.5) can be reduced to

$$\psi_{yy} - (\sigma^2 - y^2)\psi_{zz} = 0, \quad (3.6)$$

referred to as the standard equation. Using the transformation (2.20) with $\alpha_0 = n^2(\sigma^2 - y^2)$ and $\alpha_1 = 0$, the equation can be reduced to (2.21) with $q(y) = \alpha_0$, the Weber equation. Solutions are Hermite polynomials satisfying the boundary conditions $\psi = 0$ at $z = 0, 1$ and $\psi \to 0$ for $y \to \pm\infty$ (PHILANDER, 1990). However, the equation is not operator separable. Nevertheless, (3.6) has the structure of (2.18) (with $B = 0$) and characteristics correspond to energy rays.

What kind of solutions can we expect when the flat boundaries of the standard equation are perturbed? MAAS and HARLANDER (accepted) showed that the hyperbolic nature of (3.6) gives rise to focusing of characteristics and wave attractors even for small boundary perturbation. In contrast to the solutions for flat boundaries, the focusing prevents the solutions from being smooth. Although we cannot compute these solutions, neither by separation of variables nor by the characteristic web method, we still know from the characteristic-ray correspondence that the energy of short waves follows characteristics and that therefore wave attractors should coincide with regions of large kinetic energy. This demonstrates that the characteristic web alone can contain important physical information if the characteristic-ray correspondence holds.

3.2 Stern's equation

Another kind of perturbation for the boundary value problem (3.6) was discussed by STERN (1963). He perturbed the problem not via the boundaries but modified (3.6) such that it allows for non-traditional Coriolis effects. In that case wave attractors become generic too, even for non-perturbed flat boundaries. Surprisingly, wave attractor solutions can be studied explicitly for Stern's model. Stern's scaling for a homogeneous fluid beta-plane version of (3.1)–(3.5) keeps the non-traditional terms, but the hydrostatic assumption is made again; that is, just the first term in (3.3) (owing to hydrostasy) and the second term in (3.4) (owing to the fluid's homogeneity) are neglected, $\sin\phi$ is replaced by y and $\cos\phi$ by 1 (equatorial beta-plane). Introducing a meridional streamfunction, we find from (3.1)–(3.5) Stern's equation

$$\psi_{yy} - (\sigma^2 - y^2)\psi_{zz} + 2y\psi_{yz} + \psi_z = 0. \quad (3.7)$$

[1] See DUTTON (1986), page 233 for the full system on the sphere, and VERONIS (1970) or GILL (1982) for the Boussinesq version of the equatorial beta-plane equation.

In contrast to the standard equation, no smooth solutions can be found satisfying the boundary conditions $\psi = 0$ for $z = 0, 1$ and for $y \to \pm\infty$. However, the equation has coefficients (2.33) with $g(y) = y$. Thus, the operator of (3.7) is separable and the characteristics correspond to the rays of short wave energy propagation.

Using (2.35) and (2.36), the characteristic coordinates are

$$\zeta = z - \frac{1}{2}y^2 - \sigma y, \quad (3.8)$$

$$\eta = z - \frac{1}{2}y^2 + \sigma y. \quad (3.9)$$

The variables

$$\hat{Y} = 2(\eta - \zeta) = 4y\sigma \quad (3.10)$$

$$\hat{Z} = 2(\eta + \zeta - \sigma^2) = 2(2z - y^2 - \sigma^2), \quad (3.11)$$

transform (3.7) to the standard hyperbolic equation (2.37). This equation is solved by functions that are constant on straight characteristics $\hat{Y} \pm \hat{Z} = const$. Note that (3.10) and (3.11) turn the flat bottom and surface ($z = 0, 1$) into two displaced parabola. In these coordinates bottom and surface are given as

$$\hat{Z}_b(\hat{Y}) = -2(\hat{Y}^2/16\sigma^2 + \sigma^2) \quad (3.12)$$

$$\hat{Z}_s(\hat{Y}) = \hat{Z}_b(\hat{Y}) + 4. \quad (3.13)$$

In Fig. 1 we consider the characteristic web of two characteristics, launched at $(\hat{Y}, \hat{Z}) = (0, 1/2)$. The characteristics reflect at the boundaries and converge towards a limit cycle, the equatorial wave attractor. In the example shown we use $\sigma = 0.9$. Recently, HARLANDER and MAAS (submitted) presented exact solutions of Stern's equation by applying the characteristic web method of MAAS and LAM (1995). By inspecting them it can be seen that the attractor coincides with the region of strongest streamfunction gradients. An example demonstrating this is displayed in Fig. 2. Kinetic energy piles up along the wave attractor and dissipative and nonlinear effects should become important in this region, which is therefore sometimes called an internal boundary layer (STERN, 1963; HARLANDER and MAAS, submitted).

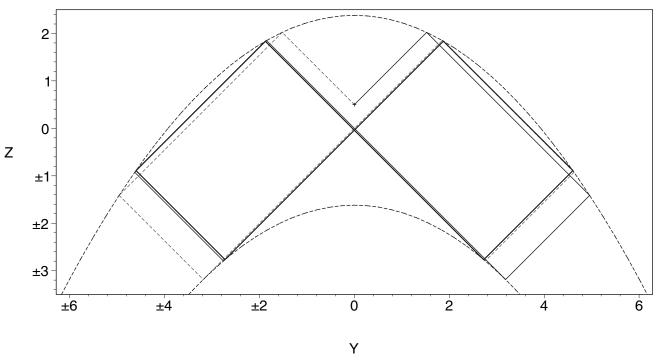

Figure 1: Equatorial wave attractor in the $(\hat{Y}, \hat{Z})$ frame for $\sigma = 0.9$.

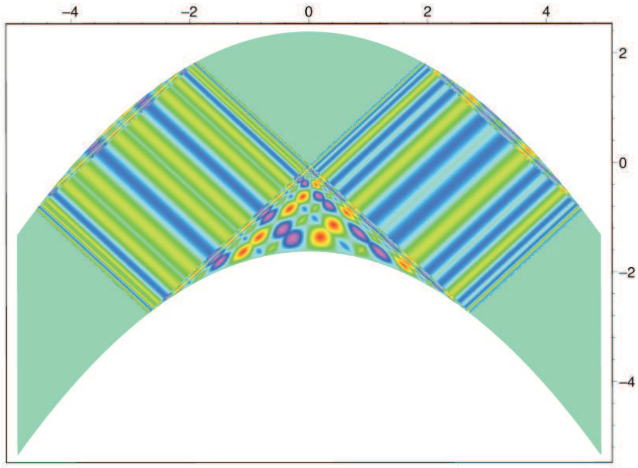

Figure 2: Exact solution of Stern's equation for $\sigma = 0.9$. The streamfunction ψ is shown in the $\hat{Y}, \hat{Z}$ frame. In the region inside the wave attractor banded flow structures can be observed. The velocity is singular along the wave attractor. More details on the solutions can be found in HARLANDER and MAAS (submitted).

In the following we extend Stern's equation. First, to include extra-tropical regions, and second, to include stratification. As will become evident, operator separability is lost again for these somewhat more general models. However, it is instructive to check whether these models keep the characteristic-ray correspondence.

3.3 'Spherical' Stern equation

Employing the same scaling as STERN (1963) (i.e. neglecting the first and last term in (3.3) and the second term in (3.4)) but avoiding the beta-plane approximation, some part of the spherical geometry, lost in (3.7), can be retained. For a thin shell with a large radius, r is approximately constant. Instead of replacing $\sin\phi$ by y (beta-plane) we replace $\sin\phi$ by $\sin y$ and $\cos\phi$ by $\cos y$. After a transformation of the velocity components and the introduction of new coordinates y', z' (see appendix A) we find for the transformed meridional velocity component

$$\hat{v}_{y'y'} - (\sigma^2 - g^2)\hat{v}_{z'z'} + 2g\hat{v}_{y'z'} + \hat{v}_{z'} = (g\hat{v})_{y'} - g^2\hat{v}_{z'}, \quad (3.14)$$

where g is a function of y' alone. This equation has no characteristic-ray correspondence since neither (2.33) nor (2.30) and (2.31) hold. On the other hand, if the right hand side of the equation is neglected (see appendix A), then the spherical Stern equation possesses a separable operator, as can be seen by comparing the coefficients on the left hand side of (3.14) with (2.33). Note that, no matter whether the right hand side of (3.14) is set to zero or not, the characteristic curves remain the same. Thus, the neglect of the right hand side of (3.14) leading to a separable operator, leaves the characteristics intact.

3.4 'Stratified' Stern equation

The last equatorial wave model discussed includes stratification which makes it the most relevant model for atmospheric and oceanic applications. We obtain it by restoring stratification ($N = const.$), keeping the non-traditional terms, assuming non-hydrostasy, but making the beta-plane assumption. In other words, we keep the system (3.1)–(3.5) but replace $\sin\phi$ by y and $\cos\phi$ by 1 (equatorial beta-plane). Introducing a streamfunction in the meridional plane, we find from (3.1)–(3.5) the stratified Stern equation (MAAS and HARLANDER, accepted)

$$\psi_{y'y'} - (\sigma^2 - \Gamma y'^2)\psi_{zz} + 2y'\psi_{y'z} + \psi_z = 0, \qquad (3.15)$$

where

$$y' = y/\Gamma^{1/2}, \qquad \Gamma = 1 + N^2 - \sigma^2. \qquad (3.16)$$

Note that (3.15) reduces to Stern's equation (3.7) for $\Gamma = 1$ (for which $y' = y$).

Comparing the coefficients of (3.7) with (2.33) we see that the operator is not separable for $\Gamma \neq 1$. Nevertheless, the characteristic-ray correspondence holds since the conditions (2.30) and (2.31) are satisfied for all Γ. (Note that the characteristics change if Γ is changed.)

Table 1 summarizes our findings on the characteristic-ray correspondence of the different equatorial wave models discussed. The standard equation and the stratified Stern equation are of special interest since they do not have a separable operator but yet possess an exact characteristic-ray correspondence.

4 Some explicit results for the stratified Stern equation

It is instructive to apply the results from section 2.3 to a concrete example. Let us therefore see how the concept works for the stratified Stern equation.

The section is organized as follows. First, we explicitly derive the characteristic-ray correspondence for the stratified Stern equation, and second, we verify the validity of the short wave solution.

4.1 Characteristic-ray correspondence

To investigate (3.15) from a short wave (or WKB) point of view it is useful first to transform the equation via (2.20) to (2.23), where α_0, α_1, and q read in 'slow' variables $(Y', Z) = \varepsilon(y', z)$

$$\alpha_0 = in - n^2(\Gamma(Y'/\varepsilon)^2 - \sigma^2),$$
$$\alpha_1 = in2Y'/\varepsilon,\ q(Y') = \frac{n^2}{\varepsilon^2}\left[Y'^2(1-\Gamma) + \sigma^2\varepsilon^2\right]$$
$$. \qquad (4.1)$$

Then the phase function (2.26) reads

$$\Omega(Z, Y') = \varepsilon n\left(Z/\varepsilon - Y'^2/\left(2\varepsilon^2\right)\right) + \theta(Y'), \qquad (4.2)$$

where $\theta(Y')$ is given by integrating the eikonal equation (2.25). Using the definition (2.27) we obtain the dispersion relation

$$l^{\pm} = -\frac{nY'}{\varepsilon} \pm q(Y')^{1/2}, \qquad (4.3)$$

that can be written as

$$(Y'^2[1 + N^2 - \sigma^2] - \sigma^2\varepsilon^2)\tilde{n}^2 + 2Y'l^{\pm}\tilde{n} + l^{\pm 2} = 0, \quad (4.4)$$

where $\tilde{n} = n/\varepsilon$. We see that the structure of this equation matches with (2.29), having coefficients $A = (Y'^2[1 + N^2 - \sigma^2] - \sigma^2\varepsilon^2)$, $B = 2Y'$, and $C = 1$.

The ray path is given from (4.4) by

$$\frac{dY'}{dZ} = \frac{\partial\sigma/\partial l^{\pm}}{\partial\sigma/\partial n} = -\frac{n}{l^{\pm}}. \qquad (4.5)$$

On the other hand, the characteristic curves are given by (2.6), where $A = \Gamma y'^2 - \sigma^2$, $B = 2y'$, and $C = 1$. Using σ^2 from (4.4), (2.6) gives the two possibilities

$$\frac{dy'}{dz} = \frac{n}{2ny' + l^{\pm}}, \quad \text{and} \quad \frac{dy'}{dz} = -\frac{n}{l^{\pm}}. \qquad (4.6)$$

From (4.3) we can see that $(2ny' + l^+) = -l^-$ and that $(2ny' + l^-) = -l^+$; i.e., the first solution of (4.6) is in fact equal to the second one. Thus we obtain actually two families of characteristics, as expected. Therefore (and consistent with our previous results in section 2.3) we can conclude that there is an exact characteristic-ray correspondence for the stratified Stern equation (3.15); that is, the characteristic curves are identical to short wave energy (group velocity) rays.

What is lacking in our analysis so far is an accuracy measure of the short wave (WKB) approximation (2.24). If the approximation does not hold, WKB ray tracing, that is the result of the section 4.1, would not be very useful. To fill this gap in our line of arguments we will compute a validity condition for the WKB approximation. Moreover, we will directly compare exact solutions with WKB approximations. The analysis will show that (2.24) is an accurate approximation for free solutions of (3.15).

4.2 Reliability of the WKB solution

Let us check under which conditions (2.24) can be expected to be an acceptable solution. A measure for accuracy of the WKB solution is given by HOLMES (1995) as

Table 1: Operator separability and characteristic-ray correspondence for the equations considered in the text. Note that a separable operator implies a characteristic-ray correspondence, but not the converse.

equation name	equation number	separable operator	char.-ray correspondence
standard equation	(3.6)	no	yes
Stern's equation	(3.7)	yes	yes
spherical Stern equation	(3.14)	no	no
stratified Stern equation	(3.15)	no	yes

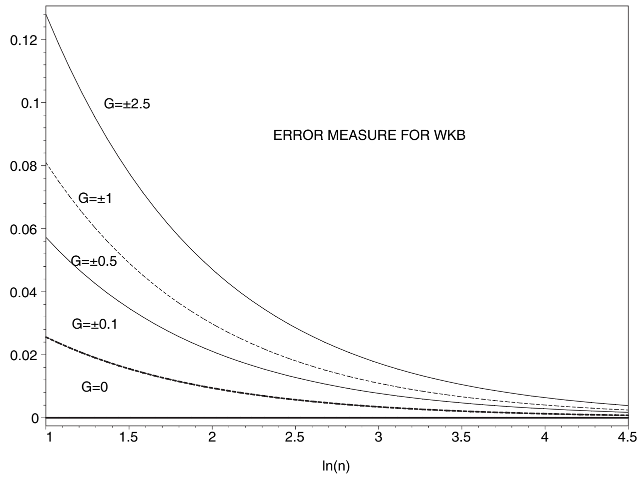

Figure 3: The WKB error measure (4.7) as a function of the vertical wavenumber n for different values of G. Here we used $c = 0, Y'_0 = -Y'_T/2, Y'_1 = Y'_T/2, \sigma = 1$.

$$\varepsilon\left[\frac{1}{32}\quad\left(\max_{Y'_0\le Y'\le Y'_1}\mid\frac{dq/dY'}{q^{3/2}}\mid\right)\left(4+\int_{Y'_0}^{Y'_1}\mid d(\ln q)/dY'\mid dY'\right)\right]\ll 1., \tag{4.7}$$

First, it is important to note that (4.7) is independent of ε when written in the fast variable y'. Second, it is obvious that the condition (4.7) is fulfilled if q' is small, or q is large, or both. Third, at so called turning latitudes given by $q(Y') = 0$, the WKB approximation breaks down. For the stratified Stern equation, turning latitudes are located at $y'_T = \pm\sigma/(-G)^{1/2}$, where $G = 1-\Gamma = \sigma^2 - N^2 < 0$.

Let us determine for which Γ and vertical wave numbers n we can expect accurate solutions. Figure 3 shows the left hand side of (4.7) for different values of G as a function of $\ln(n)$. In this example we used $Y'_0 = -Y'_T/2, Y'_1 = Y'_T/2, \sigma = 1$. For Stern's equation ($G = 0$) the measure is zero for any n. If G decreases, the measure increases. For $n < 1$ it is violated for $G < -2$. Most important, in the interval $Y' \in [-Y'_T/2, Y'_T/2]$, the condition (4.7) is fulfilled for all Γ if n is sufficiently large.

Finally, we compare exact solutions of (3.15) with corresponding WKB approximations. The boundary value problem (2.23) with the coefficient q given by (4.1) can be solved analytically. Using

$$\tilde{y} = y'\left[n^2(\Gamma-1)\right]^{1/4},\ \tilde{\phi} = \phi\exp(\tilde{y}^2/2), \tag{4.8}$$

we find

$$\frac{d^2\tilde{\phi}}{d\tilde{y}^2} - 2\tilde{y}\frac{d\tilde{\phi}}{d\tilde{y}} + (E-1)\tilde{\phi} = 0, \tag{4.9}$$

where

$$E = \frac{n^2\sigma^2}{\left[n^2(\Gamma-1)\right]^{1/2}} \tag{4.10}$$

and

$$m = \frac{1}{2}\left(\frac{n^2\sigma^2}{\left[n^2(\Gamma-1)\right]^{1/2}} - 1\right). \tag{4.11}$$

Solutions are Hermite polynomials $H_m(\tilde{y})$, where $m = 1, 2, 3, \cdots$; i.e., for any m we obtain for ϕ

$$\phi_m \sim H_m(\tilde{y})\exp(-\tilde{y}^2/2), \tag{4.12}$$

referred to as Weber-Hermite function in literature (Bell, 1968).

On the other hand, the stratified Stern equation (3.15) has the short wave solution (2.24) with the phase (2.26). To find ϕ_0 of (2.24) we have to solve the so called transport equation that follows from the order ε problem of the asymptotic expansion, reading

$$\theta''\phi_0 + 2\theta'\phi_0{}' = -\theta'^2\phi_1 + q(Y')\phi_1. \tag{4.13}$$

Obviously, by using (2.25), the right hand side of this equation vanishes and we obtain

$$\phi_0 = \frac{c}{\theta'^{1/2}}, \tag{4.14}$$

where c is an arbitrary constant. Hence, (2.24) reads

$$\phi \sim q(Y')^{-1/4}\quad\left[a_0\exp\left(\frac{i}{\varepsilon}\int^{Y'} q(s)^{1/2}ds\right) + b_0\exp\left(-\frac{i}{\varepsilon}\int^{Y'} q(s)^{1/2}ds\right)\right]. \tag{4.15}$$

Note that q is positive within the turning latitudes but becomes negative outside of them. Therefore b_0 has to

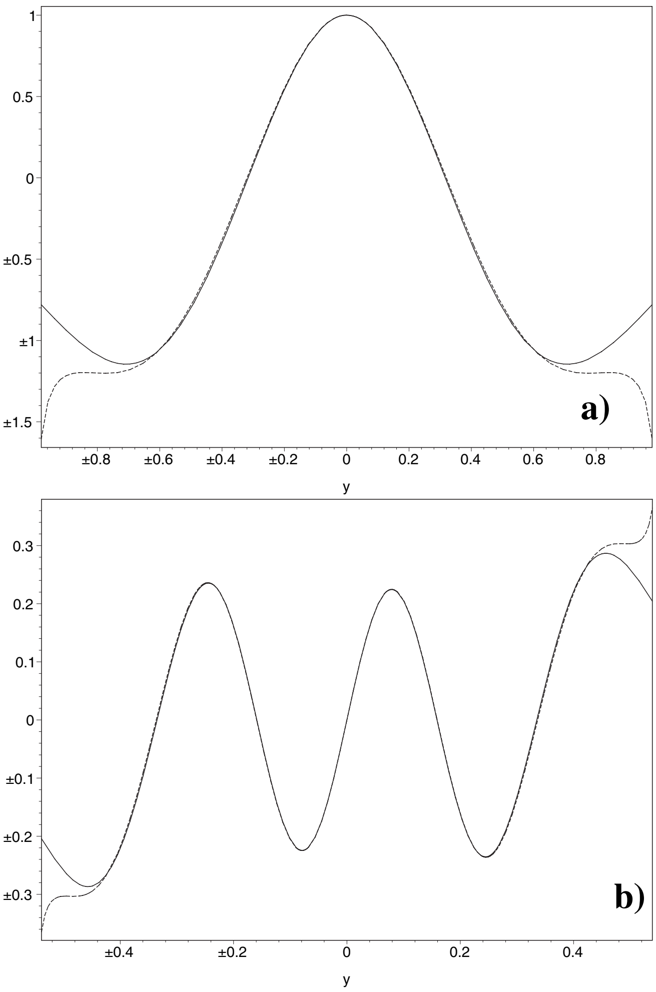

Figure 4: Comparison of exact solutions (solid line) and WKB approximations (dashed line). In a) we used $\sigma = 1$, $m = 2$, $n = 5$, which gives $G = -1$. In b) we used $\sigma = 1$, $m = 5$, $n = 20$, which gives $G = -400/121$. The solutions are plotted for $-(49/50)y'_T \leq y' \leq (49/50)y'_T$.

be zero to satisfy the boundary conditions $\phi \to 0$ for $Y' \to \pm\infty$. Fig. 4 shows a comparison between (4.12) and (4.15) for an even and an odd m. The solutions are plotted in the range $(-49/50)y'_T \leq y' \leq (49/50)y'_T$. It is obvious that the approximation (4.15) is accurate, except close to the turning points where the WKB approximation breaks down. This problem can be solved by using appropriate connection formulas (FEDORYUK, 1993; HOLMES, 1995).

5 Summary and conclusion

The solutions of Stern's (1963) boundary value problem show that wave attractors represent internal boundary layers (HARLANDER and MAAS, submitted): any energy input will lead to energy accumulation at the wave attractor. Although we considered just the linear time harmonic case here, we can tentatively say that wave breaking and fluid mixing can be expected there. Therefore we think that such layers play an important role for the equatorial dynamics.

Unfortunately, generalized models of zonally symmetric and time harmonic equatorial internal waves cannot be solved for wave attractor frequencies. The question arises if characteristics (and wave attractors) of non-solvable boundary value problems contain similar information as for Stern's solvable problem. To answer the question we considered one simplified and two generalized 'Stern' equations (describing linear zonally symmetric equatorial waves) that are, in contrast to Stern's original model, not solvable. Nevertheless, for almost all surface and bottom boundary shapes, wave attractors are common for all these models.

First, we have formulated general conditions for an exact relationship between characteristic curves and short wave energy (group velocity) rays of general spatial wave equations. Second, we have applied the conditions to the equatorial wave equations under consideration. Most important we found that for the stratified Stern equation (that includes stratification and non-hydrostasy), characteristics agree exactly with short wave energy rays, a property that holds also for Stern's original equation. From this result we can expect that equatorial wave attractors are important also for equatorial inertia-gravity waves, for which we cannot derive exact solutions from characteristic webs. Since the stratified Stern equation is a rather general model for zonally symmetric equatorial motion, including stratification and non-traditional terms, features related to such attractors should be observable in the equatorial ocean (and maybe also in the equatorial atmosphere) that shows indeed more symmetry in the zonal than in the meridional and vertical direction. We have shown that free solutions of the stratified Stern equation can be fairly well approximated by the WKB method. This forms the basis for the ray tracing analysis applied.

Several authors have recently drawn conclusions from the properties of characteristics alone without having solutions of the corresponding boundary value problem at hand (MAAS, 2001; MAAS and HARLANDER, accepted; GERKEMA and SHRIRA, 2005b). It was argued for example that energy density of near inertial waves should blow up in the deep ocean due to the existence of point attractors at intersections of the turning surface with the bottom. In contrast to Stern's equation, such point attractors show up if stratification is considered. The present analysis supports this idea and shows that indeed energy follows the characteristics and should therefore accumulate at the point attractors. The kinetic energy density $E \sim v^2 + w^2 = \phi_0(l^2 + n^2)$ reads by using the results from the WKB analysis $E \sim (l^2 + n^2)/(l + ny)^{1/2}$. Due to subsequent focusing reflections at the lower boundary, the wavenumbers blows up when a wave packet approaches one of the point attractors.

Whereas the energy density flux (i.e. the group velocity) goes to zero when the wavenumbers go to infinity, the energy density E blows up. Of course, linear theory breaks down before a wavepacket can reach an attractor. Very likely, waves will break and fluid mixing will be enhanced in the vicinity of the attractor. Therefore we can state that the idea of enhanced deep ocean mixing by near inertial waves trapped at deep ocean point attractors is consistent in the WKB sense. This concept might therefore be helpful to understand abyssally intensified near inertial motions, recently observed by VAN HAREN et al. (2002).

Finally, we should comment on the conservation properties of waves, trapped by a wave attractor. Usually, the equations describing the propagation of short wavelength waves form a Hamiltonian system (YANG, 1991). Phase space volume is conserved for such systems, a property which excludes attractors. However, the phase space of internal waves, propagating in a meridional plane is four dimensional (y position, z position, and two corresponding wavenumbers l,n). Thus, the existence of an attractor in the projection on the spatial plane does not imply a non-conservative dynamics (BADULIN and SHRIRA, 1993). In fact, wavenumbers diverge for a wave propagating towards an attractor and thus the phase space volume can still be conserved.

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A Derivation of the spherical Stern model

Introducing

$$\hat{u} = u\cos y, \quad \hat{v} = v\cos y, \quad \hat{w} = w\cos^2 y, \quad \hat{p} = p, \tag{A.1}$$

and new coordinates

$$y' = \ln\tan(y/2 + \pi/4), \quad z' = z, \tag{A.2}$$

and writing (3.1)-(3.5) in terms of $\hat{u},\hat{v},\hat{w},\hat{p}$ in the coordinates y',z' we find

$$i\sigma\hat{u} + \hat{w} - g(y')\hat{v} = 0 \tag{A.3}$$

$$i\sigma\hat{v} + g(y')\hat{u} = -\hat{p}_{y'} \tag{A.4}$$

$$-\hat{u} = -\hat{p}_{z'} \tag{A.5}$$

$$\hat{v}_{y'} + \hat{w}_{z'} + g(y')\hat{v} = 0, \tag{A.6}$$

where $g(y') = \sin y = -\tanh y'$ is the Gudermann function (ECKART, 1960). Apart from the third term in the continuity equation, this system is identical to Stern's model. If we eliminate $\hat{u}$, $\hat{w}$ and $\hat{p}$ we obtain (3.14). If we expand the velocity components as $\hat{u} = \hat{u}_0 + \varepsilon\hat{u}_1 + \cdots$, $\varepsilon \ll 1$, assume a small pressure $\hat{p} = \varepsilon\hat{p}_1 + \varepsilon^2\hat{p}_2 + \cdots$, and compress the coordinates $(y',z') = \varepsilon(\tilde{y}',\tilde{z}')$, the third term of the continuity equation can be neglected to the lowest order in ε. Then we can introduce a streamfunction $\hat{v}_0 = -\psi_{\tilde{z}}, \hat{w}_0 = \psi_{\tilde{y}}$ to find

$$\psi_{\tilde{y}\tilde{y}} - (\sigma^2 - g^2)\psi_{\tilde{z}\tilde{z}} + 2g\psi_{\tilde{y}\tilde{z}} + \psi_{\tilde{z}} = 0. \tag{A.7}$$

This is Stern's equation when g is replaced by $\tilde{y}$, but remains separable for other choices of $g(y)$.

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