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TIDE-TOPOGRAPHY INTERACTIONS IN A STRATIFIED SHELF SEA I. BASIC EQUATIONS FOR QUASI-NONLINEAR INTERNAL TIDES

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By means of a multiple-scale analysis of the shallow water equations for a uniformly rotating, stratified fluid, subject to a time-periodic advection over a small-amplitude topography, it is shown that the inclusion of quasi-nonlinear advection by the barotropic (tidal) current is a necessary ingredient of the dynamics, once the internal wave length, the barotropic tidal excursion amplitude and the topographic wave length are all of the same order of magnitude. The basic set of equations describing the generation of internal tides by the interaction of barotropic tidal currents and topography thus derived, is extended with damping both by bottom- and internal friction. The effect of bottom friction is parametrized in a Rayleigh damping term for each of the separate vertical modes, thereby allowing the vertical structure of the baroclinic tidal currents to remain expressible in terms of vertical modes. The spectral forcing equation for damped internal motions is then derived. Finally the characteristic roots (dispersion relation) of the homogeneous spectral equation are discussed and summarized in a dispersion diagram. It is shown that these consist in general of two damped gravity wave modes and a transient, the asymptotic regimes of which are discussed. The transient gives rise, among other things, to baroclinic residual currents which are the subject of a second paper, whereas the structure of the quasi-nonlinear gravity wave modes is treated in a third part.

KEY WORDS: Tide-topography interaction; stratified shelf sea, multiple-scale analysis, quasi-nonlinear advection; normal modes, modal damping, free modes.

1. INTRODUCTION

The interaction of tides and topography can be classified according to the various ratios of the horizontal length scales involved. The ratio of the topographic length scale to the tidal wave length or external Rossby deformation radius controls principally the dispersive properties of the tidal waves, giving rise to topographic wave trapping, shelf resonances and topographic Rossby waves. The topographic length scales involved in this process are typically of the order of a hundred kilometers or more. At smaller topographic length scales, from several to fifty kilometers say, dynamical processes controlled by two other tidal length scales arise. These length scales are the barotropic tidal excursion amplitude and the baroclinic tidal wave length, the ratios of which to the topographic length scale govern the two types of tide-topography interactions. Up to now these interactions have been treated in literature in two, quite separate, ways. First, it was already recognized by Zeilon (1912) that this type of interaction must be the primary source of internal tides in a stratified sea, due to a resonant matching of topographic length scales and the internal gravity wave length. Zeilon (1912) also recognized the matching of the barotropic tidal excursion amplitude and the internal wave length and thereby was the first to discuss the quasi-nonlinear advective effects of the barotropic current on the internal wave, giving rise to superharmonics in the internal tides. Whereas the internal tides discussed by Zeilon (1912) are propagating gravity waves, a second type of interaction between the barotropic tide and small-scale topographic features must be classified as topographically bounded vorticity "waves" that, by means of vorticity advection, give rise to residuals and superharmonics of the basic tidal frequency in the barotropic tidal velocity field, as first recognized by Huthnance (1973). Here again it is the matching of the barotropic tidal excursion amplitude and the topographic length-scale that is determining the response. All this suggests that there could be a unifying approach that captures these processes in a single formulation in which quasi-nonlinear advection plays a principal role. Apart from a further discussion of the influence of advection on internal tides in the form of gravity waves, such a theoretical frame could also give the baroclinic structure of residual currents, a subject on which there is no literature as far as we know.

Before seeking the most simple setting that still encompasses all these aspects, it is worthwhile to have a look at the different theoretical approaches, mainly analytical, to the problem of tide-topography interactions that have been discussed before. To that end we have devised the two literature matrices given in Tables 1a and 1b, dealing with the barotropic vorticity modes and internal tidal gravity waves. These easily show which principal choices are to be made here and have been made by others. First, there is the difference between a finite-amplitude topography and a small-amplitude topography. The first applies to the continental slope, the second more or less to topographic features on the continental shelf, like tidal sand ridges or to features at the ocean bottom. We should like to deal with both, but the analytical difficulties for a finite-amplitude topography are severe. For residual currents this approach leads only to analytical results for a step-topography (Loder, 1980) and even then, the method of approximation to be used, harmonic truncation, has its flaws (Young, 1983; Maas *et al.*, 1987). For a continuously stratified fluid the difficulties of a finite-amplitude topography, particularly a step, are augmented by the occurrence of (super)critical bottom slopes (Baines, 1974), which can perhaps only be circumvented by a two-layer approximation. For a small-amplitude topography, though, all these drawbacks disappear. The response of barotropic vorticity modes can be calculated exactly in this case (Zimmerman, 1978, 1980; Maas *et al.*, 1987), the bottom slope of a small amplitude topography is subcritical, i.e. less than internal wave particle motion (characteristic slope), implicitly and it has already been shown that the linear theory for a continuously stratified fluid (Cox and Sandstrom, 1962) can as easily be extended to the quasi-nonlinear regime (Bell, 1975; Hibiya, 1986) as for the two-layer approximation (Zeilon, 1912). Moreover, any shape of the topography can be dealt with by Fourier transformation. Thus, for simplicity, we are willing to pay a price in that our results for a small-amplitude topography, a step for instance, can at best only give a qualitative picture of effects near finite-amplitude topographies occurring in reality, as for instance at the continental slope. However, particularly for the discussion

Table 1a Matrix classifying various theoretical analytical approaches to *barotropic* tide-topography interactions, producing topographically bounded residual currents and overtides

Vertical amplitude of topography *Horizontal length scale of topography*	*Finite amplitude*	*Small amplitude*
≫ tidal excursion amplitude	Huthnance (1973)	(implicit in finite amplitude approach)
≪ tidal excursion amplitude	Loder (1980)	(idem)
all scales		Zimmerman (1978, 1980) Maas *et al.* (1987)

Table 1b Matrix classifying various theoretical approaches to *baroclinic* tide-topography interactions, producing internal gravity waves of tidal period and, in the quasi-nonlinear regime, of internal overtides. [$N^2 = \delta$ means a (two or three)-layer approximation]

Vertical amplitude of topography	*Finite amplitude*			*Small amplitude*	
stratification (N^2)	*constant*		δ	*constant*	δ
slope	subcritical "flat"	(super) critical "steep"			
linear regime (excl. barotropic advection)	Baines (1973)	Rattray *et al.* (1969) Baines (1974) Prinsenberg *et al.* (1974)	Rattray (1960)	Cox/ Sandstrom (1962)	(Implicit in finite ampl., δ)
	Sandstrom (1976)				
	Baines (1982)				
quasi nonlinear regime (incl. barotropic advection)			Pingree *et al.* (1983) Willmott/ Edwards (1987)	Bell (1975) Hibiya (1986)	Zeilon (1912)

of baroclinic residual currents, where any theory is lacking, the price seems not too high.

A second choice to be made is that between a continuously stratified fluid and a two-layer approximation. Evidently now, once a small-amplitude approximation for the topography has been chosen, the former seems a logical choice, as it is in principle able to resolve the vertical structure completely. Even when a linearly stratified fluid is adopted for simplicity, as we do here, the response of the different vertical modes gives a qualitative insight into the behaviour of the modal structure of any other form of stratification.

Finally we have to face the incorporation of advection by the barotropic tidal current as this is our principal subject. In the theory of baroclinic internal tides advection has up to now been discarded completely for a finite amplitude topography (Baines, 1973, 1974, 1982; Sandstrom, 1976; Rattray, 1960), the exception being a recent numerical study by Wilmott and Edwards (1987), extending an earlier discussion by Pingree *et al.* (1983). The option to deal with advective effects either perturbatively or by harmonic truncation in this case, as in the theory of barotropic vorticity modes—Huthnance (1973) and Loder (1980), respectively—seems never to have been considered. For a small amplitude topography, however, barotropic advection can be incorporated in full, as the expansion is basically in the topographic amplitude, rather than in a parameter characterizing non-linear advection. This applies both to the barotropic vorticity modes (Zimmerman, 1978, 1980; Maas *et al.*, 1987) as to internal gravity waves (Zeilon, 1912; Bell, 1975; Hibiya, 1986). The principal aim of the multiple-scale analysis given in chapter 2 of this paper is to derive this incorporation in a mathematically consistent way and to proof that whenever the three length-scales involved—tidal excursion amplitude, internal wavelength and topographic length scale—are of the same order of magnitude, the incorporation is necessary. This regime is called the "continental shelf regime" which should apply on the shelf or, in the context of a finite-amplitude continental slope, on the shelf-side of the slope. The regime for which the matching concerned no longer applies is termed the "deep-sea regime" and this should apply to the ocean side of the continental slope, where usually the barotropic tidal excursion amplitude is an order of magnitude smaller than on the shelf, whereas the internal wavelength is larger in the deeper fluid.

In a frictionless fluid, as discussed in the multiple-scale analysis in Section 2, the barotropic tidal current, advecting the depth-dependent baroclinic structures, is itself independent of depth. This, of course, is a very convenient property for an analytical theory, that one should like to retain when friction is included. This poses a problem, as bottom friction in a shallow sea certainly makes the barotropic current depth-dependent, whereas it creates an additional depth-dependency in the baroclinic currents. Both give rise to vertical mode coupling, that one should like to circumvent in order to retain the possibility of describing the baroclinic vertical structure in terms of superposition of mutually independent vertical modes. In two-dimensional barotropic models of tide-topography interaction, dealing with vertically integrated horizontal velocities—actually the zeroth order vertical mode—one neglects differential advection altogether, whereas the frictional effects on the topographically induced currents is parametrized by a simple Rayleigh damping term (Huthnance, 1973; Zimmerman, 1978, 1980) albeit sometimes augmented with a nonuniform damping coefficient to account for a spatially varying tidal amplitude (Loder, 1980). One should like to keep this for baroclinic tidal currents as well. In Section 3 we show that in the limit of weak friction, the effects of the bottom boundary layer on the vertical modes can indeed be parametrized by a Rayleigh damping term independent of mode number and that frictional mode coupling vanishes. As to the neglect of differential advection by the barotropic current, it has been shown (Zimmerman, 1986) that in terms of vorticity dynamics this neglect excludes the occurrence of frictionally induced horizontal vorticity, the subsequent tilting of horizontal vortex-lines in the vertical direction and the tilting of vertical vortex-lines in the horizontal direction. All this leads to the absence of any vertical structure in the barotropic topographically induced velocity field, particularly to the absence of cross-isobath residual circulation. However, in the baroclinic situation there is first the ever present solenoidal generation of horizontal vorticity that can be tilted in the vertical direction by differential vertical velocities induced by the topography, secondly the baroclinic vertical shear that may tilt vertical (planetary) vorticity in the horizontal direction and third a depth-dependent horizontal divergence that leads to depth-dependent stretching of vertical vortex lines. All this gives rise to depth-dependent, topographically induced, velocity fields even when the barotropic advecting current is only represented by its vertically averaged value. Thus, using a depth-independent barotropic current, together with a Rayleigh damping of vertical modes, means effectively that we are only looking at the vertical structure of internal tidal motions due solely to baroclinic effects, a point that is particularly relevant for the discussion of the "vorticity modes" in the second part of this study (Maas and Zimmerman, 1988), where we shall also compare frictionally and baroclinically induced vertical structure more closely.

In summary then, we discuss the response of a linearly stratified fluid to depth-independent advection by an oscillatory barotropic flow over a small-amplitude topography in terms of frictionally damped vertical modes, particularly in the regime where the tidal excursion amplitude and the internal wavelength are of the same order and match the horizontal topographic length scale. In this sense our theory is a unification of the theoretical studies encased by dashed lines in Table 1a,b. The specific spectral forcing equation is derived in Section 5, whereas the characteristic roots of its homogeneous part are discussed in Section 6, showing that the free modes consist of damped gravity waves and a topographically bounded transient, the dispersion relation of which is summarized in Figure 2. The specific results for the forced response due to the transient, the "vorticity modes", including the baroclinic residual circulation, is discussed in a second part of this study (Maas and Zimmerman, 1988; referred to later as II) and the forcing of quasi-nonlinear internal gravity waves in a third part (Maas and Zimmerman, 1989; referred to later as III). The reader who is not interested in the formal justification of our basic equations (2.45) and (4.39) may just take these as his starting point and proceed with Section 5 and the subsequent parts II and III.

2. A MULTIPLE-SCALE ANALYSIS OF THE SHALLOW WATER EQUATIONS

Our starting point is the inviscid shallow water equations for a stratified, uniformly rotating, Boussinesq fluid. This implies hydrostatic balance in the vertical direction and the neglect of density variations in the inertial terms of the momentum equations. The complete set reads:

$$\frac{\partial u_*}{\partial t_*}+u_*\frac{\partial u_*}{\partial x_*}+v_*\frac{\partial u_*}{\partial y_*}+w_*\frac{\partial u_*}{\partial z_*}-f_*v_*+\frac{1}{\bar{\rho}_*}\frac{\partial p_*}{\partial x_*}=0,$$

$$\frac{\partial v_*}{\partial t_*}+u_*\frac{\partial v_*}{\partial x_*}+v_*\frac{\partial v_*}{\partial y_*}+w_*\frac{\partial v_*}{\partial z_*}+f_*u_*+\frac{1}{\bar{\rho}_*}\frac{\partial p_*}{\partial y_*}=0,$$

$$\frac{\partial p_*}{\partial z_*}+\rho_* g=0, \tag{2.1}$$

$$\frac{\partial u_*}{\partial x_*}+\frac{\partial v_*}{\partial y_*}+\frac{\partial w_*}{\partial z_*}=0,$$

$$\frac{\partial \rho_*}{\partial t_*}+u_*\frac{\partial \rho_*}{\partial x_*}+v_*\frac{\partial \rho_*}{\partial y_*}+w_*\frac{\partial \rho_*}{\partial z_*}=0.$$

Here $\mathbf{u}_*=(u_*,v_*)$ and w_* denote the horizontal and vertical components of the velocity field given with respect to an orthogonal, rotating coordinate frame $\mathbf{x}_*=(x_*,y_*)$, z_* with z_* vertically upwards. The pressure is denoted by p_*, the density by ρ_*, which contains the constant reference density $\bar{\rho}_*$ and $\rho'_*(\mathbf{x},t)$, its spatial and temporal variation. The Coriolis frequency f_* is assumed to be constant, since the scale of the phenomena we are interested in is much smaller than the scale associated with variations in f_*. Finally, t_* denotes time. These equations are accompanied by the boundary conditions:

$$w_*=\frac{\partial \zeta_*}{\partial t_*}+u_*\frac{\partial \zeta_*}{\partial x_*}+v_*\frac{\partial \zeta_*}{\partial y_*}, \quad \text{at } z_*=\zeta_*(x_*,y_*,t_*),$$

$$p_*=p_{*\text{atmosphere}}, \tag{2.2}$$

$$w_*=-u_*\frac{\partial H_*}{\partial x_*}-v_*\frac{\partial H_*}{\partial y}, \quad \text{at } z_*=-H_*(x_*,y_*).$$

Here $\zeta_*(x_*,y_*,t_*)$ is the elevation of the surface above mean sea level, while $H_*(x_*,y_*)$ gives the bottom profile.

2.1. *Scaling of the governing equations*

Equation (2.1–2) are made non-dimensional by the following scaling:

$$\mathbf{u}_*=U\mathbf{u}, \quad w_*=\frac{UH_0}{L}w, \quad p_*=\bar{\rho}_* pgH_0, \quad \rho_*=\bar{\rho}_*\rho, \tag{2.3}$$

$$\zeta_*=\frac{UH_0}{L\sigma}\zeta, \quad H_*=H_0H, \quad t_*=\sigma^{-1}t, \quad z_*=H_0z.$$

Here U is the velocity amplitude of the barotropic tidal wave, H_0 a typical depth, σ the frequency of the barotropic tidal wave, and L the barotropic wave length scale. The latter is assumed to be much larger than the baroclinic wave length scale l_i; i.e. if

$$L=(gH_0)^{1/2}\sigma^{-1}, \quad l_i=\bar{N}_*H_0\sigma^{-1}, \quad \text{with} \quad \bar{N}_*=(g\Delta\rho_*/H_0\rho_*)^{1/2}, \tag{2.4}$$

$$\delta=\frac{l_i}{L}=\left(\frac{\Delta\rho_*}{\bar{\rho}_*}\right)^{1/2} \ll 1, \tag{2.5}$$

where $\Delta\rho_*$ is the scale of density variations ρ_*' in the vertical. Here the Brunt-Väisälä frequency, N_*, is nondimensionalized by the scale of the density variations as $N_*=\bar{N}_*N(z)$, such that in the constant Brunt–Väisälä model, to be used later, $N(z)=1$. The disparity in scales can be explored in a multiple scale analysis in the horizontal coordinates in the spirit of Pedlosky (1984) and Maas *et al.* (1987). Let

$$\mathbf{x}_*=l_i\mathbf{x}+L\mathbf{X}, \tag{2.6}$$

and define

$$\nabla\equiv\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right), \quad \tilde{\nabla}\equiv\left(\frac{\partial}{\partial X}, \frac{\partial}{\partial Y}\right). \tag{2.7}$$

Substituting (2.3) and (2.6) in (2.1–2) using (2.4) and (2.7) gives:

$$\begin{aligned}
\varepsilon\delta\frac{\partial\mathbf{u}}{\partial t}+\varepsilon^2[\mathbf{u}\cdot(\nabla+\delta\tilde{\nabla})\mathbf{u}]+\delta\varepsilon^2 w\frac{\partial\mathbf{u}}{\partial z}+\delta\varepsilon f\mathbf{j}\times\mathbf{u}+(\nabla+\delta\tilde{\nabla})p&=0,\\
\frac{\partial p}{\partial z}+\rho&=0,\\
[(\nabla+\delta\tilde{\nabla})\cdot\mathbf{u}]+\delta\frac{\partial w}{\partial z}&=0,\\
\delta\frac{\partial\rho}{\partial t}+\varepsilon[\mathbf{u}\cdot(\nabla+\delta\tilde{\nabla})\rho]+\delta\varepsilon w\frac{\partial\rho}{\partial z}&=0,
\end{aligned} \tag{2.8}$$

and

$$\left.\begin{aligned}
\delta w&=\delta\frac{\partial\zeta}{\partial t}+\varepsilon[\mathbf{u}\cdot(\nabla+\delta\tilde{\nabla})\zeta]\\
p&=p_{\text{atmosphere}},
\end{aligned}\right\} \text{ at } z=\varepsilon\zeta$$

$$\delta w=-[\mathbf{u}\cdot(\nabla+\delta\tilde{\nabla})H], \qquad \text{at } z=-H. \tag{2.9}$$

Here $\mathbf{j}$ is the vertical unit vector, $f=f_*/\sigma$ and $\varepsilon=U/\sigma L=l_0/L$, where l_0 is the barotropic tidal excursion amplitude. In fact ε is equivalent to the external Froude number. It is assumed to be a small parameter in all situations ($\varepsilon \ll 1$). We then have to make an assumption about the relative magnitude of ε and δ. This choice is a crucial one as it will turn out to separate the purely linear ("deep-sea") regime from the quasi-nonlinear ("continental shelf") regime in which advection by the barotropic tide plays an essential role in the dynamics of the internal motions. The latter occurs when ε and δ are of the same order of magnitude, which is equivalent to $l_i = O(l_0)$. On the other hand, for $\varepsilon \ll \delta$, $-\varepsilon = O(\delta^2)$, say—advective effects are negligible ($l_0 \ll l_i$). As the latter regime is a natural asymptote of the former, we shall deal with $\varepsilon = O(\delta)$ here. To this end we set $\varepsilon = \delta l_0 / l_i$ and for the moment absorb the latter ratio of length scales as nondimensional amplitude in each of the velocity components $\mathbf{u}$ and w and sealevel variations ζ. This ratio will reappear when we finally set up the spectral evolution equation for modal amplitudes in Section 5. Then (2.8) and (2.9) read:

$$\delta^2\left[\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot(\nabla + \delta\tilde{\nabla})\mathbf{u} + \delta w\frac{\partial \mathbf{u}}{\partial z} + f\mathbf{j}\times\mathbf{u}\right] + (\nabla + \delta\tilde{\nabla})p = 0, \tag{2.10a}$$

$$\frac{\partial p}{\partial z} + \rho = 0, \tag{2.10b}$$

$$(\nabla + \delta\tilde{\nabla})\cdot\mathbf{u} + \delta\frac{\partial w}{\partial z} = 0, \tag{2.10c}$$

$$\frac{\partial \rho}{\partial t} + \mathbf{u}\cdot(\nabla + \delta\tilde{\nabla})\rho + \delta w\frac{\partial \rho}{\partial z} = 0, \tag{2.10d}$$

and boundary conditions

$$\left.\begin{aligned} w &= \frac{\partial \zeta}{\partial t} + \mathbf{u}\cdot(\nabla + \delta\tilde{\nabla})\zeta \qquad (2.11\text{a}) \\ p &= p_{\text{atmosphere}} \qquad (2.11\text{b}) \end{aligned}\right\} \text{at } z = \delta\zeta,$$

$$\delta w = -\mathbf{u}\cdot\nabla + \delta\tilde{\nabla})H \qquad \text{at } z = -H. \tag{2.11c}$$

We shall often use the integrated forms of (2.10b,c) which read, making use of (2.11) and of Leibniz' rule concerning the interchange of differentiation and integration:

$$p(z) = p(\delta\zeta) + \int_z^{\delta\zeta} \rho(\mathbf{x}, z')dz', \tag{2.12}$$

$$\delta\frac{\partial \zeta}{\partial t} + (\nabla + \delta\tilde{\nabla})\cdot\left[\int_{-H}^{\delta\zeta} \mathbf{u}\,dz\right] = 0. \tag{2.13}$$

2.2. *Perturbation expansions*

As to the variations in bottom topography we shall assume that these are only functions of the "fast" coordinate, **x**, as we are particularly interested in topographic length scales of the order of the tidal excursion amplitude or the internal wave length. Moreover we shall exclusively deal with a "small-amplitude" topography here, assuming variations in depth to be of order $\delta = O(\varepsilon)$ relative to the mean depth. The latter being O(1), we then have:

$$H(\mathbf{x}) = 1 + \delta H^{(1)}(\mathbf{x}). \tag{2.14}$$

We now expand into a perturbation series

$$\phi = \sum_{n=0}^{\infty} \delta^n \phi^{(n)}, \tag{2.15}$$

where ϕ stands for each of the field variables $\mathbf{u}$, w, p, ζ, ρ and H. These series are substituted in (2.10)–(2.13). Note that (static) variations in density are given in order of magnitude by $\Delta\rho_*/\rho_*^{(0)} = \delta^2$, which implies that the series for ρ reads:

$$\rho = 1 + \delta^2 \rho^{(2)} + \delta^3 \rho^{(3)} + \cdots \tag{2.16}$$

Order δ^0

To zeroth order in δ we then have from (2.10b) and 2.12)

$$p^{(0)} = p_a - z, \tag{2.17}$$

where p_a is the atmospheric pressure at $z=0$, assumed to be constant. In the same way (2.10c) implies

$$\mathbf{u}^{(0)} = \mathbf{u}^{(0)}(\mathbf{X}, z, t). \tag{2.18}$$

Order δ^1

To first order in δ, (2.10a) with (2.17), in combination with (2.10b) and (2.12), give the equivalence of pressure perturbations and variations in sea level varying only on the "slow" coordinate:

$$p^{(1)} = \zeta^{(0)}(\mathbf{X}, t). \tag{2.19}$$

The continuity equation (2.10c) at this order reads:

$$\nabla \cdot \mathbf{u}^{(1)} + \tilde{\nabla} \cdot \mathbf{u}^{(0)} + \frac{\partial w^{(0)}}{\partial z} = 0. \tag{2.20}$$

Together with the boundary condition (2.11c),

$$w^{(0)} = -\mathbf{u}^{(0)} \cdot \nabla H^{(1)}(\mathbf{x}), \qquad z = -1, \tag{2.21}$$

this suggests to split-up the vertical velocity in parts varying on the "fast" scale and on the "slow" scale:

$$\frac{\partial w_f^{(0)}}{\partial z} + \nabla \cdot \mathbf{u}^{(1)} = 0, \tag{2.22}$$

$$\frac{\partial w_s^{(0)}}{\partial z} + \tilde{\nabla} \cdot \mathbf{u}^{(0)} = 0, \tag{2.23}$$

$$w_f^{(0)} = -\mathbf{u}^{(0)} \cdot \nabla H^{(1)}, \qquad z = -1 \tag{2.24}$$

$$w_s^{(0)} = 0, \qquad z = -1 \tag{2.25}$$

To first order in δ the integrated continuity equation (2.13) reads:

$$\frac{\partial \zeta^{(0)}}{\partial t} + \nabla \cdot \left[\int_{-1}^{0} \mathbf{u}^{(1)}\, dz + \mathbf{u}^{(0)}(-1) H^{(1)} + \mathbf{u}^{(0)}(0)\zeta^{(0)} \right] + \tilde{\nabla} \cdot \int_{-1}^{0} \mathbf{u}^{(0)}\, dz = 0, \tag{2.26}$$

which gives for variations on the "slow" coordinate:

$$\frac{\partial \zeta^{(0)}}{\partial t} + \tilde{\nabla} \cdot \int_{-1}^{0} \mathbf{u}^{(0)}\, dz = 0, \tag{2.27}$$

and on the "fast" coordinate:

$$\nabla \cdot \left[\int_{-1}^{0} \mathbf{u}^{(1)}\, dz + \mathbf{u}^{(0)}(-1) H^{(1)} \right] = 0. \tag{2.28}$$

Thus, the horizontal velocity perturbations $\mathbf{u}^{(1)}$ are induced by topographic variations on the fast coordinate.

Boundary condition (2.11a) at order δ gives, making use of (2.19),

$$w^{(0)} = \frac{\partial \zeta^{(0)}}{\partial t}, \qquad z = 0, \tag{2.29}$$

which implies that $w_s^{(0)}(0) = \partial \zeta^{(0)} / \partial t$ and $w_f(0) = 0$. The latter is the justification for the rigid lid approximation for all motion that is organized on the fast coordinate (small scale).

Order δ^2

We now turn to eq. (2.10a,b,d) to second order in δ (the continuity equation, in this higher order not being of relevance, is dropped):

$$\frac{\partial \mathbf{u}^{(0)}}{\partial t} + f\mathbf{j} \times \mathbf{u}^{(0)} + \nabla \mathrm{p}^{(2)} + \tilde{\nabla} \zeta^{(0)} = 0, \tag{2.30a}$$

$$\frac{\partial p^{(2)}}{\partial z} = -\rho^{(2)}, \tag{2.30b}$$

$$\frac{\partial \rho^{(2)}}{\partial t} + \mathbf{u}^{(0)} \cdot \nabla \rho^{(2)} = 0. \tag{2.30c}$$

Separating out the part dependent only on the "slow" coordinate, (2.30a) gives the familiar equation of motion for the barotropic flow:

$$\frac{\partial \mathbf{u}^{(0)}}{\partial t} + f\mathbf{j} \times \mathbf{u}^{(0)} + \tilde{\nabla}\zeta^{(0)} = 0. \tag{2.31}$$

As $\partial\zeta^{(0)}/\partial z = 0$ this implies the independence of $\mathbf{u}^{(0)}$ on z, hence (2.27) reads:

$$\frac{\partial \zeta^{(0)}}{\partial t} + \tilde{\nabla} \cdot \mathbf{u}^{(0)} = 0. \tag{2.32}$$

The part in (2.30a) depending only on the "fast" coordinate, $\nabla p^{(2)} = 0$, implies $p^{(2)} = p^{(2)}(\mathbf{X}, z, t)$ and therefore from (2.30b)

$$\rho^{(2)} = \rho^{(2)}(\mathbf{X}, z). \tag{2.33}$$

Thus, up to this order, perturbations in density belong to the static background that may vary, horizontally, only on large length scales.

Order δ^3

Finally we turn to order δ^3 in (2.10)–(2.13). Equation (2.10) then reads:

$$\frac{\partial \mathbf{u}^{(1)}}{\partial t} + \mathbf{u}^{(0)} \cdot \nabla \mathbf{u}^{(1)} + \mathbf{u}^{(0)} \cdot \tilde{\nabla} \mathbf{u}^{(0)} + f\mathbf{j} \times \mathbf{u}^{(1)} + \nabla p^{(3)} + \tilde{\nabla} p^{(2)} = 0, \tag{2.34a}$$

$$\frac{\partial p^{(3)}}{\partial z} + \rho^{(3)} = 0, \tag{2.34b}$$

$$\frac{\partial \rho^{(3)}}{\partial t} + \mathbf{u}^{(0)} \cdot \nabla \rho^{(3)} + \mathbf{u}^{(0)} \cdot \nabla \rho^{(2)} + \mathbf{w}^{(0)} \frac{\partial \rho^{(2)}}{\partial z} = 0, \tag{2.34c}$$

making use of (2.18) and (2.33). Also here we can make a separation between equations that apply to fast and slow scale variables. As the latter describe a nonlinear correction to the zeroth order barotropic motion given by (2.31)–(2.32) which is not of interest here, we shall only give the equations applying to the variables depending on the fast coordinate. That part of (2.34) reads:

$$\frac{\partial \mathbf{u}^{(1)}}{\partial t}+\mathbf{u}^{(0)}\cdot\nabla\mathbf{u}^{(1)}+f\mathbf{j}\times\mathbf{u}^{(1)}+\nabla p^{(3)}=0, \tag{2.35a}$$

$$\frac{\partial p^{(3)}}{\partial z}+\rho^{(3)}=0, \tag{2.35b}$$

$$\nabla\cdot\mathbf{u}^{(1)}+\frac{\partial w_f^{(0)}}{\partial z}=0, \tag{2.35c}$$

$$\frac{\partial \rho^{(3)}}{\partial t}+\mathbf{u}^{(0)}\cdot\nabla\rho^{(3)}+w_f^{(0)}\frac{\partial \rho^{(2)}}{\partial z}=0. \tag{2.35d}$$

In order now to separate vertical motions forced by the external barotropic flow (w_e) over the varying depth from those that are due to free internal waves (w_i), a further subdivision of $w_f^{(0)}$ is applied:

$$w_f^{(0)}=w_e^{(0)}+w_i^{(0)}, \tag{2.36}$$

such that the bottom boundary condition (2.24) is split-up in:

$$w_f^{(0)}(-1)=w_e^{(0)}(-1)+w_i^{(0)}(-1). \tag{2.37}$$

with

$$w_e^{(0)}(-1)=-u^{(0)}\cdot\nabla H^{(1)} \tag{2.38}$$

and

$$w_i^{(0)}(-1)=0. \tag{2.39}$$

If now the horizontal velocity vector is separated in the same way:

$$\mathbf{u}^{(1)}=\mathbf{u}_e^{(1)}+\mathbf{u}_i^{(1)}, \tag{2.40}$$

the $\mathbf{u}_e^{(1)}$ field obeys

$$\frac{\partial \mathbf{u}_e^{(1)}}{\partial t}+\mathbf{u}^{(0)}\cdot\nabla\mathbf{u}_e^{(1)}+f\mathbf{j}\times\mathbf{u}_e^{(1)}+\nabla p_e^{(3)}=0 \tag{2.41}$$

together with

$$\nabla\cdot\mathbf{u}_e^{(1)}+\frac{\partial w_e^{(0)}}{\partial z}=0. \tag{2.42}$$

Now, from (2.35b), for $\rho_e^{(3)}=0$, $p_e^{(3)}=\zeta^{(2)}$ is independent of z. Then also $\mathbf{u}_e^{(1)}$ is z-independent so that a vertical integration of (2.42) gives using (2.38)

$$\nabla\cdot\mathbf{u}_e^{(1)}=-\mathbf{u}^{(0)}\cdot\nabla H^{(1)} \tag{2.43a}$$

and

$$w_e^{(0)}(z)=z\mathbf{u}^{(0)}\cdot\nabla H^{(1)}. \tag{2.43b}$$

2.3. *Equations governing barotropic and baroclinic fields*

Taking the curl of (2.41), using (2.43), gives the barotropic equation for the vertical vorticity:

$$\frac{\partial}{\partial t}(\mathbf{j}\cdot\nabla\times\mathbf{u}_e^{(1)})+\mathbf{u}^{(0)}\cdot\nabla(\mathbf{j}\cdot\nabla\times\mathbf{u}_e^{(1)})+f(\mathbf{u}^{(0)}\cdot\nabla H^{(1)})=0, \tag{2.44}$$

which has been derived and discussed before by Zimmerman (1978, 1980) and Maas *et al.* (1987). Now, in the presence of density stratification we have a baroclinic field as well, which obeys:

$$\frac{\partial\mathbf{u}_i^{(1)}}{\partial t}+\underline{\mathbf{u}^{(0)}\cdot\nabla\mathbf{u}_i^{(1)}}+f\mathbf{j}\times\mathbf{u}_i^{(1)}+\nabla p_i^{(3)}=0, \tag{2.45a}$$

$$\frac{\partial p_i^{(3)}}{\partial z}+\rho^{(3)}=0, \tag{2.45b}$$

$$\nabla\cdot\mathbf{u}_i^{(1)}+\frac{\partial w_i^{(0)}}{\partial z}=0, \tag{2.45c}$$

$$\frac{\partial\rho^{(3)}}{\partial t}+\underline{\mathbf{u}^{(0)}\cdot\nabla\rho^{(3)}}+w_i^{(0)}\frac{d\rho^{(2)}}{dz}=-w_e^{(0)}\frac{d\rho^{(2)}}{dz}=z\mathbf{u}^{(0)}\cdot\nabla H^{(1)}N^2(z). \tag{2.45d}$$

where the latter equality follows from (2.43b) and $N^2(z)=-d\rho^{(2)}/dz$. Equations (2.45), together with the boundary conditions,

$$w_i^{(0)}=0 \quad \text{at } z=0 \quad \text{and } -1, \tag{2.46}$$

constitute the quasi-nonlinear shallow water equations for a stratified fluid. It shows particularly that whenever the expansion parameters $\varepsilon=U/\sigma L=l_0/L$ and $\delta=l_i/L$ are the same order of magnitude, i.e. if the tidal excursion amplitude of the barotropic tide and the internal wave length are of the same order of magnitude and both are comparable to the horizontal topographic length scale, the incorporation of the dashed underlined terms in (2.45) is necessary. These terms describe the advective action of the externally imposed barotropic tidal wave. Hence advection of internal motions by the barotropic tide occurs concurrently with its forcing given in the right-hand side of (2.45d). As the matching of topographic scales with those of the barotropic tidal excursion amplitude and the internal wavelength is characteristic for features on the continental shelf or near the continental slope, we shall call the regime $\varepsilon = O(\delta)$ for which (2.45) and (2.46) apply, the "continental shelf regime". For more gentle topographies in the deep sea, $\varepsilon = O(\delta^2)$ say, it can easily be shown, along the same way as that leading to (2.45), Maas (1987), that the dashed terms in (2.45) drop out. This "deep sea regime" is the one studied by Cox and Sandstrom (1962), whereas (2.45) basically is the starting point of Bell (1975), Hibiya (1986) and, in its two layer approximation, of Zeilon (1912), as well as of II and III. For completeness it should be noted that the "deep sea regime" for a finite amplitude topography (Baines, 1973, 1974, 1982; Sandstrom, 1976) is obtained by dropping the dashed terms in (2.45) and replacing one of the boundary conditions in (2.46) by

$$w_i^{(0)} = -\mathbf{u}_i^{(0)} \cdot \nabla H^{(0)} \quad \text{at} \quad z = -H^{(0)}(x),$$

as the depth is now supposed to have O(1) variations. The latter means that in (2.1) the vertical velocity component has to be scaled with UH_0/l_i rather than UH_0/L, so that w should be replaced by $\delta^{-1}w$ in (2.8) and (2.9). Setting $\varepsilon = O(\delta^2)$ then leads to the result as mentioned above (Maas, 1987), when following the same route as that from (2.10) to (2.45).

3. VERTICAL NORMAL MODES

One of the advantages of working with a small-amplitude topography is the possibility of separation of variables, by which the vertical structure of the internal motions can be described in terms of orthonormal eigenfunctions associated with the flat-bottom reference state. The theory is standard (Krauss, 1966) so that it may suffice here to give only a brief account of what we shall need furtheron.

For a flat bottom, the forcing term in the right-hand side of (2.45d) drops out, as do the dashed quasi-nonlinear terms in (2.45a,d) in a reference frame moving with the barotropic current $\mathbf{u}^{(0)}(t)$ in the absence of forcing on the small scale. In the remaining system we may treat vertical and horizontal propagation separately. The former is most conveniently described in terms of the w, p fields. Eliminating $\mathbf{u}$ and ρ we have:

$$\frac{\partial^2 p}{\partial t\, \partial z} + wN^2(z) = 0, \tag{3.1}$$

$$\left(\frac{\partial^2}{\partial t^2} + f^2\right)\frac{\partial w}{\partial z} = \nabla^2 \frac{\partial p}{\partial t}, \tag{3.2}$$

where

$$N^2(z) = -\frac{d\rho^{(2)}(z)}{dz} \tag{3.3}$$

and where we have dropped the superscripts. To this we add the boundary conditions

$$w=0, \quad z=0, -1. \tag{3.4}$$

Making the separation:

$$p=\sum_{n=1}^{\infty} p_n(\mathbf{x},t)\Pi_n(z), \tag{3.5}$$

$$w=\sum_{n=1}^{\infty} w_n(\mathbf{x},t)Z_n(z), \tag{3.6}$$

leads to the familiar Sturm-Liouville eigenvalue problem for internal waves (Groen, 1948):

$$\frac{d}{dz}\left(\frac{1}{N^2(z)}\frac{d\Pi_n}{dz}\right)+\frac{\Pi_n}{c_n^2}=0, \tag{3.7}$$

$$\frac{d\Pi_n}{dz}=0, \quad z=0, -1, \tag{3.8}$$

with

$$\Pi_n=c_n^2\frac{dZ_n}{dz}, \tag{3.9}$$

where c_n^2 is a separation constant. The orthonormal eigenfunctions Π_n and Z_n satisfy:

$$\int_{-1}^{0} \Pi_n(z)\Pi_m(z)dz=\delta_{nm}=\int_{-1}^{0} N^2(z)Z_n(z)Z_m(z)dz\,c_n c_m, \tag{3.10}$$

where δ_{nm} is the Kronecker delta. For the simple situation of $N^2(z)=$constant (and thus, as we scaled with this constant value, $N^2=1$) that we shall use here, the eigenfunctions are taken as

$$\Pi_n=\cos n\pi(z+1), \tag{3.11}$$

$$Z_n=n\pi \sin n\pi(z+1), \tag{3.12}$$

with eigenvalues:

$$c_n=\frac{1}{n\pi}. \tag{3.13}$$

The corresponding horizontal propagation problem is more conveniently described in terms of the $\mathbf{u}$, $\tilde{\zeta}$ fields, where $w=\partial\tilde{\zeta}/\partial t$. Thus $\tilde{\zeta}$ is the elevation of isopycnals with respect to their horizontal equilibrium level. With

$$\mathbf{u}=\sum_{n=1}^{\infty}\mathbf{u}_n(x,t)\Pi_n(z), \tag{3.14}$$

$$\tilde{\zeta}=\sum_{n=1}^{\infty}\zeta_n(x,t)Z_n(z), \tag{3.15}$$

elimination of ρ and p from the reduced set (2.45) gives:

$$\frac{\partial\mathbf{u}_n}{\partial t}+f\mathbf{j}\times\mathbf{u}_n+\nabla\zeta_n=0, \tag{3.16}$$

$$\frac{\partial\zeta_n}{\partial t}+c_n^2\nabla\cdot\mathbf{u}_n=0, \tag{3.17}$$

formally equivalent to the equations for the barotropic mode (2.31; 2.32). The incorporation of the barotropic mode ($n=0$) itself in the expansion in vertical eigenfunctions will sometimes be necessary, particularly in the discussion of damping and of residual currents. The incorporation has been discussed by Gill and Clarke (1974). For the barotropic mode we have

$$\Pi_0=1, \qquad Z_0=z+1 \tag{3.18}$$

with $c_0=\delta^{-1}$, δ being given by (2.5).

4. PARAMETRIZATION OF MODAL DAMPING

The inclusion of frictional damping mechanisms in the discussion of baroclinic motions in the quasi-nonlinear "continental shelf regime" is necessary for two reasons. First in shallow water, with a strong barotropic tidal current present, internal waves are propagating in a dissipative medium, primarily due to the turbulence generated by the barotropic tide. Hence the energy source of the internal waves at the same time creates their sink, which can be so strong that the e-folding distance can just be a few times the internal wave length. In an area with a complicated source topography this may lead to strongly incoherent internal wave signals at different positions. Secondly, in the case of barotropic tidal currents, it has been shown that the generation of residual currents depends in a subtle way on the inclusion of bottom friction (Huthnance, 1973, 1981; Zimmerman, 1980). Although the rectified current is strongest in the limit of weak friction, no rectification is obtained when friction is omitted in the basic equations *ab initio*. This may be expected to apply to baroclinic residual currents, as well.

The incorporation of (turbulent) viscous effects in the baroclinic shallow water equations faces two problems: First, the proper expression of the frictional force and the boundary condition to be applied at the seabed and secondly, the coupling of different vertical modes due to friction, which threatens to destroy the advantage of working in the "small amplitude" limit. We shall adopt the simplest possible way that effectively keeps the vertical modes uncoupled, but yet retains the basic feature of the damping of an individual mode, viz. that the damping rate depends both on the internal shear of the mode as on the shear induced by the collective movements of all modes near the bottom. In this, our approach for a continuous stratification resembles those of Rattray (1957) and Martinsen and Weber (1981) for a two-layer flow, particular as the relative strength of bottom versus internal friction is concerned. It also accords with the results of LeBlond (1966) and Crampin and Doré (1969) for a continuously stratified fluid, but instead of expanding the dispersion relation directly in the Ekman (or Stokes) number as these authors do, we first pose part of the vertical mode coupling problem in full, showing that the coupling vanishes in the limit of a small Ekman-number.

4.1. *Mode-dependent friction*

As to the turbulent viscous effects on the motion, we shall assume that only the divergence of the vertical turbulent momentum flux is of importance, hence a term $\partial\tau/\partial z$ must enter (2.45a), where τ is the vertical flux of horizontal momentum, which has to be specified. For this we assume that in the interior of the fluid, dimensionally,

$$\tau = K\frac{\partial \mathbf{u}}{\partial z}, \tag{4.1}$$

or nondimensionally,

$$\tau = \frac{E_s}{2}\frac{\partial \mathbf{u}}{\partial z}, \qquad E_s = \frac{2K}{\sigma H_0^2}, \tag{4.2}$$

where K is an, in general depth-dependent, eddy viscosity and E_s is the Stokes number. Near the bottom a strong shear layer is to be expected where K decreases rapidly. In order to work with a constant-K in the interior later on, and refraining from an explicit representation of the bottom shear layer we adopt a stress boundary condition at the seabed:

$$\tau_* = K\frac{\partial \mathbf{u}_*}{\partial z_*} = c_d|\mathbf{u}_*|\mathbf{u}_* \simeq r_*\mathbf{u}_*, \quad z_* = -H_* \tag{4.3}$$

with $r_* = r_*(|\mathbf{u}_*^{(0)}|)$, or non-dimensionally:

$$\frac{\partial \mathbf{u}^{(1)}}{\partial z} = s\mathbf{u}^{(1)}, \qquad z = -1 - \delta H^{(1)}, \tag{4.4}$$

with

$$s=\frac{r_*H_0}{K}. \tag{4.5}$$

The quadratic friction law in (4.3) stresses the turbulent character of the bottom boundary layer. Its linearization with the linear friction coefficient r_*, being approximately a function only of the amplitude of the barotropic tidal velocity amplitude, expresses the fact that the internal motions are thought to be of a smaller amplitude, so that the turbulent intensity in the boundary layer is primarily due to the barotropic tide. The parameter s in (4.5) measures the relative influence of bottom and internal friction. For $s=0$ bottom friction is absent and vertical modes become frictionally uncoupled for a specific expression of $K(z)$, to be discussed furtheron.

Unfortunately observations by Maas and van Haren (1987) show s to be of order 10 in a typical shelf sea, which suggests that the limit $s\to\infty$ is more appropriate. In order to keep a finite stress at the bottom this means that a (mathematically more convenient) no-slip condition can be applied:

$$\mathbf{u}=0, \qquad z=-1-\delta H^{(1)}. \tag{4.6}$$

This, in principle, produces vertical mode coupling.

Projecting the stress-divergence, $\partial\tau/\partial z$, on the vertical normal modes as discussed in Section 3, using (4.2), we obtain the mode-dependent damping terms:

$$\mathbf{T}_n=\frac{\int_{-1}^{0}\frac{\partial\tau}{\partial z}\Pi_n(z)\,dz}{\int_{-1}^{0}\Pi_n^2(z)\,dz}=\frac{\sum_{m=1}^{\infty}\frac{\mathbf{u}_m}{2}\int_{-1}^{0}\frac{d}{dz}\left[E_s(z)\frac{d\Pi_m}{dz}\right]\Pi_n\,dz}{\int_{-1}^{0}\Pi_n^2\,dz}. \tag{4.7}$$

The orthonormality of the modes, (3.10), and (3.7) guarantee that this projection produces no mode-coupling provided

$$E_s(z)N^2(z)=\text{constant},$$

or (4.8)

$$K(z)N_*^2(z)=\text{constant},$$

as has been observed by Fjeldstad (1964). Equation (4.8) is of course trivially satisfied for a linearly stratified fluid ($N=1$) if we suppose the eddy viscosity to be a function of the Brunt-Väisälä frequency only, as we shall do here. Then (4.7) reads:

$$\mathbf{T}_n=-E_n\mathbf{u}_n, \qquad E_n=\frac{E_sN^2}{2c_n^2}=\frac{n^2\pi^2}{2}E_s, \tag{4.9}$$

with c_n^2 given by (3.13). In subsequent sections we derive the damping coefficients for internal modes in the flat bottom reference state.

4.2. *The slipping velocity modes*

Incorporating (4.9) in the modal equations (3.16) and (3.17) we have

$$\frac{\partial \mathbf{u}_n}{\partial t}+f\mathbf{j}\times\mathbf{u}_n+\nabla\zeta_n+E_n\mathbf{u}_n=0, \tag{4.10}$$

$$\frac{\partial \zeta_n}{\partial t}+c_n^2\nabla\cdot\mathbf{u}_n=0. \tag{4.11}$$

Introduce the complex (rotary) velocity vectors (Prandle, 1982)

$$\mathbf{u}_{\pm n}=u_n\mp iv_n \tag{4.12}$$

and assume $\mathbf{u}_n$, $\zeta_n\propto\exp(-it)$. Then from (4.10) and (4.11) we obtain the slipping velocity modes:

$$\mathbf{u}_{-n}=\frac{1}{2i(\sigma_n-f)}\nabla_+\zeta_n, \tag{4.13}$$

$$\mathbf{u}_{+n}=\frac{1}{2i(\sigma_n+f)}\nabla_-\zeta_n, \tag{4.14}$$

where

$$\nabla_+\equiv\frac{\partial}{\partial x}+i\frac{\partial}{\partial y};\quad \nabla_-\equiv\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}, \tag{4.15}$$

and

$$\sigma_n=1+iE_n. \tag{4.16}$$

4.3. *The no-slip velocity field*

The modal solutions (4.13) and (4.14) together with their vertical eigenfunctions as given by (3.11–3.12) obey the boundary conditions at the surface, $\partial p/\partial t=w$, $\partial\mathbf{u}/\partial z=0$, as well as $w=0$ at the bottom, but *not* the no-slip condition $\mathbf{u}=0$ at $z=-1$. That is, they trigger a second velocity field, $\mathbf{u}'$, that adapts the vertical profile to the no-slip condition. This field can be dealt with as in Sverdrup (1927) for the barotropic flow. At the bottom, the field given by (4.13–4.14) gives

$$\mathbf{u}(-1)=\sum_{n=0}^{\infty}\mathbf{u}_n\Pi_n(-1)=\sum_{n=0}^{\infty}\mathbf{u}_n, \tag{4.17}$$

where we have incorporated the barotropic mode $n=0$, given by (3.18). The adaptive field, then, is the solution of

$$\frac{\partial \mathbf{u}'}{\partial t} + f\mathbf{j} \times \mathbf{u}' = \frac{E_s}{2} \frac{\partial^2 \mathbf{u}'}{\partial z^2}, \tag{4.18}$$

with boundary conditions

$$\mathbf{u}'(-1) = -\mathbf{u}(-1) = -\sum_{n=0}^{\infty} \mathbf{u}_n, \tag{4.19}$$

$$\frac{\partial \mathbf{u}'}{\partial z} = 0, \qquad z = 0. \tag{4.20}$$

The solution reads:

$$\mathbf{u}'_{\pm} = -\sum_{n=0}^{\infty} \mathbf{u}_{\pm n} F_{\pm}(z), \tag{4.21}$$

where

$$F_{\pm}(z) = \frac{\cosh \alpha_{\pm} z}{\cosh \alpha_{\pm}}, \tag{4.22}$$

$$\alpha_{\pm} = \frac{1-i}{E_{\pm}^{1/2}}, \qquad E_{\pm} = \frac{E_s}{1 \pm f}, \tag{4.23}$$

where we have adopted the same rotary decomposition for the adaptive field as in (4.12).

4.4. *Mode-coupling*

The total velocity field reads for each rotary component:

$$\mathbf{u}_{\pm} = \sum_{n=0}^{\infty} \frac{\nabla_{\mp} \zeta_n [\Pi_n - F_{\pm}(z)]}{2i(\sigma_n \pm f)}.$$

If this is substituted in the continuity equation

$$\nabla_{\pm} \cdot \mathbf{u}_{-} + \nabla_{-} \cdot \mathbf{u}_{+} + \frac{\partial w}{\partial z} = 0, \tag{4.25}$$

together with

$$w = i \sum_{n=0}^{\infty} \zeta_n \frac{d\Pi_n}{dz}, \tag{4.26}$$

we get

$$i\sum_{n=0}^{\infty}\zeta_n\frac{d^2\Pi_n}{dz^2}+\frac{1}{2i}\sum_{n=0}^{\infty}\nabla^2\zeta_n\left(\frac{\Pi_n-F_-}{\sigma_n-f}+\frac{\Pi_n-F_+}{\sigma_n+f}\right)=0. \tag{4.27}$$

Now suppose $\zeta_n\propto\exp(i\mathbf{k}\cdot\mathbf{x})$, and project (4.27) on each of the eigenmodes $\Pi_m(z)$ as given by (3.11) and (3.18). Then we have

$$\frac{\zeta_m}{c_m^2}-\frac{\kappa^2}{2}\left(\frac{1}{\sigma_m-f}+\frac{1}{\sigma_n+f}\right)\zeta_m+\frac{\kappa^2}{2}\sum_{n=0}^{\infty}\zeta_n\left(\frac{F_{-m}}{\sigma_n-f}+\frac{F_{+m}}{\sigma_n+f}\right)=0, \tag{4.28}$$

where

$$F_{\pm m}=\frac{\int_{-1}^{0}F_{\pm}(z)\Pi_m(z)dz}{\int_{-1}^{0}\Pi_m^2(z)dz}. \tag{4.29}$$

From (3.11) and (4.22) this is evaluated as

$$F_{\pm m}=\frac{F_{\pm 0}}{\sigma_{\pm m}}, \tag{4.30}$$

where

$$F_{\pm 0}=\frac{\tanh\alpha_{\pm}}{\alpha_{\pm}}, \tag{4.31}$$

$\alpha_{\pm}$ being given by (4.23). Also

$$\sigma_{\pm m}=1+\frac{im^2\pi^2E_s}{2(1\pm f)}. \tag{4.32}$$

In (4.28), where $\kappa^2=|\mathbf{k}|^2$, all modes are coupled due to the third term, which vanishes for $E_s=0$, in which case the familiar dispersion relation for undamped waves is recovered. The coupling means that due to frictional modification we need in principle all vertical modes to give a complete description of the velocity profile, no matter whether only a single mode is resonantly forced. However, the vertical modes due to frictional adaptation will not in general obey the dispersion relation for free waves, so that the coupling will not produce resonantly forced modes itself. The projection of the frictional forcing function $F(z)$ on the first four eigenmodes is shown in Figure 1 as a function of the Ekman-Stokes number, $E_{\pm}=E_s/(1\pm f)$. Apart from the vertically uniform mode, $n=0$, the projection, and thereby the mode-coupling term, is significant, in comparison to the other terms in (4.28), only in an intermediate range of values of $E_{\pm}$. For the limit $E_{\pm}\to 0$ (weak bottom friction) this means a decoupling of the modes.

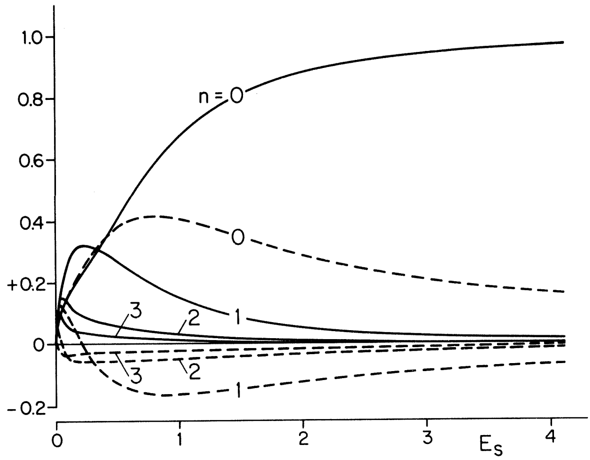

Figure 1 Real (——) and imaginary (– – –) parts of the projection of the forcing function $F_{\pm}(z)$ on the first four eigenmodes ($n=0,1,2,3$), $F_{\pm n}$, as a function of the Ekman-Stokes number $E_{\pm}=E_s/(1\pm f)$.

4.5. *Dispersion relation*

Dividing by κ^2, (4.28) constitutes a matrix equation

$$A_{ij}\zeta_j=0, \tag{4.34}$$

where

$$A_{ij}=\begin{cases} \dfrac{1}{c_i^2\kappa^2}+\dfrac{1}{2}\left(\dfrac{F_{-i}-1}{\sigma_i-f}+\dfrac{F_{+i}-1}{\sigma_i+f}\right), & i=j, \\[2ex] \dfrac{1}{2}\left(\dfrac{F_{-i}}{\sigma_j-f}+\dfrac{F_{+i}}{\sigma_j+f}\right), & i\neq j. \end{cases} \tag{4.35}$$

Equation (4.34) is satisfied once

$$\det A=0. \tag{4.36}$$

Now for small Ekman-Stokes numbers one may discard the off-diagonal terms of A_{ij} in solving (4.36), which gives effectively a decoupling of the vertical modes, analogously to the decoupling of spherical eigenfunctions in the damping of barotropic tidal waves on a rotating globe in the limit of weak dissipation (Miles, 1986). That the off-diagonal terms may be discarded can be justified by looking at (4.35) for $E_{\pm}\to 0$, so that $F_{\pm i}\sim E_{\pm}$. The products of off-diagonal terms are then always of a higher order in $E_{\pm}$ than those contained in the diagonal terms. The formal proof of the validity of this approach, for a nonrotating fluid, is given by Maas (1987), showing that the result is equal to an exact evaluation of the determinant equation upon which the limit $E_s\to 0$ is applied *a posteriori.*

Using only the diagonal terms of (4.35) in (4.36) gives the roots for κ^2 as follows

$$\kappa_n^2 = \frac{1}{c_n^2}\frac{\sigma_n^2 - f^2}{\sigma_n}\left[1 - \frac{\sigma_n + f}{\sigma_n}F_{-n} - \frac{\sigma_n - f}{\sigma_n}F_{+n}\right]^{-1}, \tag{4.37}$$

which in the limit $E_s \to 0$ reduces to

$$\kappa_n^2 = \frac{1}{c_n^2}2(1-f)\left[1 + (1+i)E_-^{1/2} + \frac{in^2\pi^2}{2}E_- + O(E_-^{3/2})\right]. \tag{4.38}$$

This is the dispersion relation, showing that the damping rate, represented by the imaginary part of κ, can be split-up in two parts which behave differently as a function of Ekman-Stokes number (eddy viscosity) and mode number. The damping rate due to internal friction is linear in the Ekman-Stokes number and increases quadratically with mode number. The damping rate due to bottom-friction is proportional to the square root of the Ekman-Stokes number and is independent of the mode number. Evidently for higher mode numbers internal friction will be dominant but for the usually more important lower modes bottom friction is the principal damping agency. These results accord with those of LeBlond (1966) and Crampin and Doré (1969).

The result given in (4.38) implies a very suitable parametrization of modal damping. If we write the divergence of vertical flux of horizontal momentum as

$$\boxed{\frac{\partial \boldsymbol{\tau}}{\partial z} = -E_-^{1/2}\mathbf{u} + \frac{E_-}{2}\frac{\partial^2 \mathbf{u}}{\partial z^2},} \tag{4.39}$$

this leads, upon projection on the vertical modes and substitution of (4.39) in the right-hand side of the modal equation, (3.16), to the proper damping rate for the individual modes without having to care of the no-slip boundary condition. That is, as long as we are only interested in that part of the vertical structure of tidal currents that is due only to baroclinic effects, and not in the part due to bottom friction, (4.39) is the proper parametrization of the damping processes that accounts implicitly for the effects of bottom friction on the internal baroclinic modes via the first term in the right-hand side. Of course, the Rayleigh damping term in (4.39) should not be interpreted as to apply in a physically strict sense. It is not meant that locally in a velocity profile the frictional force is proportional to the local velocity, but only that the very localized effects of bottom friction can be parametrized as a Rayleigh damping term for each of the vertical modes.

5. FORCING EQUATIONS FOR INTERNAL TIDES

When (4.39) is substituted in the right-hand side of (2.45a), the set of equations (2.45), together with the boundary conditions (2.46), describes the generation of internal tides by the interaction of the barotropic tidal current and the bottom topography, including the effects of quasi-nonlinear advection and frictional damping. The actual no-slip condition for the horizontal velocity at the sea bed has been removed by parametrizing its effect as a Rayleigh damping term in (4.39), at the expense of loosing any vertical structure of the tidal currents due to bottom friction. The removal of the no-slip condition makes it possible to expand all dependent variables in the vertical modes given by (3.11) and (3.12). Eliminating the density from the equations, introducing the total derivative:

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + (\mathbf{u}^{(0)} \cdot \nabla), \tag{5.1}$$

the modal amplitudes obey

$$\frac{d}{dt}\mathbf{u}_n + f\mathbf{j} \times \mathbf{u}_n + \nabla p_n = -r_n \mathbf{u}_n, \tag{5.2a}$$

$$\frac{dp_n}{dt} = w_n + w_{en}, \tag{5.2b}$$

$$w_n + c_n^2 \nabla \cdot \mathbf{u}_n = 0. \tag{5.2c}$$

These equations can be combined into one equation for the vertical velocity w_n:

$$\left[\left(\frac{d}{dt} + r_n\right)^2 + f^2\right]\frac{dw_n}{dt} - c_n^2\left(\frac{d}{dt} + r_n\right)\nabla^2(w_n + w_{en}) = 0. \tag{5.3}$$

Here

$$c_n^2 = \frac{1}{\pi^2 n^2}, \qquad r_n = E_-^{1/2} + E_- \frac{n^2\pi^2}{2}, \tag{5.4}$$

whereas the projection of the forcing w_{en} is given by

$$w_{en} = \frac{\int_{-1}^{0} w_e Z_n(z)\,dz}{\int_{-1}^{0} Z_n^2\,dz} = -(\mathbf{u}^{(0)} \cdot \nabla)H \frac{2}{n^2\pi^2}, \tag{5.5}$$

the last equality following from (2.43). (Note that we have dropped all super- and subscripts in (2.45) except for w_e and $\mathbf{u}^{(0)}$.)

Equation (5.3) is our starting point for the response of a stratified fluid to forcing over a small amplitude topography. For such a topography a horizontal Fourier transformation of (5.3) is suitable. Defining,

$$\hat{w}_n(\mathbf{k}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} w_n(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x}, \tag{5.6a}$$

$$\hat{H}(\mathbf{k}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x}, \tag{5.6b}$$

the total derivative (5.1) reads

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + i\mathbf{k}\cdot\mathbf{u}^{(0)}, \tag{5.7}$$

as $\mathbf{u}^{(0)}$ is independent of $\mathbf{x}$, being a variable dependent on the slow coordinate only, and

$$\hat{w}_{en}(\mathbf{k}) = -i\mathbf{k}\cdot\mathbf{u}^{(0)} \frac{2}{\pi^2 n^2} \hat{H}(\mathbf{k}), \tag{5.8}$$

where $\mathbf{k}$ is the topographic wave number vector.

Both from a physical and mathematical point of view it is convenient to introduce a quasi-Lagrangian reference frame, moving with the barotropic oscillatory current $\mathbf{u}^{(0)}(t)$ (Bell, 1975; Maas *et al.*, 1987). Defining

$$\hat{w}_n(\mathbf{k}, t) = \hat{W}_n(\mathbf{k}, t) \exp\left[-i \int^t \mathbf{k}\cdot\mathbf{u}^{(0)}(t')\, dt' \right], \tag{5.9}$$

$\hat{W}_n(\mathbf{k}, t)$ is the modal Fourier component of the vertical velocity as seen by an observer moving with the velocity $\mathbf{u}^{(0)}(t)$. In terms of $\hat{W}_n(\mathbf{k},t)$ the Fourier transformed eq. (5.3) reads:

$$\left[\left(\frac{\partial}{\partial t} + r_n\right)^2 + f^2\right] \frac{\partial}{\partial t} \hat{W}_n + c_n^2 |\mathbf{k}|^2 \left(\frac{\partial}{\partial t} + r_n\right) \hat{W}_n$$

$$\equiv -c_n^2 |\mathbf{k}|^2 \exp\left\{ i \int^t \mathbf{k}\cdot\mathbf{u}^{(0)}\, dt \right\} \left[\frac{\partial}{\partial t} + i(\mathbf{k}\cdot\mathbf{u}^{(0)}) + r_n \right] \hat{w}_{en} \equiv \hat{F}_n(\mathbf{k}, t). \tag{5.10}$$

Note that the transformation (5.9) has removed the quasi-nonlinear advective terms at the expense of a complicated forcing function $\hat{F}_n(\mathbf{k},t)$, in the absence of which the left-hand side of (5.10) just describes the free, frictionally damped, internal modes in a moving frame of reference.

From hereon we shall assume that the perturbations in topography, $H(\mathbf{x})$, vary only in a single direction, x, whence $\mathbf{k} = (k, 0)$. For the externally imposed barotropic tidal current $\mathbf{u}^{(0)}(t)$ we assume a simple elliptically polarized vector

$$\mathbf{u}^{(0)}(t) = \frac{l_0}{l_i} [\cos t, e\cos(t+\phi)], \tag{5.11}$$

where the ellipticity, $e \leqslant 1$ and ϕ is an arbitrary phase angle. Note that in the x-direction $\mathbf{u}^{(0)}(t)$ has an amplitude l_0/l_i in dimensionless form as explained in Section 2 [above (2.10)]. It is convenient to absorb this length scale ratio by rescaling wavenumber k to $k = k_* l_0$ and simultaneously the "fast" coordinate x to $x = x_*/l_0$. The ratio l_0/l_i is then reappearing only in the eigen wavenumber k_n, defined below. On these assumptions (5.10) finally reads

$$\left[\frac{\partial^3}{\partial t^3} + 2r_n \frac{\partial^2}{\partial t^2} + \left(f^2 + r_n^2 + \frac{k^2}{k_n^2}\right)\frac{\partial}{\partial t} + r_n \frac{k^2}{k_n^2}\right]\hat{W}_n = a_n(k) e^{ik \sin t}\left(\frac{\partial}{\partial t} + ik \cos t + r_n\right)\cos t \equiv \hat{F}_n(k,t), \quad (5.12)$$

where

$$k_n \equiv n\pi \frac{l_0}{l_i}, \quad (5.13)$$

is the ratio of the excursion amplitude l_0 to the wavelength of the internal tide of mode n: $(l_i/n\pi)$. The amplitudes $a_n(k)$ of the forcing $\hat{F}_n$ are given by:

$$a_n(k) = \frac{2}{n^2\pi^2} ik\hat{H}\frac{k^2}{k_n^2} = 2c_n^2 ik\hat{H}\frac{k^2}{k_n^2}. \quad (5.14)$$

In the absence of friction and rotation (5.12) is equivalent to Bell's (1975) equation for the generation of internal waves in a semi-infinite medium, with which it shares the complicated forcing function $\hat{F}_n(k,t)$, that gives rise to the occurrence of superharmonics of the basic driving tidal frequency. Moreover, in the limit of a gently sloping topography, $k \to 0$, in the absence of friction, (5.12) is equivalent to the basic equation of Cox and Sandstrom (1962), for which we have the linear undamped solution

$$\hat{W}_n(k,\tau) = a_n(k)\frac{\cos t}{f^2 - 1 + k^2/k_n^2}, \quad (5.15)$$

where, in this limit, $\hat{W}_n(k)$ is equivalent to $\hat{w}_n(k)$ and superharmonics are absent. The vanishing of the denominator, then, gives the dispersion relation for the resonantly forced internal gravity wave modes.

Although our basic equation thus encompasses those from previous studies, we stress here that the inclusion of the three processes: quasi-nonlinear advection, frictional damping and rotation, together is necessary in order to derive the full information about internal tidal features, particularly the occurrence of topographically bounded modes of which baroclinic residual currents are a part.

6. FREE MODES

The solution of (5.12) gives the internal response of the fluid to perturbations in topography at wavenumber k. By inverse Fourier transformation, then, the response of a topography of any shape can be constructed. The spectral Lagrangian response function $\hat{W}(k,t)$ can be obtained in a standard way once we know the solution of the homogeneous part of (5.12), which actually describes the free modes of the fluid, albeit in an oscillating frame of reference.

The homogeneous part of (5.12), setting $F=0$, is solved by substituting $\hat{W}_n=\exp(\nu t)$. Then ν must satisfy the cubic

$$\nu^3+2r_n\nu^2+(f^2+r_n^2+k^2/k_n^2)\nu+r_nk^2/k_n^2=0. \tag{6.1}$$

This cubic is a particular case of the one derived by LeBlond (1966), who included heat-, salt- and anisotropic momentum diffusion. He calculated the dependence of damping rates of the propagating inertio-gravity waves on these diffusion coeffients and wavenumber. Scaling

$$\tilde{\nu}=\nu/r_n, \qquad \tilde{f}=f/r_n, \qquad \tilde{k}=k/r_n, \tag{6.2}$$

transforms (6.1) into

$$\tilde{\nu}^3+2\tilde{\nu}^2+(1+\tilde{f}^2+\tilde{k}^2/k_n^2)\tilde{\nu}+\tilde{k}^2/k_n^2=0. \tag{6.3}$$

The two remaining independent parameters $\tilde{f}$ and $\tilde{k}/k_n$ express the ratios of the inertial frequency to the damping frequency and the eigenfrequency to the damping frequency respectively.

Together they determine the nature of the three roots. Equation (6.3) can be written as:

$$z^3-3(1-x-y)z+(3y-6x-2)=0, \tag{6.4}$$

with auxiliary variables

$$z\equiv 3\tilde{\nu}+2, \qquad x\equiv 3\tilde{f}^2, \qquad y\equiv 3\tilde{k}^2/k_n^2, \tag{6.5}$$

not to be confused with the coordinate axes.

Now for any cubic $z^3-az+b=0$, with $a<0$, there exists always only one real solution z and two complex conjugates. However, if $a>0$, z^3-az has two relative extrema, which allows the possibility of three real roots. The two complex conjugated roots in the first case belong to the internal gravity waves. If these roots become real, the waves are overcritically damped. The real third root is a transient, whose continuous presence can only be sustained by a persistent forcing. The region determined by $a<0$, a being implicitly defined by (6.4), is given by

$$x+y>1, \tag{6.6}$$

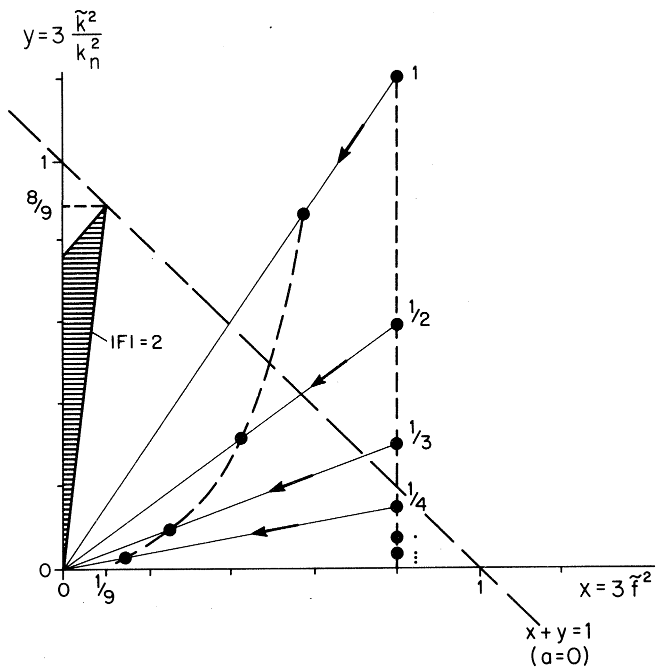

Figure 2 Parameter space of the cubic (6.4), summarizing the dispersion relations. Here $x=3f^2/r_n^2$ and $y=3k^2/k_n^2r_n^2$. The line $x+y=1$ separates the region where always two complex conjugated roots exist., together with a real root, from that where three real roots are possible. The actual region of three real roots is the hatched area. For a given first mode ($n=1$) at specific values of the parameters—dot labeled by 1—the higher modes are given by the dots labeled 1/2, 1/3 etc. and are seen to consist of a set of points that contract towards the x-axis. For weak friction these dots are approximately situated on a line $x=$constant. When friction, r_n, increases, all other parameters held fixed, the dots move so as to distort the line into a parabola due to an increase of the damping rate with mode number.

see Figure 2. Defining $z'=z/a^{1/2}$ transforms the cubic into

$$z'^3-3z'+F=0, \tag{6.7}$$

with

$$F=\frac{3y-6x-2}{(1-x-y)^{3/2}}. \tag{6.8}$$

Three real roots exist when $|F|\leqq 2$; this corresponds to the hatched region in Figure 2. The interpretation of this figure is facilitated when we assume first that the damping coefficient is independent of mode number. From (5.4) this assumption implies that $E_-\to 0$. Then, for a given f (or x), the set of 'eigenfrequencies' $[k/k_n=k/(nk_1)]$ will consist of a set of points in parameter space, which contract for increasing mode number n towards the x-axis (the dots in Figure 2). Hence the higher modes reduce to pure inertial oscillations. For finite E_- the n-dependence of the damping coefficient r_n,

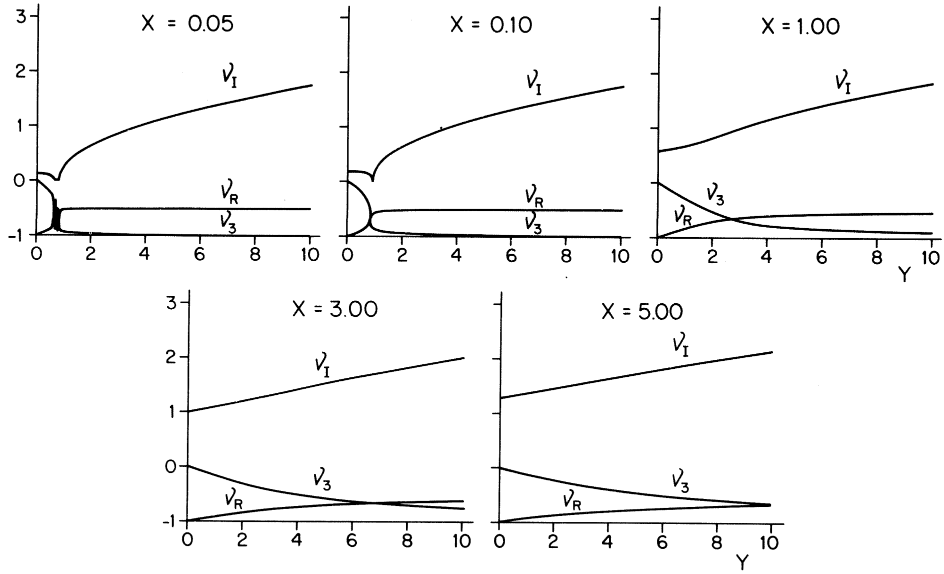

Figure 3 Behaviour of the real root ν_3 and the real, ν_r, and imaginary, ν_I, parts of the complex conjugated roots, $\nu_{1,2}$, as a function of $y = 3k^2/k_n^2 r_n^2$ for a few special values of $x = 3f^2/r_n^2$. For $x = 0.05$ and 0.1 these roots apply only in the region outside the area hatched in Figure 2.

with which the variables are scaled, assures that for increasing n contraction not only proceeds towards the x-axis ($\sim 1/n^6$), but also towards the y-axis ($\sim 1/n^4$) and thus to the origin. This is because the scaling on both axes is different for different mode numbers. The mode-number dependence of this contraction causes the original connection of a chosen set of points (dashed in Figure 2) to deform from a straight to a curved line. If, on contraction, any of the points enters the hatched area all three roots will be real, corresponding to an overdamping of the gravity wave modes.

For a few special values of x the behaviour of the roots as a function of y has been determined numerically and plotted in Figure 3. Apart from showing the possibility of three real roots for $x < 1/9$, it also reveals the asymptotic behaviour of the roots, as will be discussed analytically now.

There are a few interesting limits:

1. $\lim_{y \to 0}$, corresponding to strong damping, or weak stratification such that the topography scale (as we may interpret k^{-1} presently) is large compared with the internal wave length scale (k_n^{-1}), a situation particularly met for high modes ($n \to \infty$). Equation (6.4) reduces to

$$z^3 - 3(1-x)z - 2(1+3x) = (z-2)(z^2 + 2z + 1 + 3x) = 0,$$

which yields

$$z_{1,2} = -1 \pm i\{3x\}^{1/2} \to \nu_{1,2} = -r_n \pm if, z_3 = 2 \to \nu_3 = 0. \tag{6.9}$$

The higher mode gravity waves deform to the damped inertial oscillations referred to above, whereas the vorticity mode degenerates into a steady (geostrophic) current.

2. $\lim_{x\to 0}$, corresponding to a short spin-down time compared with the inertial time scale (or to $f\to 0$). (6.4) reduces to

$$z^3-3(1-y)z+3y-2=(z+1)(z^2-z-2+3y)=0,$$

which yields

$$\begin{aligned} z_{1,2}&=\frac{1\pm i(12y-9)^{1/2}}{2}\to v_{1,2}\simeq -1/2r_n\pm ik/k_n,\\ z_3&=-1\to v_3=-r_n, \end{aligned} \tag{6.10}$$

of which the expression for $v_{1,2}$ is valid for $y\gg 3/4$, as when the damping is weak, or the topographic wave length scale is small compared with the internal wave length scale. This is a set of two damped gravity modes, plus a third decaying transient. Of course, in the absence of frame rotation no geostrophic current results. Notice that the square-root term in $z_{1,2}$ guarantees that, in this limit, there are at most a finite number of freely propagating internal wave modes restricted by the criterion

$$y>3/4\to k/k_1>nr_n/2.$$

The modes then represent evanescent gravity 'waves'. Alternatively this condition may be interpreted as providing a low-wavenumber cut-off (LeBlond, 1966). Curiously, as can be seen in Figure 2, for x-values increasing from 0 to 1/9, the low-frequency cut-off is shrinking. Moreover, it reduces to a spectral (k) band (via y), below and above which free waves exist. This band therefore acts as a filter in wavenumber domain, separating gravity from inertial waves.

3. $\lim_{x,y\to\infty}$, corresponding to $\lim_{r_n\to 0}$ (weak friction). This limit can be directly discussed from eq. (6.1) by expanding the roots $v_j(j=1,2,3)$ in the small parameter r_n:

$$v_j=v_j^{(0)}+r_nv_j^{(1)}+r_n^2v_j^{(2)}+\dots \tag{6.11}$$

Then from (6.1) we find to zeroth order in r_n

$$v^{(0)}(v^{(0)^2}+f^2+k^2/k_n^2)=0.$$

Hence

$$\begin{aligned} v_{1,2}^{(0)}&=\pm i(f^2+k^2/k_n^2)^{1/2},\\ v_3^{(0)}&=0, \end{aligned} \tag{6.12}$$

the complex pair representing the usual dispersion relation for internal gravity waves.

The order r_n correction follows from

$$\nu^{(1)}[3\nu^{(0)^2}+f^2+k^2/k_n^2]+2\nu^{(0)^2}+k^2/k_n^2=0,$$

which shows that

$$\begin{aligned}\nu_{1,2}^{(1)}&=-\frac{(k^2/k_n^2+2f^2)}{2(k^2/k_n^2+f^2)}<0,\\ \nu_3^{(1)}&=-\frac{k^2/k_n^2}{f^2+k^2/k_n^2}<0.\end{aligned}\tag{6.13}$$

Hence we find up to order r_n:

$$\begin{aligned}\nu_{1,2}&=\pm i(f^2+k^2/k_n^2)^{1/2}-r_n\frac{(k^2/k_n^2+2f^2)}{2(k^2/k_n^2+f^2)}+O(r_n^2),\\ \nu_3&=-r_n\frac{k^2/k_n^2}{f^2+k^2/k_n^2}+O(r_n^2).\end{aligned}\tag{6.14}$$

These roots qualitatively agree with the graphs in Figure 3, expecially for $x\geqq 3$, or $f\geqq r_n$ i.e. for inertial frequencies exceeding the damping frequency, and also reduce to expression (6.9) and (6.10) when appropriate limits are taken.

The two complex roots, representing damped propagating inertio-gravity waves are of a quite different character, physically, compared with the third root, which represents an evanescent mode. The latter in combination with the forcing term in (5.12) gives rise to topographically bounded internal modes, in particular to baroclinic residual currents. These arise from quasi-nonlinear vorticity advection by the barotropic current once vorticity is produced by its interaction with the topography. It is in fact this single root which appears in barotropic rectification studies (Zimmerman, 1978; Maas *et al.*, 1987). Such "vorticity modes' are the subject of II, whereas the gravity waves are dealt with separately in III.

7. CONCLUSIONS

The equations governing motions in a stratified shelf sea in response to tidal flow over topography have been derived using a multiple-scale analysis. This approach utilizes the vast difference between barotropic wave length scale (L) compared with both the internal wave length scale, l_i, as well as the tidal excursion amplitude l_0. It is the ratio of the resulting two small parameters $\delta=l_i/L$ and $\varepsilon=l_0/L$, which determines the nature of the tidal regime one is considering. Quasi-nonlinear advective effects become important when $\varepsilon=O(\delta)$, as on continental shelves, which parameter regime has therefore been termed the "continental shelf regime". Advection can be neglected when $\varepsilon=O(\delta^2)$, referred to as the "deep-sea regime". Finite amplitude topographies, treated in literature for the latter regime almost exclusively, require an extra rescaling of the vertical velocity, effectively leading to a non-trivial bottom boundary condition for the free internal motions. The rigid-lid approximation, frequently applied in this context, is identified as the natural result of a mismatch of horizontal length scales.

In shallow shelf seas the ambient stratification often invites one to use a modal description of internal wave propagation. Any realistic description of wave motions, a study on tidal rectification in particular, requires taking proper account of frictional effects. Unfortunately, inclusion of bottom friction (as well as internal friction, once one does not adopt an idealized eddy-viscosity profile) in principle leads to a (non-resonant) coupling of all vertical modes. However, in the limit of weak friction, as when the Ekman-Stokes depth of the anticyclonic current component is small in comparison with the water depth, this coupling vanishes. Moreover we may adopt a simple parametrization of the stress-divergence, which, upon projection on the vertical modes, correctly yields a mode-independent damping rate due to bottom friction, concurrent with a mode-dependent damping term due to internal friction by the shear in the velocity modes.

Adding the parametrization of stress-divergence to the equations of motion and projection on the vertical and horizontal spectral modes leads to an evolution equation for the modal, spectral vertical velocity component in a co-oscillating, Lagrangian frame. The homogeneous part of this equation determines the free modes present in this model, which can in general (i.e. in the limit of weak damping) be identified as a set of damped, propagating inertio-gravity waves in conjunction with a (transient) geostrophic current. The response of these modes to the applied forcing is discussed in accompanying papers II and III.

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Appendix: *Glossary of notations and symbols*

ϕ_* dimensional variable ϕ

$\bar{\phi}$ average ϕ

ϕ'	varying part of ϕ
$\boldsymbol{\phi}$	vector ϕ
$\hat{\phi},\hat{\Phi}$	spectral amplitude in Eulerian ($\hat{\phi}$) and Lagrangian ($\hat{\Phi}$) frame
$\phi^{(n)}$	perturbation variable ϕ of order n
ϕ_n,Φ_n	decomposition of ϕ into a horizontal-temporal fluctuating part ϕ_n and vertical eigenfunctions Φ_n
$\tilde{\phi}$	frequencies, rescaled with r_n (section 6)
$\phi_{e,i}$	externally forced and internal free modes respectively
a_n	modal forcing amplitudes
c_d	drag coefficient
c_n	nondimensional eigen frequencies
e	ellipticity
E_s	Stokes number
$E_\pm$	Ekman-Stokes number of (anti-)cyclonic rotating current vector
f	Coriolis parameter
$F_\pm$	function describing vertical current structure due to friction
g	acceleration of gravity
H,H_0	actual depth profile and constant reference depth
i	imaginary unit
$\mathbf{j}$	vertical unit vector
$\mathbf{k}$	wavenumber (of topography)
K	eddy viscosity
l_0	tidal excursion amplitude ($U\sigma^{-1}$)
l_i	internal wave length scale
L	external wave length scale
N	Brunt–Väisälä frequency
p,p_a	pressure, atmospheric pressure
r	bottom friction coefficient
s	stress parameter expressing ratio of bottom and internal friction
t	time
T_n	modal damping terms
$u_\pm$	(anti-) cyclonic velocity component
U	scale of horizontal velocity
u,v,w	velocity components in x,y,z directions
w_f,w_s	vertical velocity varying on "fast" and "slow" scale
x,y,z	orthogonal cartesian coordinate frame ("fast" scale)
X,Y	large ("slow") scale horizontal coordinates
Z_n	vertical eigenfunctions of vertical velocity modes
δ_{nm}	Kronecker delta
δ	perturbation parameter; ratio of internal and external wave length scale
ε	ratio of tidal excursion amplitude and external length scale
ϕ	phase angle
κ	magnitude of wavenumber vector
ν	frequency
Π_n	vertical eigenfunctions for pressure and velocity modes

ρ	density
σ	tidal frequency
$\sigma_n, \sigma_{\pm n}$	nondimensional complex frequency, containing damping term as imaginary part
$\zeta, \tilde{\zeta}$	sea level elevation; internal elevation field.