13. INERTIAL WAVE PROPAGATION, FOCUSING AND MEAN FLOW GENERATION

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Abstract

Solidly rotating, homogeneous fluids are stably stratified in angular momentum in the radial direction. Therefore, they support inertial waves. These have the special property that they propagate obliquely through the fluid with an angle that depends solely on the ratio of the wave frequency to the rotation frequency. As this angle is conserved upon reflection, reflection at a sloping wall will lead to focusing or defocusing. In a two-dimensional enclosed basin, repeated reflection may lead to the appearance of wave attractors, limit cycles where all wave rays end and where the wave energy is concentrated. The concentration of energy around the attractor may lead to mixing and mean flow generation. These attractors exist over frequency intervals, as opposed to standing waves. For the latter focusing and defocusing balance exactly and no such strong energy concentration occurs. For a three-dimensional basin, wave patterns cannot be predicted exactly.
Laboratory experiments have been carried out in a rectangular basin with one sloping sidewall to study the quasi-two-dimensional wave attractors and a standing wave. The wave attractors and the standing wave have indeed been observed for the predicted frequencies. Attention has been paid to the three-dimensional structure of the wave field. In the along-tank direction, differences in strength of the motion have been observed, and for the case that the angle of propagation equals the angle of the slope also phase differences have been detected.

13.1 Inertial waves

Solidly rotating homogeneous fluids are radially stratified in angular momentum. This stable stratification enables the existence of inertial waves. Such waves are possibly relevant for the liquid outer core of the Earth (Malkus 1968, Aldridge et al. 1989), spacecraft (Aldridge et al. 1989, Manasseh 1993), the sea (Maas 2001) and stars. In the latter cases density stratification may modify the waves to inertio-gravity waves, with essentially similar features. These inertial waves have the property that the angle of propagation with respect to the rotation axis is strongly constrained: for a monochromatic wave of frequency ω and background rotation rate Ω, the angle α of the group velocity vector with the rotation axis equals

$$\frac{\omega^2}{4\Omega^2} = \sin^2 \alpha.$$

Thus, wave energy propagation is restricted to a double cone with opening angle α (Görtler 1944). Phase propagation is perpendicular to the direction of energy propagation.

This constraint has important consequences for reflection. As the wave frequency and the background rotation do not alter upon reflection, the angle α is also conserved. The double cone is symmetric with respect to walls that are parallel or perpendicular to the rotation axis, and reflection at such walls is neutral. However, for a sloping wall this symmetry is broken, which results in focusing or defocusing of the wave upon reflection. Repeated focusing in an enclosed basin, like a spherical shell, may lead to the appearance of isolated periodic wave rays (Bretherton 1964) which act as limit cycles (Stewartson 1971, 1972) to which all wave rays converge, so-called ‘wave attractors’. This implies a concentration of energy. Such limit cycles appear for linear waves and are an effect of the restricted angle of propagation and the shape of the boundary only. Apart from limit cycles, also standing waves may exist, which lack such energy concentration as every individual wave ray is periodic (in a two-dimensional description). Moreover, these standing waves exist for isolated frequencies for which the standing wave ‘fits’, whereas the attractors exist in frequency intervals (Israeli 1972). These attractors appear for many geometries, like the spherical shell (as mentioned before, and extended by Rieutord&Valdettaro 1997, Rieutord et al. 2001), a ‘lake’ with parabolic bottom shape (Maas&Lam 1995), or a basin with sloping side wall (Maas 2001), and requires no singularities (Manders et al. 2003). The principle of attraction and the difference with a standing wave is illustrated in ***Figure 13-1***.

Such attractors and standing waves were predicted in a quasi-two-dimensional approximation, for waves that do not propagate in the along-channel direction. Then, there is a direct correspondence between the wave rays and the method of characteristics that solves the wave equation. In reality, waves will propagate in three dimensions. Three-dimensional ray tracing reveals that also in a three-dimensional setting limit cycles exist for frequencies that yield attractors for the quasi-two-dimensional setting. But for the standing waves no closed ray patterns were found for rays that propagate in the along-channel direction. However, ray tracing cannot be used to construct mathematical solutions like in two dimensions. Especially where the channel is closed the wave attractor must adapt, but we have not been able to predict the exact wave patterns at such a vertical wall, as a solution in three dimensions relies heavily on the possibility of separation of variables, which is not possible for a basin with sloping side wall.

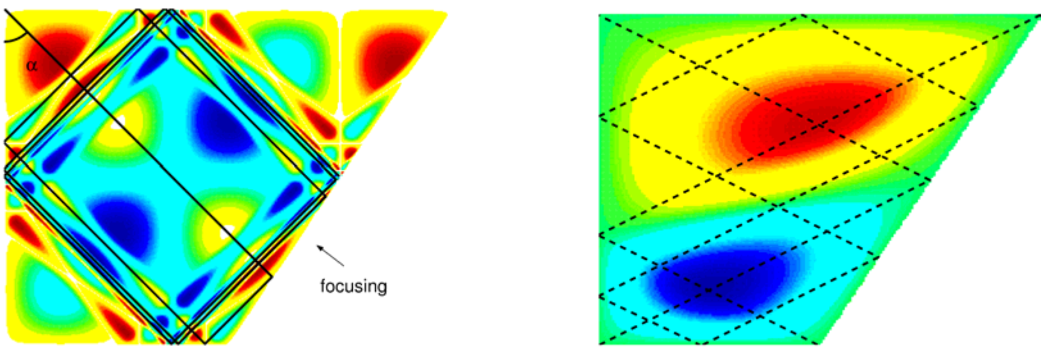

Figure 13-1. Left: a ray path towards the attractor (central square) due to reflection at the sloping wall, right: standing wave for which all wave rays are periodic. The background indicates the stream function, which shows ever-decreasing scales around the attractor, implying high velocities, and a simple cellular structure for the standing wave.

13.2 Experiments

Such wave attractors and a standing wave have been investigated at the 13 m diameter rotating platform of the Coriolis Laboratory (Grenoble, France) in a rectangular tank with one sloping side wall (***Figure 13-2***). This tank (width*length*height=107*500*80 cm^3) was filled with tap water and polystyrene particles for Particle Image Velocimetry and placed on the rotating platform. The rotation rate of the platform was modulated according to $\Omega(t) = \Omega_0(1 + \varepsilon \sin \omega t)$ to generate inertial waves of modulation frequency ω. The basic rotation rate was 0.13 rad/s, the modulation frequencies were around 0.2 rad/s. The modulation generated an alternating, horizontal, vorticity-conserving flow that generates inertial waves directly at the sloping wall near the end walls, and indirectly via Ekman pumping in viscous boundary layers. Observations have been made in various horizontal and vertical cross sections.

Six different frequencies have been investigated. Four were in the range of the simplest attractor with one reflection at the bottom and one reflection at the sloping wall, therefore classified as the (1,1)-attractor. One of them was in the lower boundary of the window of existence, where the critical slope attractor (wave rays parallel to the sloping wall) and the degenerated (1,1)-attractor come together, two of them were in the middle of the window, where the attractor forms a parallelogram, and one at the upper boundary of the window, where the attractor degenerates to a line again. The fifth frequency belonged to the (1,2) standing wave, where focusing at one (downward) reflection is balanced by a second defocusing (upward) reflection. The last frequency was for a (1,3)-attractor that is expected to be weaker.

The measurements were taken a while after the onset of the modulation, as we were interested in the spatial wave patterns, rather than the evolution of the wave field.

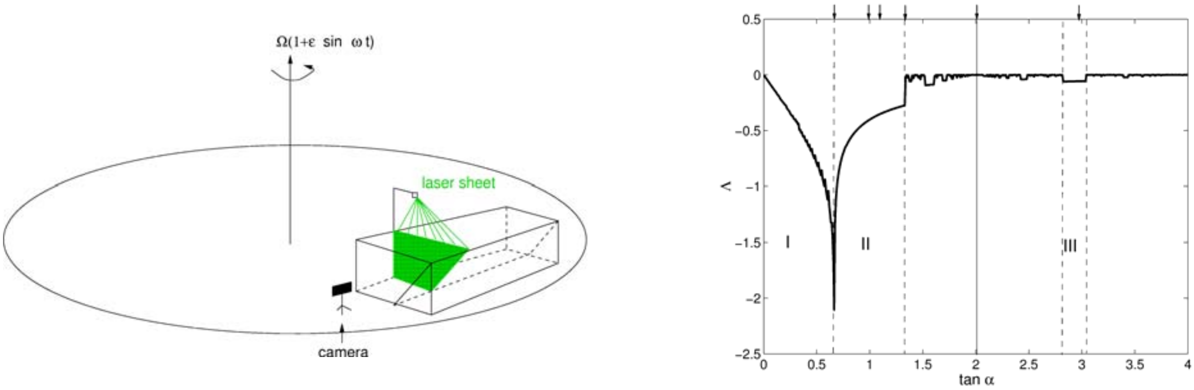

Figure 13-2. Experimental setup (left) and strength of convergence (Lyapunov-exponent) (right) for the different frequencies. Arrows indicate the values used in the experiments. In region I wave rays end in the apex, in region II the (1,1)-attractor exists, in region III the (1,3)-attractor. The solid line denotes the frequency for which the (1,2) standing wave occurs.

13.3 Results

The results consisted of time series of two-dimensional vector fields of the stationary motion. The forcing frequency (wave frequency) appeared to be dominant, and the velocity vectors describe ellipses over a period. Therefore, the measurements were conveniently summarized in terms of ellipse parameters (Maas&Van Haren 1987). In ***Figure 13-2***, the long axis of these ellipses is plotted for cross sections at ¼ of the tank's length (y=120 cm) for the 6 different frequencies.

The (1,1)-attractors are clearly the strongest. Focusing is strongest for the critical slope case, but for this frequency there is much variation in the along-slope direction. The other degenerate attractor is weaker by itself, since focusing becomes weaker for increasing α , but also since there is strong shear across this attractor. The standing wave resembles an attractor, but by studying its phase behaviour it appears that the phase does not propagate, like for wave attractors, but changes over the whole basin at once (***Figure 13-4***). No growth of amplitude and breakdown phenomena were observed that were found for standing waves in a cylinder (McEwan 1970, Manasseh 1992). The (1,3)-attractor is weak, but motion is concentrated around the lines of the theoretically predicted attractor.

In ***Figure 13-3*** all predicted wave patterns are visible. But there is variation in the along-channel direction. Only part of the channel was covered by the observations, from y=60 cm to y=210 cm. The strongest attractors (critical slope, square, self-similar) are observed to become stronger towards the end wall and weaker towards the middle of the channel. The degenerate attractor, the standing wave and the (1,3)-attractor become weaker towards the end wall at y=0 and do not become much weaker towards half way the channel (***Figure 13-5***). No resonance phenomena were observed for the standing wave. For the critical slope attractor, phase changes were observed (***Figure 13-6***), which point at energy propagation from the end wall into the interior of the tank. The wavelength in this direction was about ¼ of the tank's length. For the other (1,1)-attractors and the standing wave no propagation was found, for the (1,3)-attractor it was not possible to draw conclusions regarding propagation.

Evidence for inertial wave attractor-induced mixing is given in a similar experiment by Maas (2001). The observations presented here are described in more detail in Manders &Maas (2003).

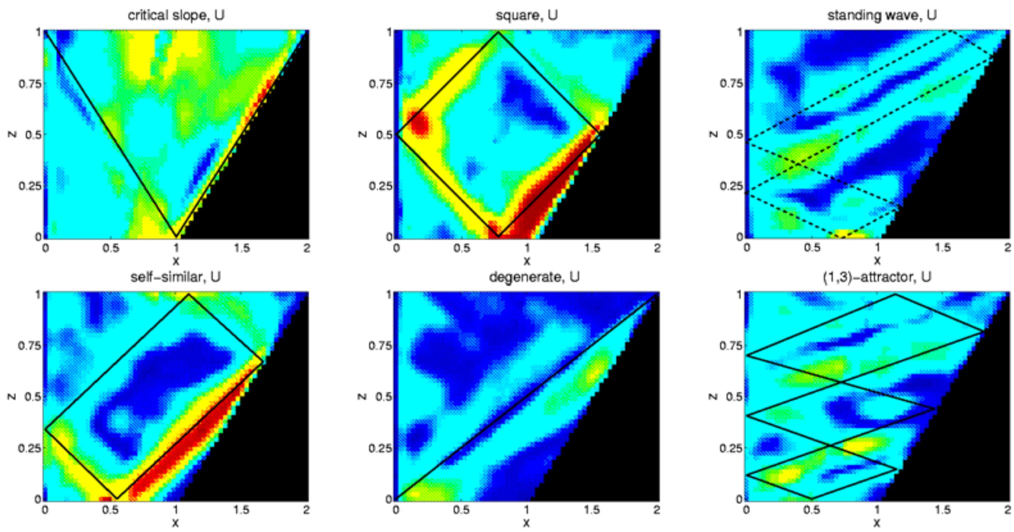

Figure 13-3. Long axis of velocity ellipse as a measure of the maximum strength of the motion over a wave period. The scaling is from blue (0 cm/s) to dark red (0.4 cm/s). Black lines indicate the theoretical attractor, the black dotted line is a wave ray path that matches the intensity pattern of the standing wave.

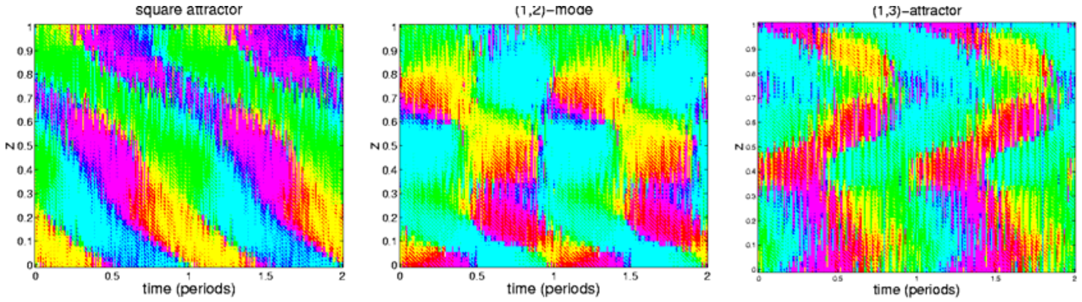

Figure 13-4. Indication of vertical phase propagation (arctan(vertical velocity/horizontal velocity)) along a vertical line over two (identical) periods. The attractors clearly exhibit phase propagation, with changes in direction over the different regions of the attractor, whereas for the standing wave the phase changes uniformly and then remains constant for half a period.

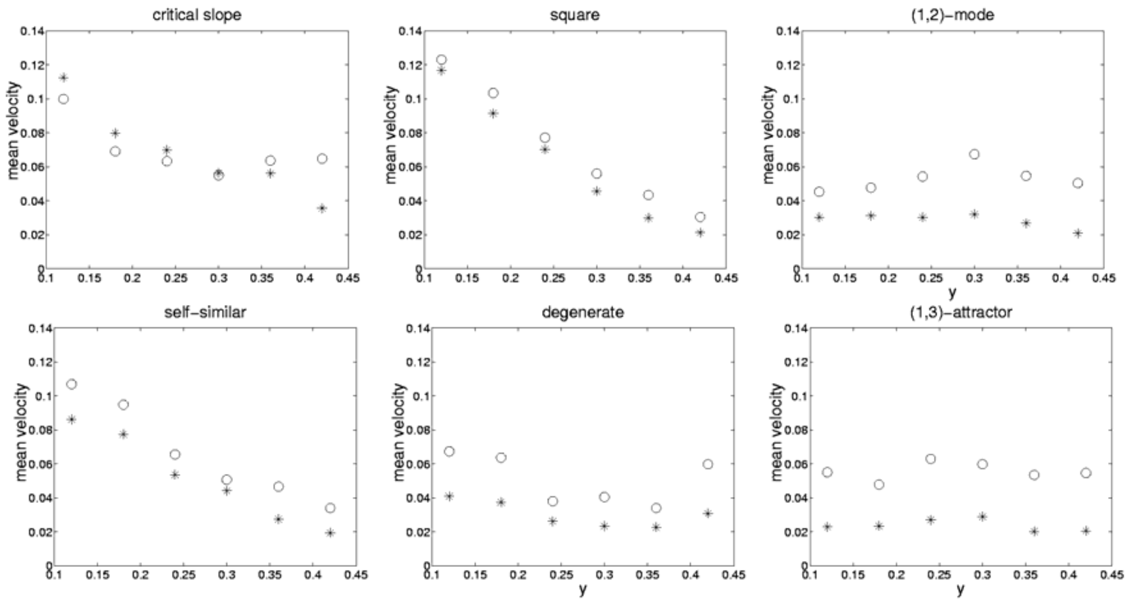

Figure 13-5. Average absolute velocity for horizontal (u, circles) and vertical (w, stars) velocity component in the vertical cross sections at different along-slope positions (normalised by the tank's length).

13.4 **Conclusions**

Wave attractors and a standing wave have been observed for the frequencies that were predicted for these wave patterns in a quasi-two dimensional channel. The presence of three-dimensional behaviour is indicated by the energy and phase changes in the along-channel direction, but the three-dimensional structure of the wave field is not resolved satisfactorily yet.

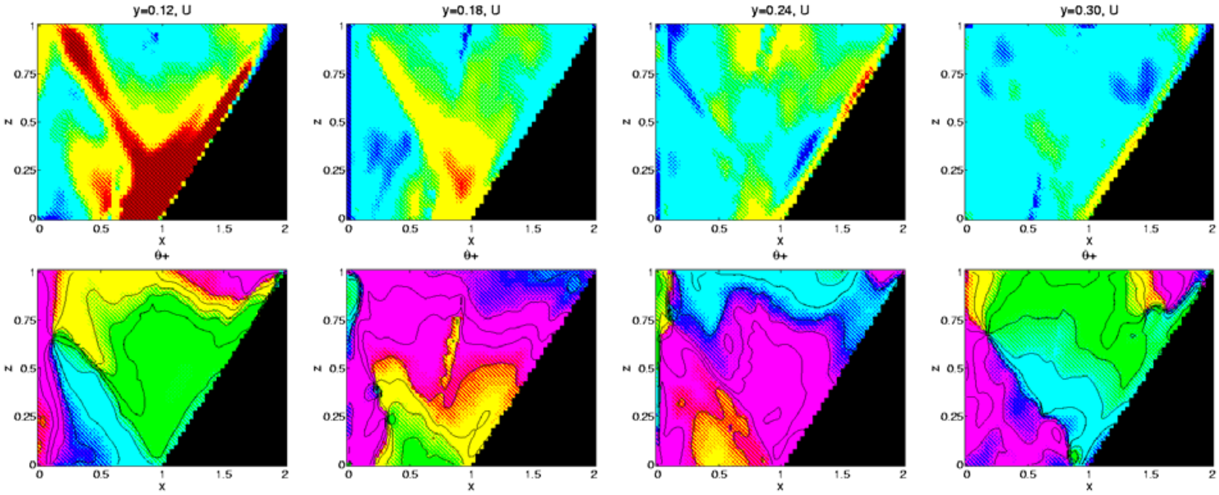

Figure 13-6. Strength of the motion of the critical slope attractor (upper panels) and indication of the phase at different along-slope directions (normalised by the tank's length). The phase pattern is nearly periodic over the 90 cm shown here, the horizontal wavelength must be slightly over 1 m.

13.5 **Acknowledgement**

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13.6 **References**

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