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MSci Project

Focusing of Inertial Waves

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Contents

Preface

This is my 4$^{\text{th}}$ year project at UCL, supervised by Prof. E. Johnson. This project started at the Royal Netherlands Institute of Oceanography (NIOZ) under the supervision of Prof. L. Maas and Miss A. Rabitti where experiments were carried out with a rotating flat box which was to extend into the full project. Not all of the instrumentation was functioning so the project evolved to include more theory and has inspired what you will see below.

The animations (obviously) cannot be seen on a piece of paper, so I've put them on a website, `http://www.ucl.ac.uk/~zcahdl5/project.html` along with all the data files from the experiments, Matlab and Mathematica code, pictures and assorted other files generated during the computations.

The project is split into the following parts:

Motivation Pictures from other experiments and an explanation of some of the different terms used during the project.

Experiments Here the analysis of the experiments at NIOZ is included, even though they didn't all go to plan they are all included for the sake of completeness.

Flat Box This utilises separation of variables to look at inertial waves inside a flat, rectangular duct. This is also the starting point for investigating inertial waves in a semi-trapezoid.

2D simple attractor in a semi-trapezium Here a similar method to [Maas and Harlander, 2007] is followed to find wave attractors for the 2D case ($k = 0$) in terms of pressure.

3D attractor in a semi-trapezium This chapter extends the previous into the third dimension for a given k, specified to be either decaying or propagating downstream.

Chapter 1

Motivation

1.1 Introduction

We will first look at inertial waves and highlight the differences between these and the more commonly seen surface and acoustic waves. Then wave focusing and ensuing wave attractors found in experiments will be introduced. From here the basic outline of how the project will develop will be described.

1.2 A Brief Introduction to Inertial waves

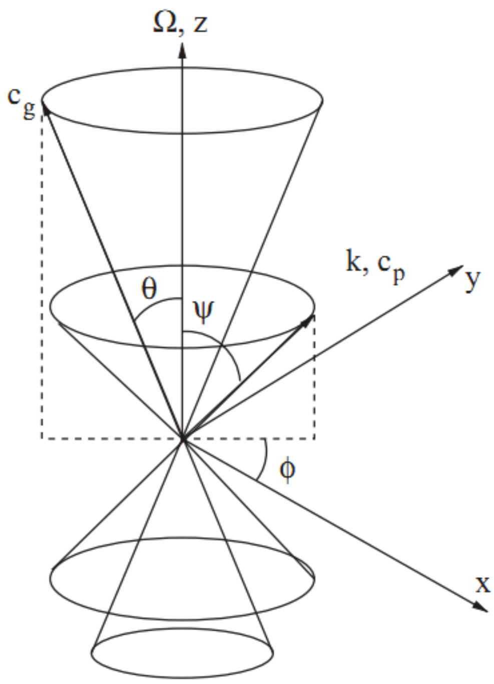
Inertial waves are also called gyroscopic waves, they are waves caused by the effects of rotation. Whereas for a surface wave the restoring force is gravity, for inertial waves the restoring force is the Coriolis force; similarly inertial waves travel through the body of the fluid, which is a notable difference to surface waves.

The frequency of inertial waves, ω, is less than the inertial frequency, f. Where f is twice the rotation rate, Ω. The dispersion relation of these inertial waves is,

$$\omega = \frac{\pm m}{\sqrt{k^2 + l^2 + m^2}} = \sin\psi \tag{1.1}$$

where k, l, m are the wavenumbers in the x, y, z directions and for simplicity denote $\boldsymbol{k} = \begin{pmatrix} k \\ l \\ m \end{pmatrix}$.

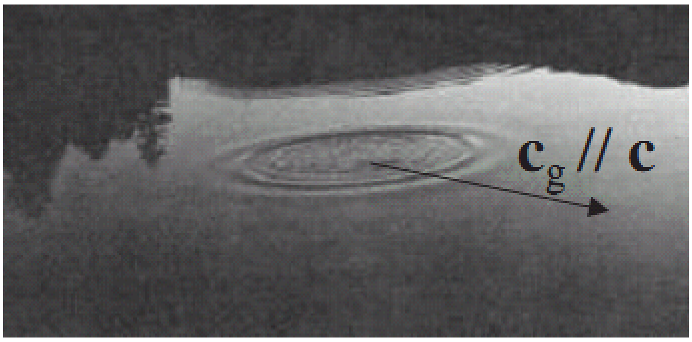
We can see that this can be written as a sine function, forming a cone. This is best demonstrated with a diagram and calculation of the group and phase velocities, c_g and c_p.



Cones of c_p, c_g and $\boldsymbol{k}$ from Manders and Maas [2004]

$$c_g = \nabla\omega \tag{1.2}$$
$$= \frac{m^2}{\omega^3}\left(-k, -l, \frac{k^2+l^2}{m}\right) \tag{1.3}$$
$$c_p = \frac{\omega}{|\boldsymbol{k}|^2}\boldsymbol{k} \tag{1.4}$$
$$= \frac{\omega}{|\boldsymbol{k}|^2}(k, l, m) \tag{1.5}$$
$$c_p \cdot c_g = \frac{m^2}{\omega^2|\boldsymbol{k}|^2}(-k^2 - l^2 + k^2 + l^2) \tag{1.6}$$
$$= 0 \tag{1.7}$$

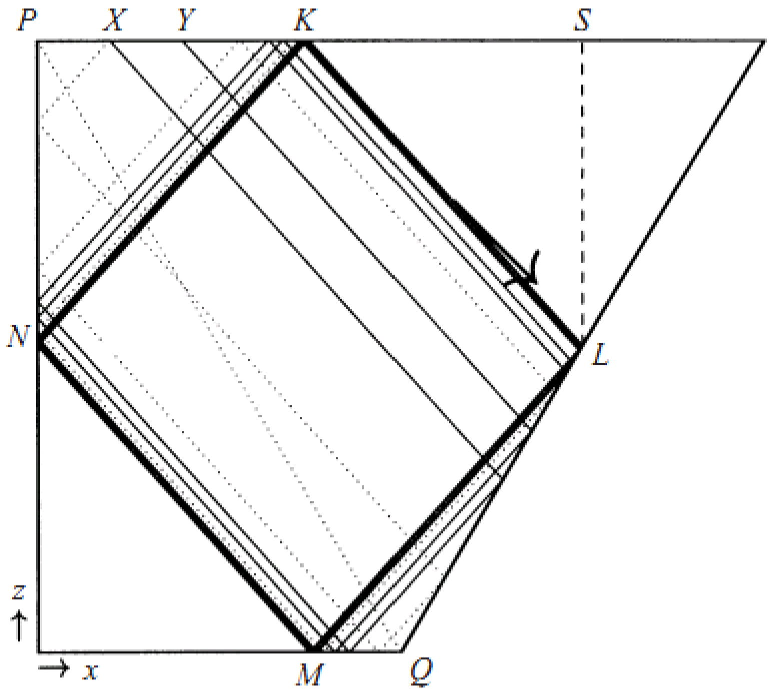
So $c_p \perp c_g$, which is the complete opposite of surface waves where $c_p || c_g$. For example,



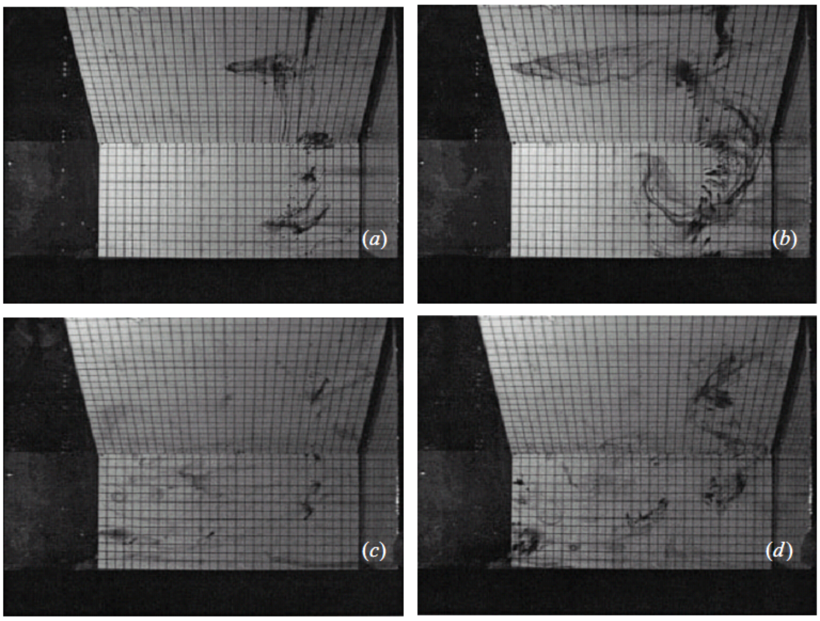
1.2 A typical surface wave, a ripple on a pond. From Maas [2005]

1.3 Focusing of Inertial Waves

Since these waves can only travel in certain directions dictated by the cones seen above, if we look in a non-symmetric domain these waves are either focused or defocused by reflection. Take the the semi-trapezoid seen in this diagram below,



1.3: Inertial wave focusing in a semi-trapezoid, from Maas [2001]

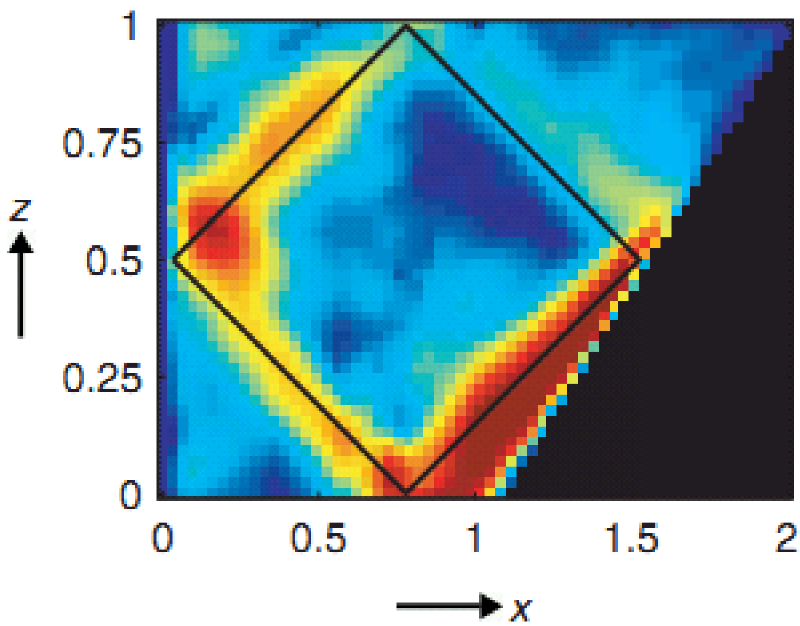


1.3: Description below, from Maas [2001].

The top figure in 1.3 is a 2D example where the cone is reduced to two directions, $z = \pm x$. All non-degenerate waves focus in the wave attractor, LKNM. Take three examples, $z = -x$ at X, $z = -x$ at Y and $z = x$ at X. It is seen by following the dashed lines on the diagram that these focus on the attractor and they all travel in the same clockwise direction, as indicated by the arrow. For a different rotation frequency the angle would be such that the degenerate attractor PQ would form. As displayed in the figures (c) and (d) of the second figure. Here it is thought viscous damping removes almost all structure.

Now for the second figure in 1.3 encompassing (a)-(d). The following description is taken almost verbatim from Maas [2001] with only minor alterations to write it as prose instead of a caption for the figure.

(a) is a top view of the right half of the container, with the sloping wall visible in the upper half of the figure. The bottom and sloping wall have been supplemented with a 2 cm grid (distorted along the slope). Dye has been injected through a number of holes (of diameter ≈ 1 mm) in the glass top lid, at 14 cm from the vertical glass wall, at the right ($y = 0$). Modulation period is 32.4 s. Dye pictures, shown here, are taken at the moment that the dye reaches its most rightward position over the slope. In (b) the picture was taken 6 modulation periods after that in (a), showing the clear leftward spreading of dye over the central part of the slope in the upper half of the figure (revealing a cyclonic mean flow, i.e. in the same direction as the anti-clockwise background rotation). (c) is as in (a), but with modulation period 37.7 s. Dye injections have additionally been made at some holes in the centre, and on the lower left of this figure. Likewise (d) is as in (c), but taken 5 periods later. Dye seems to have spread diffusively. No signs of significant mean flow were observed in this case.



1.4: PIV of $x^2 + z^2$, kinetic energy. From Manders and Maas [2003]

In 1.4 it is clear that there is a concentration of energy on the slope where the waves focus, this energy gradually dissipates as it travels around the attractor.

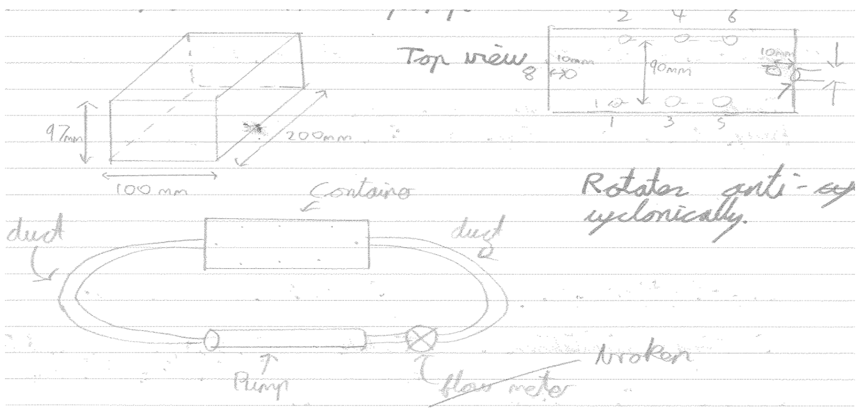
There is clearly something to look into here. Most of the papers available look into the 2D stratified, non-rotating case; however this paper will look into the three dimensional rotating case. The first step is to look at a domain where separation of variables is valid and then perturb the boundary to look at 2D rotating flow before extending this to three dimensional, rotating flow. This three dimensional approach has not been found in any of the literature to date.

Chapter 2

Experiments

2.1 Set-Up

The apparatus was set up as in the two figures of 2.1. The flow in all cases is in the direction from pressure sampling point 7 to 8 and sampling points 5 and 6 were connected to the pressure sensor. The rotation was anti-cyclonic (clockwise looking from above) and an increase in voltage output from the pressure sensor corresponds to higher pressure at sampling point 6.



Drawing of set-up of experiment

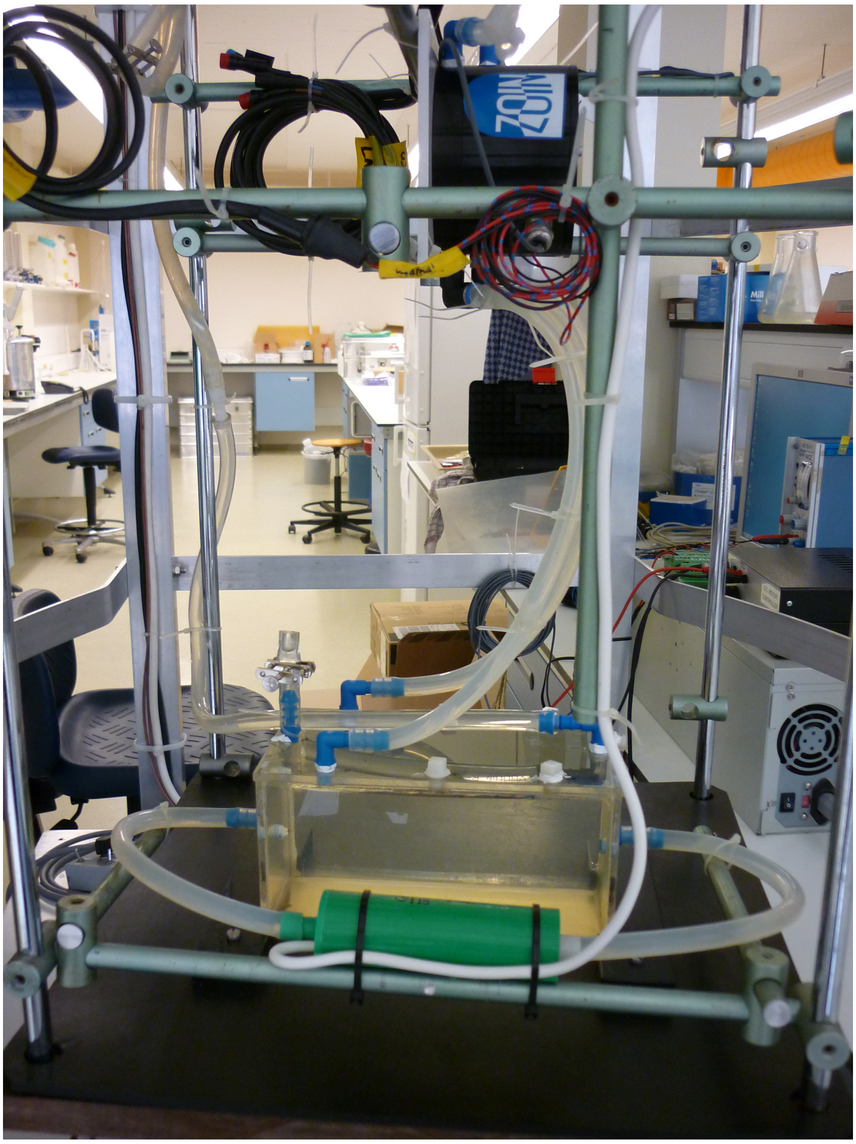


Photo of set-up of experiment

Pump	Comet - Geoinline 0-11V
Flow meter	N/A (broken)
Container	Perspex 200x100x97mm
Ducts	8mm Internal Pump→7 30cm, 8→pump 20cm
Pressure sensor	GE Druck - LPM 5480 -2 – 2 mbar, output 0-10V, precision 8x10^{-4}bar
Computer	Labview 6 (National Instruments)
Circuit	National Instruments CB-68LP
Table	NIOZ designed, motor KMF WD251 (Electro ABI)

2.1.1 Calibration

No direct calibration was done however the calibration using the same equipment done in a previous paper was

$$q[l/min] = 0.4572V_q + 0.0639$$
$$\Omega[rad/s] = 1.004V_\Omega - 0.2942$$

2.2 Results

In the following plots there is one plot per table voltage and then the output for the spectral density computed using a fast fourier transform is displaced vertically for each pump voltage. So the third line up on the second plot will be the third pump voltage and the second table voltage.

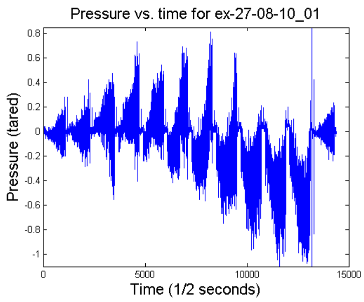
2.2.1 Good data

This was the first set of data obtained, which appears to be very good.

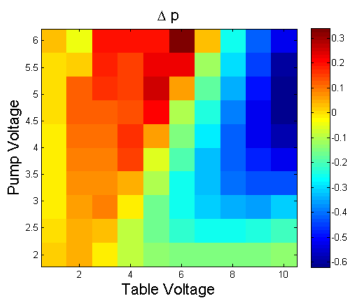
ex-27-08-10_01

2Hz 100s

Table speeds	0-10 in steps of 1, 0
Pump speeds	0, 2-6 in steps of 4/9, 0



Pressure vs. Time



Colour chart

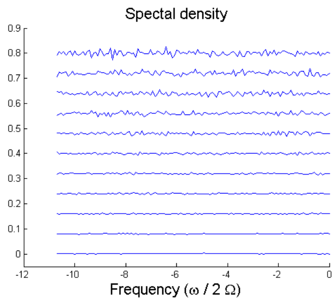


Table voltage 0

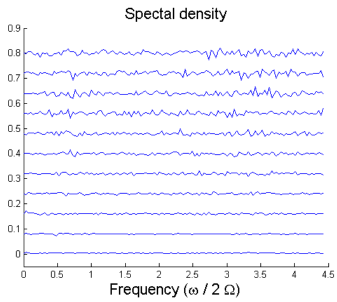


Table voltage 1

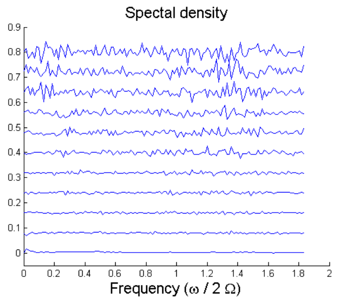


Table voltage 2

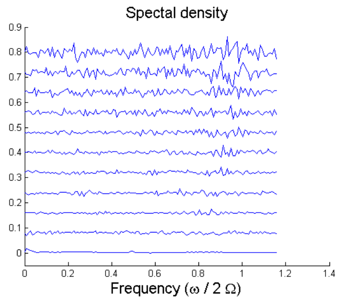


Table voltage 3

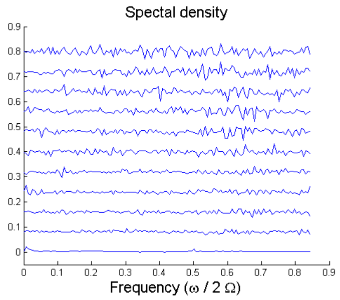


Table voltage 4

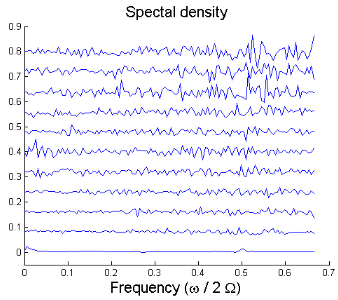


Table voltage 5

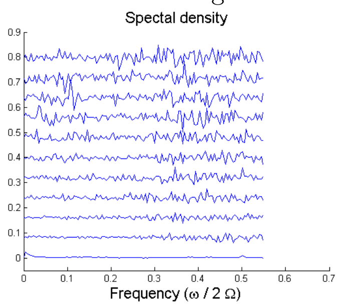


Table voltage 6

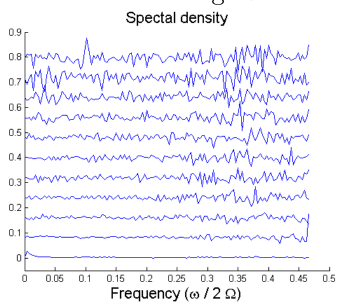


Table voltage 7

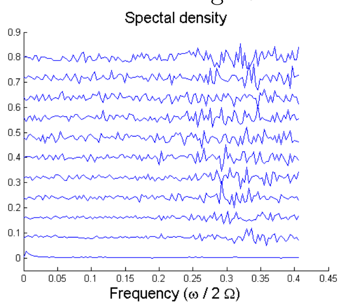


Table voltage 8

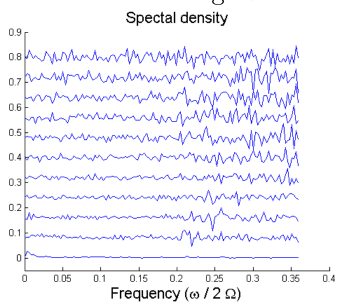


Table voltage 9

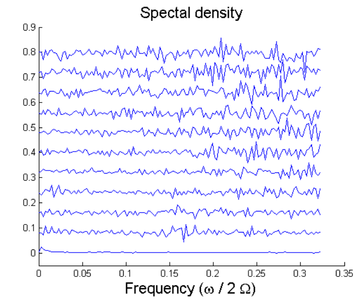


Table voltage 10

The colour chart ties in very well with [Maas, 2007] so looks to be a reliable set of data.

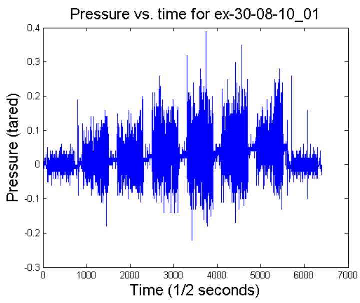
2.2.2 Different sample rates

Testing three sample rates: 2Hz; 4Hz; 8Hz for 100s

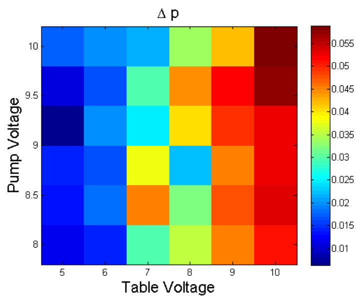
ex-30-08-10_01

2Hz, 100s.

Table speeds	0, 5-10 in steps of 1, 0
Pump speeds	0, 8-10 in steps of 2/5, 0



Pressure vs. Time



Colour Chart

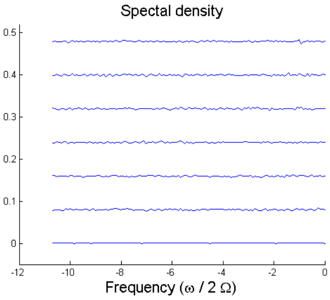


Table voltage 0

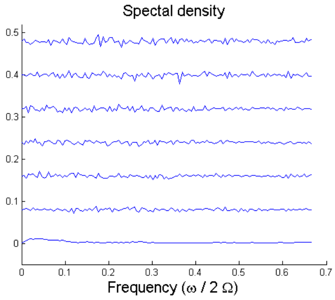


Table voltage 5

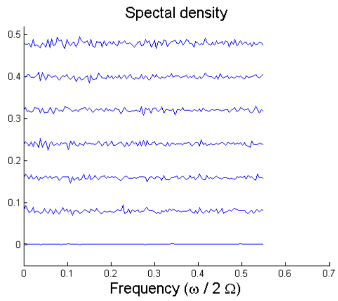


Table voltage 6

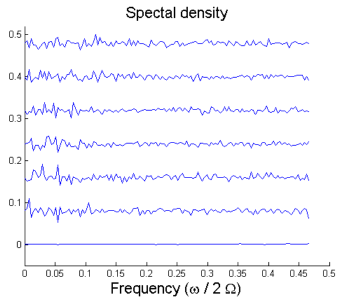


Table voltage 7

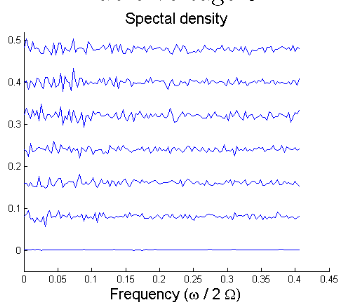


Table voltage 8

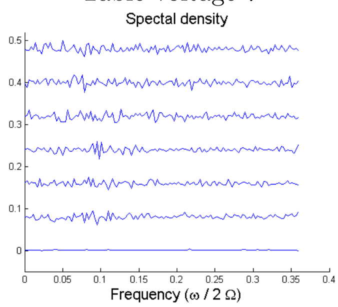


Table voltage 9

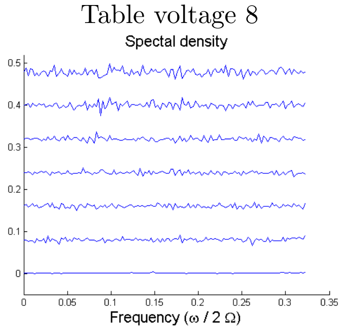


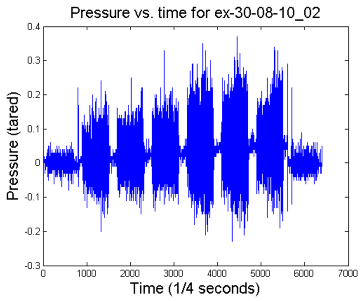
Table voltage 10

These results show that there is little variation in any of the spectral densities or in the colour charts with pump voltage. Also the pressure vs. time plot is very noisy, this is seen in all of the following experiments until it was realised that the pump was faulty. Up until that point the data is mostly useless. The useful data resumes at “Repeating ‘Good Data’ to check pump” (Pg 36) and the experiments between here and then are included only for completeness.

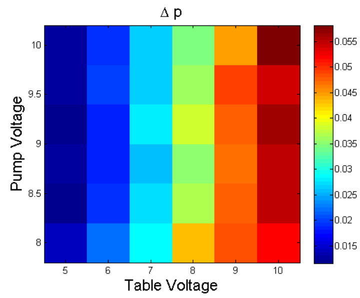
Two useful things to come out of this experiment is that it has been seen that at least 4Hz is needed to get a full range for the spectral densities and that without the excitation of the pump there are no peaks seen in the spectral density - so these can be viewed as “control” experiments.

ex-30-08-10_02

Repeat of ex-30-08-10_01 with 4Hz 100s



Pressure vs. Time



Colour Chart

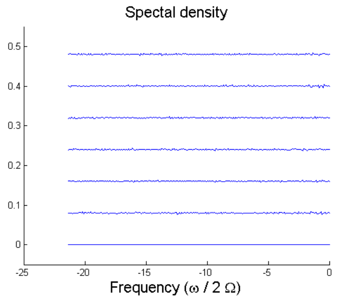


Table voltage 0

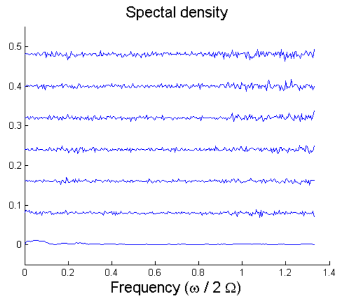


Table voltage 5

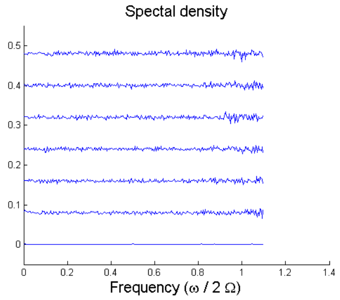


Table voltage 6

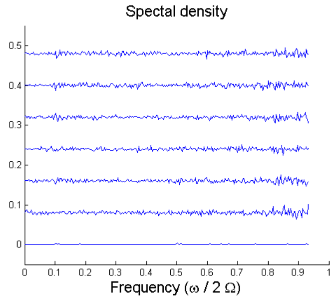


Table voltage 7

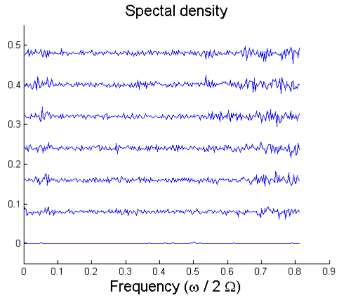


Table voltage 8

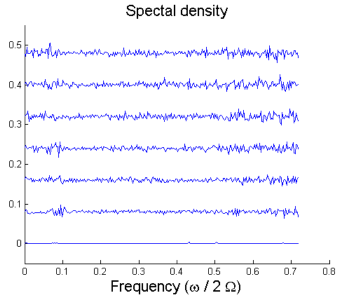


Table voltage 9

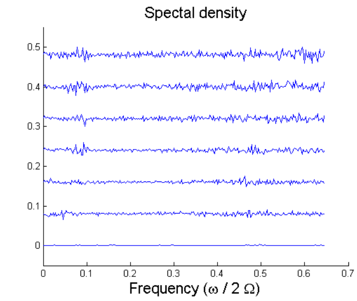
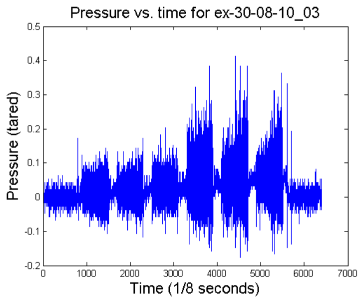


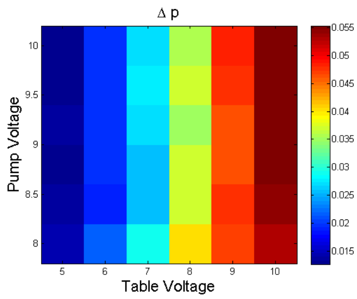
Table voltage 10

ex-30-08-10_03

Repeat of ex-30-08-10_01 with 8Hz 100s



Pressure vs. Time



Colour Chart

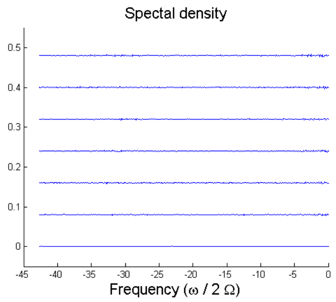


Table voltage 0

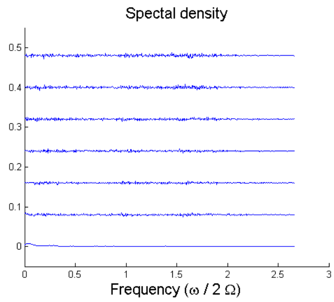


Table voltage 5

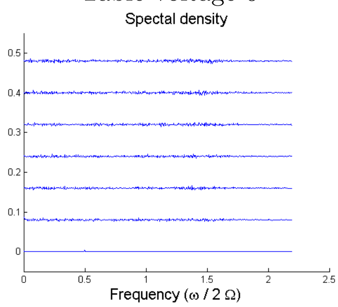


Table voltage 6

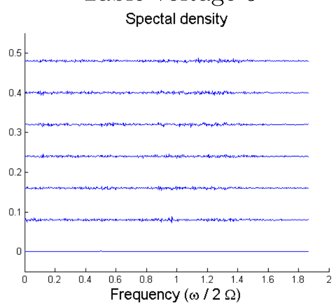


Table voltage 7

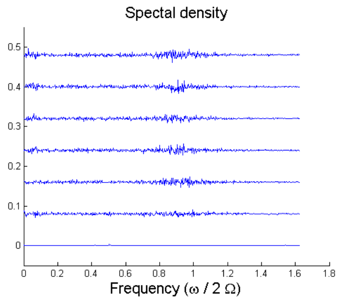


Table voltage 8

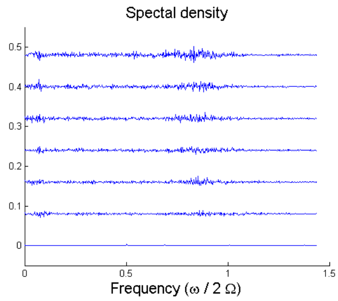


Table voltage 9

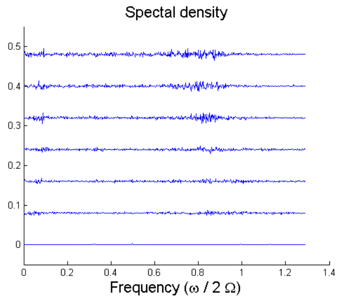


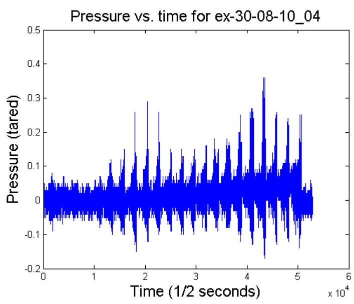
Table voltage 10

2.2.3 Wide spectrum (fine)

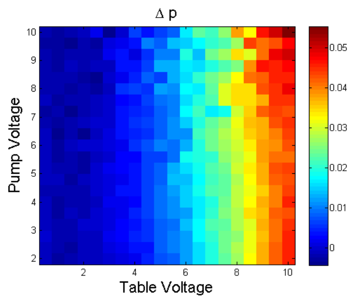
ex-30-08-10_04

A long experiment looking over the whole range available over the maximum permitted value of twenty steps. 2Hz 100s

Table speeds	0-10 in steps of 1/2, 0
Pump speeds	0, 2-10 in steps of 2/5, 0



Pressure vs. Time



Colour Chart

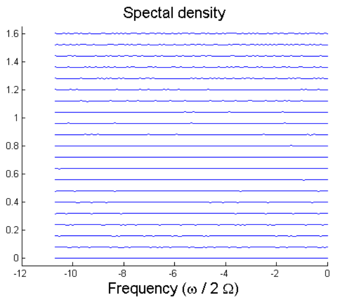


Table voltage 0

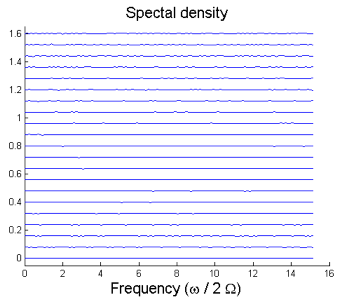


Table voltage 1

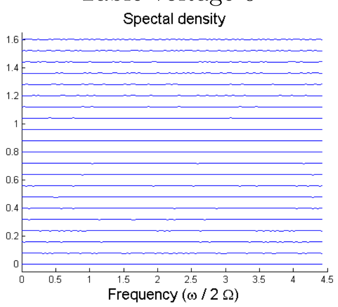


Table voltage 2

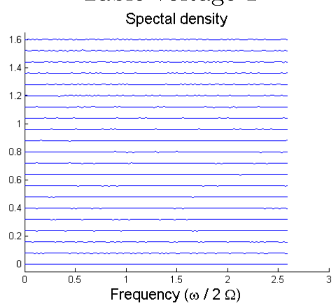


Table voltage 3

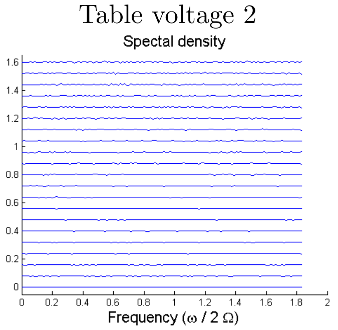


Table voltage 4

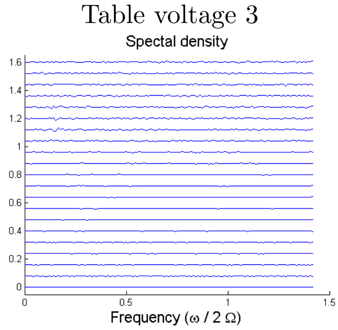


Table voltage 5

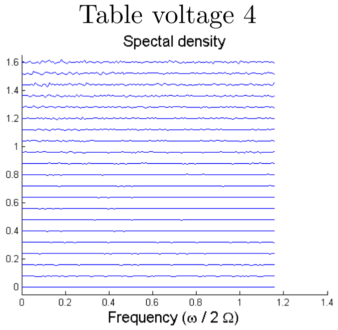


Table voltage 6

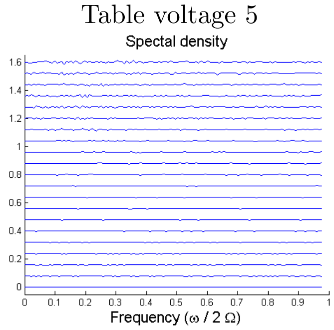


Table voltage 7

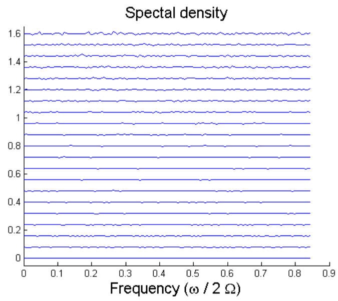


Table voltage 8

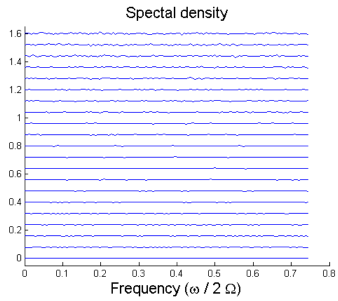


Table voltage 9

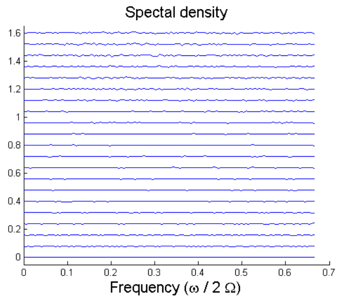


Table voltage 10

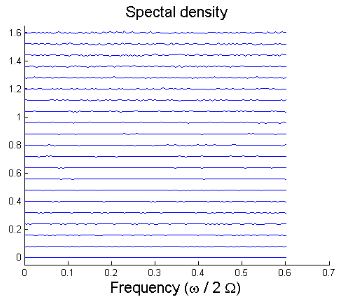


Table voltage 11

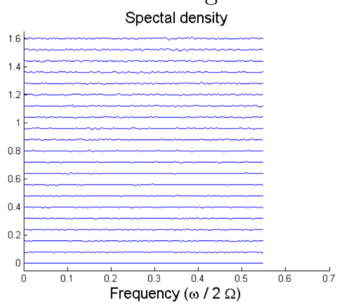


Table voltage 12

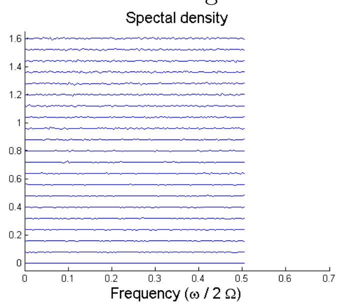


Table voltage 13

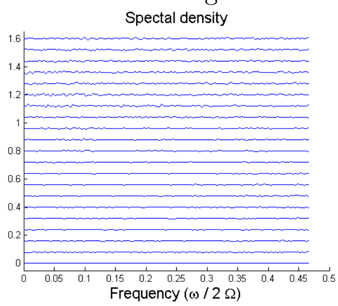


Table voltage 14

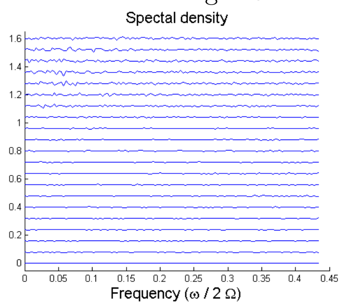


Table voltage 15

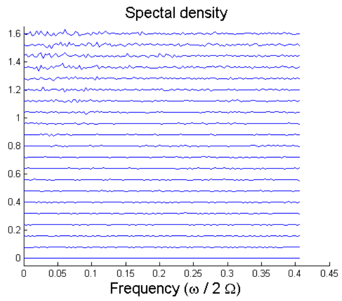


Table voltage 16

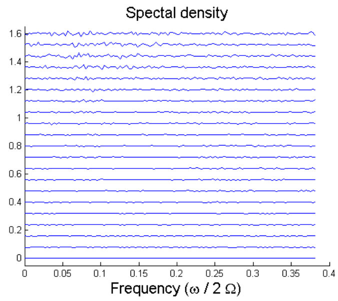


Table voltage 17

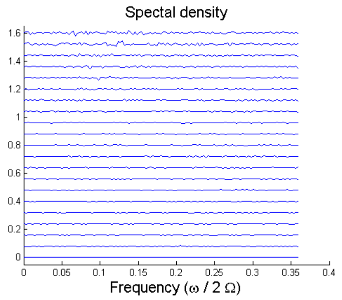


Table voltage 18

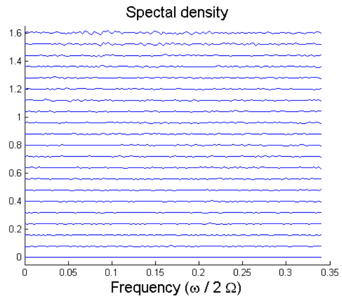


Table voltage 19

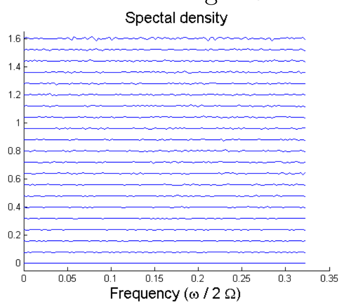


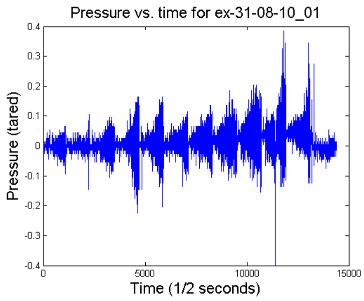
Table voltage 20

2.2.4 Wide Spectrum

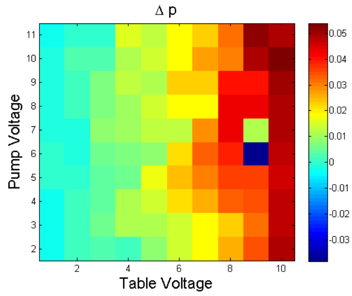
Five identical experiments designed to test the repeatability of the data. ex-01-09-10_02 was stopped early to allow time for ex-01-09-10_03. All at 2Hz 100s

Table speeds	0-10 in steps of 1, 0
Pump speeds	0 2-11 in steps of 1, 0

ex-31-08-10_01



Pressure vs. Time



Colour Chart

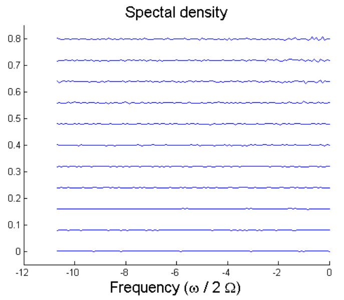


Table voltage 0

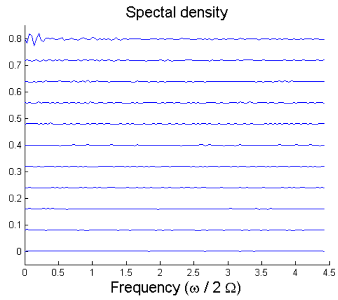


Table voltage 1

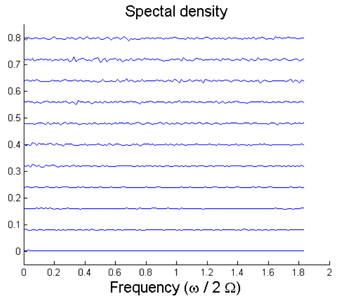


Table voltage 2

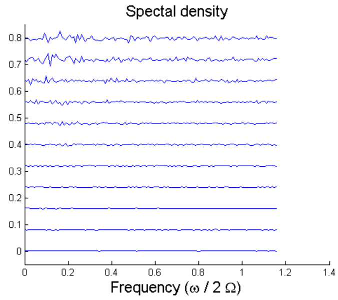


Table voltage 3

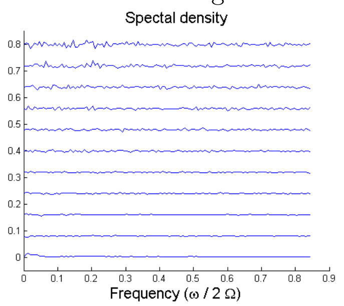


Table voltage 4

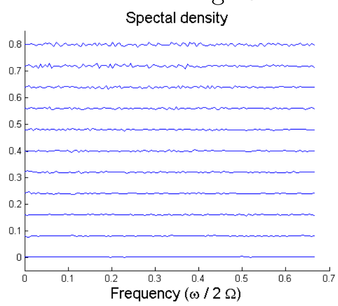


Table voltage 5

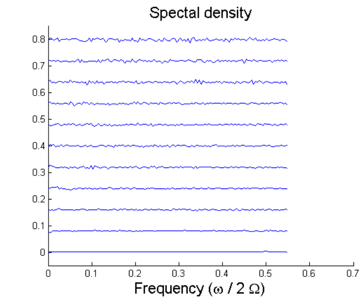


Table voltage 6

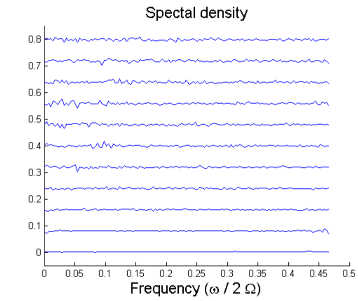


Table voltage 7

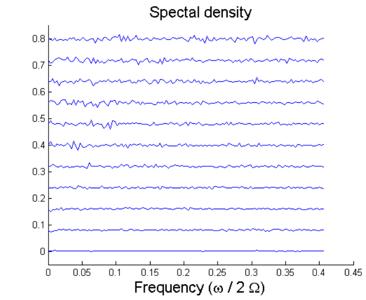


Table voltage 8

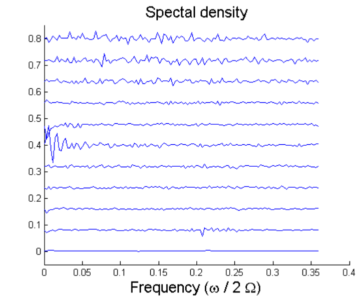


Table voltage 9

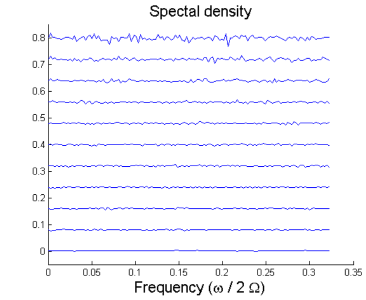
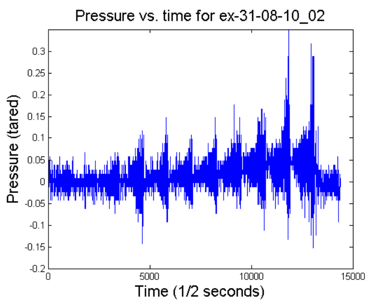
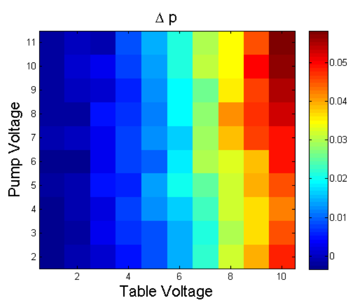


Table voltage 10

ex-31-08-10_02



Pressure vs. Time



Colour Chart

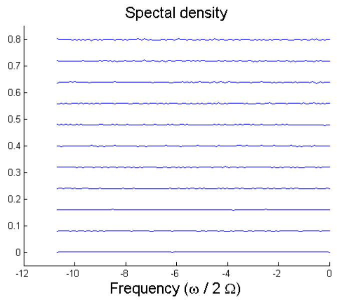


Table voltage 0

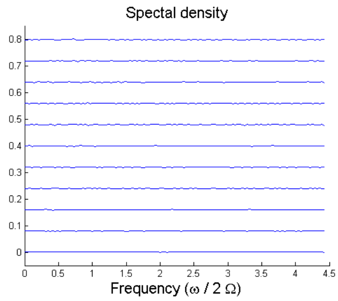


Table voltage 1

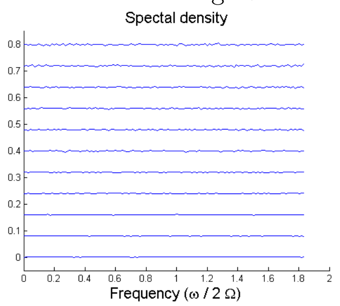


Table voltage 2

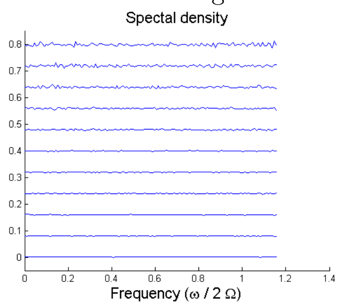


Table voltage 3

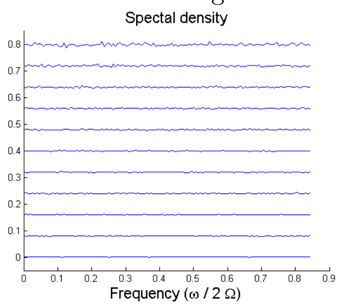


Table voltage 4

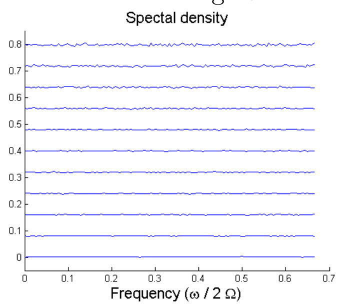


Table voltage 5

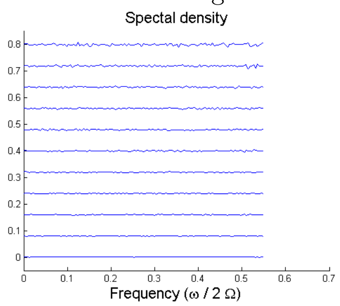


Table voltage 6

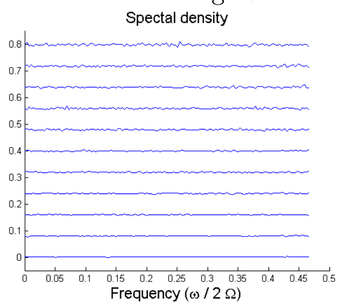


Table voltage 7

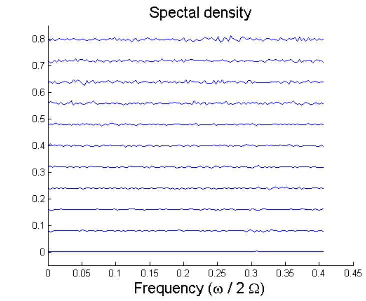


Table voltage 8

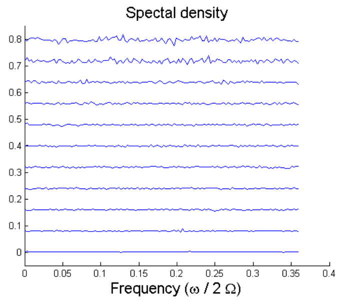


Table voltage 9

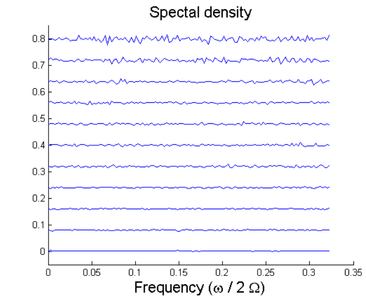
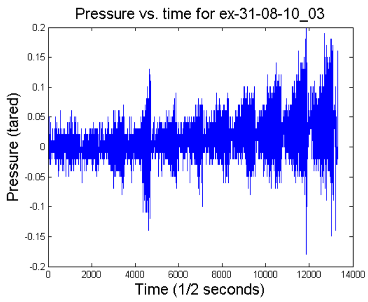
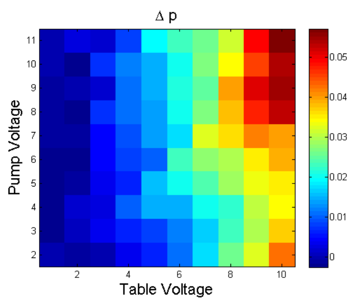


Table voltage 10

ex-31-08-10_03



Pressure vs. Time



Colour Chart

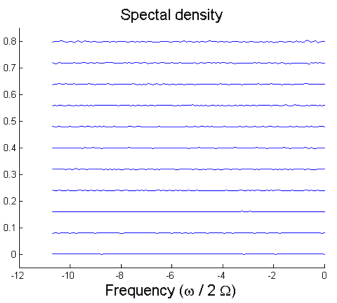


Table voltage 0

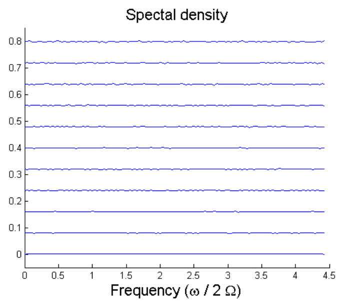


Table voltage 1

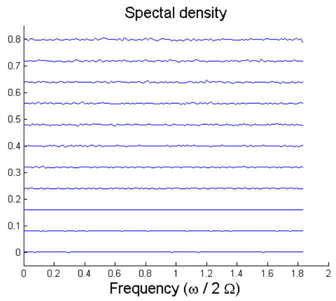


Table voltage 2

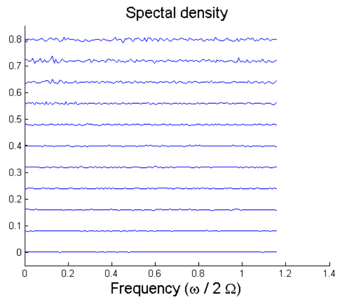


Table voltage 3

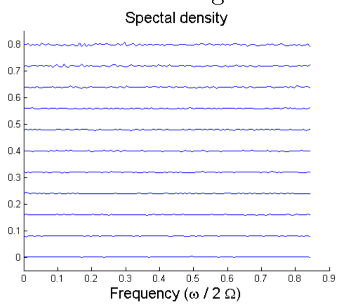


Table voltage 4

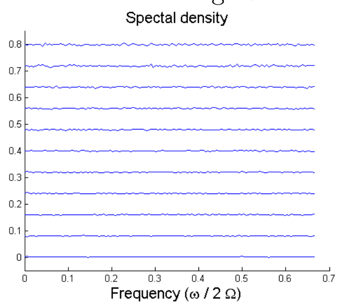


Table voltage 5

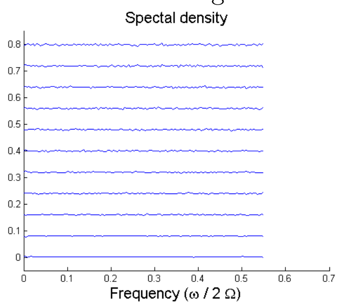


Table voltage 6

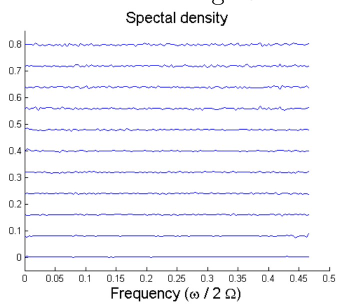


Table voltage 7

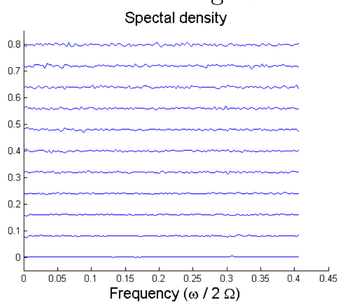


Table voltage 8

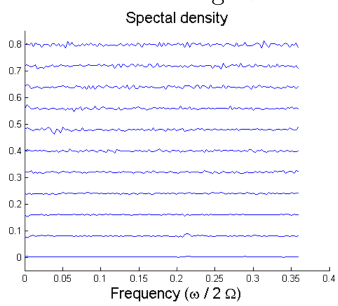


Table voltage 9

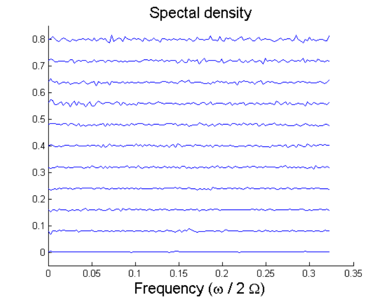
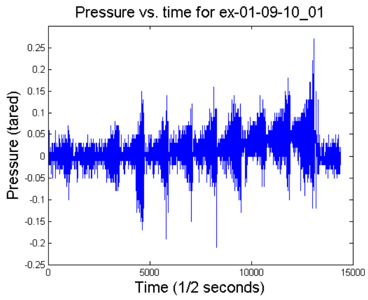
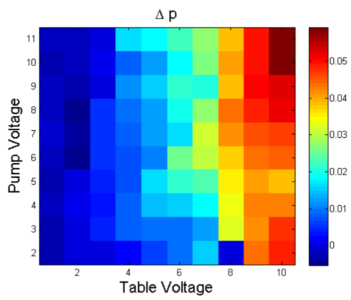


Table voltage 10

ex-01-09-10_01



Pressure vs. Time



Colour Chart

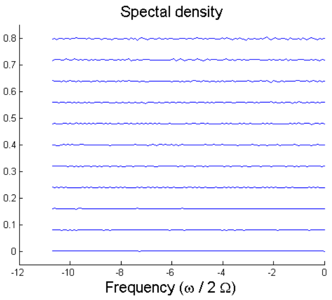


Table voltage 0

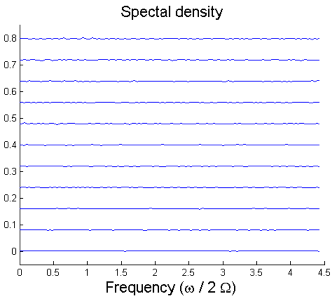


Table voltage 1

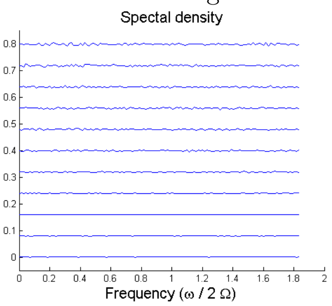


Table voltage 2

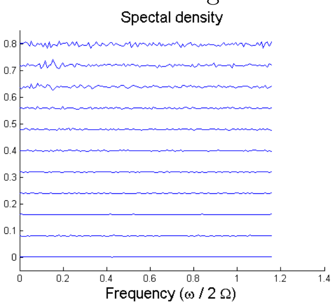


Table voltage 3

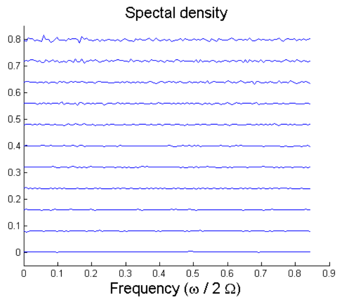


Table voltage 4

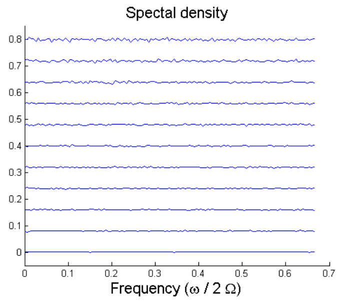


Table voltage 5

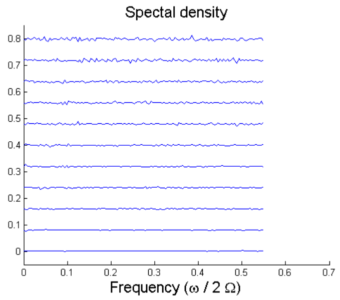


Table voltage 6

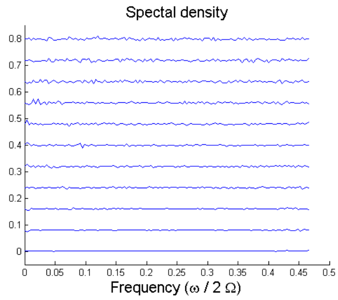


Table voltage 7

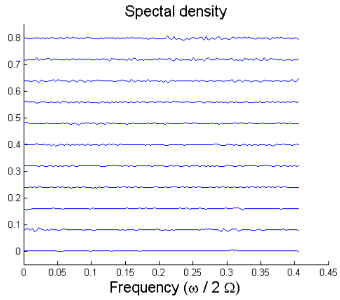


Table voltage 8

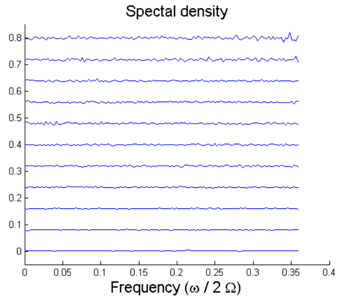


Table voltage 9

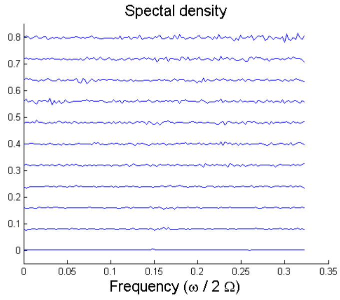
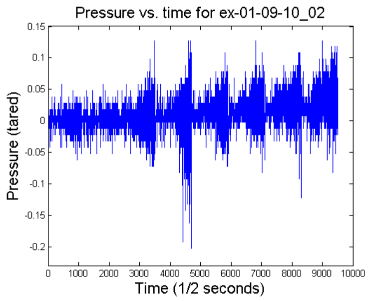
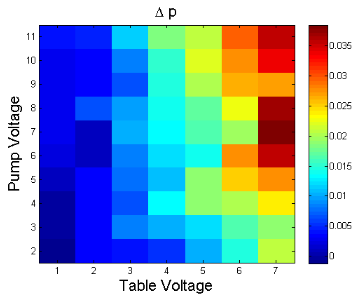


Table voltage 10

ex-01-09-10_02



Pressure vs. Time



Colour Chart

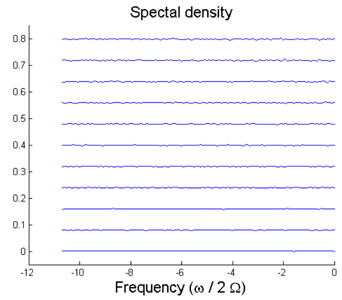


Table voltage 0

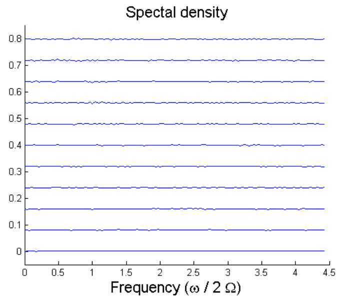


Table voltage 1

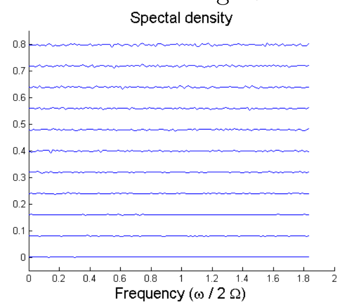


Table voltage 2

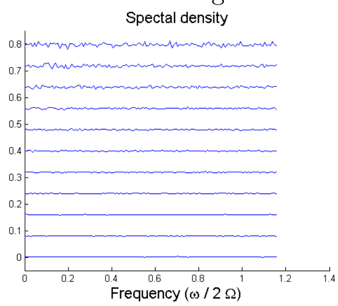


Table voltage 3

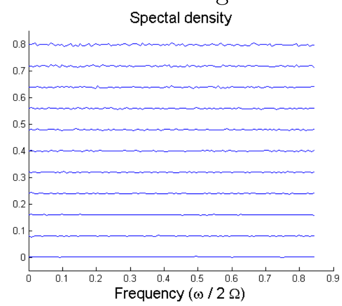


Table voltage 4

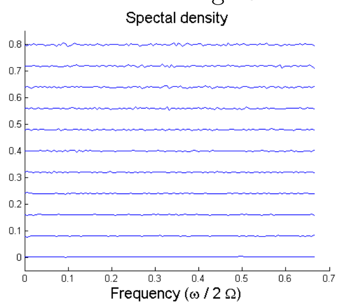


Table voltage 5

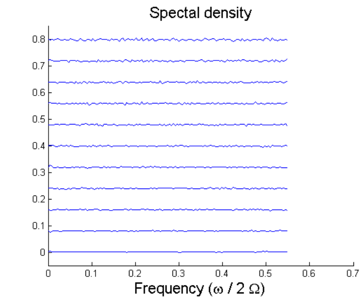


Table voltage 6

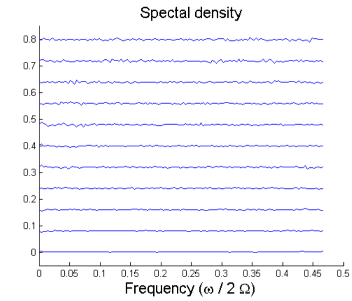


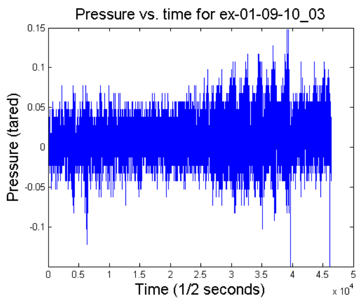
Table voltage 7

2.2.5 Focused 2-6

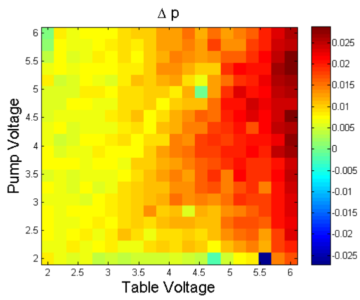
ex-01-09-10_03

A closer look at the same range used in [Maas, 2007]. 2Hz 100s

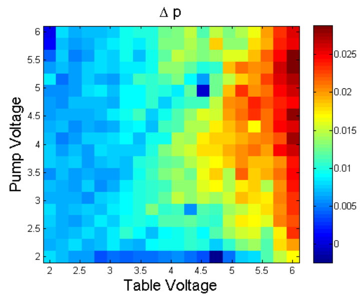
Table speeds	0, 2-6 in steps of $4/19$, 0
Pump speeds	0 2-6 in steps of $4/19$, 0



Pressure vs. Time



Colour Chart



Colour chart with average value taken for table $5^{11}/19$ pump 2

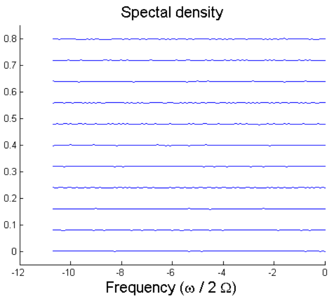
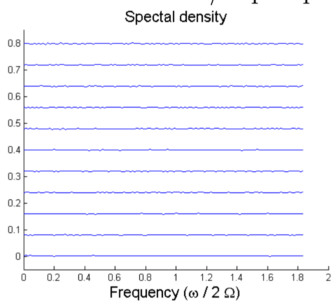


Table voltage 0



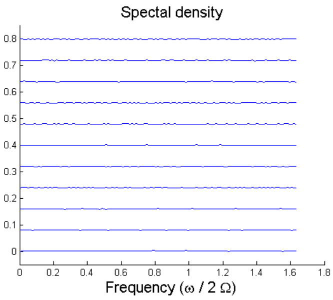


Table voltage 2

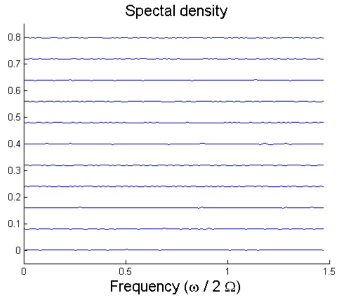


Table voltage $2^{4/19}$

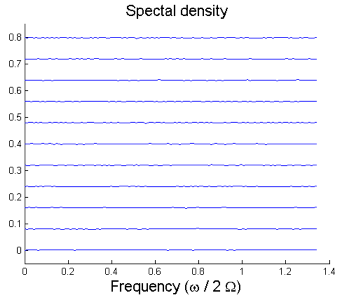


Table voltage $2^{8/19}$

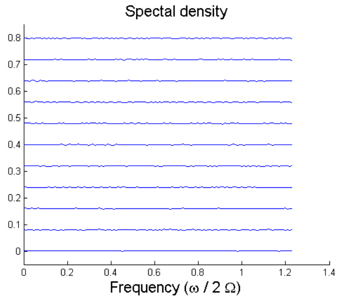


Table voltage $2^{12/19}$

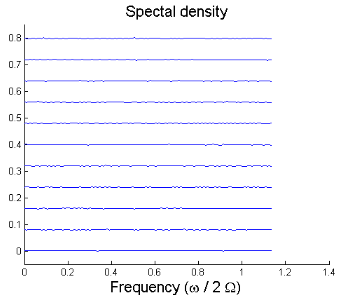


Table voltage $2^{16/19}$

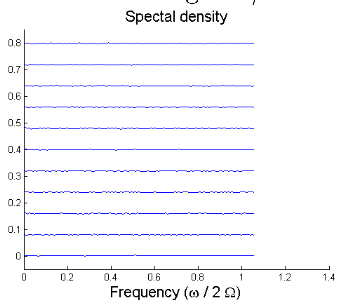


Table voltage $3^{1/19}$

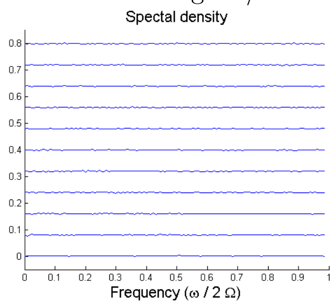


Table voltage $3^{5/19}$

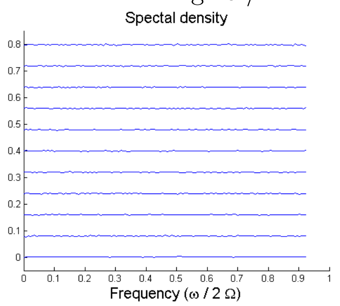


Table voltage $3^{9/19}$

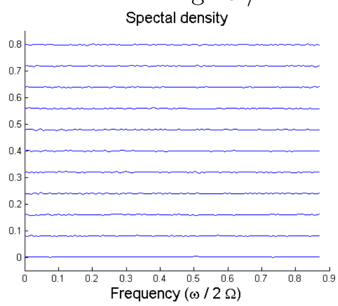


Table voltage $3^{13/19}$

Table voltage $3^{17/19}$

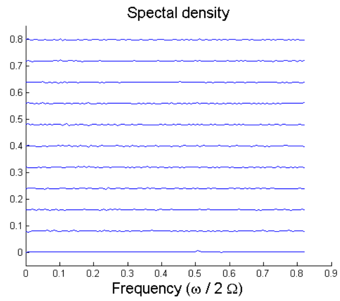


Table voltage $4\,{}^{2}/_{19}$

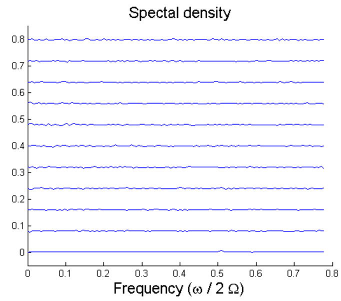


Table voltage $4\,{}^{6}/_{19}$

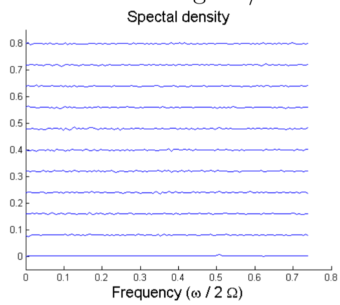


Table voltage $4\,{}^{10}/_{19}$

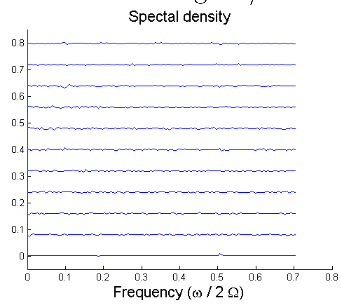


Table voltage $4\,{}^{14}/_{19}$

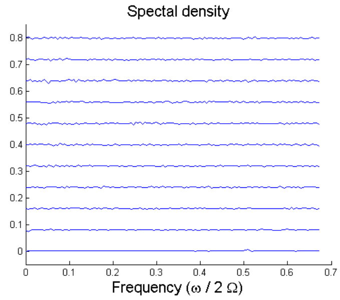


Table voltage $4\,{}^{18}/_{19}$

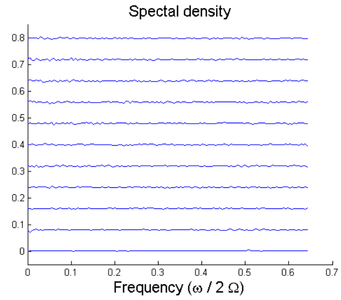


Table voltage $5\,{}^{3}/_{19}$

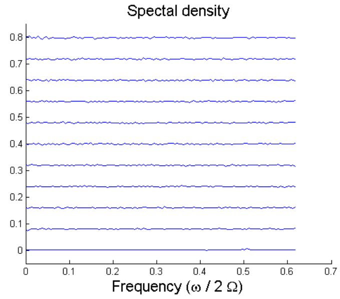


Table voltage $5\,{}^{7}/_{19}$

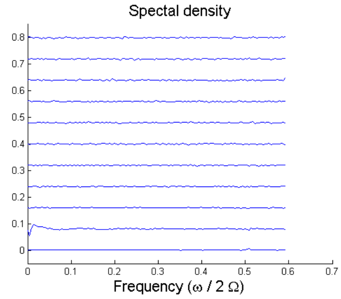


Table voltage $5\,{}^{11}/_{19}$

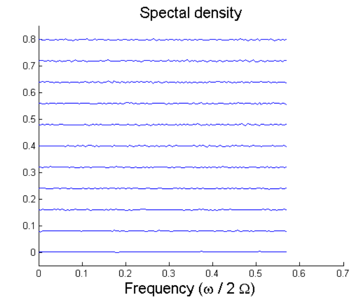


Table voltage $5\,^{15}/_{19}$

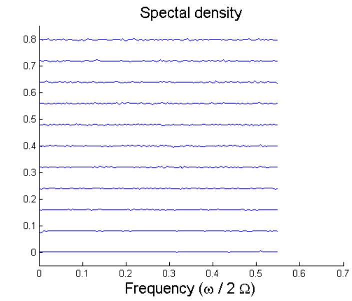


Table voltage 6

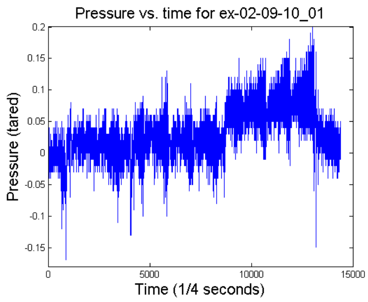
2.2.6 Displaced

ex-02-09-10_01

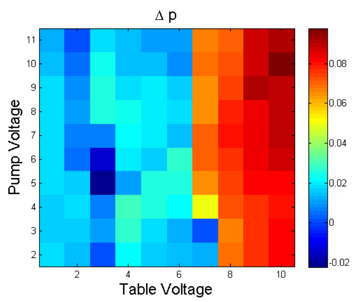
This was attempted after the tank was displaced 7cm (2.76") in the direction 6→5, initialisation file identical to ex-31-08-10_01.

At this point the analysis of the data comparing different sample rates was used and a sample rate of 4Hz was decided upon. So this experiment is taken at 4Hz, 100s.

Table speeds	0-10 in steps of 1, 0
Pump speeds	0 2-11 in steps of 1, 0



Pressure vs. Time



Colour Chart

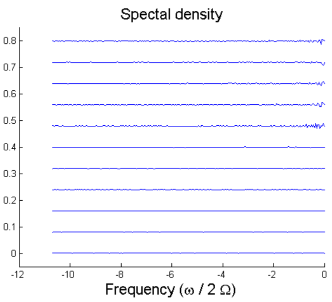


Table voltage 0

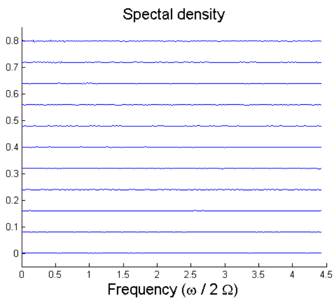


Table voltage 1

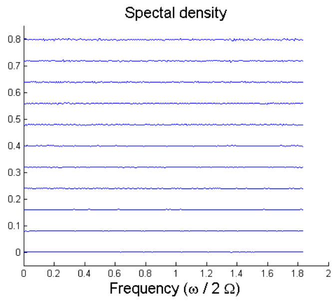


Table voltage 2

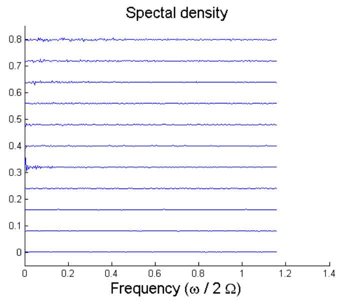


Table voltage 3

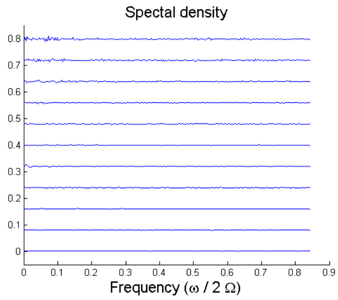


Table voltage 4

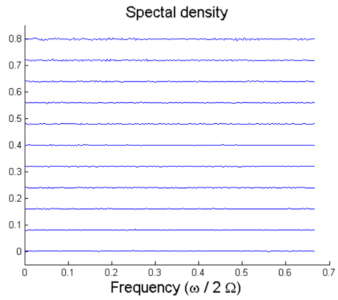


Table voltage 5

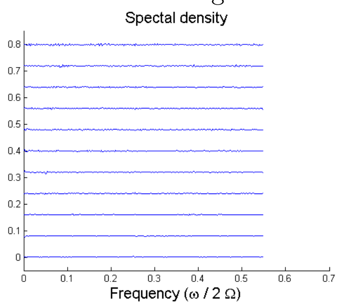


Table voltage 6

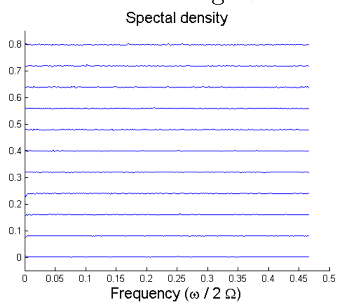


Table voltage 7

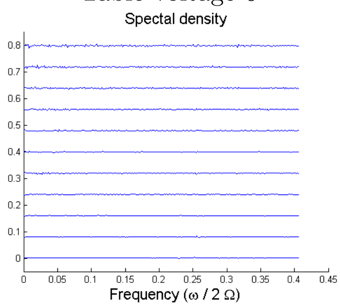


Table voltage 8

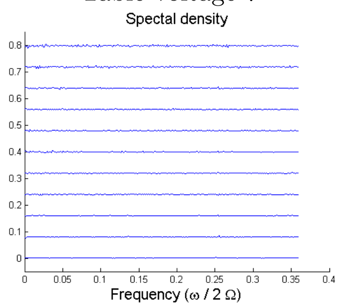


Table voltage 9

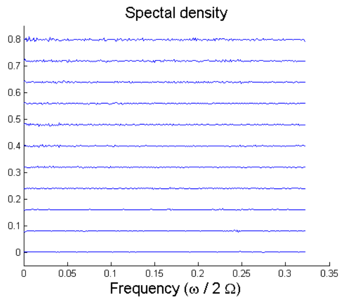


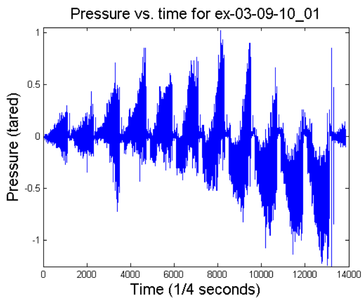
Table voltage 10

2.2.7 Repeating 'Good Data' to check pump

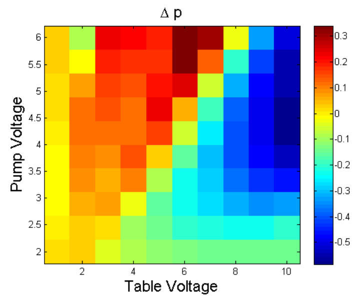
After the following it was realised that the experiment taken on Friday (ex-27-08-10_01) (Pg 13) contained useful data, however all experiments proceeding this did not due to a fault with the pump. The pump was fixed and we repeated the one experiment we had good data from, ex-27-08-10_01, to check our apparatus. This was measured at 4Hz for 100s.

Table speeds	0-10 in steps of 1, 0
Pump speeds	0, 2-6 in steps of $^4/_9$, 0

ex-03-09-10_01



Pressure vs. Time



Colour chart

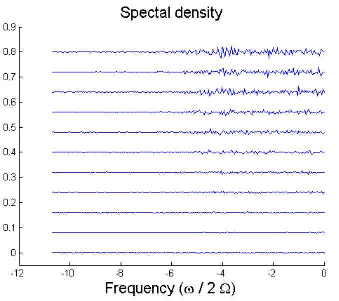


Table voltage 0

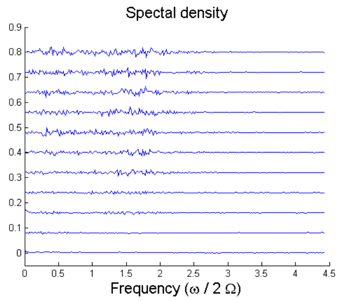


Table voltage 1

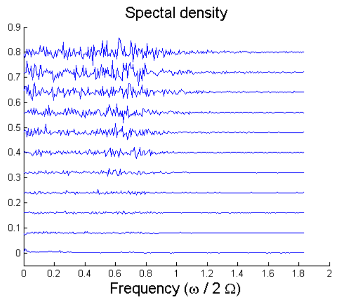


Table voltage 2

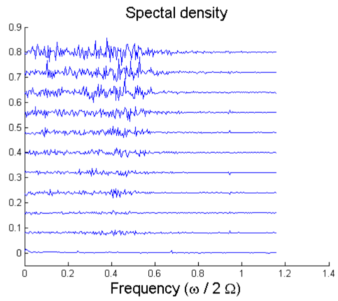


Table voltage 3

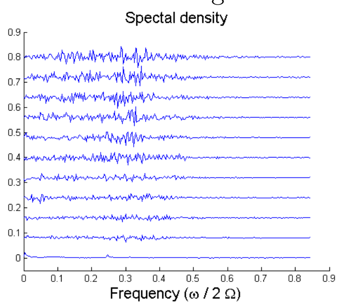


Table voltage 4

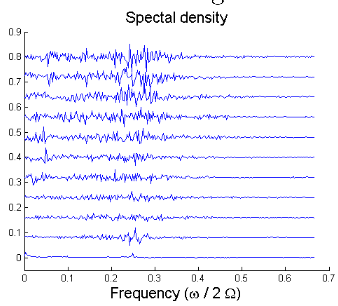


Table voltage 5

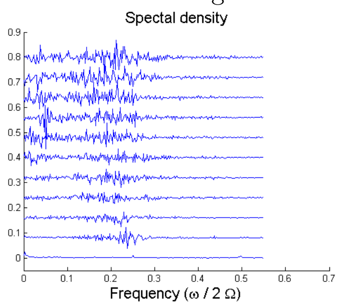


Table voltage 6

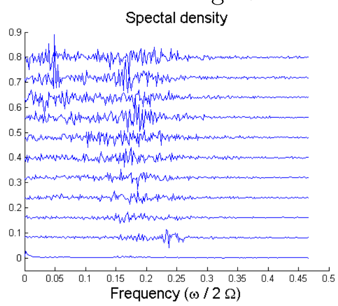


Table voltage 7

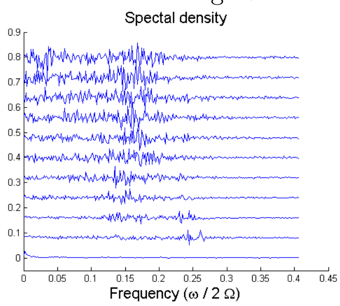


Table voltage 8

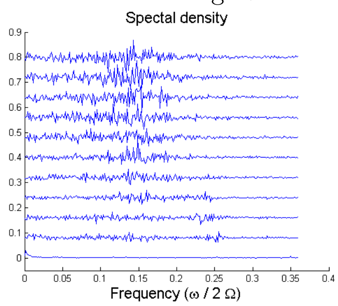


Table voltage 9

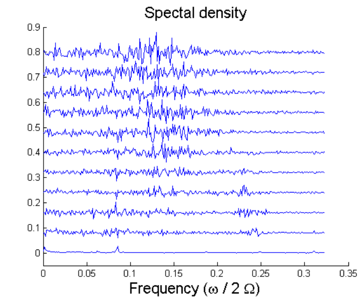
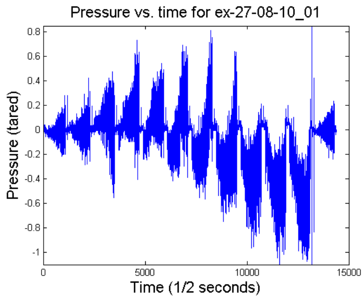


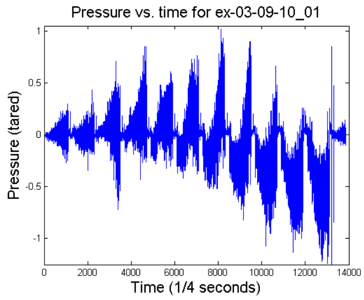
Table voltage 10

2.2.8 Comparison

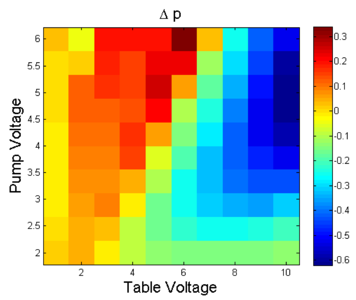
A short piece which compares "Good Data" (Pg 13) and "Repeating 'Good Data' to check pump" (Pg 36) before comparing the results.



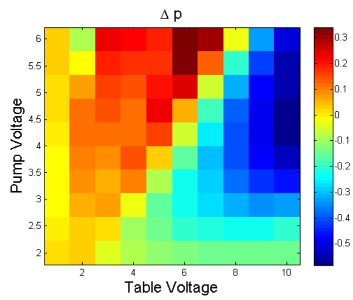
Good data
Pressure vs. Time



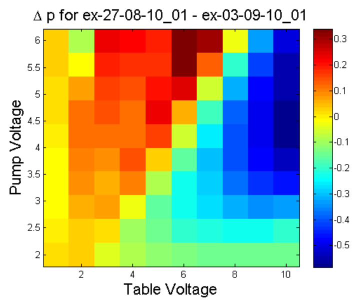
Repeating Good data
Pressure vs. Time



Good data
Colour chart



Repeating Good data
Colour chart



Difference in Δ p

When we look at the data the values are proportional to each other and when they are plotted next to each other the similarities are obvious. From this it would appear our techinal problems are solved.

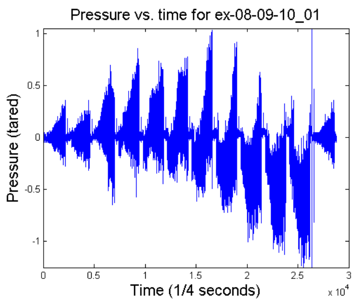
2.2.9 After I Left

This is data sent to me by Anna Rabitti after I left Holland, measured at 4Hz for 200s.

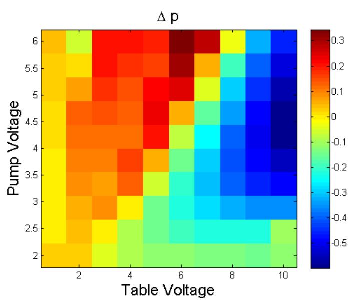
ex-08-09-10_01

Initialisation file identical to ex-27-08-10_01.

Table speeds	0-10 in steps of 1, 0
Pump speeds	0, 2-6 in steps of 4/9, 0



Pressure vs. Time



Colour chart

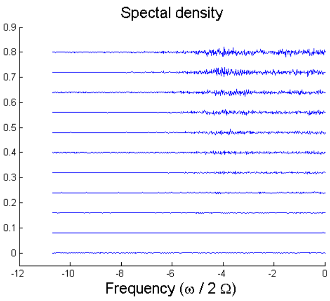


Table voltage 0

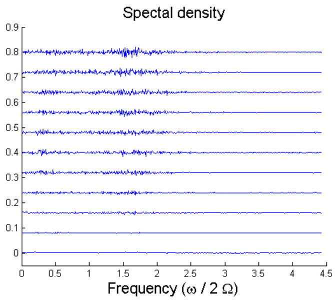


Table voltage 1

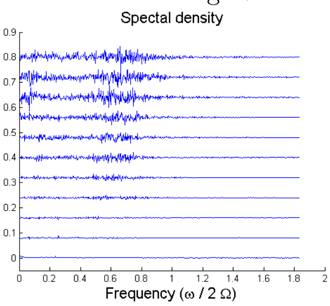


Table voltage 2

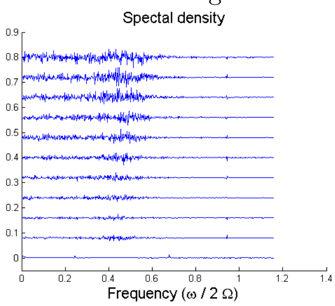


Table voltage 3

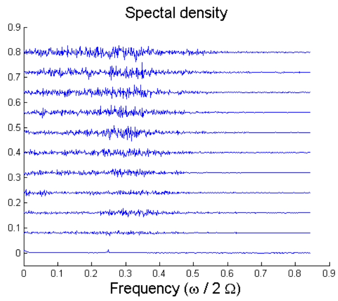


Table voltage 4

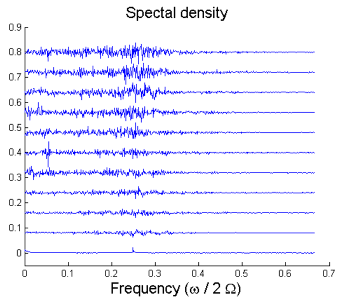


Table voltage 5

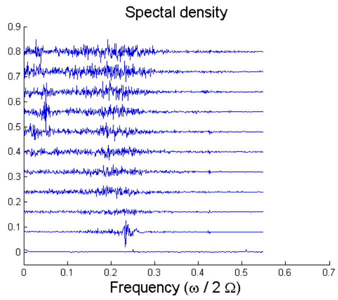


Table voltage 6

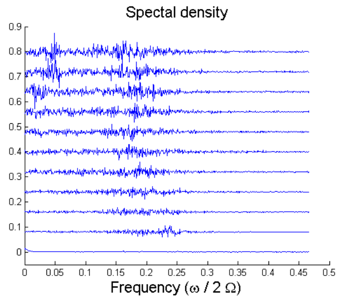


Table voltage 7

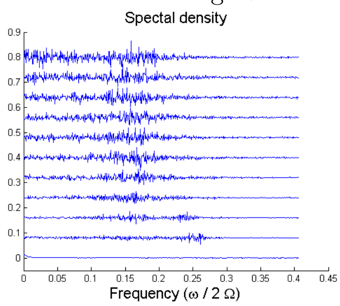


Table voltage 8

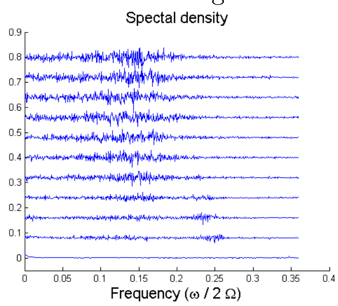


Table voltage 9

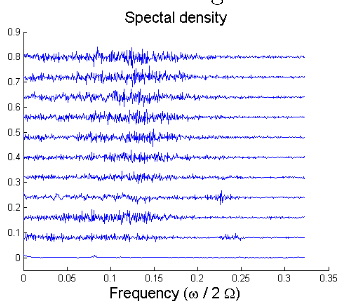


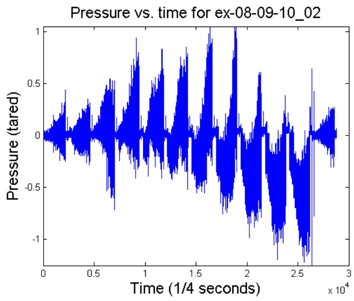
Table voltage 10

This is an excellent piece of data and there seems to be a large number of peaks between around 0.2 in all of the spectral densities.

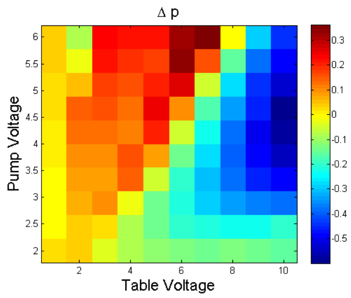
ex-08-09-10_02

Initialisation file identical to ex-27-08-10_01.

Table speeds	0-10 in steps of 1, 0
Pump speeds	0, 2-6 in steps of $4/9$, 0



Pressure vs. Time



Colour chart

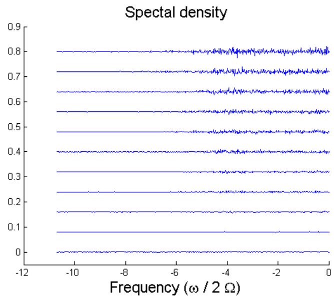


Table voltage 0

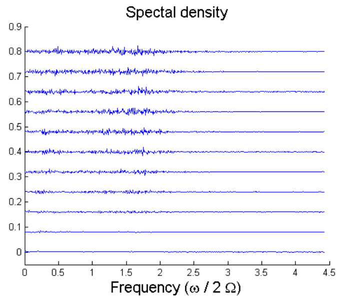


Table voltage 1

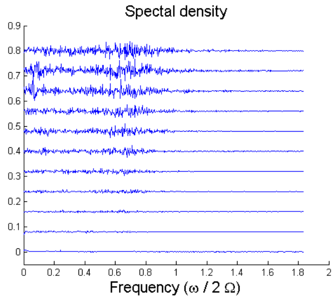


Table voltage 2

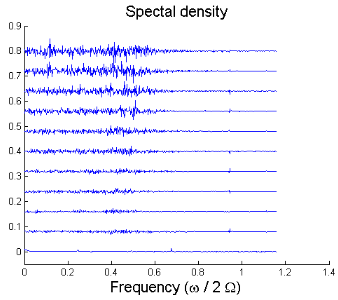


Table voltage 3

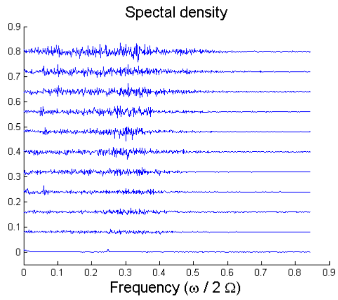


Table voltage 4

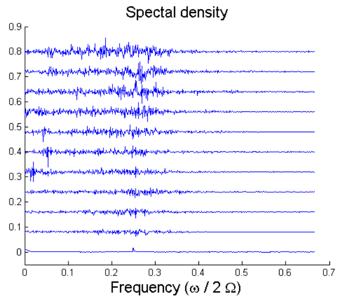


Table voltage 5

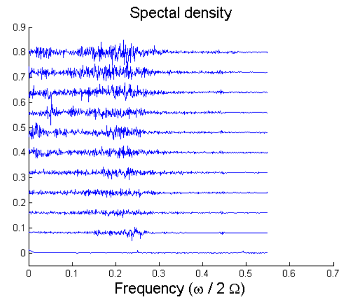


Table voltage 6

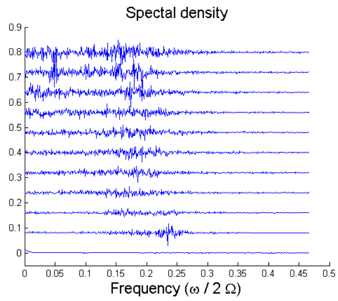


Table voltage 7

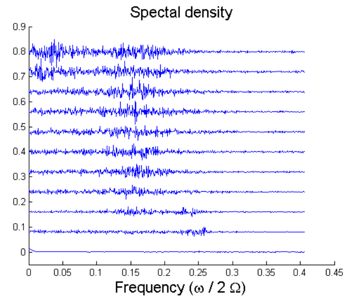


Table voltage 8

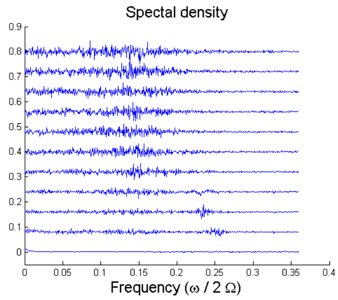


Table voltage 9

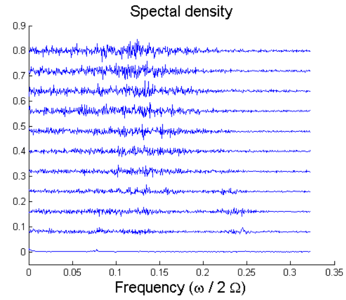


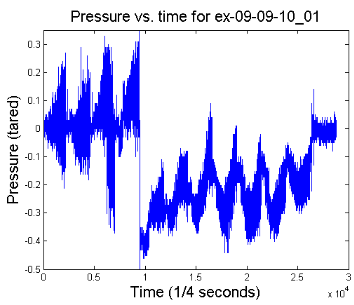
Table voltage 10

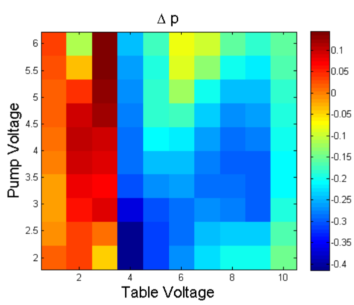
This is another top experiment, and ties in very well with the last.

ex-09-09-10_01

Initialisation file identical to ex-27-08-10_01.

Table speeds	0-10 in steps of 1, 0
Pump speeds	0, 2-6 in steps of $^4/_9$, 0





Pressure vs. Time

Colour chart

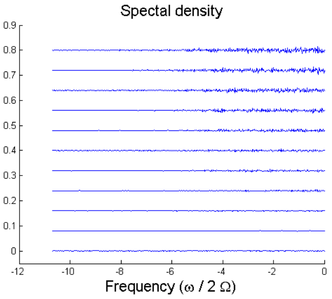


Table voltage 0

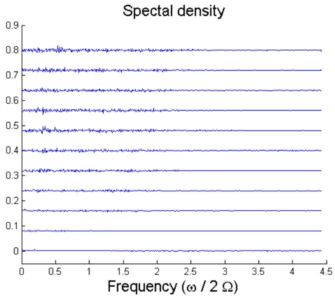


Table voltage 1

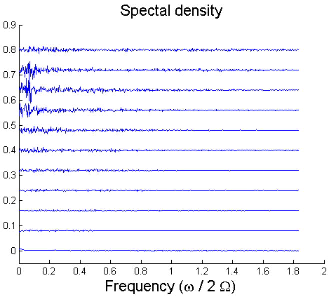


Table voltage 2

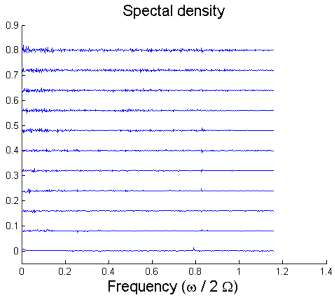


Table voltage 3

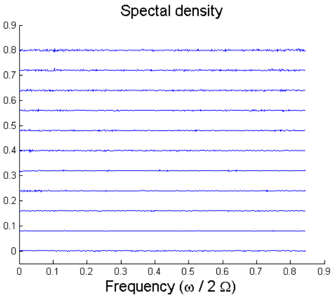


Table voltage 4

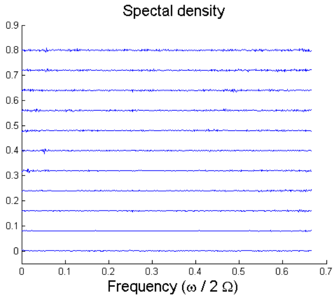


Table voltage 5

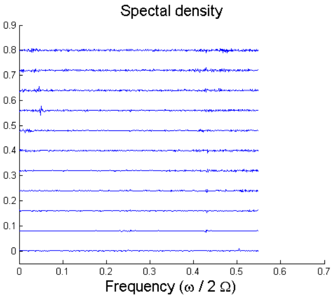


Table voltage 6

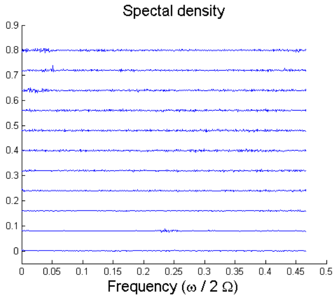


Table voltage 7

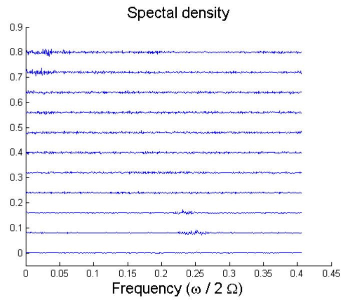


Table voltage 8

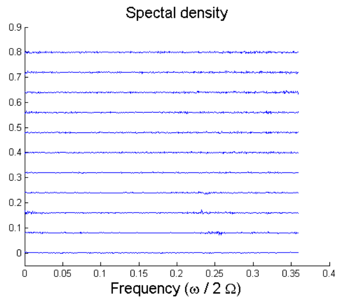


Table voltage 9

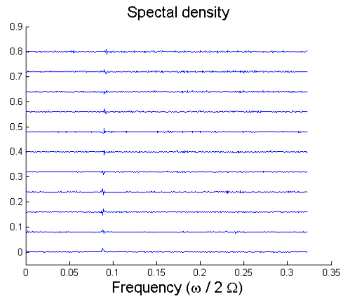


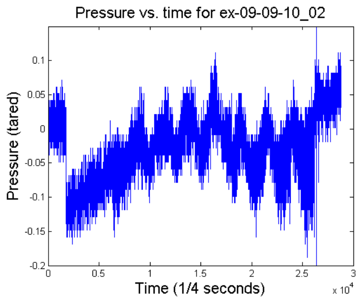
Table voltage 10

Sadly this experienced the sudden unexpected change in pressure Leo Maas found in some of the experiments deemed 'Historic Data', which were completed before I arrived. I can see little use to be obtained from this experiment.

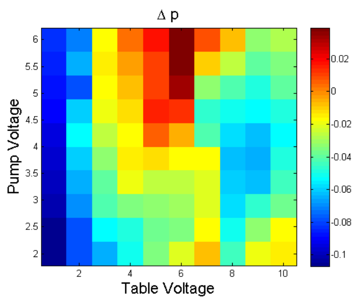
ex-09-09-10_02

Initialisation file identical to ex-27-08-10_01.

Table speeds	0-10 in steps of 1, 0
Pump speeds	0, 2-6 in steps of $4/9$, 0



Pressure vs. Time



Colour chart

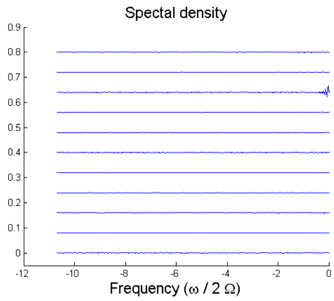


Table voltage 0

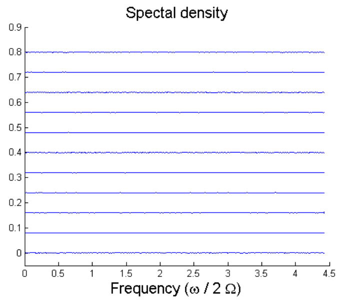


Table voltage 1

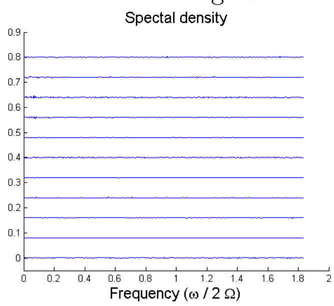


Table voltage 2

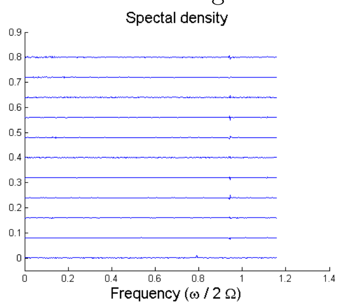


Table voltage 3

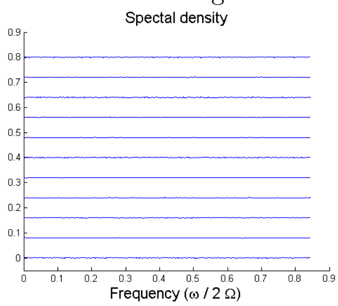


Table voltage 4

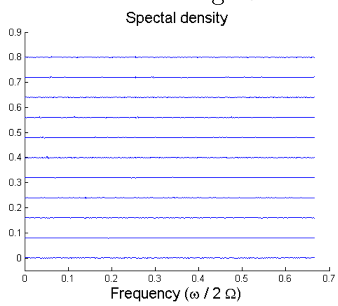


Table voltage 5

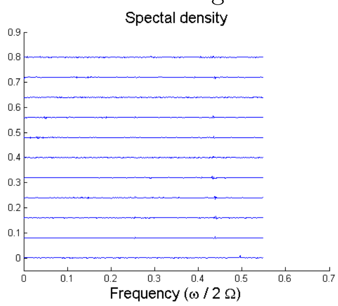


Table voltage 6

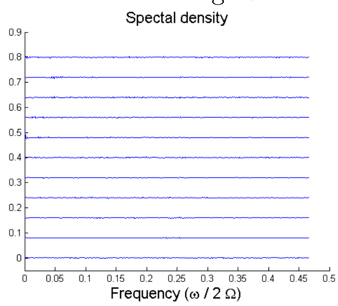


Table voltage 7

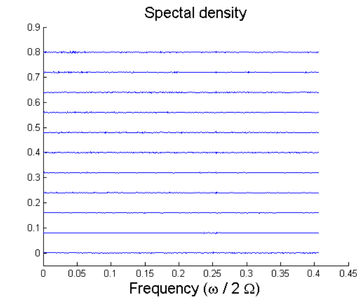


Table voltage 8

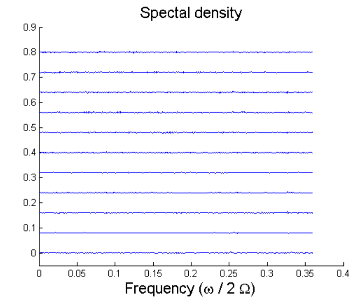


Table voltage 9

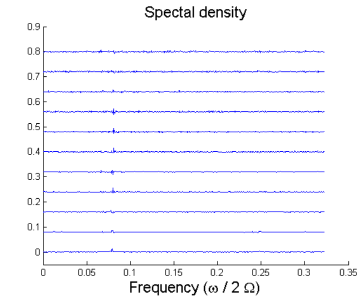


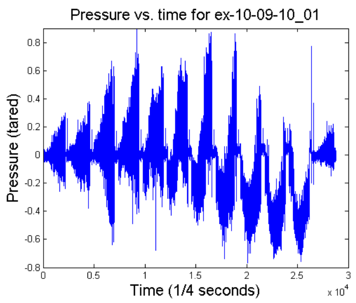
Table voltage 10

This experiment has suffered a similar fate to that of the previous experiment and the results still seem unusable.

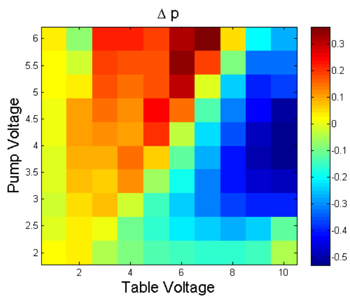
ex-10-09-10_01

Initialisation file identical to ex-27-08-10_01.

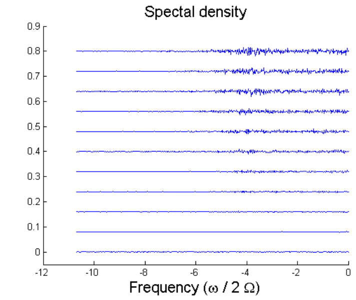
Table speeds	0-10 in steps of 1, 0
Pump speeds	0, 2-6 in steps of $4/9$, 0



Pressure vs. Time



Colour chart



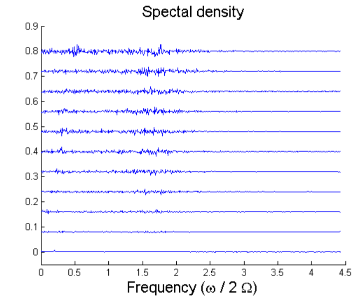


Table voltage 0

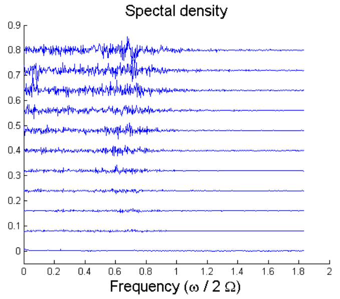


Table voltage 1

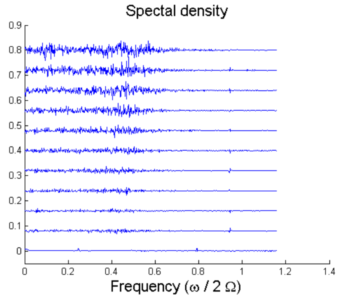


Table voltage 2

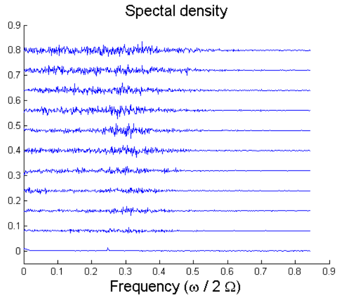


Table voltage 3

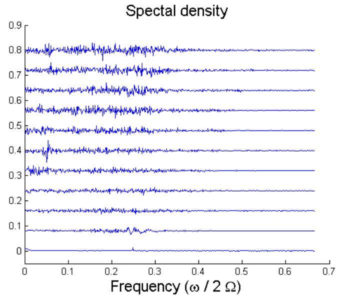


Table voltage 4

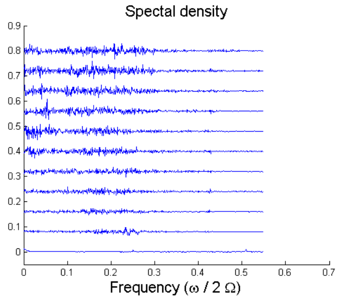


Table voltage 5

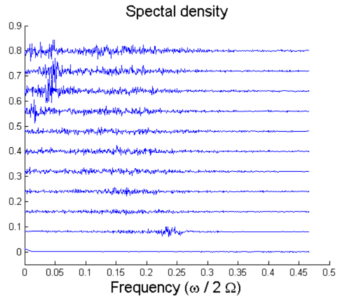


Table voltage 6

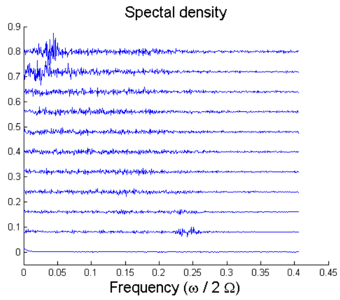


Table voltage 7

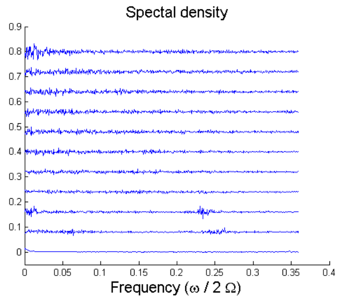


Table voltage 8

Table voltage 9

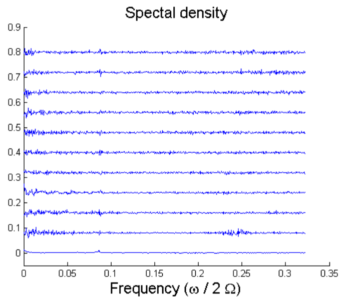


Table voltage 10

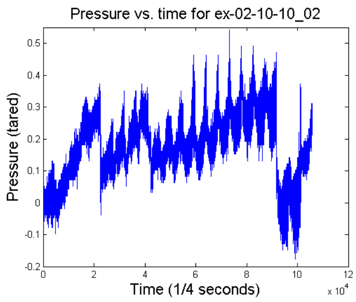
This experiment looks satisfactory. For pump rates 7 and 8 a low frequency wave manifests itself, upon closer inspection we can see that this is also included in `ex-08-09-10_01` Pg 39 and `ex-08-09-10_02` 40; however it seems much more dominant in this experiment.

ex-10-09-10_02

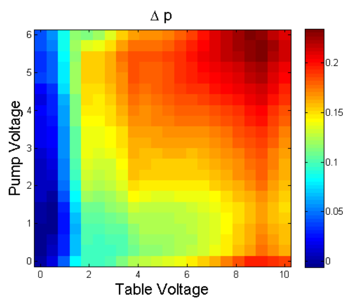
Initialisation file ex-10-09-10_02.

This was taken as a very long, very detailed experiment; which sadly did not go to plan.

Table speeds	0, 1-10 in steps of 9/20, 0
Pump speeds	0, 2-6 in steps of 1/5, 0



Pressure vs. Time



Colour chart

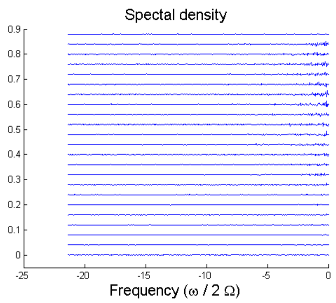


Table voltage 0

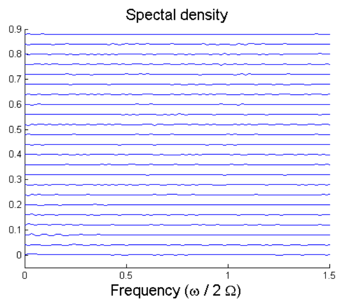


Table voltage 1

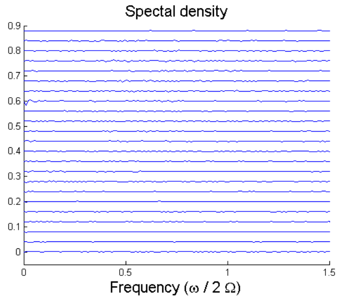


Table voltage $1\,{}^{9}/_{20}$

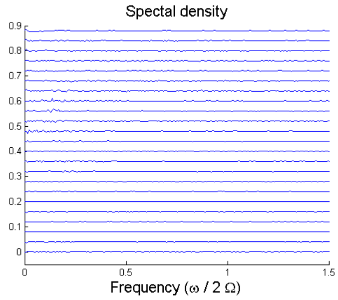


Table voltage $1\,{}^{9}/_{10}$

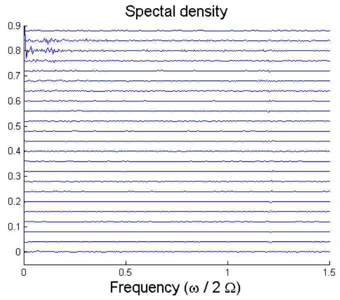


Table voltage $2\,{}^{7}/_{20}$

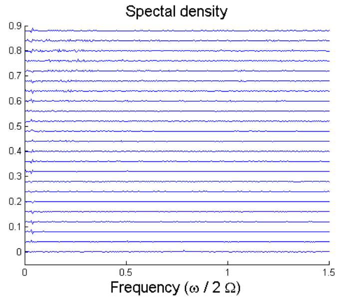


Table voltage $2\,{}^{8}/_{10}$

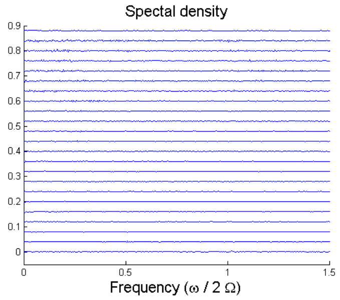


Table voltage $3\,{}^{1}/_{4}$

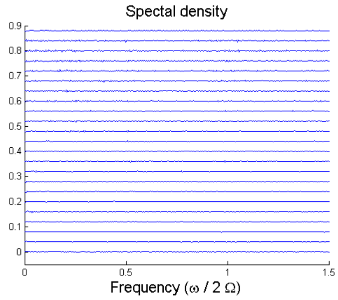


Table voltage $3\,{}^{7}/_{10}$

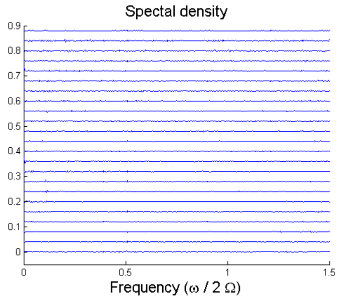


Table voltage $4\,{}^{3}/_{20}$

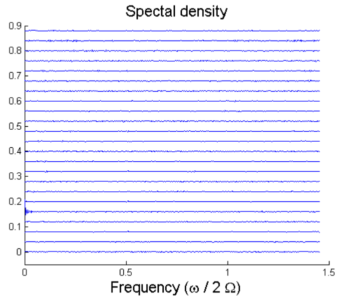


Table voltage $4\,{}^{3}/_{5}$

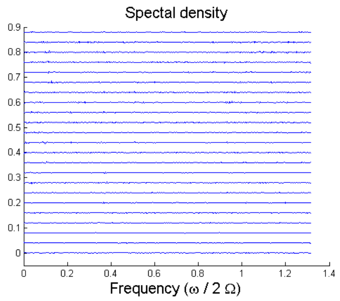


Table voltage $5\frac{1}{20}$

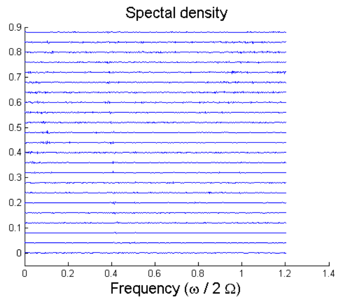


Table voltage $5\frac{1}{2}$

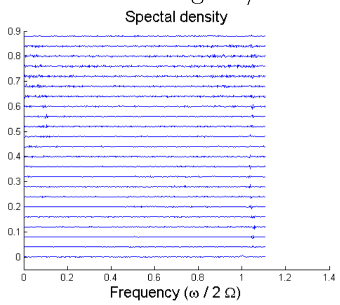


Table voltage $5\frac{19}{20}$

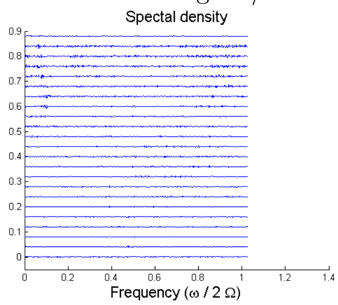


Table voltage $6\frac{2}{5}$

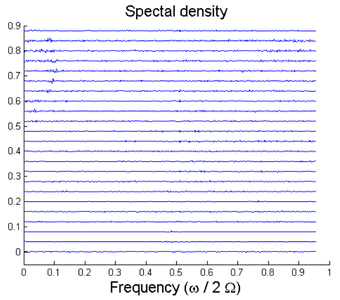


Table voltage $6\frac{17}{20}$

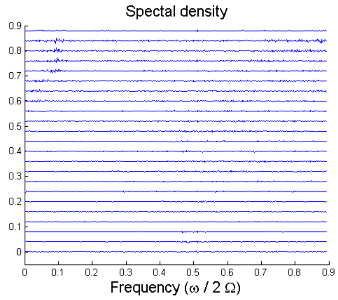


Table voltage $7\frac{3}{10}$

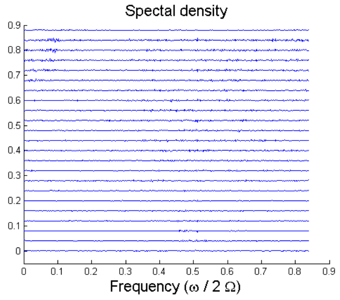


Table voltage $7\frac{3}{4}$

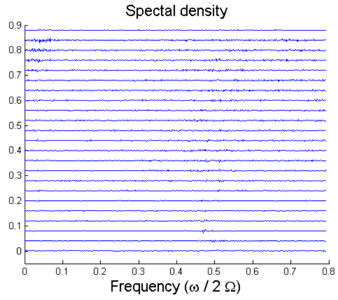


Table voltage $8\frac{1}{5}$

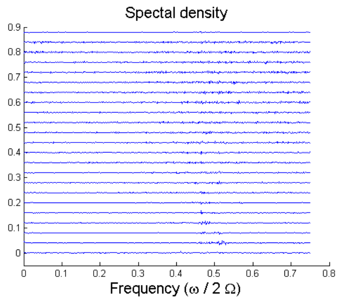


Table voltage $8\,{}^{13}/_{20}$

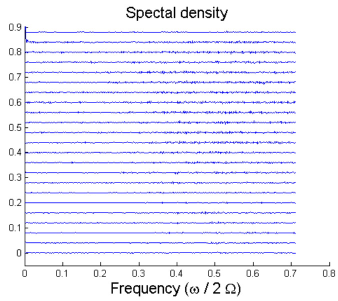


Table voltage $9\,{}^{1}/_{10}$

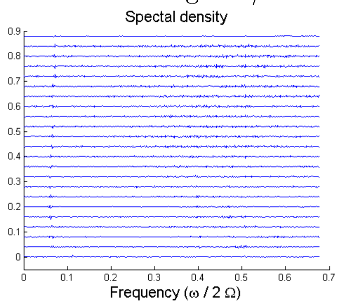


Table voltage $9\,{}^{11}/_{20}$

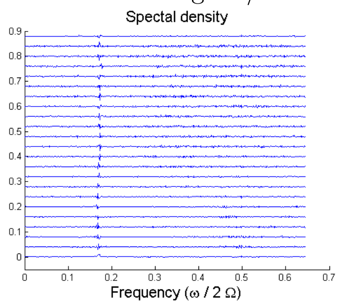


Table voltage 10

Beyond the aesthetic of the colour chart this experiment does not seem particularly useful.

2.3 Discussion

The experiments above were, in the most part, of little use. However some of them showed real promise and it was a pity that they were blighted by the difficulties that arose.

The few good experiments highlight that there are some modes which are more exciting than others and that these all seem to be of low frequency.

Chapter 3

Flat Box

3.1 Introduction

This chapter will look at a rigid rectangular duct in the x-direction and some of the waves within it.

3.2 Wave Solutions

The linearised equations to be used are:

$$u_t - fv = -p_x \tag{3.1}$$
$$v_t + fu = -p_y \tag{3.2}$$
$$w_t = -p_z \tag{3.3}$$
$$u_x + v_y + w_z = 0 \tag{3.4}$$

3.2.1 $k \neq 0$

Rescale to remove f, putting $t = f\hat{t}$ and $p = f\hat{p}$ then dividing through by f in (3.1) & (3.2).
Suppose form $\boldsymbol{u} = \hat{\boldsymbol{u}}(y, z)\, \mathrm{e}^{\mathrm{i}(kx - \omega t)}$, consequently $p = \hat{p}(y, z)\, \mathrm{e}^{\mathrm{i}(kx - \omega t)}$
Then, dropping hats

$$-\mathrm{i}\omega u - v = -p_x \tag{3.5}$$
$$-\mathrm{i}\omega v + u = -p_y \tag{3.6}$$
$$-\mathrm{i}\omega w = -p_z \tag{3.7}$$
$$\mathrm{i}ku + v_y + w_z = 0 \tag{3.8}$$

Solve for w

Remove pressure terms by cross differentiation and (for $k \neq 0$) rearrange continuity to obtain

$$(3.6)_x - (3.5)_y \quad 0 = \mathrm{i}\omega u_y + v_y + \mathrm{i}ku + \omega kv \tag{3.9}$$
$$(3.6)_z - (3.7)_y \quad 0 = u_z - \mathrm{i}\omega v_z + \mathrm{i}\omega w_y \tag{3.10}$$
$$(3.7)_x - (3.5)_z \quad 0 = \mathrm{i}\omega u_z + v_z + \omega kw \tag{3.11}$$
$$u = \frac{\mathrm{i}}{k}(v_y + w_z) \tag{3.12}$$

Remove u from (3.9) — (3.11) using continuity (3.12)

$$0 = -\frac{\omega}{k}(v_{yy} + w_{zy}) - w_z + \omega kv \tag{3.13}$$
$$0 = \frac{1}{k}(v_{yz} + w_{zz}) - \omega v_z + \omega w_y \tag{3.14}$$
$$0 = -\frac{\omega}{k}(v_{yz} + w_{zz}) + v_z + \omega kw \tag{3.15}$$

Find an equation for v in terms of w

$$\omega(3.14) + (3.15) \quad 0 = (1-\omega^2)v_z + \omega^2 w_y + \omega k w \tag{3.16}$$

$$\text{i.e.} \quad v_z = -\frac{\omega}{1-\omega^2}(\omega w_y + kw) \tag{3.17}$$

Substitute (3.17) into (3.14) to obtain

$$w_{yy} - \frac{1-\omega^2}{\omega^2} w_{zz} = k^2 w \tag{3.18}$$

Let $w = Y(y)Z(z)e^{\mathrm{i}(kx-\omega t)}$ and divide by YZ

$$\frac{Y''}{Y} - \frac{1-\omega^2}{\omega^2}\frac{Z''}{Z} = k^2 \tag{3.19}$$

Look for plane waves in y and z direction, let $Y = A_1\,\mathrm{e}^{\mathrm{i}ly}$ and $Z = A_2\,\mathrm{e}^{\mathrm{i}mz}$

The dispersion relation can be found by simply substituting Y and Z into (3.19) to give

$$k^2 = -l^2 + \frac{1-\omega^2}{\omega^2} m^2 \tag{3.20}$$

$$\omega^2 = \frac{m^2}{k^2+l^2+m^2} \tag{3.21}$$

$$\text{i.e.} \quad \omega = \frac{\pm m}{\sqrt{k^2+l^2+m^2}} = \pm\sin\theta \tag{3.22}$$

$$w = A_1\,\mathrm{e}^{\mathrm{i}(kx+ly+mz-\omega t)} \tag{3.23}$$

Solve for specific boundary conditions

Attempt a solution with w of the form:

$$w = A_1(B_1\cos ly + B_2\sin ly)(C_1\cos mz + C_2\sin mz)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)} \tag{3.24}$$

Need $w = 0$ at $z = 0, H$ so $m = \frac{n_w\pi}{H}$ where $n_w \in \mathbb{Z}$

However $\forall n_w < 0 \qquad \sin\frac{n_w\pi}{H}z = -\sin\frac{|n_w|\pi}{H}z$. So both terms will be absorbed together when finding the amplitudes, hence take $n_w \in \mathbb{N}\cup\{0\}$.

$$w = A_2(B_1\cos ly + B_2\sin ly)\sin\frac{n_w\pi}{H}z\,\mathrm{e}^{\mathrm{i}(kx-\omega t)} \tag{3.25}$$

where $A_2 = A_1 C_2$

Using (3.17)

$$v = \frac{\omega}{1-\omega^2}\frac{A_2 H}{n_w\pi}\cos\frac{n_w\pi}{H}z\left[(kB_1 + l\omega B_2)\cos ly + (kB_2 - l\omega B_1)\sin ly\right]\mathrm{e}^{\mathrm{i}(kx-\omega t)} \tag{3.26}$$

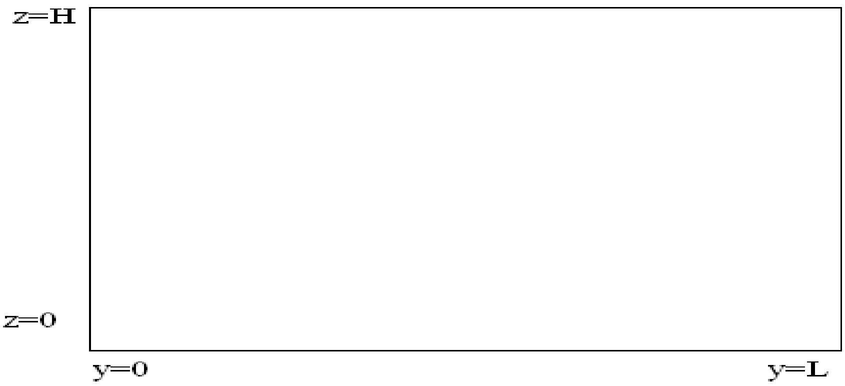


Figure 3.1: Flat rectangular duct

$$v\big|_{y=0} = 0 \quad \Longrightarrow \quad kB_1 + l\omega B_2 = 0 \quad \Longrightarrow \quad B_1 = -\frac{l\omega}{k}B_2$$

$$v = \frac{\omega}{1-\omega^2}\frac{A_3 H}{n_w \pi}(k + \frac{l^2\omega^2}{k})\sin(ly)\cos\frac{n_w\pi}{H}z \ \mathrm{e}^{\mathrm{i}(kx-\omega t)} \tag{3.27}$$

where $A_3 = A_2 B_2$

$$v\big|_{y=L} = 0 \quad \Longrightarrow \quad l = \frac{n_v\pi}{L} \text{ where } n_v \in \mathbb{N}\cup\{0\}$$

$$v = \frac{\omega}{1-\omega^2}\frac{A_3 Hk}{n_w\pi}(1 + (\frac{n_v\omega\pi}{kL})^2)\sin\frac{n_v\pi}{L}y\cos\frac{n_w\pi}{H}z \ \mathrm{e}^{i(kx-\omega t)} \tag{3.28}$$

$$w = A_3(\sin\frac{n_v\pi}{L}y - \frac{n_v\omega\pi}{kL}\cos\frac{n_v\pi}{L}y)\sin\frac{n_w\pi}{H}z \ \mathrm{e}^{\mathrm{i}(kx-\omega t)} \tag{3.29}$$

$$\text{Let} \quad \sin\frac{n_v\pi}{L}y - \frac{n_v\omega\pi}{kL}\cos\frac{n_v\pi}{L}y = r\sin(\frac{n_v\pi}{L}y - \theta) \tag{3.30}$$

$$r^2 = 1 + (\frac{n_v\omega\pi}{kL})^2 \quad \text{and} \quad \theta = \arctan\frac{n_v\omega\pi}{kL} \in [0, \frac{\pi}{2})$$

So putting (3.30) into (3.29)

$$w = A_3\, r\sin(\frac{n_v\pi}{L}y - \theta)\sin\frac{n_w\pi}{H}z \ \mathrm{e}^{\mathrm{i}(kx-\omega t)} \tag{3.31}$$

From (3.12) u can be found using (3.28) & (3.31)

$$u = A_3[\frac{i\omega}{1-\omega^2}\frac{Hn_v}{Ln_w}(1 + \left(\frac{n_v\omega\pi}{kL}\right)^2) \\ \cos\frac{n_v\pi}{L}y + \mathrm{i}\frac{rn_w\pi}{kH}\sin(\frac{n_v\pi}{L}y - \theta)]\cos\frac{n_w\pi}{H}z \ \mathrm{e}^{\mathrm{i}(kx-\omega t)} \tag{3.32}$$

$$\boldsymbol{u} = \begin{pmatrix} [\frac{\mathrm{i}\omega}{1-\omega^2}\frac{Hn_v}{Ln_w} r^2 \cos\frac{n_v\pi}{L}y + \mathrm{i}\frac{rn_w\pi}{kH}\sin(\frac{n_v\pi}{L}y - \theta)]\cos\frac{n_w\pi}{H}z \\ \frac{\omega}{1-\omega^2}\frac{Hk}{n_w\pi}(1 + (\frac{n_v\omega\pi}{kL})^2)\sin\frac{n_v\pi}{L}y\cos\frac{n_w\pi}{H}z \\ r\sin(\frac{n_v\pi}{L}y - \theta)\sin\frac{n_w\pi}{H}z \end{pmatrix} A_3\,\mathrm{e}^{\mathrm{i}(kx-\omega t)} \tag{3.33}$$

$$\omega = \frac{\pm\frac{n_w\pi}{H}}{\sqrt{k^2 + (\frac{n_v\pi}{L})^2 + (\frac{n_w\pi}{H})^2}} \tag{3.34}$$

where $r^2 = 1 + (\frac{n_v\omega\pi}{kL})^2$ and $\theta = \arctan\frac{n_v\omega\pi}{kL} \in [0, \frac{\pi}{2})$

u can be rewritten as

$$\begin{aligned} u = \mathrm{i}A_3[&\frac{\omega}{1-\omega^2}\frac{Hn_v}{Ln_w}(1 + \left(\frac{n_v\omega\pi}{kL}\right)^2)\cos\frac{n_v\pi}{L}y \\ &+ \frac{n_w\pi}{kH}(\sin\frac{n_v\pi}{L}y - \frac{n_v\omega\pi}{kL}\cos\frac{n_v\pi}{L}y)] \\ &\cos\frac{n_w\pi}{H}z\,\mathrm{e}^{\mathrm{i}(kx-\omega t)} \end{aligned} \tag{3.35}$$

i.e.

$$\begin{aligned} u = \mathrm{i}A_3[&(\frac{\omega}{1-\omega^2}\frac{Hn_v}{Ln_w}(1 + \left(\frac{n_v\omega\pi}{kL}\right)^2) - \frac{n_v n_w\omega\pi^2}{k^2LH})\cos\frac{n_v\pi}{L}y \\ &+ \frac{n_w\pi}{kH}(\sin\frac{n_v\pi}{L}y)]\cos\frac{n_w\pi}{H}z\,\mathrm{e}^{\mathrm{i}(kx-\omega t)} \end{aligned} \tag{3.36}$$

Let

$$\lambda^2 = \left(\frac{\omega}{1-\omega^2}\frac{Hn_v}{Ln_w}(1 + \left(\frac{n_v\omega\pi}{kL}\right)^2) - \frac{n_v n_w\omega\pi^2}{k^2LH}\right)^2 + \left(\frac{n_w\pi}{kH}\right)^2 \tag{3.37}$$

$$\left(\frac{\omega}{1-\omega^2}\frac{Hn_v}{Ln_w}r^2 - \frac{n_w\pi}{kH}\tan\theta\right)^2 + \left(\frac{n_w\pi}{kH}\right)^2 \tag{3.38}$$

$$\alpha = \arctan\frac{\frac{\omega}{1-\omega^2}\frac{Hn_v}{Ln_w}r^2 - \frac{n_w\pi^2}{kH}\tan\theta}{\frac{n_w\pi}{kH}} \tag{3.39}$$

Then

$$u = \mathrm{i}A_3\lambda\sin(\frac{n_v\pi}{L}y + \alpha)\cos\frac{n_w\pi}{H}z\,\mathrm{e}^{\mathrm{i}(kx-\omega t)} \tag{3.40}$$

To summarise, for $k \neq 0$ (3.33) becomes

$$\boldsymbol{u} = \begin{pmatrix} \mathrm{i}\lambda\sin(\frac{n_v\pi}{L}y + \alpha)\cos\frac{n_w\pi}{H}z \\ \frac{\omega}{1-\omega^2}\frac{Hk}{n_w\pi}r^2\sin\frac{n_v\pi}{L}y\cos\frac{n_w\pi}{H}z \\ r\sin(\frac{n_v\pi}{L}y - \theta)\sin\frac{n_w\pi}{H}z \end{pmatrix} A_3\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}$$

$$\omega = \frac{\pm\frac{n_w\pi}{H}}{\sqrt{k^2 + (\frac{n_v\pi}{L})^2 + (\frac{n_w\pi}{H})^2}}$$

where

$$\theta = \arctan\frac{n_v\omega\pi}{kL} \tag{3.41}$$

$$r^2 = 1 + (\frac{n_v\omega\pi}{kL})^2 \tag{3.42}$$

$$= 1 + \tan^2\theta \tag{3.43}$$

$$\alpha = \arctan\frac{\frac{\omega}{1-\omega^2}\frac{Hn_v}{Ln_w}r^2 - \frac{n_w n_v \pi^3\omega}{k^2HL}}{\frac{n_w\pi}{kH}} \tag{3.44}$$

$$\lambda^2 = \left(\frac{\omega}{1-\omega^2}\frac{Hn_v}{Ln_w}r^2 - \frac{n_w n_v\omega\pi^2}{k^2HL}\right)^2 + \left(\frac{n_w\pi}{kH}\right)^2 \tag{3.45}$$

$$= \left(\frac{n_w\pi}{kH}\tan\alpha\right)^2 + \left(\frac{n_w\pi}{kH}\right)^2 \tag{3.46}$$

$$\lambda = \pm\frac{n_w\pi}{kH}\sec\alpha \tag{3.47}$$

The factor i in u betrays the fact that u is $\pi/2$ out of phase with v and w.

3.2.2 $k = 0$

As before rescale to remove f, putting $t = f\hat{t}$ and $p = f\hat{p}$.

Solving for w

Suppose form $\boldsymbol{u} = \hat{\boldsymbol{u}}(y,z)\,\mathrm{e}^{-\mathrm{i}\omega t}$, consequently $p = \hat{p}(y,z)\,\mathrm{e}^{-\mathrm{i}\omega t}$
Then, dropping hats

$$\mathrm{i}\omega u + v = 0 \tag{3.48}$$

$$-\mathrm{i}\omega v + u = -p_y \tag{3.49}$$

$$-\mathrm{i}\omega w = -p_z \tag{3.50}$$

$$v_y + w_z = 0 \tag{3.51}$$

$(3.49)_z - (3.50)_y$

$$u_z - \mathrm{i}\omega v_z + \mathrm{i}\omega w_y = 0 \tag{3.52}$$

Putting (3.48) into (3.52) gives

$$\frac{1-\omega^2}{\omega^2}v_z + w_y = 0 \tag{3.53}$$

Using $(3.51)_z$ and $(3.53)_y$

$$\frac{1-\omega^2}{\omega^2}w_{zz} - w_{yy} = 0 \tag{3.54}$$

The same as above, this time with $k = 0$. Similarly take w of form $w = D_1\,\mathrm{e}^{i(ly+mz-\omega t)}$

The dispersion relation is found to be:

$$0 = \frac{1-\omega^2}{\omega^2} m^2 - l^2 \tag{3.55}$$

$$\omega^2 = \frac{m^2}{l^2 + m^2} \tag{3.56}$$

$$\omega = \frac{\pm m}{\sqrt{l^2 + m^2}} = \pm \sin\beta \tag{3.57}$$

So far it looks much the same as $k \neq 0$ however u, v and w are considerably simpler after applying boundary conditions.

Solving for specific boundary conditions

Let $w = D_1(E_1 \cos ly + E_2 \sin ly)(F_1 \cos mz + F_2 \sin mz)\,\mathrm{e}^{-i\omega t}$

$w = 0$ at $z = 0, H \Longrightarrow F_1 = 0, \quad m = \frac{n_w^o \pi}{H}, \quad n_w^o \in \mathbb{N} \cup \{0\}$

Let $D_2 = D_1 F_2$

$$w = D_2(E_1 \cos ly + E_2 \sin ly) \sin(\frac{n_w^o \pi}{H} z)\,\mathrm{e}^{-\mathrm{i}\omega t} \tag{3.58}$$

Using (3.53) to find v

$$v = \frac{\omega^2}{1-\omega^2} D_2 \frac{Hl}{n_w^o \pi}(-E_1 \sin ly + E_2 \cos ly) \cos(\frac{n_w^o \pi}{H} z)\,\mathrm{e}^{-\mathrm{i}\omega t} \tag{3.59}$$

$v = 0$ at $y = 0, L \Longrightarrow E_2 = 0, \quad l = \frac{n_v^o \pi}{L}$ where $n_v^o \in \mathbb{N} \cup \{0\}$

Let $D_3 = D_2 E_1$

$$v = -D_3 \frac{\omega^2}{1-\omega^2} \frac{H n_v^o}{L n_w^o} \sin(\frac{n_v^o \pi}{L} y) \cos(\frac{n_w^o \pi}{H} z)\,\mathrm{e}^{-\mathrm{i}\omega t} \tag{3.60}$$

Consequently from (3.48)

$$u = -\mathrm{i} D_3 \frac{\omega}{1-\omega^2} \frac{H n_v^o}{L n_w^o} \sin(\frac{n_v^o \pi}{L} y) \cos(\frac{n_w^o \pi}{H} z)\,\mathrm{e}^{-\mathrm{i}\omega t} \tag{3.61}$$

To summarise, for $k = 0$

$$\boldsymbol{u} = \begin{pmatrix} -\mathrm{i}\frac{\omega}{1-\omega^2} \frac{H n_v^o}{L n_w^o} \sin(\frac{n_v^o \pi}{L} y) \cos(\frac{n_w^o \pi}{H} z) \\ -\frac{\omega^2}{1-\omega^2} \frac{H n_v^o}{L n_w^o} \sin(\frac{n_v^o \pi}{L} y) \cos(\frac{n_w^o \pi}{H} z) \\ \cos(\frac{n_v^o \pi}{L} y) \sin(\frac{n_w^o \pi}{H} z) \end{pmatrix} D_3 \mathrm{e}^{-\mathrm{i}\omega t} \tag{3.62}$$

$$\omega = \frac{\pm n_w^o}{\sqrt{\left(\frac{H}{L} n_v^o\right)^2 + {n_w^o}^2}} = \pm \sin\beta$$

where $n_v^o, n_w^o \in \mathbb{N} \cup \{0\}$

v and w are in opposite directions and there remains a factor of i in u indicating a $\pi/2$ phase shift.

3.3 Waves of the same frequency

3.3.1 $k = 0$

If two waves are looked at, W_1 and W_2, such that W_1 has n_v, n_w j times that of W_2 it is seen

$$\omega_{W_1} = \frac{\pm j n_w^o}{\sqrt{\left(\frac{H}{L}(j n_v^o)\right)^2 + (2 n_w^o)^2}} = \frac{\pm n_w^o}{\sqrt{\left(\frac{H}{L} n_v^o\right)^2 + {n_w^o}^2}} = \omega_{W_2}$$

For unique ω take n_v^o, n_w^o coprime; or more generally take p_v^o, p_w^o such that

$$\begin{pmatrix} n_v^o \\ n_w^o \end{pmatrix} = j \begin{pmatrix} p_v^o \\ p_w^o \end{pmatrix} \text{where} \quad j = (n_v^o, n_w^o) \in \mathbb{N}$$

$$\boldsymbol{u}_j = \begin{pmatrix} -\mathrm{i}\frac{\omega}{1-\omega^2}\frac{H p_v^o}{L p_w^o}\sin\left(j\frac{p_v^o \pi}{L} y\right)\cos\left(j\frac{[p_w^o \pi}{H} z\right) \\ -\frac{\omega^2}{1-\omega^2}\frac{H p_v^o}{L p_w^o}\sin\left(j\frac{p_v^o \pi}{L} y\right)\cos\left(j\frac{p_w^o \pi}{H} z\right) \\ \cos\left(j\frac{p_v^o \pi}{L} y\right)\sin\left(j\frac{p_w^o \pi}{H} z\right) \end{pmatrix} D_{3,j}\,\mathrm{e}^{-\mathrm{i}\omega t}$$

$$\sum_j \boldsymbol{u}_j = \begin{pmatrix} -\mathrm{i}\frac{\omega}{1-\omega^2}\frac{H p_v^o}{L p_w^o}\sum_j D_{3,j}\sin\left(j\frac{p_v^o \pi}{L} y\right)\cos\left(j\frac{p_w^o \pi}{H} z\right) \\ -\frac{\omega^2}{1-\omega^2}\frac{H p_v^o}{L p_w^o}\sum_j D_{3,j}\sin\left(j\frac{p_v^o \pi}{L} y\right)\cos\left(j\frac{p_w^o \pi}{H} z\right) \\ \sum_j D_{3,j}\cos\left(j\frac{p_v^o \pi}{L} y\right)\sin\left(j\frac{p_w^o \pi}{H} z\right) \end{pmatrix} \mathrm{e}^{-\mathrm{i}\omega t}$$

Chapter 4

2D simple attractors in a semi-trapezium

4.1 Introduction

This chapter will first derive the governing equations to be used for the rest of the project and then will look into simple wave attractors in the 2D case of the semi-trapezium used in Maas and Harlander [2007].

4.2 Governing equations

Using the previous governing equations (3.5) - (3.7)

$$u = -\frac{1}{1-\omega^2}(\omega k p + p_y) \tag{4.1}$$
$$v = \frac{\mathrm{i}}{1-\omega^2}(k p + \omega p_y) \tag{4.2}$$
$$w = -\frac{\mathrm{i}}{\omega} p_z \tag{4.3}$$

Which substituted into the continuity equation (3.8) gives:

$$0 = -\frac{\mathrm{i}k}{1-\omega^2}(\omega k p + p_y) + \frac{\mathrm{i}}{1-\omega^2}(k p_y + \omega p_{yy}) - \frac{\mathrm{i}}{\omega} p_{zz} \tag{4.4}$$
$$0 = \frac{1}{1-\omega^2}(-\omega k^2 p + \omega p_{yy}) - \frac{1}{\omega} p_{zz} \tag{4.5}$$
$$0 = p_{yy} - \frac{1-\omega^2}{\omega^2} p_{zz} - k^2 p \tag{4.6}$$

Let

$$z = \frac{\sqrt{1-\omega^2}}{\omega}\hat{z} \Longrightarrow \frac{\partial}{\partial z} = \frac{\partial \hat{z}}{\partial z}\frac{\partial}{\partial \hat{z}} = \frac{\omega}{\sqrt{1-\omega^2}}\frac{\partial}{\partial \hat{z}}$$

Then (4.6) becomes

$$p_{yy} - p_{\hat{z}\hat{z}} - k^2 p = 0 \tag{4.7}$$

The Telegraph equation is:

$$p_{yy} - p_{\hat{z}\hat{z}} + \alpha^2 p = 0 \tag{4.8}$$

So for propagating waves $\left(\mathrm{e}^{\mathrm{i}kx}\right)$ $\alpha = \pm \mathrm{i}k$ and for waves that decay down channel $\left(\mathrm{e}^{-kx} \text{and} k > 0\right)$ a change that can be easily made by mapping $k \longmapsto \mathrm{i}k$ so $\pm\alpha = k \in \mathbb{R}$.

4.2.1 Simplifying to 2D

This simplification can be made by setting k=0, so the waves neither propagate nor decay downstream. This means our governing equation (4.7) reduces to the Poincaré equation:

$$p_{yy} - \frac{1-\omega^2}{\omega^2} p_{zz} = 0 \tag{4.9}$$
$$p_{yy} - p_{\hat{z}\hat{z}} = 0 \tag{4.10}$$

and our velocities (4.1) - (4.3) reduce to:

$$u = -\frac{1}{1-\omega^2}p_y \tag{4.11}$$
$$v = \frac{\mathrm{i}\omega}{1-\omega^2}p_y \tag{4.12}$$
$$w = -\frac{\mathrm{i}}{\omega}p_z = -\frac{\mathrm{i}}{\sqrt{1-\omega^2}}p_{\hat{z}} \tag{4.13}$$

For a square of side length π the flat box (3.62) provides:

$$\boldsymbol{u} = \begin{pmatrix} -\mathrm{i}\frac{\omega}{1-\omega^2}\frac{n_v}{n_w}\sin n_v y \cos n_w z \\ -\frac{\omega^2}{1-\omega^2}\frac{n_v}{n_w}\sin(n_v y)\cos n_w z) \\ \cos n_v y \sin n_w z \end{pmatrix} D_{3,n}\mathrm{e}^{-\mathrm{i}\omega t} \tag{4.14}$$

Using (4.11) − (4.14), p can be solved for in terms of y and z, the exponential term is omitted for tidiness.

$$p = -\frac{1-\omega^2}{\mathrm{i}\omega}\int \frac{\omega^2}{1-\omega^2}\frac{n_v}{n_w}\sin n_v y \cos n_w z\,\mathrm{y} + f(z) \tag{4.15}$$
$$= \mathrm{i}\omega \int \frac{n_v}{n_w}\sin n_v y \cos n_w z\,\mathrm{y} + f(z) \tag{4.16}$$
$$= -\frac{\mathrm{i}\omega}{n_w}\cos n_v y \cos n_w z + f(z) \tag{4.17}$$
$$p_z = \mathrm{i}\omega \cos n_v y \sin n_w z + f'(z) \tag{4.18}$$
$$\mathrm{i}\omega w = \mathrm{i}\omega \cos n_v y \sin n_w z \tag{4.19}$$

Yielding $f'(z) = 0$, w.l.o.g. take $f(z) = 0$.
Hence a single mode of pressure can be described as:

$$p = -\frac{\mathrm{i}\omega}{n_w}\cos n_v y \cos n_w z D_{3,n}\mathrm{e}^{-\mathrm{i}\omega t} \tag{4.20}$$

Simple substitution of p into our governing equation (4.9) yields $n_w^2 = \frac{\omega^2}{1-\omega^2}n_v^2$ which, upon writing $n = n_v$ gives us an improved p:

$$p = -\frac{\mathrm{i}\sqrt{1-\omega^2}}{n}\cos ny \cos \frac{n\omega}{\sqrt{1-\omega^2}}zD_{3,n}\mathrm{e}^{-\mathrm{i}\omega t} \tag{4.21}$$

In terms of $\hat{z}$

$$p = -\frac{\mathrm{i}\sqrt{1-\omega^2}}{n}\cos ny \cos n\hat{z}D_{3,n}\mathrm{e}^{-\mathrm{i}\omega t} \tag{4.22}$$

p can be written as a Fourier series over the domain

$$p = \left(\sum_n a_n \cos ny \cos n\hat{z}\right)\mathrm{e}^{-\mathrm{i}\omega t} \tag{4.23}$$

Satisfying governing equation (4.10)
Implying our u, v and w are

$$u = \frac{1}{1-\omega^2}\left(\sum_n n a_n \sin ny \cos n\hat{z}\right) \mathrm{e}^{-\mathrm{i}\omega t} \tag{4.24}$$

$$v = -\frac{\mathrm{i}\omega}{1-\omega^2}\left(\sum_n n a_n \sin ny \cos n\hat{z}\right) \mathrm{e}^{-\mathrm{i}\omega t} \tag{4.25}$$

$$w = \frac{\mathrm{i}}{\sqrt{1-\omega^2}}\left(\sum_n n a_n \cos ny \sin n\hat{z}\right) \mathrm{e}^{-\mathrm{i}\omega t} \tag{4.26}$$

4.3 Boundaries

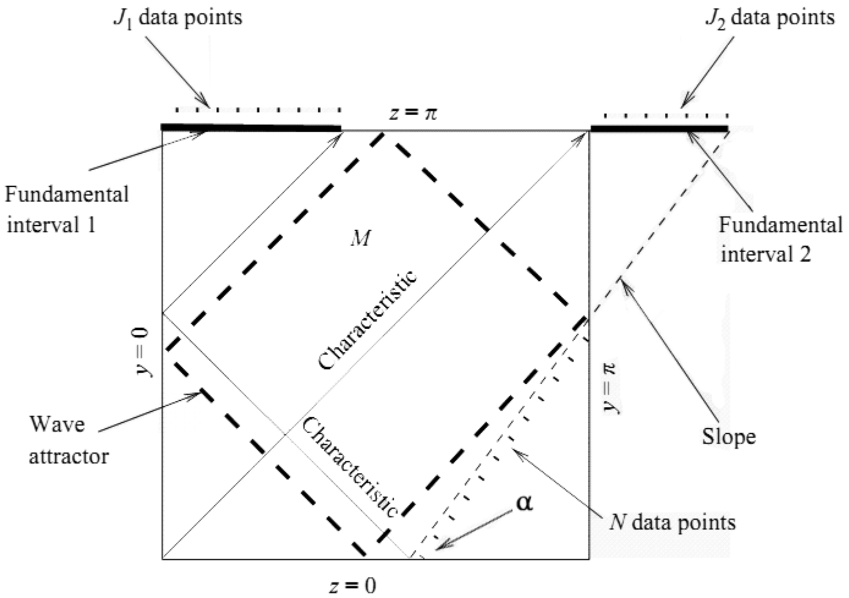


Figure 4.1: Semi-trapezium from Maas and Harlander [2007]

Let the angle between the slope and the horizontal be α.

First check boundary conditions on the flat walls at $y = 0$, $z = 0$ and $z = \pi$ using (4.25) and (4.26).

$$v = -\frac{\mathrm{i}\omega}{1-\omega^2}\left(\sum_n na_n \sin n0 \cos n\hat{z}\right)\mathrm{e}^{-\mathrm{i}\omega t} \tag{4.27}$$
$$= 0 \tag{4.28}$$
$$w = \frac{\mathrm{i}}{\sqrt{1-\omega^2}}\left(\sum_n na_n \cos ny \sin n0\right)\mathrm{e}^{-\mathrm{i}\omega t} \tag{4.29}$$
$$= 0 \tag{4.30}$$
$$w = \frac{\mathrm{i}}{\sqrt{1-\omega^2}}\left(\sum_n na_n \cos ny \sin n\pi\right)\mathrm{e}^{-\mathrm{i}\omega t} \tag{4.31}$$
$$= 0 \tag{4.32}$$

So our p satisfies the boundary conditions on the vertical and horizontal walls, $v = 0$ at $y = 0$ and $w = 0$ at $z = 0, \pi$. Now choose a_n to satisfy no normal flow on the slope,

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{i.e.} \quad w = -v \tan\alpha \tag{4.33}$$

The $\hat{z}$ co-ordinate along the slope is,

$$\hat{z} = (y - \pi + \frac{\pi}{2}\cot\alpha)\tan\alpha = (y-\pi)\tan\alpha + \frac{\pi}{2} \tag{4.34}$$

From (4.23), p along the slope is,

$$p = \left(\sum_n a_n \cos ny \cos n(y-\pi)\tan\alpha + \frac{n\pi}{2}\right)\mathrm{e}^{-\mathrm{i}\omega t} \tag{4.35}$$

Firstly putting (4.25) and (4.26) into (4.33) yields:

$$w = -v\tan\alpha \tag{4.36}$$
$$\frac{\mathrm{i}}{\sqrt{1-\omega^2}}\left(\sum_n na_n \cos ny \sin n\hat{z}\right) = \tan\alpha \frac{\mathrm{i}\omega}{1-\omega^2}\left(\sum_n na_n \sin ny \cos n\hat{z}\right) \tag{4.37}$$
$$\frac{\sqrt{1-\omega^2}}{\omega}\left(\sum_n na_n \cos ny \sin n\hat{z}\right) = \tan\alpha\left(\sum_n na_n \sin ny \cos n\hat{z}\right) \tag{4.38}$$

Then using (4.34)

$$0 = \sum_n na_n\left[\frac{\sqrt{1-\omega^2}}{\omega}\cos ny \sin n\hat{z} - \tan\alpha \sin ny \cos n\hat{z}\right] \tag{4.39}$$
$$= \sum_n na_n\Bigg[\frac{\sqrt{1-\omega^2}}{\omega}\cos ny \sin n((y-\pi)\tan\alpha + \frac{\pi}{2})$$
$$- \tan\alpha \sin ny \cos n((y-\pi)\tan\alpha + \frac{\pi}{2})\Bigg] \tag{4.40}$$

v can now be prescribe freely on the fundamental intervals, the characteristics carry this information from these three regions; completely specifying p in the whole domain.

4.4 Numerical approximation

For $\hat{N} = J_1 + J_2 + N$, where J_1 is the number of points on the first fundamental, J_2 on the second and N on the lower half of the slope then our a_n can be solved for numerically to form an approximate p:

$$p \approx \left(\sum_{n=1}^{\hat{N}} a_n \cos ny \cos n\hat{z} \right) \mathrm{e}^{-\mathrm{i}\omega t} \tag{4.41}$$

4.4.1 Example

This will be illustrated with an example, take $\tan\alpha = 3/2$ so the lower left corner is at co-ordinate $(\frac{2\pi}{3}, 0)$ and the upper right is at $(\frac{4\pi}{3}, \pi)$. Our (4.34) becomes

$$\hat{z} = (y - \pi)\frac{3}{2} + \frac{\pi}{2} = \frac{3y}{2} - \pi \tag{4.42}$$

So our (4.40) shortens to,

$$0 = \sum_{n=1}^{\hat{N}} a_n \left[\frac{\sqrt{1-\omega^2}}{\omega} \cos ny \sin(\frac{3}{2}ny - n\pi) - \frac{3}{2} \sin ny \cos(\frac{3}{2}ny - n\pi) \right] \tag{4.43}$$

v is prescribed to be $v = \sin 3y$ on the first fundamental interval and $v = -\sin 3y$ on the second.

To summarise, choose points J_1, J_2 and N and solve for the $\hat{N}$ Fourier co-efficients. The equations are reiterated here for simplicities sake with the first equation garnered from the first fundamental internal with $y \in (0, \frac{\pi}{3})$ and $z = \pi$; the second from the second fundamental interval with $y \in (\pi, \frac{4\pi}{3})$ and $z = \pi$ and the third and final equation from the lower half of the slope with $y \in (\frac{2\pi}{3}, \pi)$ and $\hat{z} = \frac{3}{2}y - \pi$.

$$0 = \frac{\mathrm{i}\omega}{1-\omega^2} \left(\sum_{n=1}^{\hat{N}} n a_n \sin ny (-1)^n \right) + \sin 3y \tag{4.44}$$

$$0 = \frac{\mathrm{i}\omega}{1-\omega^2} \left(\sum_{n=1}^{\hat{N}} n a_n \sin ny (-1)^n \right) - \sin 3y \tag{4.45}$$

$$0 = \sum_{n=1}^{\hat{N}} a_n \left[\frac{\sqrt{1-\omega^2}}{\omega} \cos ny \sin(\frac{3}{2}ny - n\pi) - \frac{3}{2} \sin ny \cos(\frac{3}{2}ny - n\pi) \right] \tag{4.46}$$

This can be solved in Matlab and then plotted in Mathematica, the code is found in the appendix or can be downloaded from the website (`http://www.ucl.ac.uk/~zcahdl5/project.html`).

Here is one example of this for $J_1 = J_2 = 40$ and $N = 8$. Notice the fine structure around the attractor. In the stratified, non-rotating case this has been solved without the approximation of inverting the large matrix numerically and a fractal structure has been found where the attractor reflects off the walls, see [Maas and Harlander, 2007].

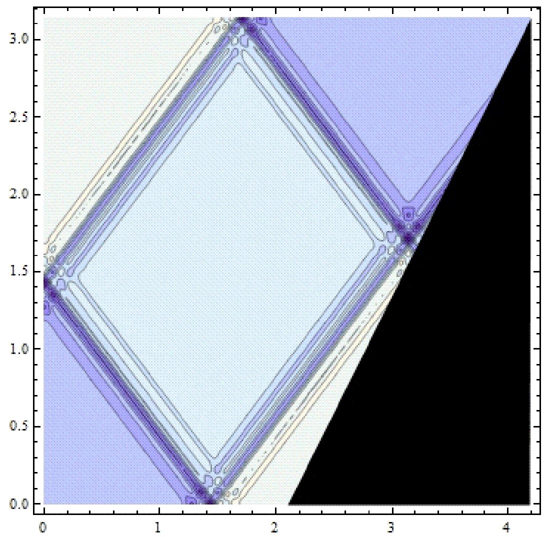


Figure 4.2: Contour plot of pressure

Chapter 5

3D simple attractors in a semi-trapezium

5.1 Introduction

This chapter will extend the previous to $k \neq 0$ and also examine the propagating and decaying modes of the waves.

5.2 Governing Equations

Using the Telegraph equation (4.7) for $k \neq 0$ our approximate solution (4.23) does not satisfy. However if p is taken of the form

$$p = \left(\sum_n a_n \cos \hat{n} y \cos n \hat{z} \right) \mathrm{e}^{-\mathrm{i}\omega t} \tag{5.1}$$

where

$$\hat{n}^2 = n^2 - k^2$$

then the Telegraph equation is satisfied.

Compute v and w from (4.2) and (4.3) it is found that

$$v = -\frac{\mathrm{i}\omega}{1-\omega^2} \left(\sum_n a_n \left[k \cos \hat{n} y - \omega \hat{n} \sin \hat{n} y \right] \cos n \hat{z} \right) \mathrm{e}^{-\mathrm{i}\omega t} \tag{5.2}$$

$$w = \frac{\mathrm{i}}{\sqrt{1-\omega^2}} \left(\sum_n n a_n \cos \hat{n} y \sin n \hat{z} \right) \mathrm{e}^{-\mathrm{i}\omega t} \tag{5.3}$$

However at $y = 0$, the vertical wall, $v \neq 0$. So our boundary conditions are not satisfied.

To take an alternative view of this, formulate v itself as

$$v = \left(\sum_n a_n \sin \hat{n} y \cos n \hat{z} \right) \mathrm{e}^{-\mathrm{i}\omega t} \tag{5.4}$$

From which p can be found by solving the differential equation (4.2) and from that w too.

$$p = \frac{\mathrm{i}(1-\omega^2)}{\omega} \left[\sum_n a_n \frac{\hat{n} \cos \hat{n} y - \frac{k}{\omega} \sin \hat{n} y}{\left(\frac{k}{\omega}\right)^2 + \hat{n}} \cos n \hat{z} \right] \mathrm{e}^{-\mathrm{i}\omega t} \tag{5.5}$$

$$w = -\frac{\sqrt{1-\omega^2}}{\omega} \left[\sum_n n a_n \frac{\hat{n} \cos \hat{n} y - \frac{k}{\omega} \sin \hat{n} y}{\left(\frac{k}{\omega}\right)^2 + \hat{n}} \cos n \hat{z} \right] \mathrm{e}^{-\mathrm{i}\omega t} \tag{5.6}$$

Clearly $v = 0$ at $y = 0$ and $w = 0$ at $z = 0, \pi$. Also the p formed still satisfies the Telegraph equation, as does the new v (5.4).

5.3 Numerical Solutions

So again look for an approximate numerical solution, this time in terms of v. Taking v of the form

$$v \approx \left(\sum_{n=1} \hat{N} a_n \sin \hat{n} y \cos n\hat{z} \right) \mathrm{e}^{-\mathrm{i}\omega t} \tag{5.7}$$

As before $w = -v \tan alpha$ on the slope. In this case skipping straight to $\tan alpha = \frac{3}{2}$ to display an example which will procure the following boundary condition on the lower half of the slope:

$$\begin{aligned} 0 = \sum_{n=1}^{\hat{N}} a_n \Bigg[& \sin\left(\sqrt{n^2 - k^2 y}\right) \cos\left(\frac{3n}{2}(y - \frac{2\pi}{3}\right) \\ & - \frac{2n}{3} \frac{\sqrt{1-\omega^2}}{\omega} \frac{\sqrt{n^2-k^2}\cos\left(\sqrt{n^2-k^2 y}\right) - \frac{k}{\omega}\sin\left(\sqrt{n^2-k^2 y}\right)}{\left(\frac{k}{\omega}\right)^2 + n^2 - k^2} \\ & \sin\left(\frac{3n}{2}(y - \frac{2\pi}{3})\right) \Bigg] \end{aligned} \tag{5.8}$$

Where $\hat{n}$ has been written in it's full form (5.2). Recall $\mathrm{e}^{\mathrm{i}kx}$ was used so k purely real corresponds to propagating waves, purely imaginary produces e^{-kx} corresponding to decaying waves and complex k gives rise to a mixture of the two.

In our fundamental intervals the same conditions as the two dimensional case are prescribed, $v = \sin 3y$ in the first and $v = -\sin 3y$ in the second, this will give us:

$$0 = \left(\sum_n a_n \sin\left(\sqrt{n^2 - k^2 y}\right) (-1)^n \right) - \sin 3y \tag{5.9}$$

$$0 = \left(\sum_n a_n \sin\left(\sqrt{n^2 - k^2 y}\right) (-1)^n \right) + \sin 3y \tag{5.10}$$

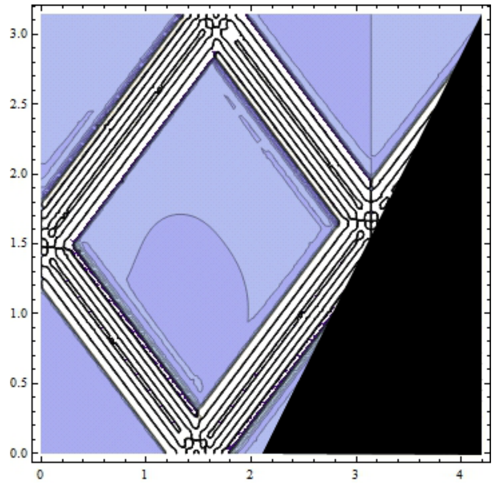
We are now in possession of three equations written in $\hat{N}$ terms which can be solved using the boundary collocation method outlined in [Maas and Harlander, 2007] for our a_n. To summarise the equations from our boundary conditions and the prescription in the fundamental intervals:

$$0 = \left(\sum_n a_n \sin\left(\sqrt{n^2 - k^2 y}\right) (-1)^n \right) - \sin 3y \tag{5.11}$$

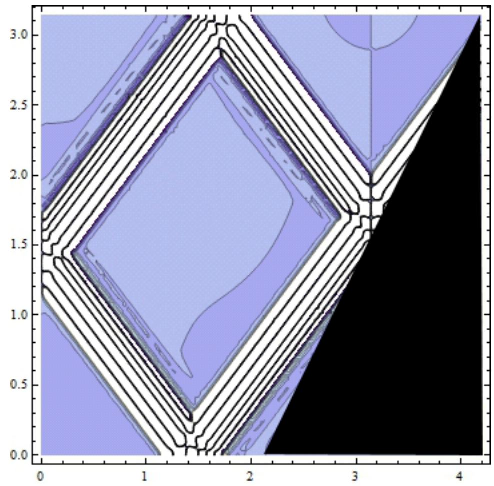
$$0 = \left(\sum_n a_n \sin\left(\sqrt{n^2 - k^2 y}\right) (-1)^n \right) + \sin 3y \tag{5.12}$$

$$\begin{aligned} 0 = \sum_{n} a_n \Bigg[& \sin\left(\sqrt{n^2 - k^2 y}\right) \cos\left(\frac{3n}{2}(y - \frac{2\pi}{3}\right) \\ & - \frac{2n}{3} \frac{\sqrt{1-\omega^2}}{\omega} \frac{\sqrt{n^2-k^2}\cos\left(\sqrt{n^2-k^2 y}\right) - \frac{k}{\omega}\sin\left(\sqrt{n^2-k^2 y}\right)}{\left(\frac{k}{\omega}\right)^2 + n^2 - k^2} \\ & \sin\left(\frac{3n}{2}(y - \frac{2\pi}{3})\right) \Bigg] \end{aligned} \tag{5.13}$$

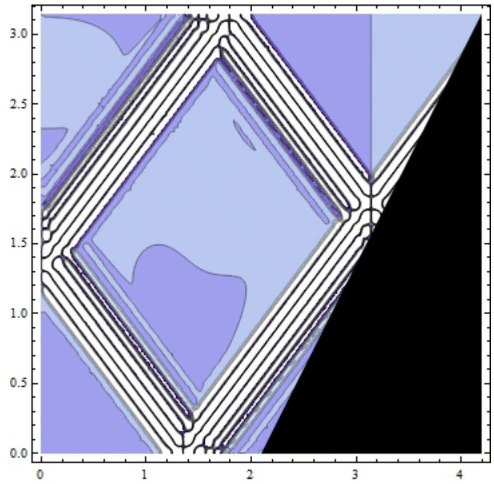
On the website (http://www.ucl.ac.uk/~zcahdl5/project.html) there are many examples of u and v for different values of k as well as animations of how contour plots and 3D plots of the velocity profile vary with time. Here I shall show just three pictures (which may look similar but have subtly different animations) for the values $k = 1.5$, $k = 1.5\text{i}$ and $k = 1.5 + 1.5\text{i}$.



$k = 1.5$



$k = 1.5\text{i}$



$k = 1.5 + 1.5\mathrm{i}$

Chapter 6

Discussion

In this project we have looked for wave attractors in a homogeneous, rotating, infinite duct. These are caused by a breaking of symmetry, which is - unfortunately - commonly not seen in most laboratory experiments where the geometry chosen precludes wave focusing. [Maas, 2001] views the more serious problem to be that most theoretical descriptions of wave motion in geophysics also eliminate focusing due to a combination of the 'traditional' and hydrostatic approximations.

Inertial waves have a dense set of frequencies which does not follow eigenmodes, strikingly different to that seen in surface or acoustic waves. Another difference is that upon reflection of surface waves the wavelength is fixed, but not the direction. Whereas for inertial waves the waves will reflect following strict rules of direction and correspondingly their wavelength will change. Inertial waves are dominated by wave attractors in all but the most symmetric domains (cylinders, spheres and cuboids). Here there is a spatial singularity where wave energy increases without bound until checked by viscous and non linear processes [Rieutord et al., 2001, Maas, 2005].

The focusing in this paper was first demonstrated using experiments before being derived for a fixed downstream wavenumber in a three dimensional domain. One problem of the method used here is that as n tends towards a value such that $n^2 - k^2 = 0$ then we tend towards a singularity. This equation is not integrable by any method found, including the use of Mathematica, which means our choice of k is restricted. Whilst this does not affect us for k imaginary or complex corresponding to decaying downstream and a mixture of decaying and propagating, it does affect us for k real; where k real corresponds for purely propagating downstream.

Further research from here seems to belong to two main paths. Firstly a better approximation to the solution, which can again be split into two sections. One of which would be to reduce the condition number of the large matrix which was inverted to find the co-efficients of the approximate solutions using p and later using v. This could possibly follow an energy minimisation method such as that of [Swart et al., 2007]. The second section of this improved solution would be to try and remove the division by zero encountered whilst trying to satisfy no normal flow on the sloped wall.

Secondly, this method could be used to generate the basic building blocks of a three dimensional closed domain. This could be accomplished though a sum over k with boundary conditions of no normal flow at the end walls.

The following is from Maas [2005]: The ubiquitous presence of inertial oscillations in the oceans might perhaps be due to wave attractors [Maas, 2001]. These attractors may play a decisive role in geophysical media as the ocean, atmosphere, liquid outer core of the earth, other planets or stars [Ogilvie and Lin, 2003], because the waves that approach them localize energy onto small scales where they contribute to mixing and drive mean ?ows. These processes may for the same reason be relevant in industrial applications, like turbomachinery.

I would like to thank my supervisor here at UCL, Ted Johnson, for all of his help and support with the project and for going above and beyond what I could ever have expected of him by setting up with Leo Maas six weeks at NIOZ over the summer. I am indebted to Leo Maas and Anna Rabitti for their continuing help and supervision not just during my stay at NIOZ but also after I left. Finally I would like to thank Sven Ober at NIOZ for his help with the experiments and the technical problems that arose.

Appendix A

Matlab and Mathematica Code

Here is the Matlab and Mathematica code for all parts of the project. The matrix "a" generated by Matlab is the matrix of co-efficients a_n which is exported under a suitable name, e.g. for $J_1 = J_2 = 40$, $N = 8$ and $k = 1.5 + 1.5\mathrm{i}$ "a" would be exported as a `.mat` file using the command `save 40-40-8_clipped_k=1.5+1.5i.mat a` with the exceptions that there is no mention of k in the 2D case and in some of the earlier examples on the website `clipped` isn't mentioned as it was a later improvement. Whilst the Mathematica code is written for the file paths on my personal computer with its own directory paths it should be obvious where to substitute in your own.

These Matlab and Mathematica as well as the pictures, animations and matrices of co-efficients generated can be found and downloaded on the project page of my website at http://www.ucl.ac.uk/~zcahdl5/project.html

A.1 Flat box

Below is the Mathematica code for the sum of different waves of the same frequency, also found here http://www.ucl.ac.uk/~zcahdl5/Project/Flat_box/Non-unique_modes_of_u_for_k=0_flat_box.nb.

Firstly a manipulate code to get a rough idea of what is going on:

```
Manipulate[Plot3D[Re[Sum[(Sin[j*m*z]*Sin[j*n*y])/(E^(I*t)*j),
{j, 1, N}]], {y, 0, Pi}, {z, 0, Pi}], {N, 1, 200, 1},
{t, 0, 2*Pi, Pi/8}, {n, 1, 10, 1}, {m, 1, 10, 1}]
```

Then export for a different ratio of sides or co-prime eigenmodes from (1,1) to (5,5).

```
picts = Table[Plot3D[Re[Sum[(Sin[j*y]*Cos[j*z])/(E^(I*t)*j),
{j, 1, 100}]], {y, 0, Pi}, {z, 0, Pi},
PlotRange -> {{0, Pi}, {0, Pi}, {-2, 2}}],
{t, 0, 2*Pi - Pi/32, Pi/32}];
Export["C:\\Temp\\k=0 non-unique modes\\1y1z.avi", picts]
Export["C:\\Temp\\k=0 non-unique modes\\1y1z.gif", picts]

picts = Table[Plot3D[Re[Sum[(Sin[j*y]*Cos[2*j*z])/(E^(I*t)*j),
{j, 1, 100}]], {y, 0, Pi}, {z, 0, Pi},
PlotRange -> {{0, Pi}, {0, Pi}, {-2, 2}}],
{t, 0, 2*Pi - Pi/32, Pi/32}];
Export["C:\\Temp\\k=0 non-unique modes\\1y2z.avi", picts]
Export["C:\\Temp\\k=0 non-unique modes\\1y2z.gif", picts]

picts = Table[Plot3D[Re[Sum[(Sin[j*y]*Cos[3*j*z])/(E^(I*t)*j),
{j, 1, 100}]], {y, 0, Pi}, {z, 0, Pi},
PlotRange -> {{0, Pi}, {0, Pi}, {-2, 2}}],
{t, 0, 2*Pi - Pi/32, Pi/32}];
Export["C:\\Temp\\k=0 non-unique modes\\1y3z.avi", picts]
Export["C:\\Temp\\k=0 non-unique modes\\1y3z.gif", picts]

picts = Table[Plot3D[Re[Sum[(Sin[j*y]*Cos[4*j*z])/(E^(I*t)*j),
{j, 1, 100}]], {y, 0, Pi}, {z, 0, Pi},
```

```
PlotRange -> {{0, Pi}, {0, Pi}, {-2, 2}}],
{t, 0, 2*Pi - Pi/32, Pi/32}];
Export["C:\\Temp\\k=0 non-unique modes\\1y4z.avi", picts]
Export["C:\\Temp\\k=0 non-unique modes\\1y4z.gif", picts]

picts = Table[Plot3D[Re[Sum[(Sin[j*y]*Cos[5*j*z])/(E^(I*t)*j),
{j, 1, 100}]], {y, 0, Pi}, {z, 0, Pi},
PlotRange -> {{0, Pi}, {0, Pi}, {-2, 2}}],
{t, 0, 2*Pi - Pi/32, Pi/32}];
Export["C:\\Temp\\k=0 non-unique modes\\1y5z.avi", picts]
Export["C:\\Temp\\k=0 non-unique modes\\1y5z.gif", picts]

picts = Table[Plot3D[Re[Sum[(Sin[2*j*y]*Cos[j*z])/(E^(I*t)*j),
{j, 1, 100}]], {y, 0, Pi}, {z, 0, Pi},
PlotRange -> {{0, Pi}, {0, Pi}, {-2, 2}}],
{t, 0, 2*Pi - Pi/32, Pi/32}];
Export["C:\\Temp\\k=0 non-unique modes\\2y1z.avi", picts]
Export["C:\\Temp\\k=0 non-unique modes\\2y1z.gif", picts]

picts = Table[Plot3D[Re[Sum[(Sin[2*j*y]*Cos[2*j*z])/(E^(I*t)*j),
{j, 1, 100}]], {y, 0, Pi}, {z, 0, Pi},
PlotRange -> {{0, Pi}, {0, Pi}, {-2, 2}}],
{t, 0, 2*Pi - Pi/32, Pi/32}];
Export["C:\\Temp\\k=0 non-unique modes\\2y2z.avi", picts]
Export["C:\\Temp\\k=0 non-unique modes\\2y2z.gif", picts]

picts = Table[Plot3D[Re[Sum[(Sin[2*j*y]*Cos[3*j*z])/(E^(I*t)*j),
{j, 1, 100}]], {y, 0, Pi}, {z, 0, Pi},
PlotRange -> {{0, Pi}, {0, Pi}, {-2, 2}}],
{t, 0, 2*Pi - Pi/32, Pi/32}];
Export["C:\\Temp\\k=0 non-unique modes\\2y3z.avi", picts]
Export["C:\\Temp\\k=0 non-unique modes\\2y3z.gif", picts]

picts = Table[Plot3D[Re[Sum[(Sin[2*j*y]*Cos[4*j*z])/(E^(I*t)*j),
{j, 1, 100}]], {y, 0, Pi}, {z, 0, Pi},
PlotRange -> {{0, Pi}, {0, Pi}, {-2, 2}}],
{t, 0, 2*Pi - Pi/32, Pi/32}];
Export["C:\\Temp\\k=0 non-unique modes\\2y4z.avi", picts]
Export["C:\\Temp\\k=0 non-unique modes\\2y4z.gif", picts]

picts = Table[Plot3D[Re[Sum[(Sin[2*j*y]*Cos[5*j*z])/(E^(I*t)*j),
{j, 1, 100}]], {y, 0, Pi}, {z, 0, Pi},
PlotRange -> {{0, Pi}, {0, Pi}, {-2, 2}}],
{t, 0, 2*Pi - Pi/32, Pi/32}];
Export["C:\\Temp\\k=0 non-unique modes\\2y5z.avi", picts]
Export["C:\\Temp\\k=0 non-unique modes\\2y5z.gif", picts]

picts = Table[Plot3D[Re[Sum[(Sin[3*j*y]*Cos[j*z])/(E^(I*t)*j),
{j, 1, 100}]], {y, 0, Pi}, {z, 0, Pi},
PlotRange -> {{0, Pi}, {0, Pi}, {-2, 2}}],
```

```
{t, 0, 2*Pi - Pi/32, Pi/32}];
Export["C:\\Temp\\k=0 non-unique modes\\3y1z.avi", picts]
Export["C:\\Temp\\k=0 non-unique modes\\3y1z.gif", picts]

picts = Table[Plot3D[Re[Sum[(Sin[3*j*y]*Cos[2*j*z])/(E^(I*t)*j),
{j, 1, 100}]], {y, 0, Pi}, {z, 0, Pi},
PlotRange -> {{0, Pi}, {0, Pi}, {-2, 2}}],
{t, 0, 2*Pi - Pi/32, Pi/32}];
Export["C:\\Temp\\k=0 non-unique modes\\3y2z.avi", picts]
Export["C:\\Temp\\k=0 non-unique modes\\3y2z.gif", picts]

picts = Table[Plot3D[Re[Sum[(Sin[3*j*y]*Cos[3*j*z])/(E^(I*t)*j),
{j, 1, 100}]], {y, 0, Pi}, {z, 0, Pi},
PlotRange -> {{0, Pi}, {0, Pi}, {-2, 2}}],
{t, 0, 2*Pi - Pi/32, Pi/32}];
Export["C:\\Temp\\k=0 non-unique modes\\3y3z.avi", picts]
Export["C:\\Temp\\k=0 non-unique modes\\3y3z.gif", picts]

picts = Table[Plot3D[Re[Sum[(Sin[3*j*y]*Cos[4*j*z])/(E^(I*t)*j),
{j, 1, 100}]], {y, 0, Pi}, {z, 0, Pi},
PlotRange -> {{0, Pi}, {0, Pi}, {-2, 2}}],
{t, 0, 2*Pi - Pi/32, Pi/32}];
Export["C:\\Temp\\k=0 non-unique modes\\3y4z.avi", picts]
Export["C:\\Temp\\k=0 non-unique modes\\3y4z.gif", picts]

picts = Table[Plot3D[Re[Sum[(Sin[3*j*y]*Cos[5*j*z])/(E^(I*t)*j),
{j, 1, 100}]], {y, 0, Pi}, {z, 0, Pi},
PlotRange -> {{0, Pi}, {0, Pi}, {-2, 2}}],
{t, 0, 2*Pi - Pi/32, Pi/32}];
Export["C:\\Temp\\k=0 non-unique modes\\3y5z.avi", picts]
Export["C:\\Temp\\k=0 non-unique modes\\3y5z.gif", picts]

picts = Table[Plot3D[Re[Sum[(Sin[4*j*y]*Cos[j*z])/(E^(I*t)*j),
{j, 1, 100}]], {y, 0, Pi}, {z, 0, Pi},
PlotRange -> {{0, Pi}, {0, Pi}, {-2, 2}}],
{t, 0, 2*Pi - Pi/32, Pi/32}];
Export["C:\\Temp\\k=0 non-unique modes\\4y1z.avi", picts]
Export["C:\\Temp\\k=0 non-unique modes\\4y1z.gif", picts]

picts = Table[Plot3D[Re[Sum[(Sin[4*j*y]*Cos[2*j*z])/(E^(I*t)*j),
{j, 1, 100}]], {y, 0, Pi}, {z, 0, Pi},
PlotRange -> {{0, Pi}, {0, Pi}, {-2, 2}}],
{t, 0, 2*Pi - Pi/32, Pi/32}];
Export["C:\\Temp\\k=0 non-unique modes\\4y2z.avi", picts]
Export["C:\\Temp\\k=0 non-unique modes\\4y2z.gif", picts]

picts = Table[Plot3D[Re[Sum[(Sin[4*j*y]*Cos[3*j*z])/(E^(I*t)*j),
{j, 1, 100}]], {y, 0, Pi}, {z, 0, Pi},
PlotRange -> {{0, Pi}, {0, Pi}, {-2, 2}}],
{t, 0, 2*Pi - Pi/32, Pi/32}];
```

```
Export["C:\\Temp\\k=0 non-unique modes\\4y3z.avi", picts]
Export["C:\\Temp\\k=0 non-unique modes\\4y3z.gif", picts]

picts = Table[Plot3D[Re[Sum[(Sin[4*j*y]*Cos[4*j*z])/(E^(I*t)*j),
{j, 1, 100}]], {y, 0, Pi}, {z, 0, Pi},
PlotRange -> {{0, Pi}, {0, Pi}, {-2, 2}}],
{t, 0, 2*Pi - Pi/32, Pi/32}];
Export["C:\\Temp\\k=0 non-unique modes\\4y4z.avi", picts]
Export["C:\\Temp\\k=0 non-unique modes\\4y4z.gif", picts]

picts = Table[Plot3D[Re[Sum[(Sin[4*j*y]*Cos[5*j*z])/(E^(I*t)*j),
{j, 1, 100}]], {y, 0, Pi}, {z, 0, Pi},
PlotRange -> {{0, Pi}, {0, Pi}, {-2, 2}}],
{t, 0, 2*Pi - Pi/32, Pi/32}];
Export["C:\\Temp\\k=0 non-unique modes\\4y5z.avi", picts]
Export["C:\\Temp\\k=0 non-unique modes\\4y5z.gif", picts]

picts = Table[Plot3D[Re[Sum[(Sin[5*j*y]*Cos[j*z])/(E^(I*t)*j),
{j, 1, 100}]], {y, 0, Pi}, {z, 0, Pi},
PlotRange -> {{0, Pi}, {0, Pi}, {-2, 2}}],
{t, 0, 2*Pi - Pi/32, Pi/32}];
Export["C:\\Temp\\k=0 non-unique modes\\5y1z.avi", picts]
Export["C:\\Temp\\k=0 non-unique modes\\5y1z.gif", picts]

picts = Table[Plot3D[Re[Sum[(Sin[5*j*y]*Cos[2*j*z])/(E^(I*t)*j),
{j, 1, 100}]], {y, 0, Pi}, {z, 0, Pi},
PlotRange -> {{0, Pi}, {0, Pi}, {-2, 2}}],
{t, 0, 2*Pi - Pi/32, Pi/32}];
Export["C:\\Temp\\k=0 non-unique modes\\5y2z.avi", picts]
Export["C:\\Temp\\k=0 non-unique modes\\5y2z.gif", picts]

picts = Table[Plot3D[Re[Sum[(Sin[5*j*y]*Cos[3*j*z])/(E^(I*t)*j),
{j, 1, 100}]], {y, 0, Pi}, {z, 0, Pi},
PlotRange -> {{0, Pi}, {0, Pi}, {-2, 2}}],
{t, 0, 2*Pi - Pi/32, Pi/32}];
Export["C:\\Temp\\k=0 non-unique modes\\5y3z.avi", picts]
Export["C:\\Temp\\k=0 non-unique modes\\5y3z.gif", picts]

picts = Table[Plot3D[Re[Sum[(Sin[5*j*y]*Cos[4*j*z])/(E^(I*t)*j),
{j, 1, 100}]], {y, 0, Pi}, {z, 0, Pi},
PlotRange -> {{0, Pi}, {0, Pi}, {-2, 2}}],
{t, 0, 2*Pi - Pi/32, Pi/32}];
Export["C:\\Temp\\k=0 non-unique modes\\5y4z.avi", picts]
Export["C:\\Temp\\k=0 non-unique modes\\5y4z.gif", picts]

picts = Table[Plot3D[Re[Sum[(Sin[5*j*y]*Cos[5*j*z])/(E^(I*t)*j),
{j, 1, 100}]], {y, 0, Pi}, {z, 0, Pi},
PlotRange -> {{0, Pi}, {0, Pi}, {-2, 2}}],
{t, 0, 2*Pi - Pi/32, Pi/32}];
Export["C:\\Temp\\k=0 non-unique modes\\5y5z.avi", picts]
```

```
Export["C:\\Temp\\k=0 non-unique modes\\5y5z.gif", picts]
```

A.2 2D

A.2.1 Matlab code

This is fairly well explained during the actual Matlab file, the actual file can be found here http://www.ucl.ac.uk/~zcahdl5/Project/2D/p2Dtrap.m

```
clear all;
close all;

nJ1=40; nJ2=40; nJ3=8; omega=0.5;
nJ12=nJ1+nJ2; N=nJ12+nJ3;
clipped=true;

%J1 y values along first fundamental interval,
%J2 along the second and J3
%is along lower half of slope.
%nJ1 is number of intervals along J1,
%likewise with nJ2 and nJ3
%Clipped true or false chooses whether to
%include the end points of the intervals or not.
%Omega is the wave frequency.

if clipped==true;
    J1=linspace(pi/(3*nJ1),pi/3-pi/(3*nJ1),nJ1);
    J2=linspace(pi+pi/(3*nJ2),4*pi/3-pi/(3*nJ2),nJ2);
    J12(1:nJ1)=J1; J12(nJ1+1:nJ12)=J2;
    J3=linspace(2*pi/3,pi,nJ3);
elseif clipped==false;
    J1=linspace(pi/(3*nJ1),pi/3,nJ1);
    J2=linspace(pi,4*pi/3,nJ2);
    J12(1:nJ1)=J1; J12(nJ1+1:nJ12)=J2;
    J3=linspace(2*pi/3,pi,nJ3);
end

% For A.a=b where A is NxN matrix of sums,
%a is the co-efficients of these
% sums (to be solved for) and b is our specified values

%Define A

A=zeros(N,N);

for n=1:N
    for m=1:nJ12
    A(m,n)=(3/2)*(1i*omega/(1-omega^2))*n*
sin(n*J12(1,m))*((-1)^n);
    end
```

```
    for m=1:nJ3
    A(nJ12+m,n)=n*(((2/3)*(sqrt(1-omega^2)/omega)*cos(n*J3(1,m))*
sin((3*n/2)*(J3(1,m)-(2*pi/3))))-
(sin(n*J3(1,m))*cos((3*n/2)*(J3(1,m)-(2*pi/3)))));
    end
end

%Define b

b=zeros(N,1);

for n=1:nJ1
    b(n,1)=sin(3*J1(n));
end
for n=1:nJ2
    b(nJ1+n,1)=-sin(3*J2(n));
end

%Zero for n=nJ12+1:N

%A.a=b implies a=A\b

a=A\b;
```

After this had been run the following command would be used to save `a` to the default folder `save 40-40-8_clipped.mat a`

A.2.2 Mathematica code

This is also at http://www.ucl.ac.uk/~zcahdl5/Project/2D/Create_plots_2D.nb and gives an output of pictures and animations. To choose between what you do and don't want simply remove the `(*` and `*)` which is the Mathematica code to ignore what is contained within. The sloped wall has to be drawn on manually.

```
a = {Flatten[Import["C:\\Temp\\40-40-8_clipped.mat"]]};
Dimensions[a]
(*set upper limit of n as dimension of a*)
f = Table[Cos[n y] Cos[n z], {n, 1, 88}];
p = a.f;
Dimensions[p] (*Check =1*)
(*density=DensityPlot[Im[p],{y,0,4 \[Pi]/3},{z,0,\[Pi]},
MaxRecursion->6]*)
(*contour = ContourPlot[Im[p], {y, 0, 4 \[Pi]/3}, {z, 0, \[Pi]},
MaxRecursion -> 4]*)
(*If MaxRecursion is increased we get a more accurate plot,
but it will take longer and require more memory*)
N[2 \[Pi]/3] (*The y value for where the wall intersects z=0*)
```

A.3 3D

A.3.1 Matlab Code

This is well described by the comments in the file and the method for exporting into Mathematica is exactly the same as before with the slightly more complicated naming criteria highlighted above. This file can be found here http://www.ucl.ac.uk/~zcahdl5/Project/3D/v3Dtrap.m

```
clear all;
close all;

nJ1=40; nJ2=40; nJ3=8; omega=0.5; clipped=true; k=0.5+0.5i;
nJ12=nJ1+nJ2; N=nJ12+nJ3;

%J1 y values along first fundamental interval,
%J2 along the second and J3
%is along lower half of slope.
%nJ1 is number of intervals along J1,
%likewise with nJ2 and nJ3.
%Clipped true or false chooses whether to include
%the end points of the intervals or not.
%Omega is the wave frequency.
%For k=... it is actually ik=... becuase we started
%off with exp{ikx) so propagating waves set k real,
%decaying waves set Re(k)=0, Im(k)>0 so we get
%exp(i^2 Im(k) x)=exp(-Im(k)x) decaying.
%So rate of decay is Im(k) hence Im(k)<0 is
%actually a growing exponential.
%Re(k^2) cannot be an integer unless Re(k^2)>N
%e.g. k=1 is propagating with wave number 1,
%k=i is decaying like e(-x), k=2+3i
%is decaying like e(-3x) and propagating with wave number 2.

if clipped==true;
    J1=linspace(pi/(3*nJ1),pi/3-pi/(3*nJ1),nJ1);
    J2=linspace(pi+pi/(3*nJ2),4*pi/3-pi/(3*nJ2),nJ2);
    J12(1:nJ1)=J1; J12(nJ1+1:nJ12)=J2;
    J3=linspace(2*pi/3,pi,nJ3);
elseif clipped==false;
    J1=linspace(pi/(3*nJ1),pi/3,nJ1);
    J2=linspace(pi,4*pi/3,nJ2);
    J12(1:nJ1)=J1; J12(nJ1+1:nJ12)=J2;
    J3=linspace(2*pi/3,pi,nJ3);
end

%For A.a=b where A is NxN matrix of sums,
%a is the co-efficients of these sums
%(to be solved for) and b is our specified values

%Define A
```

```
A=zeros(N,N);



for n=1:N
    nhat=sqrt(n^2 - k^2);
    for m=1:nJ12
    y=J12(1,m);
    A(m,n)=sin(nhat*y)*((-1)^n);
    end
    for m=1:nJ3
    y=J3(1,m);
    A(nJ12+m,n)=sin(nhat*y)*cos((3*n/2)*y - n*pi)
- (2*n/3)*((sqrt(1 - omega^2))/omega)*(nhat*cos(nhat*y)
- (k/omega)*sin(nhat*y))/((k^2)/(omega^2) + nhat^2)
*sin((3*n/2)*y - n*pi);
    end
end

%Define b

b=zeros(N,1);

for n=1:nJ1
    b(n,1)=sin(3*J1(n));
end
for n=1:nJ2
    b(nJ1+n,1)=-sin(3*J2(n));
end

%Zero for n=nJ12+1:N

%A.a=b implies a=A\b

a=A\b;
```

After this a is exported again using a similar command to above which can then be imported using Mathematica to create plots.

A.3.2 Mathematica code

This can also be found here http://www.ucl.ac.uk/~zcahdl5/Project/3D/Create_plots_3D.nb and can produce contours plots - which were found to be far superior to density plots - and animations of these with respect to time. It can also produce animations of 3D plots of the velocity profile. Plots of the pressure term were found to be inferior to those of the horizontal velocity, v, and plots of the downstream velocity profile, u, have some of the best detail.

```
a =
```

```
{Flatten[Import["C:\\Temp\\40-40-8_clipped_k=1.5+1.5i.mat"]]};
Dimensions[a]

(*set upper limit of n as dimension of a*)
\[Omega] = 0.5; k = 1.5 + 1.5*I;
vtrig = Table[Sin[n*y]*Cos[Sqrt[n^2 - k^2]*z], {n, 1, 88}];

ptrig =
Table[((I*(1 - \[Omega]^2))/\[Omega])
*((Sqrt[n^2 - k^2]*Cos[Sqrt[n^2 - k^2]*y]
- (k/\[Omega])*Sin[Sqrt[n^2 - k^2]*y])/
      ((k/\[Omega])^2 + n^2 - k^2))*Cos[n*z],
{n, 1, 88}];

utrig = Table[I*((k*Sqrt[n^2 - k^2]
*((1 - \[Omega]^2)/\[Omega]^2)*Cos[Sqrt[n^2 - k^2]*y]
+ (n^2/\[Omega])*Sin[Sqrt[n^2 - k^2]*y])/
      ((k/\[Omega])^2 + n^2 - k^2))*Cos[n*z], {n, 1, 88}];

(*For pictures use these two*)
contourv = ContourPlot[Re[v*E^(I*\[Omega])],
{y, 0, (4*Pi)/3}, {z, 0, Pi}, MaxRecursion -> Automatic];

contouru = ContourPlot[Re[u*E^(I*\[Omega])],
{y, 0, (4*Pi)/3}, {z, 0, Pi}, MaxRecursion -> Automatic];


(*For 2D animations use these*)
contourv = Table[ContourPlot[Re[v*E^(I*\[Omega]*t)],
{y, 0, (4*Pi)/3}, {z, 0, Pi}, MaxRecursion -> Automatic],
    {t, 0, (2*Pi)/\[Omega] - Pi/8, Pi/8}];
Export
["R:\\Project\\40-40-8_clippedv_k=1.5+1.5i_omega=0.5.gif",
contourv]
contourv =.

contourp = Table[ContourPlot[Re[p*E^(I*\[Omega]*t)],
{y, 0, (4*Pi)/3}, {z, 0, Pi}, MaxRecursion -> Automatic],
    {t, 0, (2*Pi)/\[Omega] - Pi/8, Pi/8}];
Export
["R:\\Project\\40-40-8_clippedp_k=1.5+1.5i_omega=0.5.gif",
contourp]
contourp =.

contouru = Table[ContourPlot[Re[u*E^(I*\[Omega]*t)],
{y, 0, (4*Pi)/3}, {z, 0, Pi}, MaxRecursion -> Automatic],
    {t, 0, (2*Pi)/\[Omega] - Pi/8, Pi/8}];
Export
["R:\\Project\\40-40-8_clippedu_k=1.5+1.5i_omega=0.5.gif",
contourp]
```

```
contouru =.
```

For a 3D animation of the velocity profile, run this

```
a={Flatten[Import["C:\\Temp\\40-40-8_clipped_k=1.5+1.5i.mat"]]};
\[Omega] = 0.5; k = 1.5 + 1.5*I;
vtrig = Table[Sin[n*y]*Cos[Sqrt[n^2 - k^2]*z], {n, 1, 88}];
v = a . vtrig;
Export["C:\\Temp\\3D_40-40-8_clippedv_k=1.5+1.5i_omega=0.5.gif",
  Table[Plot3D[Re[v*E^(I*\[Omega]*t)], {y, 0, (4*Pi)/3},
{z, 0, Pi}, PlotRange -> {-5000, 5000}],
{t, 0, (2*Pi)/\[Omega] - Pi/8, Pi/8}]]
```

Bibliography

B Dintrans and M Rieutord. Gravito-inertial waves in a rotating stratified sphere or spherical shell. Journal of Fluid Mechanics, 398:271–297, 1999. URL http://www.ast.obs-mip.fr/users/rieutord/articles/1999JFM.pdf.

Sybren Drijfhout and Leo R. M. Maas. Impact of Channel Geometry and Rotation on the Trapping of Internal Tides. Journal of Physical Oceanography, 37(11):2740, 2007. ISSN 0022-3670. doi: 10.1175/2007JPO3586.1. URL http://journals.ametsoc.org/doi/abs/10.1175/2007JPO3586.1.

Uwe Harlander and Leo R. M. Maas. Two alternatives for solving hyperbolic boundary value problems of geophysical fluid dynamics. Journal of Fluid Mechanics, 588:331–351, September 2007. ISSN 0022-1120. doi: 10.1017/S0022112007007574. URL http://www.journals.cambridge.org/abstract_S0022112007007574.

David Houcque. Introduction To Matlab For Engineering Students.

Cyril Lamriben, Pierre-philippe Cortet, Frederic Moisy, and Leo R. M. Maas. Excitation of inertial modes in a closed grid turbulence experiment under rotation. Physics of Fluids, 23(1):015102, 2011. ISSN 10706631. doi: 10.1063/1.3540660. URL http://link.aip.org/link/PHFLE6/v23/i1/p015102/s1&Agg=doi.

Leo R. M. Maas. Wave focusing and ensuing mean flow due to symmetry breaking in rotating fluids. Journal of Fluid Mechanics, 437:13–28, June 2001. ISSN 0022-1120. doi: 10.1017/S0022112001004074. URL http://www.journals.cambridge.org/abstract_S0022112001004074.

Leo R. M. Maas. On the amphidromic structure of inertial waves in a rectangular parallelepiped. Fluid Dynamics Research, 373:373–401, 2003. doi: 10.1016/j.fuiddyn.2003.08.003.

Leo R. M. Maas. Wave attractors: linear yet nonlinear. Int. J. Bifurcation Chaos, 15(9):2757–2782, 2005. URL http://www.nioz.nl/public/fys/staff/leo_maas/publications/maasijbc05.pdf.

Leo R. M. Maas. Experiments on rotating flows: Impact of rotation on flow through tilted rectangular ducts. Fluxes and Structures in Fluids, 2007.

Leo R. M. Maas. Exact analytic self-similar solution of a wave attractor field. Physica D: Nonlinear Phenomena, 238(5):502–505, March 2009. ISSN 01672789. doi: 10.1016/j.physd.2008.11.006. URL http://linkinghub.elsevier.com/retrieve/pii/S0167278908004156.

Leo R. M. Maas and U Harlander. Two alternatives for solving hyperbolic boundary value problems of geophysical fluid dynamics. Journal of Fluid Mechanics, 588:331–351, 2007. doi: 10.1017/S0022112007007574. URL http://journals.cambridge.org/abstract_S0022112007007574.

Leo R. M. Maas and FPA Lam. Geometric focusing of internal waves. Journal of Fluid Mechanics, 300(1):1–41, April 1995. ISSN 0022-1120.

L.R.M. Maas, Dominique Benielli, Joel Sommeria, and FPA Lam. Observation of an internal wave attractor in a confined, stably stratified fluid. Letters to Nature, 388(August):557–561, 1997.

Lorenz Magaard. Ein Beitrag zur Theorie der internen Wellen als Störungen geostrophischer Strömungen. Ocean Dynamics, 21(6):241–278, 1968. ISSN 1616-7341.

A.M.M. Manders and L.R.M. Maas. On the three-dimensional structure of the inertial wave field in a rectangular basin with one sloping boundary. Fluid dynamics research, 35(1):1–21, 2004. ISSN 0169-5983. doi: 10.1016/j. URL http://linkinghub.elsevier.com/retrieve/pii/S0169598304000334.

Astrid M. M. Manders. Internal wave patterns in enclosed density-stratified and rotating fluids. 2003. URL http://igitur-archive.library.uu.nl/dissertations/2003-1022-161652/UUindex.html.

Astrid M. M. Manders and Leo R. M. Maas. Observations of inertial waves in a rectangular basin with one sloping boundary. Journal of Fluid Mechanics, 493: 59–88, October 2003. ISSN 00221120. doi: 10.1017/S0022112003005998. URL http://www.journals.cambridge.org/abstract_S0022112003005998.

T. Oetiker, Hubert Partl, Irene Hyna, and Elisabeth Schlegl. The not so short introduction to LATEX.

G I Ogilvie and D N C Lin. Tidal dissipation in rotating giant planets. The Astrophysical Journal, 610(1):74, 2003. URL http://arxiv.org/abs/astro-ph/0310218.

Gordon I. Ogilview. Wave attractors and the asymptotic dissipation rate of tidal disturbances. Journal of Fluid Mechanics, 543:19, November 2005. ISSN 0022-1120. doi: 10.1017/S0022112005006580. URL http://www.journals.cambridge.org/abstract_S0022112005006580.

J. Proudman. On the Dynamical Equations of the Tides. Proceedings of the London Mathematical Society, s2-34(1):293–304, January 1932. ISSN 0024-6115. doi: 10.1112/plms/s2-34.1.293. URL http://plms.oxfordjournals.org/cgi/doi/10.1112/plms/s2-34.1.293.

M Rieutord, B Georgeot, and L Valdettaro. Wave attractors in rotating fluids: A paradigm for ill-posed cauchy problems. Physical review letters, 85(20): 4277–80, November 2000. ISSN 1079-7114. URL http://www.ncbi.nlm.nih.gov/pubmed/11060617.

M. Rieutord, B. Georgeot, and L. Valdettaro. Inertial waves in a rotating spherical shell: attractors and asymptotic spectrum. Journal of Fluid Mechanics, 435:103–144, June 2001. ISSN 0022-1120. doi: 10.1017/S0022112001003718. URL http://www.journals.cambridge.org/abstract_S0022112001003718.

A Swart, G Sleijpen, L Maas, and J Brandts. Numerical solution of the two-dimensional Poincaré equation. Journal of Computational and Applied Mathematics, 200(1):317–341, March 2007. ISSN 03770427. doi: 10.1016/j.cam.2005.12.024. URL http://linkinghub.elsevier.com/retrieve/pii/S0377042706000185.

Geoffrey W. Wake, GREGORY N. Ivey, JÖRG Imberger, N. ROBB McDonald, and ROMAN Stocker. Baroclinic geostrophic adjustment in a rotating circular basin. Journal of Fluid Mechanics, 515(515):63–86, September 2004. ISSN 0022-1120. doi: 10.1017/S0022112004000230. URL http://www.journals.cambridge.org/abstract_S0022112004000230.

Carl Wunsch. Progressive internal waves on slopes. Journal of Fluid Mechanics, 66(01):3821, January 1969. doi: 10.1017/S0022112069001005.