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Exact analytic self-similar solution of a wave attractor field

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ABSTRACT

Stratified and rotating fluids support obliquely propagating internal waves. A symmetry-breaking shape of the fluid domain focuses them on a wave attractor. For a trapezoidal basin, it is here shown how to determine the internal wave field analytically. This requires solving the wave equation on a closed domain – an ill-posed Cauchy problem – whose solution exhibits a remarkable self-similar spatial structure. These results are relevant for mixing and mean flow generation in oceans, atmospheres and stars whose symmetry is generally broken and where internal waves are tidally forced.

Internal gravity waves play an important role in transporting energy, momentum, heat and chemicals in naturally stratified fluids as lakes, oceans and planetary and stellar atmospheres. Moreover, these waves have a great similarity to inertial waves arising in rotating homogeneous fluids, to electron-cyclotron waves in magnetized fluids, and to any combination of these [1]. In all these media, waves propagate obliquely relative to the stratifying direction (set by gravity, rotation, or magnetic field vector), under an angle that is fixed by the wave's frequency. Therefore the shape of the fluid domain exerts a big influence on these waves [2]. Generally, symmetry-breaking produced by any boundary that is neither parallel nor perpendicular to the stratifying direction, leads to wave focusing and the formation of a singularity of the wave field, which acts as a wave attractor [3]. Wave attractors, derived in an ideal fluid context, have relevance for real, viscous fluids too [4–6]. They enforce localized mixing that generates mean transport of matter, heat and momentum [5,7]. Wave attractors were originally constructed by an algorithmic method [2], which precludes the exact computation of derivatives that determine the local fluid velocity field. They have subsequently been found in numerical (viscous) computations, both on small [8] and large scales [7,9,10], but these results are necessarily approximative.

In this paper, an exact analytic wave attractor field is determined for the first time, enabling computation of its velocity field. As an analytical novelty, a self-similar spatial structure is obtained. This echoes a remarkable periodicity of its Fourier expansion coefficients, related to the presence of the Weierstrass function [11].

Let x, z denote horizontal and vertical Cartesian coordinates, where z is antiparallel to gravity. In the vertical plane, upon scaling, the streamfunction of a monochromatic internal wave field, $\psi(x, z)$, is governed by the wave equation [3]:

$$\psi_{xx} - \psi_{zz} = 0. \tag{1}$$

The streamfunction needs to vanish ($\psi = 0$) at the boundary ∂D, but this requirement leaves the streamfunction underdetermined. When also its normal derivative is prescribed, this (Cauchy) boundary value problem (BVP) becomes ill-posed [10]. The wave equation is here cast strictly in spatial coordinates, to be solved over a finite region of space. Interestingly, the companion *initial* value problem (replacing z by time t), leaving future behavior unspecified, was found to feature wave focusing as well, as when the wave equation is solved for $x \in [0, L(t)]$, on a one-dimensional string of variable length $L(t)$ [12,13]. Exact results

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show the collapse of the initial wave function due to focusing reflections at the moving boundary [14–16]. However, applied to electromagnetic waves, the rate at which the wall moves is required to be less than the speed of light ($\mathrm{d}L/\mathrm{d}t < 1$, in scaled coordinates). In contrast, in the present physical setting, reflection towards negative z is not prohibited, and, in fact, needs to occur on a domain of finite size, which gives a twist to the present problem. Therefore, alternative solution procedures were looked at, such as a Fourier expansion method [17,18]. This led to an infinite matrix equation for the Fourier coefficients that was truncated and inverted numerically. That numerical solution mimicked the algorithmic one quite well, with the bonus of yielding analytic information on the corresponding velocity field, but at the expense of having to solve a large matrix equation. The aim of the present paper is to show that this infinite matrix equation can, in fact, be solved exactly.

The trapezoidal domain D, considered here, consists of a horizontal rigid-lid surface, $z = 0$, a vertical side wall, $x = -1$, a horizontal bottom, $z = -\tau$, when $x \in [-1, 0]$, and a single sloping side wall, $z = \tau(x-1)$, when $x \in (0, 1]$. Wave frequency, stratification rate and aspect ratio are all lumped together in the single parameter τ which one can think of simply as the depth [2]. Consider a single particular depth, $\tau = 3/2$, which yields a symmetric, square-shaped wave attractor [4] and look for a Fourier series solution [17]:

$$\psi = \sum_{n=1}^{\infty} a_n \sin n\pi \frac{(x+1)}{\tau} \sin n\pi \frac{z}{\tau}. \tag{2}$$

In terms of characteristic coordinates, $\zeta^{\pm} = x + 1 \pm z$, employed in the algorithmic approach, this can be written as:

$$\psi = \frac{1}{2} \sum_{n=1}^{\infty} a_n \left(\cos n\pi \left(\frac{\zeta^-}{\tau} \right) - \cos n\pi \left(\frac{\zeta^+}{\tau} \right) \right),$$

which shows that the solution is of type $\psi(x, z) = f(\zeta^-) - g(\zeta^+)$, for arbitrary functions f and g. Following these characteristics, it was found that the BVP posed so far is underdetermined [2]. One still needs to prescribe functions f and g, related to the normal derivative of the streamfunction, in two so-called fundamental intervals [2]. This feature will be recovered in the present analytic derivation too. Clearly, each Fourier mode not only satisfies the governing equation, but also correctly vanishes at surface ($z = 0$), vertical wall ($x = -1$), and bottom ($z = -\tau$). Therefore, Fourier coefficients a_n are determined by vanishing of the streamfunction along the slope $z = \tau(x-1)$, $x \in (0, 1)$. Inserting $\tau = 3/2$ and transforming $x = (1+\xi)/2$, this interval maps to $\xi \in (-1, 1)$ and the origin $\xi = 0$ shifts to the reflection point of the attractor. The latter boundary condition then reads

$$\sum_{n=1}^{\infty} a_n \left(\cos n\pi \left(\frac{1}{2} + \frac{\xi}{6} \right) - \cos n\pi \left(\frac{1}{2} + \frac{5\xi}{6} \right) \right) = 0.$$

By denoting

$$F(\xi) = \sum_{n=1}^{\infty} a_n \cos n\pi \left(\frac{1}{2} + \frac{\xi}{6} \right) \tag{3}$$

this shows that $F(\xi)$ satisfies a linear functional equation:

$$F(\xi) = F(5\xi). \tag{4}$$

This equation implies that once F is given for $\xi \in (1/5, 1)$, $F(\xi) \equiv F_0^+(\xi)$ say, its shape is repeated in compressed form in subsequent intervals, $\xi \in (5^{-(n+1)}, 5^{-n})$, in which $F(\xi) \equiv F_n^+(\xi) = F_0^+(5^n\xi)$. A second such interval exists for negative ξ, with similar implications given any arbitrary prescription of $F(\xi) \equiv F_0^-(\xi)$ for $\xi \in (-1, -1/5)$. Note that recursive determination of F never brings us quite to the asymptote, $\xi = 0$. Hence, $F(0)$ is left unspecified, entirely in correspondence with the algorithmic solution procedure.

Each solution, $F(\xi) = F^a(\xi) + F^s(\xi)$, separates into an antisymmetric, $F^a(\xi)$, and a symmetric part, $F^s(\xi)$, where

$$F^a(\xi) = \frac{1}{2}(F(\xi) - F(-\xi)), \qquad F^s(\xi) = \frac{1}{2}(F(\xi) + F(-\xi)),$$

for $\xi > 0$. Taking $F_0^-(-\xi) = F_0^+(\xi)$, for $\xi \in (1/5, 1)$, and applying recursion equation (4) for $|\xi| < 1/5$, we get $F^a(\xi) = 0$, so that $F(\xi)$ is symmetric. This implies $a_{2k+1} = 0$, for $k \in \mathbb{N}$, and F simplifies to

$$F^s(\xi) = \sum_{k=1}^{\infty} (-1)^k a_{2k} \cos k\pi \frac{\xi}{3}. \tag{5}$$

Conversely, taking $F_0^-(-\xi) = -F_0^+(\xi)$ we obtain $F^s(\xi) = 0$ and $F(\xi)$ is antisymmetric, implying $a_{2k} = 0$, $k \in \mathbb{N}$, and

$$F^a(\xi) = \sum_{k=0}^{\infty} (-1)^{k+1} a_{2k+1} \sin \left((2k+1)\pi \frac{\xi}{6} \right). \tag{6}$$

For this particular choice of τ, even and odd indexed coefficients a_n neatly associate with even and odd functions $F^s(\xi)$ and $F^a(\xi)$ respectively.

Specifying $F_0^{\pm}(\xi)$, for $\xi \in (1/5, 1)$ and $\xi \in (-1, -1/5)$ respectively, equivalent to specifying $F^{a,s}(\xi)$ for $\xi \in (1/5, 1)$, one may now regard $F(\xi)$ as given for all $\xi \in (-1, 1)$, excluding $\xi = 0$. The remaining task is to extract its Fourier coefficients a_n from (3). Transform to a symmetric description on $\xi \in (1/5, 1)$ by letting $s = (5\xi - 3)/2$. This interval is then situated in $s \in (-1, 1)$, on which

$$F^{a,s}(s) = \sum_{l=0}^{\infty} (\alpha_l^{a,s} \sin l\pi s + \beta_l^{a,s} \cos l\pi s)$$

is specified by choosing $\alpha_l^{a,s}$ and $\beta_l^{a,s}$ arbitrarily. However, all $\alpha_l^{a,s} = 0$. This follows by inserting the inverse relation, $\xi = (2s+3)/5$, on interval $s \in (-1, 1)$ into $F(5\xi)$ which, by (4), is identical to $F(\xi)$, as then

$$F(5\xi) = \sum_{n=1}^{\infty} a_n \cos n\pi \left(\frac{1}{2} + \frac{5\xi}{6} \right) = \sum_{n=1}^{\infty} (-1)^n a_n \cos n\pi \left(\frac{s}{3} \right).$$

This is a sum over even functions of s, requiring the vanishing of odd expansion coefficients, $\alpha_l^{a,s} = 0$. This implies one is free to prescribe $F^{a,s}(s)$ in $s \in (0, 1)$ only. The original problem becomes well-posed when $F_0^{\pm}(\xi)$ are prescribed over $\xi \in (3/5, 1)$, and $(-1, -3/5)$ respectively, which were therefore termed *fundamental intervals* [2]. In the algorithmic solution procedure, the symmetry on the interval $s \in (-1, 1)$ follows directly from the fact that the characteristic emanating from $s = 0$ connects upon a single surface reflection to a corner point, so that characteristics on either side reflect symmetrically. Here, consider a particular, symmetric case, looked at before numerically [17], obtained by choosing all coefficients zero, except for $\beta_1^s = 1$. Given this choice, $F_0^s(\xi) = -\sin(5\pi\xi/2)$, and using the recurrence and symmetry relations, $F^s(\xi)$ is determined for all $\xi \in (-1, 1)$, with the exception of $\xi = 0$.

The Fourier coefficients need to be recovered from Eq. (5). However, the $\cos k\pi\xi/3$, that multiply these coefficients, are not a complete set of Fourier modes on $\xi \in (-1, 1)$. But, using a well-known expansion ([19], Eq. (1.60)), that employs the convention $\epsilon_0 = 1$, $\epsilon_{1,2,3,\ldots} = 2$, one has

$$\cos k\pi \frac{\xi}{3} = \sum_{n=0}^{\infty} (-1)^k \Gamma_{nk} \cos n\pi\xi, \tag{7}$$

where

$$\Gamma_{nk} \equiv -\frac{3k}{\pi} \sin\left(\frac{k\pi}{3}\right) \frac{(-1)^n \epsilon_n}{k^2 - 9n^2}.$$

Thus, expressing each of the fractional Fourier modes in terms of orthonormal Fourier modes on $\xi \in (-1, 1)$, by projection one obtains a matrix equation

$$\sum_{k=1}^{\infty} \Gamma_{nk} a_{2k} = F_n, \tag{8}$$

with F_n the Fourier coefficients of the known function $F(\xi)$. Truncating and numerically inverting Eq. (8), an approximate solution of this semi-infinite dimensional matrix equation was found for a finite number of a_{2k} [17].

However, one can in fact circumvent this truncation and inversion and obtain exact expressions for Fourier coefficients a_{2k}. Using functional equation (4), one may specify $F(\xi) = -\sin(\pi\xi/2)$ for $1 < \xi < 3$ too. This extension allows transforming $\xi = 3\nu$, so that the cosine expansion

$$\sum_{k=1}^{\infty} (-1)^k a_{2k} \cos k\pi\nu = F(3\nu)$$

is now cast as a complete set of Fourier modes on $\nu \in (-1, 1)$. Hence, the Fourier modes follow by projection:

$$a_{2m} = 2(-1)^m \int_0^1 \cos(m\pi\nu) F(3\nu) d\nu,$$

where integration is over positive ν only, using the symmetry of $F(\nu)$. Returning to integration over ξ, this splits into two integrals, one, S_1, between 0 and 1, and an end term, S_2, over the region with which ξ was extended. By using the fact that the prescribed functions in $\xi \in (5^{-(n+1)}, 5^{-n})$ are compressed copies of the interval $\xi \in (1/5, 1)$, the first integral is written as an infinite sum, so that

$$a_{2m} = \frac{2}{3}(-1)^{m+1}(S_1 + S_2),$$

where

$$S_1 = \sum_{n=0}^{\infty} \int_{5^{-(n+1)}}^{5^{-n}} \cos(m\pi\xi/3) \sin(5^{n+1}\pi\xi/2) d\xi$$

and

$$S_2 = \int_1^3 \cos(m\pi\xi/3) \sin(\pi\xi/2) d\xi.$$

When the integrals over $\xi \in (5^{-(n+1)}, 5^{-n})$ are mapped to $\xi' \equiv 5^n\xi \in (1/5, 1)$, one finds, by exchanging integration and summation,

$$S_1 = \int_{1/5}^{1} \left[\sum_{n=0}^{\infty} 5^{-n} \cos(5^{-n} m\pi\xi'/3)\right] \sin(5\pi\xi'/2) d\xi'.$$

The sum within square brackets is a Weierstrass function

$$W(x) = \sum_{n=0}^{\infty} a^n \cos b^n \pi x,$$

celebrated as an example of a continuous, nowhere differentiable function for $1 > a > 0$, $ab > 1$ [11]. In the present application, $a = b = 1/5$, rendering $ab < 1$, and one thus samples this Weierstrass function outside its nondifferentiable regime. Nevertheless, as seen later on, the resulting Fourier spectrum still retains a special structure, which must lie at the origin of the *spatial* self-similarity of the streamfunction field [3].

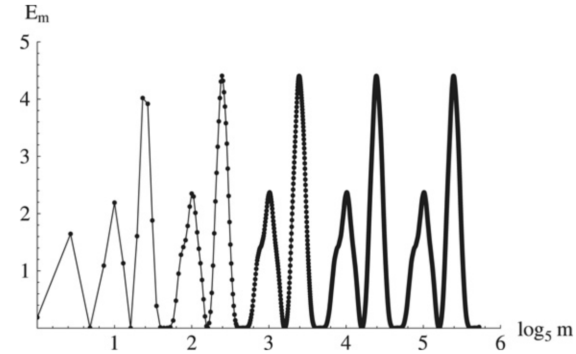

Fig. 1. Spectral energy contained in wave number m, $E_m = m^2 a_{2m}^2$, versus $\log_5 m$, for $m = 1, 2, \ldots, 10\,000$ (dots). For clarity, the dots are connected by straight lines.

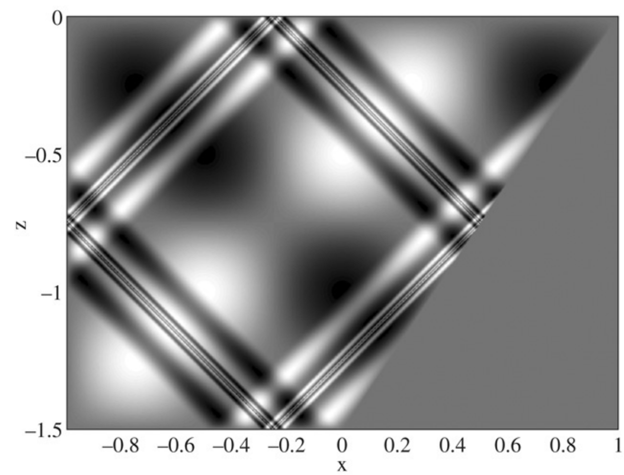

Fig. 2. Streamfunction field (2) computed on a 500 × 500 grid for $\tau = 3/2$, employing 1000 Fourier modes (10). Each Fourier mode consists of an infinite sum, truncated at 10 terms.

Returning to the original order of integration and summation, and carrying out the two remaining integrations, incorporating the last term into the first sum and by shifting its index, this can, using

$$b_n \equiv \frac{3}{2} 5^n, \tag{9}$$

be expressed as

$$a_{2m} = \frac{2m(-1)^m}{\pi} \sum_{n=0}^{\infty} \sin\left(\frac{m\pi}{2b_n}\right) \left(\frac{1}{m^2 - b_n^2} - \frac{1}{m^2 - b_{n+1}^2}\right). \tag{10}$$

This series converges rapidly and the first 75 terms are identical to those determined numerically [17]. Using (2), the total kinetic energy, $E \equiv \int_D \nabla\psi^2 dxdz = 2\pi^2 \sum_{m=1}^{\infty} E_m$, equals the sum of the spectral contributions $E_m = m^2 a_{2m}^2$, proportional to the squared Fourier coefficients, see Fig. 1. Apart from confirming a m^{-1} drop-off of Fourier coefficients a_{2m} [17], this displays an interesting log 5 periodicity. Inspection reveals that for $m = 3k$, $k = 1, 2, 4, \ldots$, $a_{2m} = 5a_{10m}$ exactly. For $m = 3k - 1$ and $m = 3k - 2$ this is only approximately so (and better so in the limit $k \to \infty$).

With even Fourier coefficients given by (10) and vanishing odd coefficients, Eq. (2) represents the first exact analytic self-similar solution of a boundary value problem for the wave equation. Fig. 2 displays the streamfunction, which is similar to that obtained by the algorithmic [4] and numerical methods [17]. It is continuous and differentiable over domain D, although in this inviscid description its derivative grows without bound on approach of the square-shaped attractor that spawns from the undetermined midpoint, $x = 1/2$. As is evident from Fig. 1, its total energy diverges with increasing number of modes taken into account. In real fluids, viscosity will regularize the wave field and the infinite sum (2) will be effectively truncated at some maximum mode number, M, determined by viscosity, stratification and basin size [6,10].

Interestingly, inserting (10) into streamfunction field (2), the sum over the Fourier modes can in fact be evaluated, albeit leading to discontinuous functions. While the ability to obtain velocity fields (by computing derivatives) is then lost, it leads to a more rapid evaluation of the streamfunction field and also reveals the connection with the characteristic (algorithmic) method. This derivation rests on using trigonometric identities to obtain eight sums of type akin to that in Eq. (7), ([19], Eq. (1.62)),

$$\frac{2}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m m}{m^2-b^2}\sin m\pi\sigma = \frac{-\sin b\pi\sigma}{\sin b\pi}S(\sigma), \tag{11}$$

where $S(\sigma) = \text{sign}[\text{mod }(\sigma+1,4)/2-1]$ and $\text{mod }(a,b) = a \text{ mod } b$. When applying (11) to (10), and writing b_n in (9) as $b_n = 3/2 + 6k_n$, with $k_n \in \mathbb{N}$, the denominator of its right-hand side, $\sin(b_n\pi) = -1$, always. Inserting the resulting expressions for a_{2m}, as well as $a_{2m+1} = 0$, for $m \in \mathbb{N}$, into (2), one obtains, using trigonometric identities, and exchanging the order of the two remaining summations,

$$\psi(x,z) = \frac{1}{4}\sum_{n=0}^{\infty}(f_n(x+1-z) - f_n(x+1+z)), \tag{12}$$

where

$$f_n(\zeta) \equiv (\cos(2\pi 5^n\zeta) - \cos(2\pi 5^{n+1}\zeta)) \times\left[S\left(\frac{5^{-n}+4\zeta}{3}\right) + S\left(\frac{5^{-n}-4\zeta}{3}\right)\right].$$

As is evident from the presence of block function $S(\sigma)$, this contains discontinuous functions, leading in general to aliasing problems upon evaluating this expression. However, over a 2×2 square encompassing the fluid domain, aliasing can be avoided if one takes grid points at $(x_i+1, z_j+2) = (1/4+i, j)5^{-N}$, for integers i, j running from 0 to 2×5^N, where $N = n_{\max}+1$ and $n_{\max}$ denotes the maximum n at which (12) is truncated.

In summary, it has been shown how an exact, analytic wave attractor field can be constructed. Remarkably, this possesses a self-similar spatial structure which echoes a log-periodicity of its Fourier coefficients, related to the presence of the Weierstrass function. While this attractor field is here obtained for a particular trapezoidal fluid domain only, it uncovers how such self-similar patterns might be constructed under more general circumstances. This generalization is, however, beyond the scope of this paper. Apart from offering a benchmark against which to test the accuracy of numerical methods [8,17], the analytic method, described in this paper, promises to be useful in situations that are beyond the realm covered by the exact algorithmic method [2], such as in three-dimensional, viscous and nonlinear settings. It may help establish the strength and location of mixing and mean transport generation in oceans, atmospheres and stars, as e.g. due to focusing of tidally generated waves [5,7]. A case in point is that given the log-periodicity of the total energy, its truncation at the actual truncation scale M might explain why experimentally observed wave attractors seem to lack fine structure and might explain their saturation level [4,6].

Finally, the presence of singularities in waves in stratified, rotating or magnetized media, considered here, is analogous to those arising in classical wave systems, such as the vibrating cavity [14–16]. Interestingly, the vibrating cavity was argued to be relevant for black hole (Hawking) radiation, the nonstationary Casimir effect and photon creation by vacuum fluctuations [20]. This analogy suggests a deeper connection between these wave types, that should be exploited. Based on former results [2,4], it predicts e.g. a fractal dependence of wave solutions on the parameters specifying the motion $L(t)$ of the cavity wall.

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