Synthetic seismograms for a spherically symmetric, non-rotating, elastic and isotropic Earth model

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Abstract

A synthetic seismogram is a predicted seismogram, based on an assumed Earth model. Synthetic seismograms are a valuable tool for investigating the Earth's interior, because synthetics can be compared with real data. The misfit between real data and synthetics contains information about the errors in the Earth model.
In this study, the theory of the free oscillations of a spherically symmetric, non-rotating, elastic and isotropic Earth (SNREI) model is discussed, followed by a derivation of an expression for a synthetic seismogram, by using a normal mode summation. A point source approximation is used, in which force components as well as (not necessarily symmetric) moment tensor components are included.
The main research question is how well the synthetics coincide with the real data.
To address this question, a synthetic seismogram is calculated for the 9.0 MW earthquake on 11-03-2011 near the east coast of Honshu, Japan. This synthetic seismogram is compared to real data and the accuracy of the synthetic seismogram is discussed.
It turns out that the synthetics resemble the data, but several mismatches can be identified: the amplitude of the synthetic seismogram is large compared to the real data and the arrival time of the surface waves is slightly different. The reason for these mismatches could be the failure of the point source approximation for large earthquakes as the 9.0 MW earthquake in Japan and the lateral heterogeneity of the Earth's lithosphere respectively.

Acknowledgements

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I have used many sources while writing the theory chapter of this thesis. My most valuable source were lecture notes written by John Woodhouse [1] on the solution of the equations of motion in spherical coordinates. Dog-ears appeared on my copy a week after I obtained them. An other very important source was the book 'Theoretical Global Seismology' by Dahlen and Tromp, especially for their complete historical overview on normal modes, the derivation of basic results in continuum mechanics and for their extensive appendix filled with handy mathematical identities. For the excitation problem, one paper by Dziewonski and Woodhouse [2] was exceptionally useful. Without all these sources I would not have been able to give any review on the theory.

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1 Introduction

The energy released by an earthquake of magnitude 8 is comparable to the energy released by the detonation of approximately thousand Hiroshima atomic bombs [3]. This energy is converted into fracturing, heat and seismic waves radiating from the earthquake location.

The seismic waves radiating from a source can be seen as a sum of of standing elastic-gravitational waves of the whole Earth. Just like a string or a drum, the boundary conditions allow only for distinct eigenfrequencies for these oscillations in the solid parts of the Earth.

Seismograms measured routinely all around the world, contain valuable data that can be used to probe the Earth's interior. It is essential for any seismological study of the Earth's interior to be able to calculate theoretical (also called synthetical) seismograms for a given earthquake. By examining the differences between synthetic seismograms and real data, algorithms can be developed to update the Earth model to obtain a better correspondence between synthetics and reality.

This study focuses on the calculation of a synthetic seismogram for a spherically symmetric, non-rotating, elastic and isotropic Earth model. To address this question, first the displacement field of the free oscillations of the Earth and their eigenfrequencies have to be found. Then a representation of the source must be given and a way has to be found to calculate the relative contributions of each mode to the seismogram. If all these questions are resolved, a synthetic seismogram will be calculated and compared with a real seismogram. Subsequently, the accuracy of the result will be discussed.

Before we start with the theory of the normal modes of a spherically symmetric Earth, first a short historical introduction in the subject will be given.

The history of the research of the Earth's normal modes can be divided in two periods: from 1828 to 1960 and from 1960 until now. In the first period, observations of the spectrum of normal mode frequencies of the Earth were not available. Hence all studies were purely theoretical. In the second period, it became possible to measure the long-period free oscillations of the Earth. Also, computers capable of Fourier-transforming long time series just became available. This was the beginning of the observational period.

The theoretical research in the normal modes of the Earth was initiated by the French mathematician Poisson. In 1828 he developed a general theory of deformation for solid materials and applied it to determine the frequencies of the purely radial oscillations of a homogeneous, non-gravitating sphere [4]. Since Poisson was not directly interested in the normal modes of oscillation of the Earth, he did not calculate a frequency for a sphere with the dimensions and density of the Earth. However, together with his contemporaries Navier and Cauchy he laid the foundations for the modern theory of linear elasticity.

In 1863, Lord Kelvin was the first person to make a numerical estimate of a vibrational eigenfrequency of the Earth [5]. For a self-gravitating fluid Earth he found a period of 94 minutes for this mode. He obtained this result by means of a dynamical analysis of a homogeneous, incompressible ($\kappa = \infty$) fluid ($\mu = 0$) sphere, where κ is the incompressibility and μ is the rigidity [6]. The mode he investigated is now designated ${}_0S_2$. For a solid Earth with the same rigidity as steel, he stated that the period would be approximately 69 minutes. This estimate was obtained by calculating the time required for a shear wave to cross the entire Earth.

Lamb was the first person to give a complete description of the free oscillations of a non-gravitating sphere in 1882 [7]. He distinguished two the types of modes: spheroidal and toroidal modes; which he called 'vibrations of the first and second class'. He concluded that the period of the ${}_0S_2$ mode for a steel sphere the size of the Earth should be 65 minutes in the case $\kappa = \infty$.

The implementation of gravitation led to a problem that confounded researchers at that time: the classical theory of elasticity dealt with deformations away from an unstressed and unstrained equilibrium configuration. However, the total stress exerted by gravity was far too great to be related to an infinitesimal strain by Hooke's law. Lord Rayleigh proposed a solution to this problem in 1906 [8]. He decomposed the total stress into a large initial stress, balanced by the self gravitation, and an incremental stress related to the strain by Hooke's law.

Love [9] used Rayleigh's idea to derive a system of equations in 1907, but he failed to distinguish between the incremental stress at a fixed point in space and the incremental stress experienced by a material particle. He realized his error and derived and solved the correct system of equations in 1911 [10]. He found a period almost exactly 60 minutes for the ${}_0S_2$ mode for a homogeneous, self-gravitating Poisson-solid with the rigidity of steel. Love's equations were for a homogeneous sphere only, although the correct equations for radially variable density and elastic parameters are not fundamentally different from his equations.

The general equations were stated first explicitly by Hoskins in 1920 [11]. Up until this point, the normal modes of oscillations of the Earth were not related to seismology. Jeans [12] was the first person in 1927 to show that the superposition of free oscillations excited by an earthquake source could represent a seismogram, and hence that a theoretical or synthetic seismogram can be constructed by performing a sum over normal modes.

The laws governing the free oscillations of the Earth can be stated as a variational principle, known as Hamilton's principle in the time domain. The first correct variational determination of the eigenfrequencies of a radially variable, realistic, self gravitating Earth model were done independently and simultaneously by Jobert [13], [14], [15], Pekeris and Jarosch [16] and Takeuchi [17] in the period from 1956 to 1961. The period of the ${}_0S_2$ mode was found to be approximately 52 minutes.

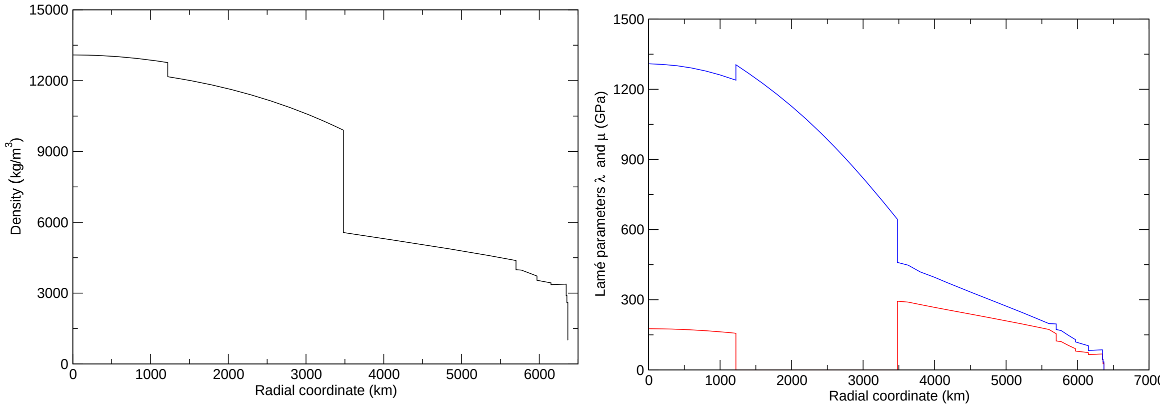
As soon as modern computational techniques came widely available, in 1959, Alterman, Jarosch and Pekeris [18] reformulated the second order differential equations governing the normal mode oscillations of the Earth into sets of first order differential equations, which were easier to solve with numerical integration. The Runge-Kutta method was used to calculate the eigenfrequencies and eigenfunctions of a couple of modes. They found a period of 53.7 minutes for the ${}_0S_2$ mode, in very good agreement with modern determinations.

In 1960 on May 22, the largest earthquake of the 20th century took place. At that moment, technology was advanced enough to be able to record the frequencies of the long period free oscillations of the Earth. Seismologists from Caltech and UCLA managed to obtain frequencies of many fundamental modes of the Earth barely two months later. Pekeris, Alterman and Jarosch [19] showed that the observations agreed with theoretical calculations for a realistic Earth model in 1961.

In the two decades after these measurements, observations of fundamental modes and their overtones were made. Gilbert and Dziewonski made a set of 1064 measured eigenfrequencies of the free oscillations in 1975 [20]. This was approximately 60 percent of all modes with periods longer than 80 seconds.

As soon as observations could be done more precisely and routinely, the study of geophysical inverse problems became more prominent. These studies focused on the determination of the Earth's internal structure and the determination of the earthquake source mechanism. The Preliminary Reference Earth Model (PREM, see figure 1), developed by Dziewonski and Anderson in 1981 was the culmination of the research to the internal structure of the Earth by using free oscillations [21].

Gilbert [22] and Kostrov [23] introduced the concept of a seismic moment tensor independently in 1970, to represent the earthquake source. Gilbert and Dziewonski [20] were able to reconstruct the moment tensor for two events in 1975. Dziewonski, Chou and Woodhouse [24] generalized the moment-tensor analysis allowing a spatial and temporal shift in the location of the point source, since the point of initiation of the earthquake is not necessarily the point which represents the earthquake best. This study stood at the beginning of the Harvard centroid-moment tensor project, which now automatically determines the source mechanisms of sufficiently strong earthquakes, which happens two or three times a day. The resulting global source catalogue accumulated from 1977 up to the present is, together with the PREM model, one of the most widely used results of normal mode studies.



(a) The density as a function of radial coordinate r.

(b) Lamé's elasticity parameters λ (blue) and μ (red).

Figure 1: The parameters of the PREM model.

In the mean time, attention was also given to the fact that the Earth is in fact not spherically symmetric. Effects of rotation, ellipticity, anisotropy and lateral heterogeneity were studied. Modern seismological studies since 1980 focus not only on the resolution of the three dimensional structure of the interior Earth, but also on the geodynamical and compositional causes of heterogeneity.

In recent history, the research focus is mainly on the non-spherically symmetric properties of the Earth. This is less relevant for the study presented here, which is about the spherically symmetric Earth. In the next chapter, an expression for a synthetic seismogram on a simple Earth model will be derived from first principles.

2 Theory

The main goal of this chapter is to derive an expression for a theoretical seismogram at any point on a spherically symmetric, elastic, non-rotating and isotropic Earth model due to a point source term, such as an earthquake, which generates mechanical waves small compared to the dimensions of the Earth.

The approach I will use, is to find the normal modes of vibration of the Earth model. Subsequently, the theoretical seismogram can be found by performing a normal mode sum. The coefficients of the individual modes in the sum depend on their excitation by the point source.

Hence the outline of this chapter will be as follows: first the conventions and notation will be stated and the characteristics of the Earth model will be specified. Then the equations governing the Earth's free oscillations will be derived from first principles and a (necessarily) numerical way to obtain the solutions to these equations for the Earth model will be discussed. Finally a mathematical description of a point source and an expression of the theoretical seismogram will be derived.

2.1 Notation

Before a discussion of the theory of standing waves on the Earth is given, some notation will be introduced here.

Any vector, denoted by $\vec{v}$ or its components v_i will be represented by an array. Any second, third or fourth order tensor will be represented by a matrix of the form T_{ij}, T_{ijk} and T_{ijkl} respectively.

To denote a position, a vector $\vec{r}$ or r_i is used, which points to the position from the origin. The Earth has a certain equilibrium state or initial state at $t = 0$, in which there is no relative motion between the particles constituting the Earth. Every material particle in the Earth is labeled uniquely by its position $\vec{x}^0$ in the equilibrium state. As soon as this equilibrium state is disturbed (for example by an earthquake), deformation takes place and the material particles initially at $\vec{x}^0$ move to a new position which is given by $\vec{r}\left(\vec{x}^0, t\right) = \vec{x}^0 + \vec{s}\left(\vec{x}^0, t\right)$, where $\vec{s}\left(\vec{x}^0, t\right)$ is the displacement of a material particle initially at $\vec{x}^0$.

Now a very important notational convention will be introduced: if a certain property ζ is evaluated *for a material particle* initially in $\vec{x}^0$ at a certain time t, it will be denoted by $\zeta\left(\vec{x}^0, t\right)$. Thus $\zeta\left(\vec{x}^0, t\right)$ is the state of the property ζ for particle $\vec{x}^0$ at time t. If the property is evaluated not for a material particle but for an arbitrary position $\vec{r}$ in space, it will be denoted by $\zeta\left(\vec{r}, t\right)$. For example, $\vec{s}\left(\vec{x}^0, t\right)$ is the total displacement experienced by a material particle at time t, with respect to its initial position $\vec{x}^0$. The displacement field (not necessarily of a particle) at a certain position $\vec{r}$ in the Earth model is given by $\vec{s}\left(\vec{r}, t\right)$.

A superscript zero will be used to denote that a certain model parameter (scalar, vector or tensor) is evaluated in equilibrium, as a function of initial position $\vec{x}^0$. For example, the density scalar field at point $\vec{x}^0$ in the Earth in equilibrium is denoted by $\rho^0\left(\vec{x}^0, 0\right)$ or simply ρ^0.

If the equilibrium state is disturbed, the model parameter fields are perturbed. The delta-notation will be used for a perturbation. For example, if the scalar field of the density is perturbed from the equilibrium state, we get that the density field in the perturbed state is the density field in the equilibrium state plus a perturbation $\delta\rho$: $\rho\left(\vec{r}, t\right) = \rho\left(\vec{r}, 0\right) + \delta\rho\left(\vec{r}, t\right)$. Note that this is a perturbation at an arbitrary point in space, not necessarily at a material particle. One can also have a perturbation at a material particle: $\rho\left(\vec{x}^0, t\right) = \rho^0 + \delta\rho\left(\vec{x}^0, t\right)$. Both perturbations will have a different value.

The gradient operator will be denoted by ∂_μ, or sometimes $\vec{\nabla}$. The gradient is always taken with respect to the corresponding spatial coordinates. Hence, $\partial_\mu \varphi^0$ is a gradient with respect to $\vec{x}_0$, and $\partial_\mu \varphi(\vec{r}, t)$ is a gradient with respect to $\vec{r}$. The Einstein summation convention will be used throughout, unless stated otherwise. For example, the divergence of a vector field $\vec{v}$ is given by $\partial_\mu v_\mu$. The trace of a second order tensor T_{ij} is given by T_{kk}. For the Laplacian operator $\partial_\mu \partial_\mu$, the shorthand ∇^2 will be used.

The difference D between the value of a function $\varphi\left(\vec{r}\right)$ just above and below a surface S will be denoted by $\left[\varphi\left(\vec{r}\right)\right]_-^+ = D$ on S. If a function is continuous on S, D will be zero. It will also occur that $\left[\varphi\left(\vec{r}\right)\right]_-^+$ on S is integrated over a surface S. This will be denoted by $\int_S \left[\varphi\left(\vec{r}\right)\right]_-^+ dS$.

Fourier transformation in time for a function $g\left(t\right)$ is defined as:

$$\mathcal{F}\left(g\right)\left(\omega\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} g\left(t\right) dt \tag{1}$$

where $\mathcal{F}\left(g\right)\left(\omega\right)$ is the Fourier transform of $g\left(t\right)$. The Fourier transformed function $\mathcal{F}\left(g\right)\left(\omega\right)$ will be denoted by $\tilde{g}$.

2.2 SNREI Earth model

The Earth model considered here is a mechanical representation of the Earth. It gives the shape, symmetries, mechanical properties, constitutive equations, initial conditions and boundary conditions of the Earth. The real Earth has a complex geometry and its mechanical properties, such as the density, have a complicated dependence on location. However, an Earth model closely resembling the real Earth can be obtained if a few assumptions are made.

For example, the shape of the Earth is approximately spherical, therefore the shape of the Earth is taken as a sphere in most models, which allows the usage of the spherical harmonics as a basis for the solutions. The radial variation of all properties of the Earth is much larger than the lateral variation, which is the main reason that most Earth models are spherically symmetric. In reality, non-hydrostatic initial stress exists, but because the hydrostatic stress arising due to the overburden of material at a point in the Earth predominates the initial stress, in most models the initial stress is simply taken to be hydrostatic.

We are now about to introduce an Earth model which satisfies the following conditions:

(i) The Earth model is spherically symmetric and non-rotating in its equilibrium state.

(ii) The material particles of the Earth form a continuum.

(iii) The material particles behave linearly and isotropically, which means that the relationship between applied stress on the material particle and deformation is linear and directionally independent: the material does not have a preferential direction of deformation.

(iv) The initial stress is a hydrostatic pressure p^0 due to the force of gravity. Atmospheric pressure is neglected, hence the outer surface of the Earth model is stress free.

(v) In its initial state, the Earth model is in equilibrium under self-gravitation alone. The gravitational influence of other astronomical bodies, such as the sun and moon, is hence neglected.

An Earth model satisfying these conditions is called a SNREI (spherical, non-rotating, elastic, isotropic) Earth model. Obviously, the real Earth does not satisfy these conditions, but since the deviation of the real Earth from these conditions is not very large, the standing waves of the real Earth do not differ greatly from the standing waves of the SNREI model.

Now let us interpret the model conditions mathematically.

(i) The first condition implies that all model parameters in the initial state are dependent on the radial coordinate r only. However, the spherical symmetry does not exclude radial discontinuities for model parameters. If the Earth is non-rotating, it is in an inertial frame of reference, hence there are no fictitious forces present, such as the Coriolis force.

(ii) The second condition allows the introduction of tensor fields to describe the mechanical behaviour of the Earth model, such as the deformation tensor, the stress tensor and strain tensor. I will only give the definitions here, a more thorough discussion can be found in [25].

The deformation tensor relates the current and initial positions of particles. Two particles initially located at $\vec{x}^0$ and $\vec{x}^0 + d\vec{x}^0$ move to $\vec{r}\left(\vec{x}^0, t\right)$ and $\vec{r}\left(\vec{x}^0, t\right) + d\vec{r}$ at time t. The relative current and initial position vectors $d\vec{r}$ and $d\vec{x}^0$ are related by:

$$dr_i = \partial_j r_i(\vec{x}^0, t) dx_j^0 \tag{2}$$

where $\partial_j r_i(\vec{x}^0, t) = F_{ij}(\vec{x}^0, t)$ or simply F_{ij} is the deformation tensor, or in mathematical terms the Jacobian matrix of the transformation from initial coordinates to current coordinates.

The second order stress tensor is a measure of the average force per unit area of a surface within a continuum. It has six independent components, because the tensor is symmetric. The strain tensor describes deformation: it gives the relative displacement of material particles in in any direction in a continuum. This tensor is symmetric as well. In the equilibrium configuration, there is an equilibrium stress distribution T_{ij}^0. The equilibrium situation is taken as a reference for the strain tensor, hence the strain tensor is zero in equilibrium. The strain tensor for small deformations is defined as:

$$e_{ij}(\vec{x}^0, t) = \frac{1}{2}\left(\partial_j s_i(\vec{x}^0, t) + \partial_i s_j(\vec{x}^0, t)\right) \tag{3}$$

As soon as the equilibrium is disturbed, the stress at a material particle changes from an initial stress T_{ij}^0 to a new stress $T_{ij}\left(\vec{x}^0, t\right)$. This change is called the incremental stress, and is given by the incremental stress tensor $\delta T_{ij}\left(\vec{x}^0, t\right)$. This is the total change in stress experienced by the material particle during deformation. Compared to the equilibrium state, there is relative motion between material particles, hence the strain tensor becomes non-zero as well.

(iii) The third condition implies that there is a linear relationship between the incremental stress tensor and strain tensor at a material particle, this means that there is a fourth order tensor, called the elasticity tensor c_{ijkl} relating incremental stress to strain: $\delta T_{ij}\left(\vec{x}^0, t\right) = c_{ijkl}\left(\vec{x}^0, t\right) e_{kl}\left(\vec{x}^0, t\right)$. Since the relationship between applied stress on the material particle and deformation is directionally independent, $c_{ijkl}\left(\vec{x}^0, t\right)$ should be the same tensor in each orthogonal basis you choose at the material particle. It turns out that the most general tensor satisfying this property is $c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk}$, where c_{ijkl}, λ, μ, and ν are scalar functions of $\vec{x}^0$ and t. [26] Because the strain tensor is symmetric, we can interchange indices k and l, giving the possibility to redefine the elasticity tensor as $c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + 2\mu\delta_{ik}\delta_{jl}$. Hence we can give the following relation between stress increment and strain:

$$\delta T_{ij}\left(\vec{x}^0, t\right) = \lambda\left(\vec{x}^0, t\right) e_{kk}\left(\vec{x}^0, t\right)\delta_{ij} + 2\mu\left(\vec{x}^0, t\right) e_{ij}\left(\vec{x}^0, t\right) \tag{4}$$

where λ and μ are called Lamé parameters and e_{kk} is the trace of the strain tensor, which is the relative volume change during deformation. This relation is the continuum analogue of Hooke's Law for a spring, which gives a linear dependence between extension and load of a spring. Note that the position $\vec{x}^0$ in this equation is the position of a material particle.

(iv) The fourth condition gives a diagonal initial stress tensor:

$$T_{ij}^0 = -p^0\delta_{ij} \tag{5}$$

where δ_{ij} is the Kronecker delta. The minus sign arises because compression (a pressure opposed to the radial unit vector) is defined to be negative.

(v) The fifth condition implies that the difference in force induced by pressure between the top and the bottom of a thin spherical shell centered around the origin must equal the gravity force on this shell, since the sum of all forces is zero in equilibrium:

$$-\partial_i p^0 = \rho^0\partial_i\varphi^0 \tag{6}$$

where ρ^0 is the density and φ^0 is the gravitational potential, which satisfies Poisson's equation:

$$\nabla^2\varphi^0 = 4\pi G\rho^0 \tag{7}$$

where $G = 6.673 \cdot 10^{-11}\ m^3\ kg^{-1}\ s^{-2}$ is the gravitational constant. Equations (6) and (7) are subject to the boundary conditions that φ^0, $\partial_r\varphi^0$ and p^0 are continuous, even if ρ^0 is discontinuous. Hence:

$$\left[\varphi^0\right]_-^+ = 0, \qquad \left[\partial_r\varphi^0\right]_-^+ = 0, \qquad \left[p^0\right]_-^+ = 0 \text{ for every surface.} \tag{8}$$

The continuity of $\partial_r\varphi^0$ follows from integrating equation (7) over the volume of a thin disc and applying Gauss's theorem on the left hand side of the equation.

Further, $p^0 = 0$ at the outer surface of the Earth and φ^0 and g^0 vanish as r approaches infinity. Under these conditions, we can find φ^0, g^0 and p^0 in terms of ρ^0 by integration. Hence φ^0, g^0 and p^0 are no independent model parameters. They can be calculated once the density distribution is known.

Hence the total number of independent model parameter functions is three: the density ρ^0 and the Lamé parameters λ and μ. Often, μ is called the shear modulus, since it governs the relation between shear stress and shear strain.

Since the Earth has internal boundaries or interfaces, such as the boundary between the core and the mantle, the value of the parameters λ, μ and ρ_0 can be discontinuous across these boundaries. We therefore divide the model into a set of regions $\{V_1, V_2, ..., V_n\}$ together with V_{n+1}, which is the space not occupied by the model. The boundaries Σ_i between the regions are spherical surfaces. There are three kinds of boundaries:

(i) The outer surface of the Earth, which we will call the free surface Σ_n.

(ii) Welded boundaries Σ_{ss}, these are boundaries between two solid regions.

(iii) Fluid-solid boundaries Σ_{fs}, these are boundaries between a fluid and a solid region.

The union of all surfaces will be called Σ, the whole of space not occupied by Σ will be called the volume V. In fluid regions, the shear modulus μ vanishes, because a fluid does not support shear stress.

There is one major problem with the model introduced so far. It will remain in equilibrium forever if no force acts upon it. We will now see what happens if the equilibrium situation is disturbed.

2.3 Free oscillations

The equations of motion of a system describe the motion of a system as a function of time. If the equilibrium situation of the Earth model is disturbed by a displacement field $\vec{s}$, particles change their position accordingly. The goal of this section is to derive linear differential equations for non-driven oscillations of the Earth, for displacements which are small compared to the size of the Earth, which is the case for the displacement caused by earthquakes. Non-driven oscillations of an object are called *free oscillations.*

Because the oscillations are relatively small, all terms containing second order perturbations can be neglected. Hence terms containing products of perturbations, a product of a perturbation and the displacement field $\vec{s}$, or terms that are of second order in $\vec{s}$ can be ignored.

2.3.1 Description of motion

When analyzing the deformation of solids such as the Earth, it is necessary to describe the evolution of the position and the physical properties (such as the velocity or the density) of the particles. There are two ways of doing this: one could label all particles uniquely with their equilibrium position $\vec{x}^0$ and follow their position in time, while determining the properties at the material location. This is typically the approach in rigid body mechanics and in seismology, since a seismometer is attached to the particles in motion and measures the properties (such as the displacement) at a material particle. This is the so called *Lagrangian* description of motion. This corresponds to the notation of any arbitrary parameter ζ as a function of initial position and time: $\zeta(\vec{x}^0, t)$. Lagrangian perturbations $\delta\zeta(\vec{x}^0, t)$ are changes at the material particle. We need this description because many properties of the system are inherently Lagrangian. For example the relationship between stress and strain is most naturally given in the Lagrangian description: the strain caused to a material particle is related to the incremental stress *at the same material particle.* In fact, we already have used the notation for this description for the incremental stress at a material particle, which was denoted by $\delta T_{ij}\left(\vec{x}^0, t\right)$.

In normal mode theory, the Lagrangian description leads to a problem: gravity depends on the density distribution, and it is not straightforward to give a Lagrangian description of Poisson's equation, since in the deformed state the density in this equation is not a function of the position of a material particle. There is another approach which leads to a straightforward treatment of the density, which is the so called *Eulerian* approach. Then the particles are not followed individually, but the properties of the particles will be given as a field which is a function of position $\vec{r}$, not necessarily the position of a material particle. The density then simply becomes a function of space and time, and we do not have to bother with the density at a material particle anymore. This corresponds to the notation of any arbitrary parameter ζ as a function of fixed position and time: $\zeta(\vec{r}, t)$. Hence Eulerian perturbations $\delta\zeta(\vec{r}, t)$ are changes at a certain position, not necessarily the particle position. Thus, Eulerian and Lagrangian perturbations are not equal in general.

2.3.2 Area and volume

To obtain equations of motion it turns out that expressions are needed relating volume and surface elements in the deformed and initial state. A volume element in the initial state dV^0 will be rotated and deformed into a new volume element dV because particles move according to a mapping from $\vec{x}^0$ to $\vec{x}^0 + s(\vec{x}^0, t)$. The ratio between the deformed and undeformed volume element is given by the

Jacobian determinant J of this mapping. Hence $dV = JdV^0$.

Now we consider an infinitesimal surface area element in the initial configuration: $\hat{n}^0 d\Sigma^0$; and in the deformed state: $\hat{n}d\Sigma = \hat{n}(\vec{x}^0,t)d\Sigma(\vec{x}^0,t)$. The relationship between these surface elements can be obtained by considering the surface of a deformed and undeformed parallellogram, since any arbitrary surface can be built up from infinitesimally small parallellograms. Take an undeformed parallellogram with sides $d\vec{x}^0$ and $\delta\vec{x}^0$ which deforms into a parallellogram with sides $d\vec{r} = d\vec{r}(d\vec{x}^0,t)$ and $\delta\vec{r} = \delta\vec{r}(\delta\vec{x}^0,t)$. Then we have:

$$\hat{n}^0 d\Sigma^0 = d\vec{x}^0 \times \delta\vec{x}^0 \tag{9}$$
$$\hat{n} d\Sigma = d\vec{r} \times \delta\vec{r} \tag{10}$$

Writing the cross products with the Levi-Civita symbol we obtain:

$$\hat{n}_i^0 d\Sigma^0 = \epsilon_{ijk} dx_j^0 \delta x_k^0 \tag{11}$$
$$\hat{n}_i d\Sigma = \epsilon_{ijk} dr_j \delta r_k \tag{12}$$

where dx_k^0, δx_k^0, dr_k and δr_k are the Cartesian components of the vectors. By using the inverse deformation tensor the initial parallellogram sides $d\vec{x}^0$ and $\delta\vec{x}^0$ can be related to the current parallellogram sides $d\vec{r}$ and $\delta\vec{r}$.

$$dx_j^0 = F_{jm}^{-1} dr_m \tag{13}$$
$$\delta x_k^0 = F_{kn}^{-1} \delta r_n \tag{14}$$

Upon substituting these expressions into the expression of the undeformed surface element, we obtain:

$$\hat{n}_i^0 d\Sigma^0 = \epsilon_{ijk} F_{jm}^{-1} F_{kn}^{-1} dr_m \delta r_n \tag{15}$$

Both sides of this equation are multiplied with F_{il}^{-1} and the identity $J = \det F = \frac{1}{6}\epsilon_{ijk}\epsilon_{lmn}F_{il}F_{jm}F_{kn}$ is used, together with the identity $\epsilon_{ijk}\epsilon_{ijk} = 6$. Then one obtains:

$$F_{il}^{-1}\hat{n}_i^0 d\Sigma^0 = J^{-1}\epsilon_{lmn} dr_m \delta r_n = J^{-1}\hat{n}_l d\Sigma \tag{16}$$

and hence the relationship between deformed and undeformed surface area elements is:

$$\hat{n}_j d\Sigma = J d\Sigma^0 \hat{n}_i^0 F_{ij}^{-1} \tag{17}$$

2.3.3 Conservation of mass

Suppose we take an arbitrary volume in the initial configuration of the model and follow this volume in time. Then the total mass within this volume is conserved, since there is no flux of material in or out this volume. Hence:

$$\int_V \rho(\vec{x}^0,t)dV = \int_{V^0} \rho^0 dV^0 \tag{18}$$

But the integral on the left can also be transformed into an integral in the initial coordinate system:

$$\int_V \rho(\vec{x}^0,t)dV = \int_{V^0} \rho(\vec{x}^0,t) J dV^0 \tag{19}$$

Hence $\rho^0 = J\rho(\vec{x}^0,t)$ is the mathematical formulation of the conservation of mass in the Lagrangian description of motion.

2.3.4 Equations of motion

The main goal of this section is to obtain linear differential equations determining the time evolution of the free oscillations of the SNREI model. Linearization is possible, because the displacements are small compared to the size of the Earth. A linear differential equation for the displacement $\vec{s}$ of the form

$$\mathcal{L}_i(\vec{s}) = \rho^0\left(\vec{x}^0,t\right)\frac{\partial^2 s_i\left(\vec{x}^0,t\right)}{\partial t^2} \tag{20}$$

will be derived, where $\mathcal{L}_i(\vec{s})$ is a linear operator acting on $\vec{s}\left(\vec{x}^0, t\right)$, depending only on known model parameters. The right hand side of the equation is the net force density on a material particle.

To get the equation of motion, we take an arbitrary volume V with boundary Σ in a continuum at time t and apply Newton's second law on it. The result we will obtain is known as Cauchy's equation of motion for a continuum.

Two types of forces act on the volume: forces acting on all particles in the volume V and forces acting on the boundary Σ of the volume. These forces are called body forces and surface forces respectively. In the case of the Earth, gravity acts as a body force. The gravitational acceleration g_i can be given as the derivative of a potential field: $g_i = -\partial_i \varphi\left(\vec{x}^0, t\right)$. All surface forces on a surface with normal $\hat{n}$ are given by the traction vector $\vec{\tau}$, which is related to the normal $\hat{n}$ through the stress tensor: $\tau_i = T_{ij} n_j$.

Hence in this case Newton's second law gives:

$$\frac{\partial}{\partial t}\int_V \rho(\vec{x}^0,t)v_i(\vec{x}^0,t)dV = \int_\Sigma T_{ij}(\vec{x}^0,t)\hat{n}_j(\vec{x}^0,t)d\Sigma - \int_V \rho(\vec{x}^0,t)\partial_i\varphi(\vec{x}^0,t)dV \tag{21}$$

where $\vec{v}$ is the particle velocity.

To proceed and obtain the equations of motions, the idea is to transform the volume and surface integrals in the expression above to the initial coordinate system $\vec{x}^0$ and to approximate the terms for small displacements. By using the expressions derived in the section 2.3.2 on area and volume, one obtains:

$$\frac{\partial}{\partial t}\int_{V^0} \rho(\vec{x}^0,t)Jv_k(\vec{x}^0,t)dV^0 = \int_{\Sigma^0} T_{ij}(\vec{x}^0,t)J\hat{n}^0_k(\vec{x}^0,t)F^{-1}_{kj}d\Sigma^0 - \int_{V^0} \rho(\vec{x}^0,t)J\partial_i\varphi(\vec{x}^0,t)dV^0$$

Now conservation of mass can be used:

$$\frac{\partial}{\partial t}\int_{V^0} \rho^0 v_k(\vec{x}^0,t)dV^0 = \int_{\Sigma^0} T_{ij}(\vec{x}^0,t)J\hat{n}^0_k(\vec{x}^0,t)F^{-1}_{kj}d\Sigma^0 - \int_{V^0} \rho^0\partial_i\varphi(\vec{x}^0,t)dV^0 \tag{22}$$

Now the definition of the first Piola-Kirchhoff stress tensor can be recognized. The definition of this tensor is given on page 35 of [27]. The first Piola-Kirchhoff stress tensor gives the force in the deformed state per unit area in the equilibrium state. The Piola-Kirchhoff stress will be denoted by $\overline{T}_{ij}$ and it's definition is $\overline{T}_{ij} = JT_{ik}F^{-1}_{kj}$. Hence, using Gauss's theorem for the first term on the right hand side of equation (22), this equation can be rewritten as:

$$\frac{\partial}{\partial t}\int_{V^0} \rho^0 v_i(\vec{x}^0,t)dV^0 = \int_{V^0} \partial_j\overline{T}_{ij}(\vec{x}^0,t)dV^0 - \int_{V^0} \rho^0\partial_i\varphi(\vec{x}^0,t)dV^0 \tag{23}$$

The differential operator on the left can be flipped with the integral operator, since there is no time dependence in the integration bounds, because they are given in the initial coordinate system. Because this equation holds for any arbitrary initial volume V^0, we might as well drop the integral signs and we obtain:

$$\rho^0\frac{\partial^2 s_i(\vec{x}^0,t)}{\partial t^2} = \partial_j\overline{T}_{ij}(\vec{x}^0,t) - \rho^0\partial_i\varphi(\vec{x}^0,t) \tag{24}$$

Now this equation is approximated for small displacements by discarding all second order perturbation terms. Hence the first Piola-Kirchhoff stress tensor and gravitational potential have to be approximated to first order. Starting with the Piola-Kirchhoff stress tensor, notice that because $\vec{r}\left(\vec{x}^0,t\right) = \vec{x}^0 + \vec{s}\left(\vec{x}^0,t\right)$ the deformation tensor has the form $F_{ij} = \delta_{ij} + \partial_j s_i(\vec{x}^0,t)$ and its inverse can be approximated to first order in $\vec{s}$ with $F^{-1}_{ij} = \delta_{ij} - \partial_j s_i(\vec{x}^0,t)$. The relative volume change J is, to first order in $\vec{s}$, equal to $1 + \partial_i s_i(\vec{x}^0,t)$. Hence the first Piola-Kirchhoff stress tensor approximated to first order becomes:

$$\overline{T}_{ij}(\vec{x}^0,t) = T_{ij}(\vec{x}^0,t)\left(1 + \partial_k s_k(\vec{x}^0,t)\right) - T_{ik}\partial_j s_k(\vec{x}^0,t) \tag{25}$$

The gradient of the gravitational potential at a material particle in the deformed state can be approximated by expressing it in terms of the gradient of the potential in the initial state $\partial_i\varphi^0$ plus the total change experienced by the material particle during deformation $\delta\partial_i\varphi\left(\vec{x}^0,t\right)$. The latter term consists of two parts: a change due to movement through the initial potential gradient field and a local perturbation due to the change in density distribution caused by the deformation. Hence:

$$\begin{aligned}\partial_i\varphi\left(\vec{x}^0,t\right) &= \partial_i\varphi^0+\delta\partial_i\varphi\left(\vec{x}^0,t\right) \quad (26)\\ &= \partial_i\varphi^0+s_j(\vec{x}^0,t)\partial_j\partial_i\varphi^0+\delta\partial_i\varphi(\vec{r},t) \quad (27)\end{aligned}$$

where the first term gives the initial field at $t=0$, the second term gives the change due to the movement through the initial field by using a linear Taylor approximation. The last term gives the change in the field at time t at the current position of the particle $\vec{r}=\vec{r}(\vec{x}^0,t)$, relative to the value of the field in the initial situation *at the same spatial coordinate*. Hence $\delta\partial_i\varphi(\vec{r},t)$ is an Eulerian perturbation (see the beginning of section 2.3.1 on page 10). Hence we obtain the following approximation for the potential gradient field:

$$\partial_i\varphi\left(\vec{x}^0,t\right)=\partial_i\varphi^0+s_j(\vec{x}^0,t)\partial_j\partial_i\varphi^0+\partial_i\delta\varphi(\vec{r},t) \quad (28)$$

Finally, the Cauchy stress tensor can be rewritten as the sum of the initial stress and incremental stress at the material particle:

$$T_{ij}\left(\vec{x}^0,t\right)=-p^0\delta_{ij}+\delta T_{ij}\left(\vec{x}^0,t\right) \quad (29)$$

Hence the first Piola-Kirchhoff stress tensor can be rewritten as

$$\overline{T}_{ij}=-p^0\delta_{ij}+\delta\overline{T}_{ij}\left(\vec{x}^0,t\right) \quad (30)$$

where

$$\delta\overline{T}_{ij}\left(\vec{x}^0,t\right)=p^0\partial_i s_j\left(\vec{x}^0,t\right)-p^0\partial_k s_k(\vec{x}^0,t)\delta_{ij}+\delta T_{ij}\left(\vec{x}^0,t\right) \quad (31)$$

is the incremental first Piola-Kirchhoff stress tensor. If we apply all these approximations in the general equation of motion, we obtain:

$$\rho^0\frac{\partial^2 s_i}{\partial t^2}=-\partial_j p^0\delta_{ij}+\partial_j\delta\overline{T}_{ij}-\rho^0\left(\partial_i\varphi^0+s_j\partial_j\partial_i\varphi^0+\partial_i\delta\varphi(\vec{r},t)\right) \quad (32)$$

Note that the dependence $(\vec{x}^0,t)$ is omitted from now on to make the equations more readable. All quantities without a superscript zero or other explicitly given dependence, are dependent of $(\vec{x}^0,t)$. Because of the equilibrium equation (6), the first and third term of the right hand side of the equation vanish, and one obtains:

$$\rho^0\frac{\partial^2 s_i}{\partial t^2}=\partial_j\delta\overline{T}_{ij}-\rho^0\left(s_j\partial_j\partial_i\varphi^0+\partial_i\delta\varphi(\vec{r},t)\right) \quad (33)$$

This differential equation contains the divergence of the incremental first Piola-Kirchhoff stress. In order to find its relationship to the displacement, the divergence term is calculated explicitly:

$$\begin{aligned}\partial_j\delta\overline{T}_{ij} &= \partial_j\delta T_{ij}+\partial_j p^0\partial_i s_j+p^0\partial_i\left(\partial_j s_j\right)-\partial_i p^0\left(\partial_k s_k\right)-p^0\partial_i\left(\partial_k s_k\right)\\ &= \partial_j\delta T_{ij}-\rho^0\partial_j\varphi^0\partial_i s_j+\rho^0\partial_i\varphi^0\left(\partial_k s_k\right)\\ &= \partial_j\delta T_{ij}-\rho^0\partial_i\left(s_j\partial_j\varphi^0\right)+\rho^0 s_j\partial_i\partial_j\varphi^0+\rho^0\partial_i\varphi^0\left(\partial_k s_k\right) \quad (34)\end{aligned}$$

where we have used the equilibrium equation (6). If we plug this into the differential equation (33), we get:

$$\rho^0\frac{\partial^2 s_i}{\partial t^2}=\partial_j\delta T_{ij}-\rho^0\partial_i\left(s_j\partial_j\varphi^0\right)+\rho^0\partial_i\varphi^0\left(\partial_k s_k\right)-\rho^0\partial_i\delta\varphi(\vec{r},t) \quad (35)$$

This equation contains the Lagrangian incremental stress tensor, which is related to $\vec{s}$ through the strain tensor. To have an explicit dependence on $\vec{s}$, it will be convenient to write:

$$\delta T_{ij}=\gamma_{ijkl}\partial_k s_l \quad (36)$$

where $\gamma_{ijkl}=\mu\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right)+\lambda\delta_{ij}\delta_{kl}$.

Hence we obtain:

$$\rho^0\frac{\partial^2 s_i}{\partial t^2}=\partial_j\gamma_{ijkl}\partial_k s_l-\rho^0\partial_i\left(s_j\partial_j\varphi^0\right)+\rho^0\partial_i\varphi^0\left(\partial_k s_k\right)-\rho^0\partial_i\delta\varphi(\vec{r},t) \quad (37)$$

Now three differential equations have been obtained for four unknown functions: the three components of the displacement and the gravitational potential perturbation. Hence one more equation is needed. The density perturbation $\delta\rho\left(\vec{r},t\right)$ can be expressed in terms of $\vec{s}$. Consider a volume V, fixed in space. The increase in mass within V is equal to the influx of mass through the surface ∂V enclosing V:

$$\int_V \rho\left(\vec{r},t\right) dV = \int_V \rho^0(\vec{r},0) dV - \int_{\partial V} \left(\rho^0(\vec{r},0) + \delta\rho\left(\vec{r},t\right)\right) s_k\left(\vec{r},t\right) n_k\left(\vec{r},t\right) dS \tag{38}$$

where n_k is a unit normal vector on the surface element dS. If we consider this equation in first order, the product between $\delta\rho$ and s_k can be neglected, and by using the divergence theorem (and dropping the volume integral since the equation is valid for an arbitrary volume) we find:

$$\delta\rho\left(\vec{r},t\right) = -\partial_j\left(\rho^0\left(\vec{r},0\right) s_j\left(\vec{r},t\right)\right) \tag{39}$$

The potential field of gravity φ^0 is governed by Poisson's equation. Let us see how this equation is affected by the disturbance $\vec{s}$. The incremental Poisson's equation is satisfied by $\delta\varphi$:

$$\nabla^2\delta\varphi(\vec{r},t) = 4\pi G\delta\rho(\vec{r},t) = -4\pi G\partial_j\left(\rho^0\left(\vec{r},0\right) s_j(\vec{r},t)\right) \tag{40}$$

In this equation, $s_j(\vec{r},t)$ can be replaced by $s_j(\vec{x}^0,t) = s_j$, since their difference is of second order in $\vec{s}$. The density $\rho^0\left(\vec{r},0\right)$ can also be replaced by $\rho^0\left(\vec{x}^0,0\right) = \rho^0$ since this replacement only has a second order effect on the equation. Hence:

$$\nabla^2\delta\varphi(\vec{r},t) = 4\pi G\delta\rho(\vec{r},t) = -4\pi G\partial_j\left(\rho^0 s_j\right) \tag{41}$$

Now we are in the situation to be able to write down a system of differential equations for $\vec{s}$ and $\delta\varphi(\vec{r},t)$, in terms of the known initial model parameters ρ^0, λ and μ:

$$\left.\begin{array}{rcl} \rho^0\partial_t^2 s_i &=& \rho_0\left[\left(\partial_k s_k\right)\partial_i\varphi^0 - \partial_i\delta\varphi(\vec{r},t) - \partial_i\left(s_k\partial_k\varphi^0\right)\right] + \partial_j\left(\gamma_{ijkl}\partial_k s_l\right) \\ \nabla^2\delta\varphi(\vec{r},t) &=& -4\pi G\partial_j\left(\rho^0 s_j\right) \end{array}\right\} \tag{42}$$

For theoretical purposes, it would be nice if we could write the system of equations (42) as a single linear differential equation for $\vec{s}$. This is possible if we see that $\delta\phi$ can be solved uniquely through the second equation in (42), with appropriate boundary conditions (49) which will be derived in the next section. Further, the differential equation for $\delta\phi$ is linear in $\vec{s}$, hence we can write:

$$\delta\phi = \delta\Phi\left(\vec{s}\right) \tag{43}$$

where $\delta\Phi$ is a linear operator on $\vec{s}$.

Now we can define a linear operator $\mathcal{L}_i$ acting on a displacement field $\vec{s}$ which helps us to summarize the system of differential equations (42):

$$\mathcal{L}_i\left(\vec{s}\right) = \rho^0\left[\left(\partial_k s_k\right)\partial_i\phi^0 - \partial_i\delta\Phi\left(\vec{s}\right) - \partial_i\left(s_k\partial_k\phi^0\right)\right] + \partial_j\left(\gamma_{ijkl}\partial_k s_l\right) \tag{44}$$

This operator depends on known model parameters only. The system of differential equations can now be reduced to the desired form given in the beginning of this section, namely as a single linear vector differential equation:

$$\mathcal{L}_i\left(\vec{s}\right) = \rho^0\partial_t^2 s_i \tag{45}$$

2.3.5 Boundary conditions

As we discussed, the Earth model is composed of layers with possible discontinuities in density and elastic parameters on the boundaries. To solve the differential equations derived in the previous section, we need conditions for $\vec{s}$ and $\delta\varphi$ on the boundaries.

Firstly, the potential perturbation $\delta\varphi$ is continuous. But if there is a density jump on a boundary, the slope of $\delta\varphi$ changes at the boundary, thus $\partial_i\delta\varphi$ is not necessarily continuous on a boundary. We therefore seek a quantity related to $\partial_i\delta\varphi$, which is continuous. The incremental Poisson equation, which is the second of our system of equations (42), can be rewritten as:

$$\partial_i\left(\partial_i\delta\varphi - 4\pi G\rho^0 s_i\right) = 0 \tag{46}$$

since the Laplacian operator ∇^2 is the same as $\partial_i\partial_i$. Remember that we derived the continuity of g^0 by integrating the Poisson equation over a thin spherical shell in section 2.2 on page 9. We will now use a similar argument. If we integrate equation (46) over the volume of a thin disc containing a piece of the interface between two regions in the model and apply Gauss's theorem, we find that $\left(\partial_i\delta\varphi - 4\pi G\rho^0 s_i\right)\hat{n}_i$ must be the same on either side of the deformed interface, where $\hat{n}_i = \hat{n}_i(\vec{x}^0, t)$ is the unit normal vector on the deformed interface. This is our second constraint on $\delta\varphi$.

In this constraint, the unit normal to the deformed interface is $\hat{n}_i$, which is slightly changed with respect to the unit normal $\hat{n}_i^0$ in the equilibrium state. The influence of the deflection of the unit normal vector on the constraint is an effect of second order in $\vec{s}$, since the constraint in the undeformed state only consists of terms of first order. Therefore, we can replace the unit normal in the deformed state with the unit normal in the undeformed state. Hence from now on, the unit normal in the deformed state is equal to the unit normal in the undeformed state.

At this point it is convenient to use a spherical polar coordinate system (r, θ, ϕ) where r is the radius, θ is the colatitude and ϕ is the longitude. Hence any vector is denoted by v_i or $\vec{v}$ and any tensor is denoted by T_{ij} in the following way:

$$\vec{v} = \hat{r}\, v_r + \hat{\theta}\, v_\theta + \hat{\phi}\, v_\phi = v_i = (v_r, v_\theta, v_\phi) \tag{47}$$

$$T_{ij} = \begin{pmatrix} T_{rr} & T_{r\theta} & T_{r\phi} \\ T_{\theta r} & T_{\theta\theta} & T_{\theta\phi} \\ T_{\phi r} & T_{\phi\theta} & T_{\phi\phi} \end{pmatrix} \tag{48}$$

Hence the indices i, j can be r, θ or ϕ.
In spherical polar coordinates, the unit normal on the interface is simply $\hat{r}$. Summarizing:

$$[\delta\varphi]_-^+ = 0, \qquad \left[\partial_r\delta\varphi - 4\pi G\rho^0 s_r\right]_-^+ = 0 \text{ on } \Sigma \tag{49}$$

Because ϕ and ϕ^0 vanish at infinity, $\delta\varphi$ must vanish at infinity as well.

The boundary conditions on $\vec{s}$ are different for different kinds of interfaces. On the free surface, the force per unit area, also called the traction, is zero. Hence:

$$\delta T_{ij} n_j = \gamma_{ijkl}\partial_k s_l n_j = 0 \text{ on } \Sigma_n \tag{50}$$

At fluid-solid boundaries, the traction is continuous. Continuity of traction is a fundamental result in continuum mechanics. [25] The traction is also normal to the boundary since on the fluid side, no shear stress is allowed.

Because of the continuity of traction, shear stress is zero on the solid side of the boundary as well.

The displacement normal to the boundary is also continuous, since the neighbouring regions do not separate or interpenetrate during deformation. If we again use that the deflection of the unit normal due to the displacement has no first order influence on the conditions, we get:

$$[\delta T_{ir}]_-^+ = 0, \quad [s_r]_-^+ = 0 \text{ on } \Sigma_{fs} \tag{51}$$

Note that there can be a discontinuity of displacement perpendicular to the boundary, since the fluid can flow freely across the boundary and the particles are not attached to each other. This is not the case if we consider a solid-solid boundary.

At solid-solid boundaries, the traction and displacement are continuous, hence:

$$[\delta T_{ir}]_-^+ = 0, \quad [s_i]_-^+ = 0 \text{ on } \Sigma_{ss} \tag{52}$$

Now we have a set of differential equations with the needed boundary conditions to determine the displacement $\vec{s}$ due to a given force field $\vec{f}$.

2.3.6 Statement of the mathematical problem for free oscillations

We define a set of vector fields $\mathcal{S}$ which consists of all vector fields $\vec{s}$ satisfying the boundary conditions discussed in the previous section:

$$[s_r]_-^+ = 0 \text{ on } \Sigma_{fs} \quad (53)$$

$$[s_i]_-^+ = 0 \text{ on } \Sigma_{ss} \quad (54)$$

Now we can state the problem of determining the displacement field of Earth normal modes as follows:

Find the displacement field $\vec{s}\left(\vec{x}^0, t\right) \in \mathcal{S}$, for each fixed t, which satisfies:

$$\mathcal{L}_i\left(\vec{s}\right) = \rho^0 \partial_t^2 s_i \quad (55)$$

$$[\gamma_{ijkl} \partial_k s_l]_-^+ = 0 \text{ on all boundaries} \quad (56)$$

Note that the free surface condition is included in these equations by defining $\gamma_{ijkl} = 0$ outside the Earth model. The fluid-solid boundary condition of no shear stress is included by setting $\mu = 0$ in the fluid.

2.4 Solution of the equations of motion for free oscillations

The mathematical problem stated in the previous section is a differential equation. Fourier analysis can be used to transform this differential equation into algebraic equations. By Fourier transforming (55) the following algebraic equations are obtained:

$$\mathcal{L}_i\left(\tilde{\vec{s}}\right) + \rho^0 \omega^2 \tilde{s}_i = 0 \quad (57)$$

where the tilde denotes Fourier transformation. The standard way of solving these kind of problems is to find the eigenvalues and eigenfunctions of the operator $\mathcal{L}_i$.

$$\vec{\mathcal{L}}\left(\vec{s}_k\right) + \rho^0 \omega_k^2 \vec{s}_k = 0 \quad (58)$$

where $\vec{s}_k = \vec{s}_k\left(\vec{x}^0, \omega_k\right)$ is an eigenfunction of $\vec{\mathcal{L}}$, ω_k^2 is the corresponding eigenvalue and k is an index labeling the eigenfunctions.

A solution of the mathematical problem for the displacement (55) is given by $\vec{s} = e^{i\omega_k t}\vec{s}_k$.

To solve the eigenvalue equation, first some properties of the operator $\mathcal{L}_i$ will be derived, which will lead to the conclusion that $\mathcal{L}_i$ is a self-adjoint operator on the space of all possible displacement fields obeying the boundary conditions. Then the mathematical problem for the displacement (55) will be written as a set of scalar equations in spherical coordinates, expanded into spherical harmonics. This will yield a set of scalar differential equations dependent on the radius r only. Finally, methods of solution of these radial equations will be discussed.

2.4.1 Properties of the operator $\mathcal{L}_i$

The main goal of this chapter is to prove certain useful facts about the operator $\mathcal{L}_i$, which will lead to a proof in the following chapter that the operator $\mathcal{L}$ is self adjoint, if $\mathcal{S}$ is endowed with a suitable inner product. The derivation of the properties $\mathcal{L}_i$ satisfies is slightly simplified if the operator $\mathcal{L}_i$ is written in terms of the first Piola-Kirchhoff stress tensor $\delta\overline{T}_{ij}$. The simplification of the derivation is because the operator $\mathcal{L}_i$ takes a more concise form if we use this definition. Note that $\delta\overline{T}_{ij}$ is not a symmetric tensor.

The first Piola-Kirchhoff stress tensor can be written as:

$$\delta\overline{T}_{ij} = d_{ijkl} \partial_l s_k \quad (59)$$

where $d_{ijkl} = \gamma_{ijkl} + p^0\left(\delta_{il}\delta_{kj} - \delta_{ij}\delta_{kl}\right)$. Note that this gives the symmetry $d_{ijkl} = d_{klij}$, a fact we need later on.

Therefore the operator $\mathcal{L}_i$ can be written as:

$$\mathcal{L}_i(\vec{s}) = \partial_j (d_{ijkl}\partial_l s_k) - \rho^0 \left(\partial_i \delta\Phi(\vec{s}) + s_j \partial_j \partial_i \varphi^0\right) \tag{60}$$

Now three results will be proven, which will lead to the proof that $\mathcal{L}_i$ is self-adjoint.

2.4.1.1 Result 1 For any two displacement fields $\vec{s}$ and $\vec{s}'$ with associated gravitational potentials $\delta\varphi = \delta\Phi(\vec{s})$ and $\delta\varphi' = \delta\Phi(\vec{s}')$

$$\begin{aligned} s_i'\mathcal{L}_i(\vec{s}) &= \partial_j \left[s_i' \delta\overline{T}_{ij} + \delta\varphi' \left(\frac{1}{4\pi G}\partial_j \delta\varphi + \rho^0 s_j \right) \right] \\ &\quad - \left[\delta\overline{T}_{ij}\partial_j s_i' + \rho^0 s_i' \partial_i \delta\varphi + \rho^0 s_i \partial_i \delta\varphi' + \rho^0 s_i s_j' \partial_i \partial_j \varphi^0 + \frac{1}{4\pi G}\partial_i \delta\varphi \partial_i \delta\varphi' \right] \end{aligned} \tag{61}$$

Proof
From (60):

$$\begin{aligned} s_i' L_i(\vec{s}) &= s_i' \partial_j \delta\overline{T}_{ij} - \rho^0 s_i' \partial_i \delta\varphi - \rho^0 s_i' s_j \partial_i \partial_j \delta\varphi^0 \\ &\quad -\rho^0 s_i \partial_i \delta\varphi' + \rho^0 s_i \partial_i \delta\varphi' \end{aligned} \tag{62}$$

where the same term is first subtracted and then added on the last line. The first term on the right hand side of (62) is:

$$s_i'\partial_j \delta\overline{T}_{ij} = \partial_j \left(s_i' \delta\overline{T}_{ij}\right) - \delta\overline{T}_{ij}\partial_j s_i' \tag{63}$$

and the last term is:

$$\begin{aligned} \rho^0 s_i \partial_i \delta\varphi' &= \partial_i \left(\rho^0 s_i \delta\varphi'\right) - \delta\varphi' \partial_i \left(\rho^0 s_i\right) \\ &= \partial_i \left(\rho^0 s_i \delta\varphi'\right) + \delta\varphi' \delta\rho \\ &= \partial_i \left(\rho^0 s_i \delta\varphi'\right) + \delta\varphi' \frac{1}{4\pi G}\nabla^2 \delta\varphi \\ &= \partial_i \left(\rho^0 s_i \delta\varphi' + \frac{1}{4\pi G}\delta\varphi' \partial_i \delta\varphi\right) - \frac{1}{4\pi G}\partial_i \delta\varphi' \partial_i \delta\varphi \end{aligned} \tag{64}$$

Using (63) and (64) in (62) gives (61) and this completes the proof of Result 1.

2.4.1.2 Result 2 For any two displacement fields $\vec{s}$ and $\vec{s}'$ with associated gravitational potentials $\delta\varphi = \delta\Phi(\vec{s})$ and $\delta\varphi' = \delta\Phi(\vec{s}')$

$$s_i'\mathcal{L}_i(\vec{s}) - s_i \mathcal{L}_i(\vec{s}') = \partial_j \left[s_i' \delta\overline{T}_{ij} - s_i \delta\overline{T}'_{ij} + \delta\varphi' \left(\frac{1}{4\pi G}\partial_j \delta\varphi + \rho^0 s_j\right) - \delta\varphi \left(\frac{1}{4\pi G}\partial_j \delta\varphi' + \rho^0 s_j'\right) \right] \tag{65}$$

where $\delta\overline{T}_{ij} = d_{ijkl}\partial_l s_k$ and $\delta\overline{T}'_{ij} = d_{ijkl}\partial_l s_k'$.

Proof
The term $\delta\overline{T}_{ij}\partial_j s_i'$ occurring in the second bracket of Result 1 is given by $\delta\overline{T}_{ij}\partial_j s_i' = d_{ijkl}\partial_l s_k \partial_j s_i'$. Because of the symmetry property of d_{ijkl} we have:

$$d_{ijkl}\partial_l s_k \partial_j s_i' = d_{klij}\partial_l s_k \partial_j s_i' = d_{ijkl}\partial_l s_k' \partial_j s_i \tag{66}$$

hence this term is unaffected if $\vec{s}$ and $\vec{s}'$ are swapped. The rest of the terms in the second bracket of Result 1 are also unchanged by this operation, which gives us Result 2.

2.4.1.3 Result 3

For any two vector fields $\vec{s}$ and $\vec{s}'$

$$\int_V s_i' \cdot \mathcal{L}_i(\vec{s})\, dV + \int_\Sigma [s_i' \delta T_{ir}]_-^+ \, d\Sigma = \int_V s_i \cdot \mathcal{L}_i(\vec{s}')\, dV + \int_\Sigma [s_i \delta T_{ir}']_-^+ \, d\Sigma \tag{67}$$

where V and Σ are the same volume and surface we defined in section 2.2.

Proof

To obtain this result, we have to integrate Result 2 over all of V and apply Gauss's theorem to obtain surface integrals over Σ. To make the proof easy to follow, first two lemmas will be proved. The first lemma is a more general form of Gauss's theorem. Since we have a discontinuous displacement field, we need a form of Gauss's theorem which is also valid for discontinuous fields.

Lemma 1, alternate form of Gauss's theorem

For any vector field $\vec{u}(\vec{r})$ we have

$$\int_V \partial_j u_j dV = -\int_\Sigma [u_r]_-^+ \, d\Sigma \tag{68}$$

where n_j is the unit normal vector on Σ and the discontinuity $[u_r]_-^+$ is $u_r^+ - u_r^-$, where the positive contribution is from the displacement furthest from the origin. Recall that the union of all surfaces of discontinuity is called Σ.

Proof. For any vector field $\vec{u}(\vec{r})$ we have

$$\int_V \partial_j u_j dV = \sum_{k=1}^{n+1} \int_{V_k} \partial_j u_j dV \tag{69}$$

and if we apply Gauss's theorem on each spherical shell V_k we obtain

$$\int_V \partial_j u_j dV = \sum_{k=1}^{n+1} \int_{S_k} u_j n_j^{(k)} dV \tag{70}$$

where S_k is the boundary surface of V_k and $n_j^{(k)}$ is the outward normal on S_k. The surface S_k consists of the inner spherical boundary Σ_{k-1} and the outer spherical boundary Σ_k. The outward normal on Σ_{k-1} is $-\hat{r}$ and on Σ_k it is $\hat{r}$. Note that in the right hand side of equation (70) the limit of u_j on the boundary has to be taken approaching from the inside of the volume V_k. Therefore we obtain:

$$\int_V \partial_j u_j dV = -\sum_{k=1}^{n} \int_{\Sigma_k} [u_r]_-^+ \, dS \tag{71}$$

where $[u_j]_-^+$ denotes the discontinuity in u_j across Σ_k, the positive contribution arising from the side of Σ_k farthest from the origin. Thus we obtain

$$\int_V \partial_j u_j dV = -\int_\Sigma [u_r]_-^+ \, d\Sigma \tag{72}$$

which proves Lemma 1. □

Lemma 2, Tangential derivative
The tangential derivative operator $\vec{\nabla}_t$ is defined as

$$\vec{\nabla}_t = \vec{\nabla} - \hat{r}\left(\hat{r}\cdot\vec{\nabla}\right) = \partial_i - \hat{r}\partial_r \tag{73}$$

$$= \frac{1}{r}\left(\hat{\theta}\frac{\partial}{\partial\theta} + \csc\theta\hat{\phi}\frac{\partial}{\partial\phi}\right) \tag{74}$$

thus for any quantity η, $\vec{\nabla}_t\eta$ is simply the projection of the gradient onto a spherical surface with normal $\vec{r}$. The lemma which will be proven is:

For any vector field $\vec{u}$, defined and differentiable on the spherical surface Σ

$$\int_\Sigma \left(\vec{\nabla}_t\cdot\vec{u}\right) d\Sigma = \int_\Sigma \frac{2u_r}{r} d\Sigma \tag{75}$$

Proof. From the definition of $\vec{\nabla}_t$ (74) we have

$$\vec{\nabla}_t\cdot\vec{u} = \vec{\nabla}\cdot\vec{u} - \partial_r u_r \tag{76}$$

since $\partial_r\hat{r} = 0$. If we now use the standard expression for the divergence of a vector field in spherical polar coordinates and integrate over a spherical surface, we obtain

$$\int_\Sigma \left(\vec{\nabla}_t\cdot\vec{u}\right) d\Sigma = \int_\Sigma \frac{2u_r}{r} d\Sigma + \int_0^{2\pi}\int_0^{\pi} r^2\sin\theta\left(\frac{\partial_\theta\left(\sin\theta u_\theta\right)}{r\sin\theta} + \frac{\partial_\phi u_\phi}{r\sin\theta}\right) d\theta d\phi \tag{77}$$

The last integral can be integrated directly and gives zero, which proves Lemma 2. □

Now we can proceed with the proof of result 3. If we integrate result 2 over the volume V and use Lemma 1, we get

$$\begin{aligned}
&\int_V \left[s_i'\mathcal{L}_i\left(\vec{s}\right) - s_i\mathcal{L}_i\left(\vec{s}'\right)\right] dV \\
=& -\int_\Sigma \left[s_i'\delta\overline{T}_{ir} - s_i\delta\overline{T}'_{ir} + \delta\varphi'\left(\frac{1}{4\pi G}\partial_r\delta\varphi + \rho^0 s_r\right) - \delta\varphi\left(\frac{1}{4\pi G}\partial_r\delta\varphi' + \rho^0 s_r'\right)\right]_-^+ d\Sigma
\end{aligned} \tag{78}$$

The last two terms in this integral are zero because of the boundary conditions (49) for $\delta\varphi$, since the product of two continuous functions is continuous as well. Hence if we use $\delta\overline{T}_{ij} = \delta T_{ij} + p^0\partial_i s_j - p^0\delta_{ij}\left(\partial_k s_k\right)$ and its equivalent for $\delta\overline{T}'_{ij}$, we obtain

$$\begin{aligned}
&\int_V \left[s_i'\mathcal{L}_i\left(\vec{s}\right) - s_i\mathcal{L}_i\left(\vec{s}'\right)\right] dV \\
=& -\int_\Sigma \left[s_i'\left(\delta T_{ir} + p^0\partial_i s_r - p^0\delta_{ir}\left(\partial_k s_k\right)\right) - s_i\left(\delta T'_{ir} + p^0\partial_i s_r' - p^0\delta_{ir}\left(\partial_k s_k'\right)\right)\right]_-^+ d\Sigma \\
=& -\int_\Sigma \left[s_i'\delta T_{ir} - s_i\delta T'_{ir}\right]_-^+ d\Sigma - \int_\Sigma \left[s_i'p^0\partial_i s_r - s_r'p^0\left(\partial_k s_k\right) - s_i p^0\partial_i s_r' + s_r p^0\left(\partial_k s_k'\right)\right]_-^+ d\Sigma
\end{aligned} \tag{79}$$

To prove Result 3 we have to show that the last surface integral on the right is zero. To do this, we rewrite first two terms of the last integral in terms of the surface integral definition (74).

$$\begin{aligned}
&\int_\Sigma \left[s_i'p^0\partial_i s_r - s_r'p^0\left(\partial_k s_k\right)\right]_-^+ d\Sigma \\
=& \int_\Sigma \left[s_i'p^0\nabla_i^t s_r + s_r'p^0\partial_r s_r - s_r'p^0\left(\nabla_k s_k\right) - s_r'p^0\left(\partial_r s_r\right)\right]_-^+ d\Sigma \\
=& \int_\Sigma \left[s_i'p^0\nabla_i^t s_r - s_r'p^0\left(\nabla_k s_k\right)\right]_-^+ d\Sigma
\end{aligned} \tag{80}$$

Using the result above and the corresponding result with $\vec{s}$ and $\vec{s}'$ interchanged, the last integral of equation (79) becomes

$$\begin{aligned}\int_\Sigma \left[s_i' p^0 \partial_i s_r - s_r' p^0 \left(\partial_k s_k\right)\right]_-^+ d\Sigma &= \int_\Sigma \left[s_i' p^0 \nabla_i^t s_r - s_r' p^0 \left(\nabla_k s_k\right) - s_i' p^0 \nabla_i^t s_r + s_r' p^0 \left(\nabla_k s_k\right)\right]_-^+ d\Sigma \\ &= \int_\Sigma \nabla_i^t \left[s_i' p^0 s_r - s_i p^0 s_r'\right]_-^+ d\Sigma \end{aligned} \quad (81)$$

where we used the product rule for the gradient in the last step, together with the fact that the initial pressure gradient has only a radial component, which makes the tangential derivative zero. If we now use Lemma 2, we get

$$\int_\Sigma \left[s_i' p^0 \partial_i s_r - s_r' p^0 \left(\partial_k s_k\right)\right]_-^+ d\Sigma = \int_\Sigma \frac{2}{r} \left[s_r' p^0 s_r - s_r p^0 s_r'\right]_-^+ d\Sigma = 0 \quad (82)$$

which completes the proof.

2.4.2 Properties of eigenfunctions and eigenvalues

Previously, the vector space $\mathcal{S}$ was defined, containing all the vector fields $\vec{s}$ satisfying the boundary conditions governing the continuity of displacement. Now a subspace $\mathcal{J}$ of $\mathcal{S}$ is defined, containing the vector fields which also obey the traction conditions, namely:

$$\left[\gamma_{ijkl} \partial_k s_l\right]_-^+ = 0 \text{ on all boundaries} \quad (83)$$

Hence a displacement field $\vec{s} \in \mathcal{J}$ simply satisfies all the boundary conditions appropriate for a displacement field. Suppose we take two displacement fields $\vec{s}$ and $\vec{s}'$, both in $\mathcal{J}$. Then result 3, equation (67), becomes:

$$\int_V s_i' \cdot \mathcal{L}_i \left(\vec{s}\right) dV = \int_V s_i \cdot \mathcal{L}_i \left(\vec{s}'\right) dV \quad (84)$$

Thus $\mathcal{L}$ is a self adjoint operator on $\mathcal{J}$. Now we will prove two important properties of self-adjoint operators, namely that its eigenvalues are real and its eigenfunctions orthogonal.

Take two eigenfunctions $\vec{s}_k$ and $\vec{s}_{k'}$ with eigenvalues ω_k^2 and $\omega_{k'}^2$:

$$\mathcal{L}\left(\vec{s}_k\right) = -\rho^0 \omega_k^2 \vec{s}_k \quad (85)$$

$$\mathcal{L}\left(\vec{s}_{k'}\right) = -\rho^0 \omega_{k'}^2 \vec{s}_{k'} \quad (86)$$

Taking the complex conjugate of (86) one finds:

$$\mathcal{L}\left(\vec{s}_{k'}^*\right) = -\rho^0 \omega_{k'}^{2*} \vec{s}_{k'}^* \quad (87)$$

since $\mathcal{L}$ is a real operator. Because the eigenfunctions satisfy all boundary conditions, we can use the self-adjointness of $\mathcal{L}$:

$$\int_V \vec{s}_{k'}^* \cdot \vec{\mathcal{L}}\left(\vec{s}_k\right) dV = \int_V \vec{s}_k \cdot \vec{\mathcal{L}}\left(\vec{s}_{k'}^*\right) dV \quad (88)$$

Hence one obtains:

$$\left(\omega_k^2 - \omega_{k'}^{2*}\right) \int_V \rho^0 \vec{s}_{k'}^* \cdot \vec{s}_k dV = 0 \quad (89)$$

If the eigenfunctions are identical, $\omega_k^2 = \omega_{k'}^{2*}$ and hence the eigenvalues ω_k^2 are all real. If the eigenvalues ω_k^2 and $\omega_{k'}^{2*}$ are different, the eigenfunctions have to be orthogonal in the sense that

$$\int_V \rho^0 \vec{s}_{k'}^* \cdot \vec{s}_k dV = 0 \quad (90)$$

2.4.3 Scalar equations in spherical coordinates

If the differential equations (57) and the incremental Poisson equation for the Fourier transformed gravity field are given in the spherical coordinate system, three equations are obtained. The tilde arising from the Fourier transformation has been omitted in this section for clarity:

$$-\rho^0\omega^2 s_r = -\rho^0\frac{\partial\delta\varphi}{\partial r} + \rho^0\Delta\frac{\partial\varphi^0}{\partial r} - \rho^0\frac{\partial}{\partial r}\left(s_r\frac{\partial\varphi^0}{\partial r}\right) + t_r \tag{91}$$

$$-\rho^0\omega^2 s_\theta = \frac{1}{r}\left(-\rho^0\frac{\partial\delta\varphi}{\partial\theta} - \rho^0\frac{\partial}{\partial\theta}\left(s_r\frac{\partial\varphi^0}{\partial r}\right)\right) + t_\theta \tag{92}$$

$$-\rho^0\omega^2 s_\phi = \frac{1}{r\sin\theta}\left(-\rho^0\frac{\partial\delta\varphi}{\partial\phi} - \rho^0\frac{\partial}{\partial\phi}\left(s_r\frac{\partial\varphi^0}{\partial r}\right)\right) + t_\phi \tag{93}$$

$$0 = \left(\hat{r}\frac{\partial}{\partial r} + \vec{\nabla}_t\right)\cdot\left(\hat{r}\frac{\partial\delta\varphi}{\partial r} + \vec{\nabla}_t\delta\varphi + 4\pi G\rho^0\vec{s}\right) \tag{94}$$

where $t_i = \partial_j T_{ij}$ is the divergence of the stress tensor, $T_{ij} = 2\mu e_{ij} + \lambda\Delta\delta_{ij}$ is the relationship between stress and strain, $\Delta = \nabla\cdot\vec{s}$ is the dilation and $e_{ij} = \frac{1}{2}\left(\partial_i s_j + \partial_j s_i\right)$ is the strain tensor. I used the alternative version of the Poisson equation, given in equation (46).

To reduce these equations completely into spherical coordinates, the quantities defined above (strain tensor, dilation and divergence of the stress tensor) have to be given explicitly in terms of s_r, s_θ and s_ϕ. To avoid needless lengthy calculations, I will just state the results here. A derivation can be found in appendix A of [27], equation A.139 for the strain tensor, A.140 for the dilation and A.144 for the divergence of the stress tensor. When deriving the expressions, the most important fact to be considered is that the direction of the unit vectors depends on their position, hence their derivative is generally nonzero.

The strain tensor has the components:

$$e_{rr} = \frac{\partial s_r}{\partial r} \tag{95}$$

$$e_{\theta\theta} = \frac{1}{r}\frac{\partial s_\theta}{\partial\theta} + \frac{s_r}{r} \tag{96}$$

$$e_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial s_\phi}{\partial\phi} + \frac{s_r}{r} + \frac{s_\theta\cot\theta}{r} \tag{97}$$

$$e_{r\theta} = \frac{1}{2}\left(\frac{\partial s_\theta}{\partial r} + \frac{1}{r}\frac{\partial s_r}{\partial\theta} - \frac{s_\theta}{r}\right) \tag{98}$$

$$e_{r\phi} = \frac{1}{2}\left(\frac{s_\phi}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial s_r}{\partial\phi} - \frac{s_\phi}{r}\right) \tag{99}$$

$$e_{\theta\phi} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial s_\phi}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial s_\theta}{\partial\phi} - \frac{s_\phi\cot\theta}{r}\right) \tag{100}$$

The dilation is the trace of the strain tensor, hence:

$$\Delta = \frac{1}{r^2}\frac{\partial\left(r^2 s_r\right)}{\partial r} + \frac{1}{r\sin\theta}\left(\frac{\partial\left(\sin\theta s_\theta\right)}{\partial\theta} + \frac{\partial s_\phi}{\partial\phi}\right) \tag{101}$$

The divergence of the stress tensor is:

$$t_r = \frac{\partial T_{rr}}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial T_{\phi r}}{\partial\phi} + \frac{1}{r}\left(\frac{\partial T_{\theta r}}{\partial\theta} + 2T_{rr} - T_{\theta\theta} - T_{\phi\phi} + T_{\theta r}\cot\theta\right) \tag{102}$$

$$t_\theta = \frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\theta}}{\partial\phi} + \frac{1}{r}\left(\frac{\partial T_{\theta\theta}}{\partial\theta} + \left(T_{\theta\theta} - T_{\phi\phi}\right)\cot\theta + 3T_{r\theta}\right) \tag{103}$$

$$t_\phi = \frac{\partial T_{r\phi}}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi}}{\partial\phi} + \frac{1}{r}\left(\frac{\partial T_{\theta\phi}}{\partial\theta} + 3T_{r\phi} + 2T_{\theta\phi}\cot\theta\right) \tag{104}$$

For Poisson's equation, the brackets have to be worked out. This gives:

$$
\begin{aligned}
0 &= \left(\hat{r}\frac{\partial}{\partial r} + \vec{\nabla}_t\right) \cdot \left(\hat{r}\frac{\partial \delta\varphi}{\partial r} + \vec{\nabla}_t \delta\varphi + 4\pi G \rho^0 \vec{s}\right) && (105)\\
&= \frac{\partial}{\partial r}\left(\frac{\partial \delta\varphi}{\partial r} + 4\pi G \rho^0 s_r\right) + \left(\vec{\nabla}_t \cdot \hat{r}\right)\frac{\partial \delta\varphi}{\partial r} + \vec{\nabla}_t \cdot \left(4\pi G \rho^0 \vec{s}\right) + \nabla_t^2 \delta\phi && (106)\\
&= \frac{\partial}{\partial r}\left(\frac{\partial \delta\varphi}{\partial r} + 4\pi G \rho^0 s_r\right) + \frac{2}{r}\frac{\partial \delta\varphi}{\partial r} + 4\pi G \rho^0 \left(\nabla \cdot \vec{s} - \frac{\partial s_r}{\partial r}\right) + \nabla_t^2 \delta\varphi && (107)
\end{aligned}
$$

Using these expressions together with the relation between stress and strain, the equations of motion in spherical coordinates are obtained:

$$
\begin{aligned}
-\rho^0\omega^2 s_r &= \rho^0 \left(\Delta \frac{\partial \varphi^0}{\partial r} - \frac{\partial \delta\varphi}{\partial r} - \partial_r \left(\frac{\partial \varphi^0}{\partial r} s_r\right)\right) + \partial_r \left(\lambda\Delta + 2\mu e_{rr}\right)\\
&+ \frac{2\mu}{r}\left(\frac{e_{\theta r}}{\partial \theta} + 2e_{rr} - e_{\theta\theta} - e_{\phi\phi} + \cot\theta e_{\theta r}\right) + \frac{2\mu}{r\sin\theta}\frac{\partial e_{\phi r}}{\partial \phi} && (108)\\
-\rho^0\omega^2 s_\theta &= -\frac{\rho^0}{r}\frac{\partial \delta\varphi}{\partial \theta} + \frac{1}{r}\frac{\partial}{\partial \theta}\left(-s_r \rho^0 \frac{\partial \varphi^0}{\partial r} + 2\mu e_{\theta\theta} + \lambda\Delta\right) + \frac{\partial \left(2\mu e_{r\theta}\right)}{\partial r}\\
&+ \frac{2\mu}{r}\left((e_{\theta\theta} - e_{\phi\phi})\cot\theta + 3e_{r\theta}\right) + \frac{2\mu}{r\sin\theta}\frac{\partial e_{\phi\theta}}{\partial \phi} && (109)\\
-\rho^0\omega^2 s_\phi &= -\frac{\rho^0}{r\sin\theta}\frac{\partial \delta\varphi}{\partial \phi} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(-s_r \rho^0 \frac{\partial \varphi^0}{\partial r} + 2\mu e_{\phi\phi} + \lambda\Delta\right) + \frac{\partial \left(2\mu e_{r\phi}\right)}{\partial r}\\
&+ \frac{2\mu}{r}\left(\frac{\partial e_{\theta\phi}}{\partial \theta} + 3e_{r\phi} + 2e_{\theta\phi}\cot\theta\right) && (110)\\
0 &= \frac{\partial}{\partial r}\left(\frac{\partial \delta\varphi}{\partial r} + 4\pi G \rho^0 s_r\right) + \frac{2}{r}\frac{\partial \delta\varphi}{\partial r} + 4\pi G \rho^0 \left(\nabla \cdot \vec{s} - \frac{\partial s_r}{\partial r}\right) + \nabla_t^2 \delta\varphi && (111)
\end{aligned}
$$

where the relations between the dilation and the strain tensor and the displacement field are given in expressions (95-101).

2.4.4 Spherical harmonic expansion of the equations of motion and boundary conditions

The differential equations for the Fourier transform of $\vec{s}$ that have been derived, prove to be solved in an efficient way if the solution is expanded in spherical harmonics. Spherical harmonics are the angular part of the solution to Laplace's equation in spherical coordinates. In the following two paragraphs the differential equations and their boundary conditions will be expanded in spherical harmonics. This will result in a set of differential equations and boundary conditions for the spherical harmonic expansion coefficients.

2.4.4.1 Spherical harmonic representation of the equations of motion

Now that we have obtained the equations of motion in spherical coordinates, the displacement $\tilde{\vec{s}}$ shall be expanded into *vector spherical harmonics* $\vec{P}_l^m(\theta,\phi)$, $\vec{B}_l^m(\theta,\phi)$, $\vec{C}_l^m(\theta,\phi)$, defined by:

$$
\begin{aligned}
\vec{P}_l^m(\theta,\phi) &= \hat{r} Y_l^m(\theta,\phi) && (112)\\
\vec{B}_l^m(\theta,\phi) &= \vec{\nabla}_{t_1} Y_l^m(\theta,\phi) && (113)\\
\vec{C}_l^m(\theta,\phi) &= -\hat{r} \times \vec{\nabla}_{t_1} Y_l^m(\theta,\phi) && (114)
\end{aligned}
$$

where $\vec{\nabla}_{t_1}$ is the tangential derivative operator on the unit sphere ($r = 1$) and $Y_l^m(\theta,\phi)$ is a scalar spherical harmonic, defined by:

$$
Y_l^m(\theta,\phi) = (-)^m \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(\cos\theta) e^{im\phi} \qquad (115)
$$

The Fourier transform of the displacement field can be expanded into vector spherical harmonics [28] as

$$\tilde{\vec{s}} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} U_l^m(r)\vec{P}_l^m(\theta,\phi) + V_l^m(r)\vec{B}_l^m(\theta,\phi) + W_l^m(r)\vec{C}_l^m(\theta,\phi) \tag{116}$$

Or, alternatively

$$\tilde{\vec{s}} = U\hat{r} + \vec{\nabla}_{t_1} V - (\hat{r} \times \vec{\nabla}_{t_1})W \tag{117}$$

where

$$U = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} U_l^m(r) Y_l^m(\theta,\phi) \tag{118}$$

$$V = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} V_l^m(r) Y_l^m(\theta,\phi) \tag{119}$$

$$W = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} W_l^m(r) Y_l^m(\theta,\phi) \tag{120}$$

Working out equation (117) in spherical coordinates:

$$\tilde{s}_r = U \tag{121}$$

$$\tilde{s}_\theta = \frac{\partial V}{\partial \theta} + \csc\theta \frac{\partial W}{\partial \phi} \tag{122}$$

$$\tilde{s}_\phi = \csc\theta \frac{\partial V}{\partial \phi} - \frac{\partial W}{\partial \theta} \tag{123}$$

By substituting these vector spherical harmonic expansions into the differential equations, ordinary differential equations for U, V and W can be found. First the Fourier transform of the strain tensor and dilation in terms of U, V and W have to be determined.

$$\tilde{e}_{rr} = \frac{\partial U}{\partial r} \tag{124}$$

$$\tilde{e}_{\theta\theta} = \frac{1}{r}\left(\frac{\partial^2 V}{\partial \theta^2} - \cot\theta\csc\theta\frac{\partial W}{\partial \phi} + \csc\theta\frac{\partial^2 W}{\partial\phi\partial\theta} + U\right) \tag{125}$$

$$\tilde{e}_{\phi\phi} = \frac{1}{r}\left(\csc^2\theta\frac{\partial^2 V}{\partial\phi^2} - \csc\theta\frac{\partial^2 W}{\partial\phi\partial\theta} + U + \cot\theta\frac{\partial V}{\partial\theta} + \cot\theta\csc\theta\frac{\partial W}{\partial\phi}\right) \tag{126}$$

$$\tilde{e}_{r\theta} = \frac{1}{2}\left(\frac{\partial^2 V}{\partial\theta\partial r} + \csc\theta\frac{\partial^2 W}{\partial\phi\partial r} + \frac{1}{r}\left(\frac{\partial U}{\partial\theta} - \frac{\partial V}{\partial\theta} - \csc\theta\frac{\partial W}{\partial\phi}\right)\right) \tag{127}$$

$$\tilde{e}_{r\phi} = \frac{1}{2}\left(\csc\theta\frac{\partial^2 V}{\partial\phi\partial r} - \frac{\partial^2 W}{\partial\theta\partial r} + \frac{1}{r}\left(\csc\theta\frac{\partial U}{\partial\phi} - \csc\theta\frac{\partial V}{\partial\phi} + \frac{\partial W}{\partial\theta}\right)\right) \tag{128}$$

$$\tilde{e}_{\theta\phi} = \frac{1}{2r}\left(2\csc\theta\frac{\partial^2 W}{\partial\phi\partial\theta} - 2\cot\theta\csc\theta\frac{\partial V}{\partial\phi} - \frac{\partial^2 W}{\partial\theta^2} + \csc^2\theta\frac{\partial^2 W}{\partial\phi^2} + \cot\theta\frac{\partial W}{\partial\theta}\right) \tag{129}$$

The dilation is the trace of the strain tensor and becomes:

$$\tilde{\Delta} = \frac{\partial U}{\partial r} + \frac{1}{r}\left(\nabla_{t_1}^2 V + 2U\right) \tag{130}$$

where $\nabla_{t_1}^2 = \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \csc^2\theta\frac{\partial^2}{\partial\phi^2}$ is the surface Laplacian on the unit sphere: $\nabla_{t_1}^2 = \left[\vec{\nabla}_t \cdot \vec{\nabla}_t\right]_{r=1}$.

The Fourier transform of the perturbation in the gravitational potential $\delta\tilde{\varphi}$ will be expanded in ordinary spherical harmonics:

$$\delta\tilde{\varphi} = \sum_l \sum_m \delta\varphi_l^m Y_l^m(\theta,\phi) \tag{131}$$

Substituting the spherical harmonic representations into the equations of motion, after simplifying algebraically, the following equations are obtained:

$$
\begin{aligned}
-\rho^0\omega^2 U = \rho^0 \left(\tilde{\Delta}\frac{\partial\varphi^0}{\partial r} - \frac{\partial\delta\varphi}{\partial r} - \frac{\partial}{\partial r}\left(\frac{\partial\varphi^0}{\partial r}U\right)\right) + \frac{\partial}{\partial r}\left(\lambda\tilde{\Delta} + 2\mu\frac{\partial U}{\partial r}\right) \\
+ \frac{\mu}{r^2}\left(r\nabla^2_{t_1}\frac{\partial V}{\partial r} + \nabla^2_t(U-3V) + 4r\frac{\partial U}{\partial r} - 4U\right)
\end{aligned}
\tag{132}
$$

$$
\begin{aligned}
\vec{\nabla}_{t_1}\Bigg\{\rho^0\left(\omega^2 V - \frac{\delta\varphi}{r} - \frac{\partial\varphi^0}{\partial r}\frac{U}{r}\right) + \frac{\lambda\tilde{\Delta}}{r} + \frac{\partial}{\partial r}\left(\mu\left(\frac{\partial V}{\partial r} + \frac{U-V}{r}\right)\right) \\
+\frac{\mu}{r^2}\left(2\nabla^2_{t_1}V + 5U - V + 3r\frac{\partial V}{\partial r}\right)\Bigg\} \\
-\hat{r}\times\vec{\nabla}_{t_1}\Bigg\{\rho^0\omega^2 W + \frac{\partial}{\partial r}\left(\mu\left(\frac{\partial W}{\partial r} - \frac{W}{r}\right)\right) \\
+\frac{\mu}{r^2}\left(\nabla^2_{t_1}W + 3r\frac{\partial W}{\partial r} - W\right)\Bigg\} = 0
\end{aligned}
\tag{133}
$$

$$
\frac{\partial}{\partial r}\left(\frac{\partial\delta\tilde{\varphi}}{\partial r} + 4\pi G\rho^0 U\right) = -\frac{2}{r}\frac{\partial\delta\tilde{\varphi}}{\partial r} - \frac{4\pi G\rho^0}{r}\left(2U + \nabla^2_{t_1}V\right) - \frac{\nabla^2_{t_1}\delta\tilde{\varphi}}{r^2}\delta\tilde{\varphi}
\tag{134}
$$

Since $\nabla^2_{t_1}Y_l^m(\theta,\phi) = -l(l+1)Y_l^m(\theta,\phi)$ equations (132) and (134) are sums over spherical harmonics and equation (133) is a sum over vector spherical harmonics of the form (116), with $U_l^m(r)$, $V_l^m(r)$ and $W_l^m(r)$ as coefficients. Because of the uniqueness of the (vector) spherical harmonic representations, equations (132), (133) and (134) are valid for the coefficients as well. In other words: if a (vector) spherical harmonic expansion of a (vector) field is zero, it must be the zero (vector) field.

Hence now equations for each coefficient can be obtained. For clarity the coefficients will be denoted by $U = U_l^m(r)$, $V = V_l^m(r)$, $W = W_l^m(r)$ and $\delta\varphi = \delta\varphi_l^m$. Now, using Poisson's equation for φ^0 in spherical coordinates and using the fact that $\nabla^2_{t_1}Y_l^m(\theta,\phi) = -l(l+1)Y_l^m(\theta,\phi)$, we obtain the system of equations for the spherical harmonic coefficients.

$$
\begin{aligned}
\frac{\partial}{\partial r}\left(\lambda\tilde{\Delta} + 2\mu\frac{\partial U}{\partial r}\right) &= \rho^0\left(4\pi G\rho^0 - \frac{2}{r}\frac{\partial\varphi^0}{\partial r} - \omega^2\right)U + \rho^0\left(\frac{\partial\varphi^0}{\partial r}\frac{\partial U}{\partial r} + \frac{\partial\delta\varphi}{\partial r} - \tilde{\Delta}\frac{\partial\varphi}{\partial r}\right) \\
&\quad + \frac{\mu}{r^2}\left(4U - 4r\frac{\partial U}{\partial r} + l(l+1)\left(U - 3V + r\frac{\partial V}{\partial r}\right)\right)
\end{aligned}
\tag{135}
$$

$$
\begin{aligned}
\frac{\partial}{\partial r}\left(\mu\left(\frac{\partial V}{\partial r} + \frac{U-V}{r}\right)\right) &= -\rho^0\omega^2 V + \frac{\rho^0}{r}\left(\frac{\partial\delta\varphi}{\partial r} + U\frac{\partial\varphi^0}{\partial r}\right) - \frac{\lambda\tilde{\Delta}}{r} \\
&\quad + \frac{\mu}{r^2}\left(V - 5U - 3r\frac{\partial V}{\partial r} + 2l(l+1)V\right)
\end{aligned}
\tag{136}
$$

$$
\frac{\partial}{\partial r}\left(\mu\left(\frac{\partial W}{\partial r} - \frac{W}{r}\right)\right) = -\rho^0\omega^2 W + \frac{\mu}{r^2}\left(W - 3r\frac{\partial W}{\partial r} + l(l+1)W\right)
\tag{137}
$$

$$
\frac{\partial}{\partial r}\left(\frac{\partial\delta\varphi}{\partial r} + 4\pi G\rho^0 U\right) = -\frac{2}{r}\frac{\partial\delta\varphi}{\partial r} - \frac{4\pi G\rho^0}{r}\left(2U - l(l+1)V\right) + \frac{l(l+1)}{r^2}\delta\varphi
\tag{138}
$$

2.4.4.2 Spherical harmonic representation of the boundary conditions

The boundary conditions valid for solutions of the differential equations are continuous traction on all interfaces, continuous displacement on all solid-solid boundaries and continuous radial displacement on all fluid-solid boundaries. Traction is zero at the free surface, and there is no shear stress at a fluid-solid boundary. Furthermore, on all interfaces, the boundary conditions for the perturbation of the gravitational field are:

$$
[\delta\phi]_-^+ = 0, \qquad \left[\partial_r\delta\phi - 4\pi G\rho^0 s_r\right]_-^+ = 0
\tag{139}
$$

The Fourier transform of the traction vector field $\vec{\tau}_r$ on a surface with normal $\hat{r}$ can also be expanded into vector spherical harmonics. First, using the stress-strain relationship and the spherical harmonic representation of the strain tensor given in equations (124-129) we obtain:

$$\tilde{\vec{\tau}}_r = \tilde{T}_{rr}\hat{r} + \tilde{T}_{r\theta}\hat{\theta} + \tilde{T}_{r\phi}\hat{\phi} \tag{140}$$

$$= \left(\lambda\tilde{\Delta} + 2\mu\frac{\partial U}{\partial r}\right)\hat{r} + \mu\left(\vec{\nabla}_{t_1}\left(\frac{\partial V}{\partial r} + \frac{U-V}{r}\right) - \hat{r}\times\vec{\nabla}_{t_1}\left(\frac{\partial W}{\partial r} - \frac{W}{r}\right)\right) \tag{141}$$

Using the stress-strain relationship and the spherical harmonic representation of the strain tensor given in equations (124-129) we obtain:

$$\tilde{\vec{\tau}}_r = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} P_l^m(r)\vec{P}_l^m(\theta,\phi) + S_l^m(r)\vec{B}_l^m(\theta,\phi) + T_l^m(r)\vec{C}_l^m(\theta,\phi) \tag{142}$$

where $P = P_l^m(r)$, $S = S_l^m(r)$ and $T = T_l^m(r)$ are related to $U = U_l^m$, $V = V_l^m$ and $W = W_l^m$ by:

$$P = \lambda\tilde{\Delta} + 2\mu\frac{\partial U}{\partial r} \tag{143}$$

$$S = \mu\left(\frac{\partial V}{\partial r} + \frac{U-V}{r}\right) \tag{144}$$

$$T = \mu\left(\frac{\partial W}{\partial r} - \frac{W}{r}\right) \tag{145}$$

Since the traction is continuous throughout the model, P, S and T must be continuous across all surfaces of discontinuity and be zero on the free surface.

The boundary conditions for the gravitational potential perturbation also hold for each spherical harmonic component seperately. Since $\delta\varphi = \delta\varphi_l^m$ vanishes at $r = \infty$ and satisfies Laplace's equation outside the Earth-model, it takes the form

$$\delta\varphi = Ar^{-l-1} \tag{146}$$

outside the model, where A is a constant. Thus

$$\frac{\partial\delta\varphi}{\partial r} + (l+1)\frac{\delta\varphi}{r} = 0 \tag{147}$$

outside the model. Thus we can define

$$\delta\psi = \frac{\partial\delta\varphi}{\partial r} + (l+1)\frac{\delta\varphi}{r} + 4\pi G\rho^0 U \tag{148}$$

and by virtue of the boundary conditions on $\delta\varphi$ and equation (147) $\delta\psi$ is continuous and zero at the free surface.

To summarize, all boundary conditions in terms of spherical harmonics will be given now. These conditions hold for each mode (n, l, m) separately. At the free surface:

$$P = S = T = \delta\psi = 0 \tag{149}$$

At a solid-solid boundary:

$$[P]_-^+ = 0,\ [S]_-^+ = 0,\ [T]_-^+ = 0,\ [\delta\psi]_-^+ = 0,\ [U]_-^+ = 0,\ [V]_-^+ = 0,\ [W]_-^+ = 0,\ [\delta\varphi]_-^+ = 0 \tag{150}$$

At a fluid-solid boundary:

$$S = T = 0 \text{ and } \quad [P]_-^+ = 0, \quad [\delta\psi]_-^+ = 0, \quad [U]_-^+ = 0, \quad [\delta\varphi]_-^+ = 0 \tag{151}$$

2.4.5 Classification of eigenfunctions

Three important observations can be made regarding the system of ordinary differential equations that have been derived for the spherical harmonic coefficients or *mode shapes* U, V, W and $\delta\varphi$. Firstly every spherical harmonic coefficient belonging to (l, m) is completely decoupled from every other. Hence it is a fundamental property of a SNREI Earth that no mode coupling takes place. Further, the differential equations are independent of m.

Finally, the equations for (U, V) are completely decoupled from the equations for W. Hence there are two types of modes, one governed by U and V and the other determined by W. If $W = 0$ a mode is called spheroidal or poloidal, if $U = V = \delta\varphi = 0$ then a mode is called toroidal. Spheroidal and toroidal modes are the normal mode equivalent of Rayleigh waves and Love waves respectively.

The eigenvalues ω^2 of the differential equations (135-138) are the squared angular frequencies of the oscillations subject to the boundary conditions (149-151). For some fixed l the eigenfrequencies will be denoted by ${}_n\omega_l$, where $n \in \mathbb{N}_0$ labels the eigenfrequencies for fixed l in increasing order. The fundamental mode is designated by $n = 0$, overtones have $n > 0$. For each pair (n, l) there are $2l + 1$ vector eigenfuncties ${}_n\vec{s}_l^m$, where m goes in integer steps from $-l$ to l. Tho specify an eigenfunction, (n, l, m) has to be specified, known as the radial, angular and azimuthal order respectively.

For fixed (n, l) the $2l+1$ eigenfunctions form a multiplet and the eigenvalue is $(2l+1)$-fold degenerate.

To get an overview of the displacement patterns of the modes, a very good source are animations of the modes, which can be found on the internet page of the nanomaterials department of the Laboratoire Interdisciplinaire Carnot de Bourgogne **??**.

2.4.5.1 Toroidal modes

Toroidal oscillations are characterized by motion perpendicular to straight lines through the center of the Earth. Hence they exhibit no radial motion. There is no volume change, because $\nabla \cdot \vec{s} = 0$. Hence the density distribution is not perturbed and there are no gravitational effects. There are no toroidal modes of angular order $l = 0$. The modes of angular order $l = 1$ and $m = -1, 0, 1$ represent rigid body rotations about the $\hat{x}$, $\hat{z}$ and $\hat{y}$ axes respectively. The displacement vector of an eigenfunction with arbitrary l and m is given by:

$$ {}_n\vec{s}_l^m = -{}_nW_l(r) \left(\hat{r} \times \vec{\nabla}_{t_1} \right) Y_l^m(\theta, \phi) \tag{152} $$

or explicitly, in terms of the s_r, s_θ and s_ϕ components of the eigenfunction:

$$ s_r = 0 \tag{153} $$

$$ s_\theta = {}_nW_l(r) \csc\theta \frac{\partial Y_l^m(\theta, \phi)}{\partial \phi} \tag{154} $$

$$ s_\phi = -{}_nW_l(r) \frac{\partial Y_l^m(\theta, \phi)}{\partial \theta} \tag{155} $$

2.4.5.2 Spheroidal modes

In general, spheroidal modes exhibit both radial and tangential motion. If the angular order $l = 0$ the motion is purely radial. Spheroidal modes can be characterized by the fact that $\vec{\nabla} \times \vec{s}$ has no radial component. The modes angular order $l = 1$ and $m = -1, 0, 1$ represent rigid body translations along the $\hat{x}$, $\hat{z}$ and $\hat{y}$ axes respectively. The displacement vector of an eigenfunction with arbitrary l and m is given by:

$$ {}_n\vec{s}_l^m =_n U_l(r) \hat{r} Y_l^m(\theta, \phi) +_n V_l(r) \vec{\nabla}_{t_1} Y_l^m(\theta, \phi) \tag{156} $$

or explicitly, in terms of the s_r, s_θ and s_ϕ components of the eigenfunction:

$$s_r = {}_nU_l(r)Y_l^m(\theta,\phi) \tag{157}$$

$$s_\theta = {}_nV_l(r)\frac{\partial Y_l^m(\theta,\phi)}{\partial\theta} \tag{158}$$

$$s_\phi = {}_nV_l(r)\csc\theta\frac{\partial Y_l^m(\theta,\phi)}{\partial\phi} \tag{159}$$

2.4.6 Calculation of eigenfunctions

The initial value problem for the mode shapes U, V, W and $\delta\varphi$ is now completely specified. We shall now briefly discuss how this initial value problem can be solved. The differential equations are of second order. It is most convenient to replace this second order problem with an equivalent coupled first order inital value problem, since there are many numerical techniques for solving coupled first order differential equations, for example Runge-Kutta methods or the propagator matrix method.

Since all differential equations are of second order, there are two linearly independent solutions for each mode shape. However, it appears that one of these solutions is singular at the Earth's origin and hence does not obey the boundary conditions.

The outline of this section will be to derive the coupled first order initial value problems for the mode shapes and to give a 'walk-through' of the numerical solution process.

For toroidal modes, equations (145) and (137) are used to derive the coupled first order initial value problem:

$$\frac{\partial W}{\partial r} = r^{-1}W + \mu^{-1}T \tag{160}$$

$$\frac{\partial T}{\partial r} = \left(\mu r^{-2}(l-1)(l+2) - \rho^0\omega^2\right)W - 3r^{-1}T \tag{161}$$

In the fluid outer core of the Earth toroidal modes do not exist. Hence the inner core and the mantle have independent toroidal oscillations. Since there is no known way of exciting or observing inner core modes, these are commonly left out of the set of normal modes of the Earth. Hence by ${}_nT_l$ only the toroidal modes of the mantle are labeled. The stress $T = 0$ at the inner core boundary and at the free surface, T and W must be continuous across the mantle.

Equations (160) and (161) may be written as a single matrix-vector equation as:

$$\frac{d}{dr}\begin{bmatrix} W \\ T \end{bmatrix} = \begin{bmatrix} r^{-1} & \mu^{-1} \\ \mu r^{-2}(l-1)(l+2) - \rho^0\omega^2 & -3r^{-1} \end{bmatrix}\begin{bmatrix} W \\ T \end{bmatrix} \tag{162}$$

At the core-mantle interface the vector $\begin{bmatrix} W \\ T \end{bmatrix}$ is proportional to the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ since the stress is zero at this boundary. Because the eigenfunction is defined up to a normalization factor, we can choose this normalization to be one at the core-mantle boundary. Thus the solution can be obtained by choosing some trial value for ω and set $\begin{bmatrix} W \\ T \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ at the core-mantle interface. Then equation (162) can be integrated up to the surface. Generally, T will be non-zero there, so ω is varied and a root-finding algorithm is used until a value of ω is found for which $T = 0$. This value of ω is an eigenvalue and the corresponding $W(r)$ is the scalar eigenfunction. In this way all eigenfunctions ${}_nW_l(r)$ can be found, for each fixed l.

For spheroidal modes, equations (135), (136), (138), together with (143), (145) and (148) give the following matrix equation in solid regions:

$$\frac{d}{dr}\begin{bmatrix} U \\ V \\ P \\ S \\ \delta\varphi \\ \delta\psi \end{bmatrix} = \begin{bmatrix} -\frac{2\lambda}{\sigma r} & \frac{l(l+1)\lambda}{\sigma r} & 1/\sigma & 0 & 0 & 0 \\ -1/r & 1/r & 0 & 1/\mu & 0 & 0 \\ -\rho^0\omega^2 + \frac{4}{r}\left(\frac{\gamma}{r} - \rho^0\frac{\partial\varphi^0}{\partial r}\right) & \frac{l(l+1)}{r}\left(\rho^0\frac{\partial\varphi^0}{\partial r} - 2\gamma/r\right) & \frac{2\lambda}{\sigma r} - 2/r & l(l+1)/r & -(l+1)\rho^0/r & \rho^0 \\ r^{-1}\rho^0\frac{\partial\varphi^0}{\partial r} - 2\gamma/r^2 & -\rho^0\omega^2 + \left(l(l+1)(\gamma+\mu) - 2\mu\right)/r^2 & -\frac{\lambda}{\sigma r} & -3/r & \rho^0/r & 0 \\ -4\pi G\rho^0 & 0 & 0 & 0 & -(l+1)/r & 1 \\ -4\pi G\rho^0(l+1)/r & 4\pi G\rho^0 l(l+1)/r & 0 & 0 & 0 & (l-1)/r \end{bmatrix}\begin{bmatrix} U \\ V \\ P \\ S \\ \delta\varphi \\ \delta\psi \end{bmatrix} \tag{163}$$

In a fluid region, the left hand side of (136) vanishes, since $\mu = 0$, and thus $S = 0$ and equation (136) gives a relation between U, V, P and $\delta\varphi$:

$$V - \frac{U}{r\omega^2}\frac{\partial\varphi^0}{\partial r} - \frac{P}{\rho^0\omega^2 r} + \frac{\delta\varphi}{\omega^2 r} \tag{164}$$

The remaining equations reduce to the following matrix equation in fluid regions:

$$\frac{d}{dr}\begin{bmatrix} U \\ P \\ \delta\varphi \\ \delta\psi \end{bmatrix} = \begin{bmatrix} -\frac{2}{r} + \frac{l(l+1)}{r^2\omega^2}\frac{\partial\varphi^0}{\partial r} & -\frac{l(l+1)}{r^2\rho^0\omega^2} + 1/\sigma & \frac{l(l+1)}{r^2\omega^2} & 0 \\ -\rho^0\omega^2 - \frac{4\rho^0}{r}\frac{\partial\varphi^0}{\partial r} + \rho^0\frac{l(l+1)}{r^2\omega^2}\left(\frac{\partial\varphi^0}{\partial r}\right)^2 & -\frac{l(l+1)}{r^2\omega^2}\frac{\partial\phi^0}{\partial r} & \rho^0\left(\frac{l(l+1)}{r^2\omega^2}\frac{\partial\phi^0}{\partial r} - (l+1)/r\right) & \rho^0 \\ -4\pi G\rho^0 & 0 & -(l+1)/r & 1 \\ 4\pi G\rho^0\left(\frac{l(l+1)}{r^2\omega^2}\frac{\partial\varphi^0}{\partial r} - \frac{l+1}{r}\right) & -4\pi G\frac{l(l+1)}{r^2\omega^2} & 4\pi G\rho^0\frac{l(l+1)}{r^2\omega^2} & (l-1)/r \end{bmatrix}\begin{bmatrix} U \\ P \\ \delta\varphi \\ \delta\psi \end{bmatrix} \tag{165}$$

To find the eigenfrequencies and eigenfunctions for spheroidal equations is more involved than for toroidal modes. There are three independent solution vectors of equation (163) that are regular at the origin. These can be found by examining the exact solutions of equation (163) for a homogeneous Earth, which can be found in chapter 8 of [27].

Using each of these three vectors as an initial value, equation (163) can be integrated to the inner-outer core boundary. Since V is not linearly independent from U, P and $\delta\varphi$ there, only two linear combinations of these three vectors will satisfy the boundary condition $S = 0$ at this boundary, say $\begin{pmatrix} U_1 & P_1 & \delta\varphi_1 & \delta\psi_1 \end{pmatrix}$ and $\begin{pmatrix} U_2 & P_2 & \delta\varphi_2 & \delta\psi_2 \end{pmatrix}$.

These two linear combinations are used as initial value for the solution of equation (165) in the outer core and integrated up to the mantle, leading to two vectors $\begin{pmatrix} U_3 & 0 & P_3 & 0 & \delta\varphi_3 & \delta\psi_3 \end{pmatrix}$ and $\begin{pmatrix} U_4 & 0 & P_4 & 0 & \delta\varphi_4 & \delta\psi_4 \end{pmatrix}$ at the mantle bottom. A linearly independent third starting condition can be given by $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$, since in the mantle, V is linearly independent. The third starting condition allows tangential slip at the core-mantle boundary.

Integrating equation (163) three times through the mantle up to the free surface, three linearly independent solutions at the surface are now obtained, say $\begin{pmatrix} U_5 & V_5 & P_5 & S_5 & \delta\varphi_5 & \delta\psi_5 \end{pmatrix}$, $\begin{pmatrix} U_6 & V_6 & P_6 & S_6 & \delta\varphi_6 & \delta\psi_6 \end{pmatrix}$ and $\begin{pmatrix} U_7 & V_7 & P_7 & S_7 & \delta\varphi_7 & \delta\psi_7 \end{pmatrix}$. A linear combination of these vectors obeying the free traction condition at the surface gives the solution for the mode shapes. Hence:

$$\det \begin{bmatrix} P_5 & P_6 & P_7 \\ S_5 & S_6 & S_7 \\ \psi_5 & \psi_6 & \psi_7 \end{bmatrix} = 0 \tag{166}$$

In general, this will not hold, hence, as with the toroidal modes, this process must be repeated for different ω using a root finding algorithm. The solution value of ω is an eigenfrequency and the corresponding linear combinations of the solution vectors in the various regions yield the corresponding eigenfunctions U, V and $\delta\varphi$, using equation (164) in the fluid core.

2.5 The excitation problem

2.5.1 Seismic source representation

Earthquakes and other driving forces of the oscillations of the Earth are all characterized by a permanent deformation of the Earth at the region where the driving force acts. This means that the linear elastic relationship between stress and strain *fails* locally and transiently. This idea was introduced by Backus and Mulcahy in 1976. [29]

The failure of the consitutive equations can be represented by a tensor quantity $\Gamma_{ij} = \Gamma_{ij}(\vec{x}^0, t)$. This tensor is called the 'stress glut' and is the difference between the model stress given by the constitutive equation and the real stress in the source region. Both stresses are symmetric hence the stress glut Γ_{ij} is symmetric as well.

The stress glut vanishes outside the source area, is zero at times before the earthquake and is constant at times after the source has ceased to act. This means that the stress glut rate $\dot{\Gamma}_{ij}$ is zero outside the region and time interval of the source.

These considerations lead to a change in the source region of equation (55) to:

$$\rho^0 \left[\frac{\partial^2 s_i}{\partial t^2} + \partial_i \left(\delta\varphi(\vec{r}, t) + s_k \partial_k \varphi^0 \right) - (\partial_k s_k) \partial_i \varphi^0 \right] = \partial_j \left(\gamma_{ijkl} \partial_k s_l - \Gamma_{ij} \right) \tag{167}$$

where we have used the definition (44) of the operator $\mathcal{L}_i$. This means Γ_{ij} is symmetric and

$$\rho^0 \frac{\partial^2 s_i}{\partial t^2} - \mathcal{L}_i\left(\vec{s}\right) = -\partial_j \left(\Gamma_{ij}\right) \tag{168}$$

Hence the source term can be represented by $-\partial_j \Gamma_{ij}$. For a point source in space and time the stress glut rate is simply a delta function in space and time:

$$\dot{\Gamma}_{ij} = M_{ij} \delta(\vec{x}^0 - \vec{x}_S) \delta(t) \tag{169}$$

where M_{ij} is the *moment tensor* and $\vec{x}_S$ is the source point and the source acts at time $t = 0$. Note that since Γ_{ij} is symmetric, M_{ij} is symmetric as well. Hence 9 independent parameters are needed to describe a source which is caused by failure of the constitutive equations, six for the moment tensor and three for the location. This means that the source term representation $-\partial_j \Gamma_{ij}$ becomes:

$$-\partial_j \Gamma_{ij} = -M_{ij}\partial_j \delta(\vec{x}^0 - \vec{x}_S)H(t) \tag{170}$$

where $H(t)$ is the Heaviside step function.

Failure of the constitutive equation is a process which takes place within the earth, hence no external forces play a role. If external forces are taken into account, Γ_{ij} is not necessarily symmetric anymore, since there are additional, external force terms making up the difference between real stress and model stress. An example of an external force causing an earthquake is a meteorite striking the Earth. For theoretical purposes, the seismogram due to an external force is important, because it allows the calculation of Green's functions which are of fundamental interest for many seismological problems. Hence a more general source representation, taking into account external point forces is:

$$-\partial_j \Gamma_{ij} = (f_i - M_{ij}\partial_j)\delta(\vec{x}^0 - \vec{x}_S)H(t) \tag{171}$$

where M_{ij} is not necessarily symmetric. Equation (168) now becomes:

$$\rho^0 \frac{\partial^2 s_i}{\partial t^2} - \mathcal{L}_i(\vec{s}) = q_i \delta(\vec{x}^0 - \vec{x}_S)H(t) \tag{172}$$

where $q_i = q_i(\vec{x}^0, t) = (f_i - M_{ij}\partial_j)$.

The extra force terms represent a net force acting on the Earth, a non-symmetric moment tensor represents an external torque, as can be seen on page 149-150 in [27]. Now 15 parameters are needed to specify this source term, three for the location, three force components and nine moment tensor components. This is the most general point source exerting a net force and torque on the Earth. We regard the forces and torques applied on the Earth small enough to be able to neglect the effects of rotation caused by the source. In the next section the synthetic seismogram for this point source will be calculated.

2.5.2 Synthetic seismograms

The main goal of this chapter is to derive an expression for a synthetic seismogram of the form:

$$\vec{s}(\vec{x}^0, t) = \sum_k \frac{1 - e^{\alpha_k t}\cos(\omega_k t)}{\omega_k^2} \sum_m E_{km} \vec{s}_k^m(\vec{x}^0) \tag{173}$$

where k is an index incorporating n and l, α_k is a parameter governing the attenuation of a mode k and E_{km} are excitation coefficients of a mode (k, m), dependent on the source parameters. As defined earlier, ω_k is the angular eigenfrequency of mode k and $\vec{s}_k^m$ is an eigenfunction corresponding to eigenfrequency ω_k.

As we have seen in chapter 2.5.1 on the source the seismic displacement $\vec{s}$ at the material particle $\vec{x}^0$ at time t is governed by:

$$\rho^0 \frac{\partial^2 s_i}{\partial t^2} - \mathcal{L}_i(\vec{s}) = q_i \delta(\vec{x}^0 - \vec{x}_S)H(t) \tag{174}$$

satisfying boundary conditions (56). In this equation, ρ^0 is the unperturbed density distribution, the terms on the right hand size represent a seismic source and $\mathcal{L}$ is a linear, self adjoint operator defined in equation (60) having real eigenvalues and orthogonal eigenfunctions in the sense:

$$\int_{V^0} \rho^0 \vec{s}_{k'}^* \cdot \vec{s}_k dV^0 = \delta_{kk'} \tag{175}$$

One could object that not all eigenfunctions necessarily have to be orthogonal, since for each l there are $2l + 1$ degenerate eigenfunctions. Luckily, the eigenfunctions are expanded into (vector) spherical harmonics, which are orthogonal in the sense:

$$\int_0^{2\pi} \int_0^{\pi} Y_l^m(\theta, \phi) Y_{l'}^{m'}(\theta, \phi) \sin\theta \, d\theta d\phi = \delta_{ll'}\delta_{mm'} \tag{176}$$

This means that equation (175) certainly holds for all eigenfunctions.

Now a solution of equation (174) is sought in terms of an eigenfunction expansion:

$$\vec{s}(\vec{x}^0, t) = \sum_{k,m} a_k^m(t)\vec{s}_k^m\left(\vec{x}^0\right) \tag{177}$$

This is possible because the eigenfunctions of $\mathcal{L}_i$ form a complete set of basis functions for the displacement. We will not prove this fact here, but it stems from the completeness of the vector spherical harmonic expansion. Substituting (177) in differential equation (174) we find:

$$\sum_{k,m} \rho^0 \left(\frac{\partial^2 a_k^m}{\partial t^2} + \omega^2 a_k^m\right) \vec{s}_k^m(\vec{x}^0) = \vec{q}\delta(\vec{x}^0 - \vec{x}_S)H(t) \tag{178}$$

Taking the inner product with $\vec{s}_k^*$ and integrating over V^0, making use of the orthonormality (175) we get:

$$\frac{\partial^2 a_k^m}{\partial t^2} + \omega_k^2 a_k^m = F_k^m \tag{179}$$

where

$$F_k^m = F_k^m(t) = \int_{V^0} \left(\vec{s}_k^{*m} \cdot \vec{q}\delta(\vec{x}^0 - \vec{x}_S)H(t)\right) dV^0 \tag{180}$$

Equation (179) is a differential equation which can be solved by standard methods, for example Laplace transformation. Its solution is:

$$a_k^m(t) = \frac{1}{\omega_k}\int_{-\infty}^{t} \sin\left(\omega_k(t - t')\right) F_k^m(t')dt' \tag{181}$$

assuming that $F_k^m(t) = a_k^m(t) = 0$ for sufficiently small t. Integrating by parts yields:

$$a_k^m(t) = \frac{1}{\omega_k^2}\int_{-\infty}^{t} (1 - \cos\left(\omega_k(t - t')\right) \dot{F}_k^m(t')\, dt' \tag{182}$$

This form is more convenient since $\dot{F}_k^m(t) = 0$ if the source is inactive. Thus, after the force has ceased to act:

$$a_k^m(t) = \frac{1}{\omega_k^2}\int_{-\infty}^{\infty} h_k(t - t')\dot{F}_k^m(t')\, dt' \tag{183}$$

where $h_k(t) = 1 - \cos\left(\omega_k(t - t')\right)$. Because attenuation takes place for each mode seperately, these results have to be modified to take this effect into account. It appears [22] that it is necessary to replace $h_k(t)$ by:

$$h_k(t) = 1 - e^{-\alpha_k t}\cos(\omega_k t) \tag{184}$$

where α_k is a decay constant characteristic of mode k. When α_k, the eigenfunctions s_k and their corresponding eigenfrequencies ω_k are known and the source parameters are known, equation (177) and (183) completely specify the seismogram.

Now let us work out equation (180):

$$\begin{aligned} F_k^m &= \int_{V^0} \left(\vec{s}_k^{*m} \cdot \vec{q}\delta(\vec{x}^0 - \vec{x}_S)H(t)\right) dV^0 \\ &= \vec{s}_k^{*\,m}(\vec{x}_S) \cdot \vec{q}H(t) \end{aligned} \tag{185}$$

Hence $\dot{F}_k^m$ is:

$$\dot{F}_k^m = \vec{s}_k^{*m}(\vec{x}_S) \cdot \vec{q}\,\delta(t) \tag{186}$$

If this result is used in equation (183) $a_k^m(t)$ becomes:

$$a_k^m(t) = \frac{h_k(t)}{\omega_k^2}\vec{q} \cdot \vec{s}_k^{*m}(\vec{x}_S) \tag{187}$$

Using this result in the normal mode expansion (177) we obtain the following expression for the seismogram:

$$\vec{s}(\vec{x}^0, t) = \sum_{k,m} \frac{h_k(t)}{\omega_k^2} \vec{q} \cdot \vec{s}_k^{*m}(\vec{x}_S) \vec{s}_k^m \left(\vec{x}^0\right) \tag{188}$$

Note that this is essentially equation (173) from the beginning of this chapter, where $\vec{q} \cdot \vec{s}_k^{*m}(\vec{x}_s^0)$ are the excitation coefficients E_{km}. Hence if the excitation coefficients are obtained explicitly in spherical polar coordinates we will have an explicit formula to calculate seismograms. The remainder of this chapter is exactly about this subject.

Let us start working out the excitation coefficients. In this derivation $\vec{s}^* = \vec{s}_k^{*m}(\vec{x}_S)$ to avoid unclear expressions. For the expression $M_{ij}\partial_j s_i^*$ we need the expression for the gradient of a vector field in spherical polar coordinates. A straight forward calculation of the gradient of a vector field can be found in [27] appendix A, equation (A.138).

$$\begin{aligned}
E_{km} &= \vec{q} \cdot \vec{s}^* && (189)\\
&= f_r s_r^* + f_\theta s_\theta^* + f_\phi s_\phi^* \\
&\quad - M_{rr} \frac{\partial s_r^*}{\partial r} - \frac{M_{r\theta}}{r} \left(\frac{\partial s_r^*}{\partial \theta} - s_\theta^* \right) - \frac{M_{r\phi}}{r} \left(\csc\theta \frac{\partial s_r^*}{\partial \phi} - s_\phi^* \right) \\
&\quad - M_{\theta r} \frac{\partial s_\theta^*}{\partial r} - \frac{M_{\theta\theta}}{r} \left(\frac{\partial s_\theta^*}{\partial \theta} + s_r^* \right) - \frac{M_{\theta\phi}}{r} \left(\csc\theta \frac{\partial s_\theta^*}{\partial \phi} - s_\phi^* \cot\theta \right) \\
&\quad - M_{\phi r} \frac{\partial s_\phi^*}{\partial r} - \frac{M_{\phi\theta}}{r} \frac{\partial s_\phi^*}{\partial \theta} - \frac{M_{\phi\phi}}{r} \left(\cot\theta \frac{\partial s_\phi^*}{\partial \phi} + s_r^* + s_\theta^* \cot\theta \right) && (190)
\end{aligned}$$

The next step is to express the eigenfunction $\vec{s}_k^{*m}(\vec{x}_S)$ in terms of the eigenfunctions U, V and W and spherical harmonics, all evaluated at the source. We can combine the results for spheroidal and toroidal modes if we add for $\vec{s}_k^{*m}$ equations (153)-(155) with equations (157)-(159) and obtain:

$$\begin{aligned}
s_r(\vec{x}) &= {}_nU_l(r) Y_l^m(\theta, \phi) && (191)\\
s_\theta(\vec{x}) &= {}_nV_l(r) \frac{\partial Y_l^m(\theta,\phi)}{\partial \theta} + {}_n W_l(r) \csc\theta \frac{\partial Y_l^m(\theta,\phi)}{\partial \phi} && (192)\\
s_\phi(\vec{x}) &= {}_nV_l(r) \csc\theta \frac{\partial Y_l^m(\theta,\phi)}{\partial \phi} - {}_n W_l(r) \frac{\partial Y_l^m(\theta,\phi)}{\partial \theta} && (193)
\end{aligned}$$

where if the excitation coefficients are to be evaluated for a toroidal mode, U and V have to be put to zero. For a spheroidal mode $W = 0$. Substituting this expression for $\vec{s}_k^{*m}(\vec{x}_S)$ into the expression for the excitation coefficients and writing $U = {}_nU_l(r_S)$, $V = {}_nV_l(r_S)$, $W = {}_nW_l(r_S)$, $Y = Y_l^m(\theta_S, \phi_S)$, $Y' = \left[\frac{\partial Y}{\partial \theta}\right]_{(\theta,\phi)=(\theta_S,\phi_S)}$, using the fact that $\frac{\partial Y^*}{\partial \phi} = -imY^*$ and the fact that Y satisfies Legendre's equation $Y'' = -\cot\theta Y' - \left(l(l+1) - m^2 \csc^2\theta\right) Y$ to eliminate Y'' and denoting a partial derivative of U, V, W with respect to r with an overhead dot we obtain:

$$\begin{aligned}
E_{km} &= f_r U Y^* + f_\theta \left(V Y'^* - im\csc\theta W Y^*\right) - f_\phi \left(im\csc\theta V Y^* + W Y'^*\right) \\
&\quad - M_{rr} \dot{U} Y^* - \frac{M_{r\theta}}{r} \left((U - V) Y'^* + im\csc\theta W Y^*\right) \\
&\quad - \frac{M_{r\phi}}{r} \left(im\csc\theta \left(V - U\right) Y^* + W Y'^*\right) - M_{\theta r} \left(\dot{V} Y'^* - im\csc\theta \dot{W} Y^*\right) \\
&\quad - \frac{M_{\theta\theta}}{r} \left(V \left(m^2\csc^2\theta - l(l+1)\right) Y^* - V\cot\theta Y'^* + im\csc\theta W \left(\cot\theta Y^* - Y'^*\right) + U Y^*\right) \\
&\quad - \frac{M_{\theta\phi}}{r} \left(-im\csc\theta V Y'^* - m^2\csc^2 W Y^* + im\cot\theta\csc\theta V Y^* + \cot\theta W Y'^*\right) \\
&\quad + M_{\phi r} \left(im\csc\theta \dot{V} Y^* + \dot{W} Y'^*\right) \\
&\quad - \frac{M_{\phi\theta}}{r} \left(im\cot\theta\csc\theta V Y^* - im\csc\theta V Y'^* + W \left(\cot\theta Y'^* + \left(l(l+1) - m^2\csc^2\theta\right) Y^*\right)\right) \\
&\quad - \frac{M_{\phi\phi}}{r} \left(\cot\theta \left(-m^2\csc\theta V Y^* + imW Y'^* + V Y'^* - im\csc\theta W Y^*\right) + U Y^*\right) && (194)
\end{aligned}$$

Rearranging terms, we obtain:

$$
\begin{aligned}
E_{km} &= Y^* \left(f_r U - M_{rr}\dot{U} + l(l+1)\frac{M_{\theta\theta}}{r}V - \frac{M_{\theta\theta}+M_{\phi\phi}}{r}U - \frac{M_{\phi\theta}}{r}l(l+1)W \right) \\
&+ \left(\cot\theta Y'^* - m^2\csc^2\theta Y^*\right)\left(\frac{M_{\theta\theta}}{r}V - \frac{M_{\theta\phi}+M_{\phi\theta}}{r}W\right) \\
&+ \cot\theta\left(m^2\csc\theta Y^* - Y'^*\right)\frac{M_{\phi\phi}}{r}V \\
&+ Y'^*\left(f_\theta V - f_\phi W - \frac{M_{r\theta}}{r}(U-V) - \frac{M_{r\phi}}{r}W - M_{\theta r}\dot{V} + M_{\phi r}\dot{W}\right) \\
&- im\csc\theta Y^*\left(f_\theta W + f_\phi V + \frac{M_{r\theta}}{r}W + \frac{M_{r\phi}}{r}(V-U) - M_{\theta r}\dot{W} - M_{\phi r}\dot{V}\right) \\
&- im\csc\theta\left(\cot\theta Y^* - Y'^*\right)\left(\frac{M_{\theta\theta}}{r}W + \frac{M_{\theta\phi}+M_{\phi\theta}}{r}V\right) \\
&- im\cot\theta\left(Y'^* - \csc\theta Y^*\right)\frac{M_{\phi\phi}}{r}W
\end{aligned}
\tag{195}
$$

The idea is now to put the source in the north pole ($\theta_S = 0$). This will simplify the expression for E_{km} greatly. Since we have a non-rotating Earth, where the poles are on the axis of rotation, this is not a problem. To do this, we take the limit of the expression above as the source point approaches the pole along the meridian $\phi = 0$. It is important to define a meridian along which we approach the pole, because the unit vector $\hat{\phi}$ will tend to different limits allong different meridians. This all stems from the fact that the unit vectors $\hat{\theta}$ and $\hat{\phi}$ are not well defined at the pole.

Now we take the limits of the terms depending on the angular coordinates, for $\phi_S = 0$. This means that the unit vectors $(\hat{r}, \hat{\theta}, \hat{\phi})$ and the source direction (in the components of the force vector f_i and moment tensor M_{ij}), point in the directions (up, south, east):

$$
\lim_{\theta_S\to 0} Y^* = \begin{cases} \nu & \text{if } m=0 \\ 0 & \text{if } m=\pm1 \\ 0 & \text{if } m=\pm2 \end{cases} \tag{196}
$$

$$
\lim_{\theta_S\to 0} \left(\cot\theta Y'^* - m^2\csc^2\theta Y^*\right) = \begin{cases} -\nu\zeta^2/2 & \text{if } m=0 \\ 0 & \text{if } m=\pm1 \\ -\nu\zeta\sqrt{\zeta^2-2}/4 & \text{if } m=\pm2 \end{cases} \tag{197}
$$

$$
\lim_{\theta_S\to 0} \cot\theta\left(m^2\csc\theta Y^* - Y'^*\right) = \begin{cases} \nu\zeta^2/2 & \text{if } m=0 \\ 0 & \text{if } m=\pm1 \\ \nu\zeta\sqrt{\zeta^2-2}/4 & \text{if } m=\pm2 \end{cases} \tag{198}
$$

$$
\lim_{\theta_S\to 0} Y'^* = \begin{cases} 0 & \text{if } m=0 \\ \mp\nu\zeta/2 & \text{if } m=\pm1 \\ 0 & \text{if } m=\pm2 \end{cases} \tag{199}
$$

$$
\lim_{\theta_S\to 0} -im\csc\theta Y^* = \begin{cases} 0 & \text{if } m=0 \\ i\nu\zeta/2 & \text{if } m=\pm1 \\ 0 & \text{if } m=\pm2 \end{cases} \tag{200}
$$

$$
\lim_{\theta_S\to 0} -im\csc\theta\left(\cot\theta Y^* - Y'^*\right) = \begin{cases} 0 & \text{if } m=0 \\ 0 & \text{if } m=\pm1 \\ \pm i\nu\zeta\sqrt{\zeta^2-2}/4 & \text{if } m=\pm2 \end{cases} \tag{201}
$$

$$
\lim_{\theta_S\to 0} -im\cot\theta\left(Y'^* - \csc\theta Y^*\right) = \begin{cases} 0 & \text{if } m=0 \\ 0 & \text{if } m=\pm1 \\ \mp i\nu\zeta\sqrt{\zeta^2-2}/4 & \text{if } m=\pm2 \end{cases} \tag{202}
$$

In these expressions, $\nu = \sqrt{\frac{2l+1}{4\pi}}$ and $\zeta^2 = l(l+1)$. For $|m| > 2$ all limits tend to zero, hence if the limits are put into the expression for the spherical harmonic coefficients, we obtain after reworking some terms:

$$
E_{km} = \begin{cases} \nu \Big(f_r U_k - M_{rr}\dot{U}_k + \big((V\zeta^2 - 2U)/2r_S\big)\left(M_{\theta\theta} + M_{\phi\phi}\right) \\ \qquad + \big(W\zeta^2/2r_S\big)\left(M_{\theta\phi} - M_{\phi\theta}\right)\Big) & \text{if } m = 0 \\ \frac{\nu\zeta}{2}\Big((V \mp iW)(\mp f_\theta + i f_\phi) + (\dot{V} \mp i\dot{W})(\pm M_{\theta r} - iM_{\phi r}) \\ \qquad + (U - V \pm iW)(\pm M_{r\theta} - iM_{r\phi})/r_S\Big) & \text{if } m = \pm 1 \\ \frac{\nu\zeta\sqrt{\zeta^2-2}}{4r_S}(V \mp iW)(M_{\phi\phi} - M_{\theta\theta} \pm iM_{\theta\phi} \pm iM_{\phi\theta}) & \text{if } m = \pm 2 \end{cases} \tag{203}
$$

In this expression, the θ-direction corresponds to the south, the ϕ-direction is east, because we chose $\phi = 0$ as our approaching meridian.

Of course, for practical reasons, it is not really useful to have an expression for a seismogram with the source at the North pole, since you want to be able to produce a seismogram with an arbitrary source location. The easiest way to solve this problem is to define a new spherical polar coordinate system, the *epicentral coordinate system*, with the source location as the north pole. The colatitude then becomes the angular epicentral distance between the source and receiver, the longitude is the azimuth to the receiver, measured in counterclockwise sense from due south at the source. More about the epicentral coordinate system can be found in [27], Chapter 10.1. I will now state the results expressing the epicentral distance Θ and azimuth Φ in terms of the source location (r_S, θ_S, ϕ_S) and receiver location (r, θ, ϕ) in ordinary spherical polar coordinates.

$$
\begin{aligned}
\cos\Theta &= \sin\theta_S \sin\theta \cos(\phi_S - \phi) + \cos\theta_S \cos\theta \\
\cos\Phi &= \csc\Theta \left(\cos\theta_S \sin\theta \cos(\phi - \phi_S) - \sin\theta_S \cos\theta\right) \\
\sin\Phi &= \csc\Theta \sin\theta \sin(\phi - \phi_S)
\end{aligned} \tag{204}
$$

The seismograms s_Θ and s_Φ produced with this coordinate system are in the longitudinal and transverse direction respectively. If we want to calculate seismograms in the northern and eastern direction we just have to apply a rotation to the results:

$$
s_N = -s_\Theta \cos\xi - s_\Phi \sin\xi \tag{205}
$$

$$
s_E = -s_\Theta \sin\xi + s_\Phi \cos\xi \tag{206}
$$

where subscripts N and E refer to the components in the directions North and East and where ξ is the conventional azimuth of the source from the receiver:

$$
\begin{aligned}
\cos\xi &= \csc\Theta \left(\cos\theta_S \sin\theta - \sin\theta_S \cos\theta \cos(\phi - \phi_S)\right) \\
\sin\xi &= \csc\Theta \sin\theta_S \sin(\phi_S - \phi)
\end{aligned} \tag{207}
$$

Now all ingredients for the calculation of a synthetic seismogram on a SNREI earth have been assembled. In the Results chapter the computational procedure is discussed and one example calculation is presented and will be compared to real data and other synthetic seismograms in the Discussion section.

3 Results

In this chapter, a description of the numerical procedure used to calculate synthetic seismograms is given, together with the resulting synthetic seismograms and real data acquired from the seismograph at Fort Hoofddijk, Utrecht to be able to make a comparison between the theory and reality.

To do this, one particular event is analyzed, the 9.0 MW earthquake on 11-03-2011 near the east coast of Honshu, Japan has been chosen. This event, which had a very destructive impact on Japan, caused maximum displacements in the Netherlands of approximately 1 centimeter.

Many moment tensor inversions have been performed for this earthquake, I use the USGS Centroid moment tensor inversion. [30] It gives the following earthquake parameters:

$$\begin{cases} \text{depth} & = 10 \text{ km} & \text{latitude} & = 38.308° \\ \text{longitude} & = 142.383° & & \\ M_{rr} & = 2.03 \cdot 10^{22} \text{ Nm} & M_{\theta\theta} & = -0.16 \cdot 10^{22} \text{ Nm} \\ M_{\phi\phi} & = -1.87 \cdot 10^{22} \text{ Nm} & M_{r\theta} & = 2.06 \cdot 10^{22} \text{ Nm} \\ M_{r\phi} & = 3.49 \cdot 10^{22} \text{ Nm} & M_{\theta\phi} & = -0.60 \cdot 10^{22} \text{ Nm} \end{cases} \tag{208}$$

The moment tensor is symmetric and the force terms are zero, since an earthquake does not apply a net force or torque on the Earth.

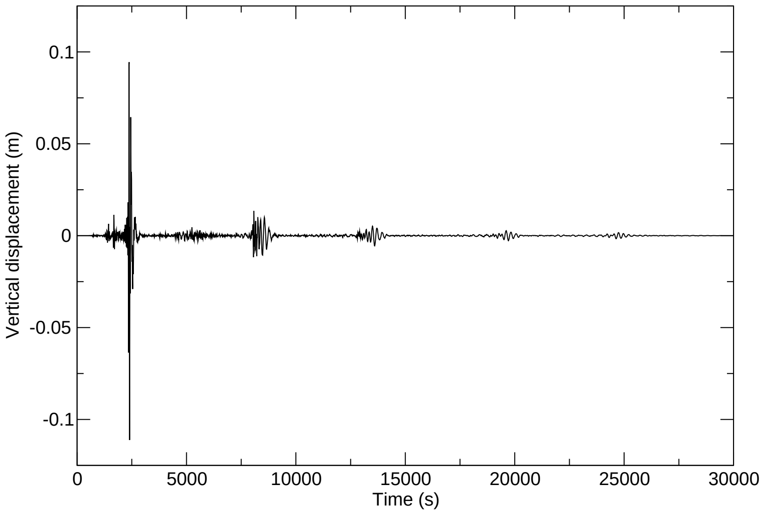
With all earthquake parameters known, two more things have to be known to be able to calculate a seismogram: the model parameters and the receiver location.

For the model parameters, I use the PREM Earth model [21]. The eigenfrequencies and mode shapes for the PREM model have been previously calculated using the standard code for this type of calculation `MINEOS` [32]. These are regarded as input for this calculation. The receiver location is $(\text{latitude}, \text{longitude}) = (52.088°, 5.172°)$

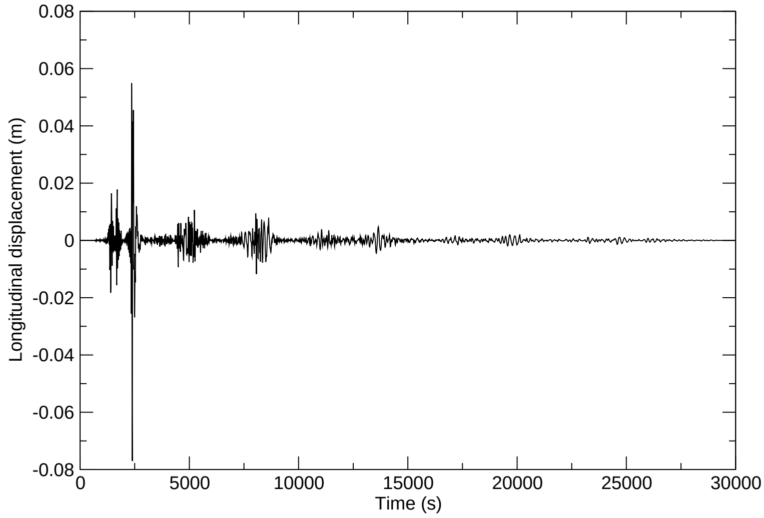
The numerical procedure to calculate synthetic seismograms is as follows:
First the eigenfrequencies and mode shapes for the PREM model are read in from a file. Then the program reads in the source parameters, source and receiver location are read in from an input file. Then the excitation coefficients E_{km} from equation (203) are calculated for each mode.
Then the actual seismogram can be calculated. For this, the epicentral coordinates (204) of the receiver are calculated first.
Then, within a sum over k containing the time dependent part $\frac{1-e^{\alpha_k t}\cos(\omega_k t)}{\omega_k^2}$, the sum over m in equation (173) is calculated, where the eigenfunctions $\vec{s}_k^m\left(\vec{x}^0\right)$ are calculated at the receiver in the epicentral coordinate system. Then, rotation (207) is applied and all seismograms are filtered with a tapered cosine window with a pass band from 3-29 mHz and a cutoff frequency of 32 mHz.

The resulting synthetic seismograms can be found in figure 2. As a comparison, the ground motion measured at Fort Hoofddijk, Utrecht, is presented in figure 3. The vertical displacement is parallel to the unit vector $\hat{r}$. The longitudinal displacement is parallel to $\hat{\Theta}$, in the direction of the source-receiver great circle. The transverse displacement is parallel to $\hat{\Phi}$, perpendicular to the vertical and longitudinal directions. The raw data has been convolved with the instrument response and filtered for low frequencies with a tapered cosine window with a pass band from 3-29 mHz and a cutoff frequency of 32 mHz. This is done to make it possible to compare the synthetic seismograms with the data, since the synthetic seismogram uses normal modes with a frequency up to 32 mHz.

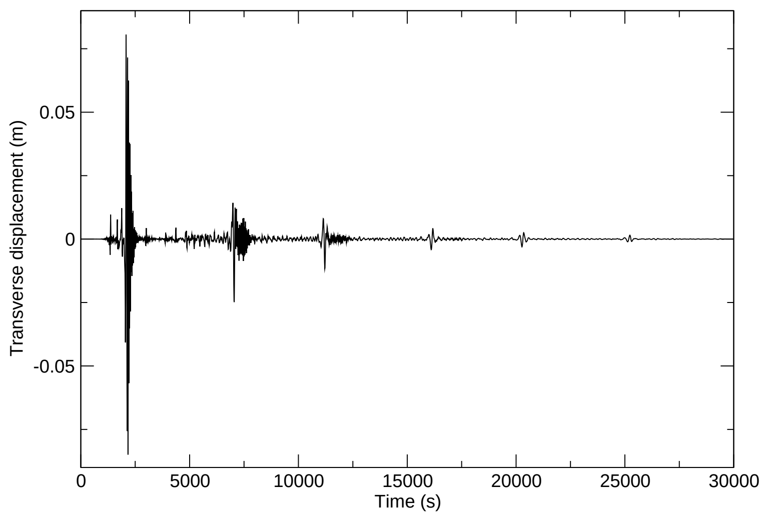
Roughly speaking, a seismogram consists of three important arrivals: the P-wave arrival, which is the first wave to reach the seismograph; the S-wave, which is the second arrival; and the surface waves, which have the largest amplitude. The most obvious difference between synthetics and data is their mismatch in amplitude. The second, third, fourth and fifth arrival of the surface waves (a surface wave has multiple arrival times because it stays on the surface, which is closed) is also clearly visible, both in the data and the synthetics. To see the other differences more clearly, a renormalized synthetic seismogram has been made so that the amplitudes match. A shorter time period is chosen as well. Both the synthetic and the real seismograms can be found in figure 4. It can be seen that the P-wave arrival of data and synthetics coincide almost precisely. The surface waves in the synthetics arrive significantly earlier than in the real data. After the first surface wave arrival, the data is more 'messy' than the synthetic signal. The possible reasons for these discrepancies will be discussed in the next section.



(a) Vertical displacement

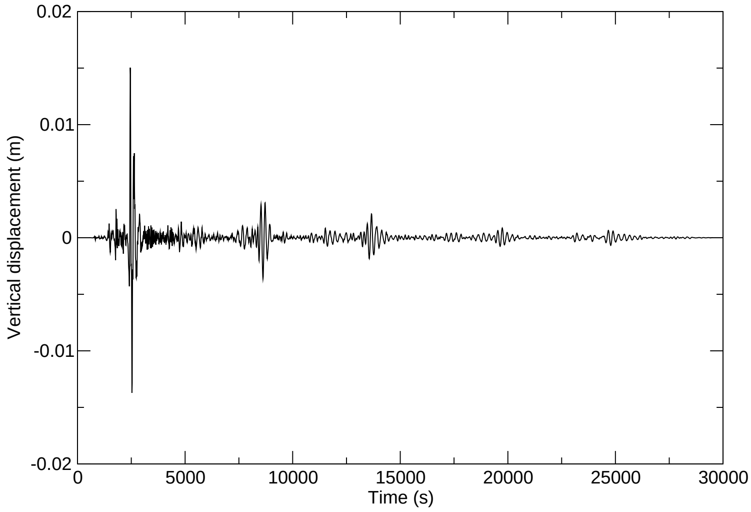


(b) Longitudinal displacement

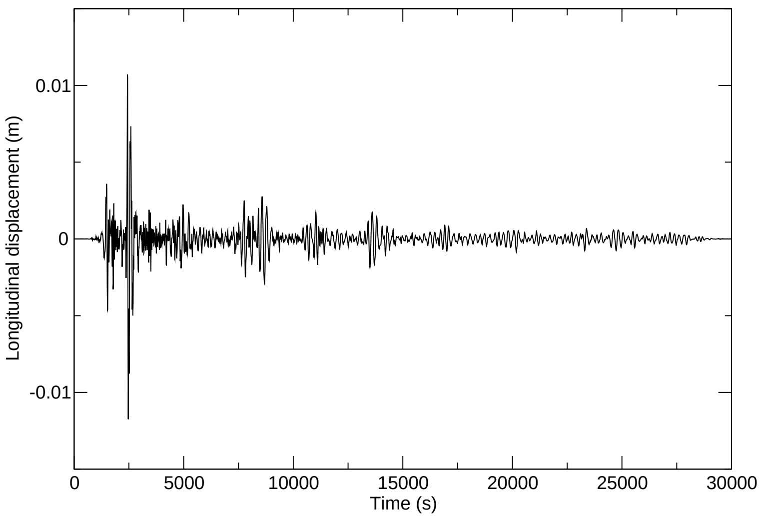


(c) Transverse displacement

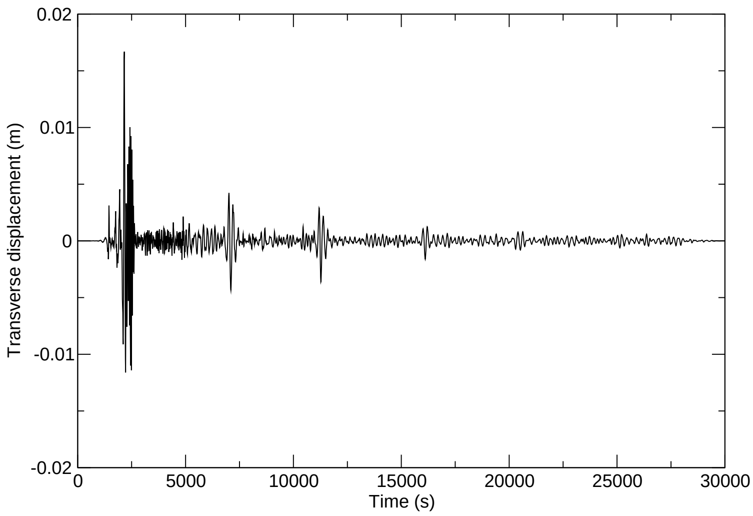
Figure 2: Synthetic displacement at Fort Hoofddijk, Utrecht in meters. The time at the origin is the time of the event.



(a) Vertical displacement

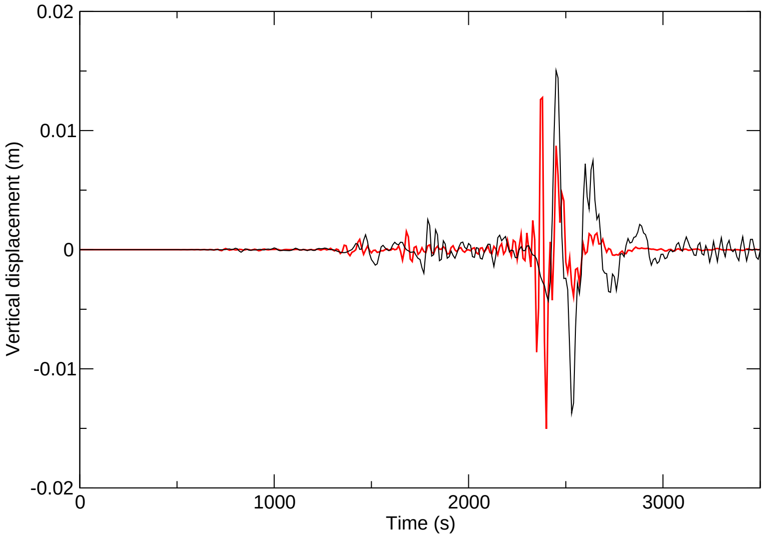


(b) Longitudinal displacement

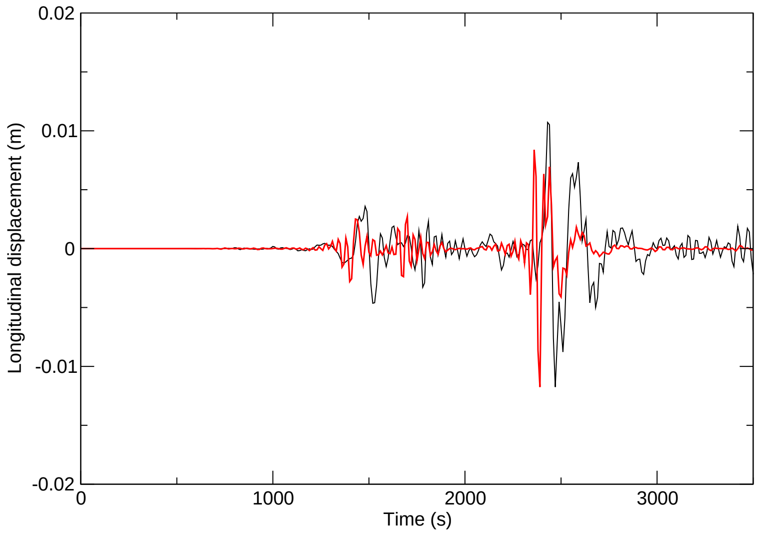


(c) Transverse displacement

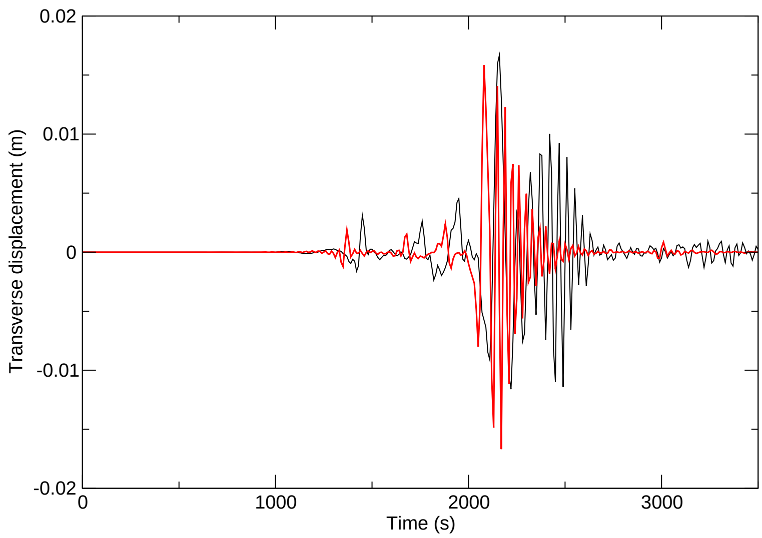
Figure 3: Displacement at Fort Hoofddijk, Utrecht in meters. The time at the origin is the time of the event.



(a) Vertical displacement



(b) Longitudinal displacement



(c) Transverse displacement

Figure 4: Displacement at Fort Hoofddijk, Utrecht in meters. The time at the origin is the time of the event.

4 Discussion

The outline of this section is as follows. First a very short description of this study will be given, followed with a discussion of the results. Then the importance of synthetic seismograms will be discussed, and directions of further research will be given.

The main goal of this study is to give an introduction into the theory of normal mode seismology from first principles. For this, the free oscillations of a spherically symmetric, non-rotating, elastic and isotropic earth are obtained. The excitation problem is also discussed and an expression for a theoretical seismogram for a point source is derived. As an application, this expression is used to generate a synthetic seismogram for the 9.0 MW earthquake on 11-03-2011 near the east coast of Honshu, Japan. This synthetic seismogram is compared with real data.

The main difference between synthetics and data is their amplitude. The amplitude of the synthetics is approximately 6 times bigger than the amplitude of the data. The wave form is much better in agreement, the P-wave arrival at 1300 seconds is however better in agreement than the surface wave arrival, for which the arrival time in the theoretical seismogram is earlier, at 2400 seconds. The real data show an arrival time of the surface waves at approximately 2550 seconds. The theoretical seismogram is also less noisy after the surface wave arrival.

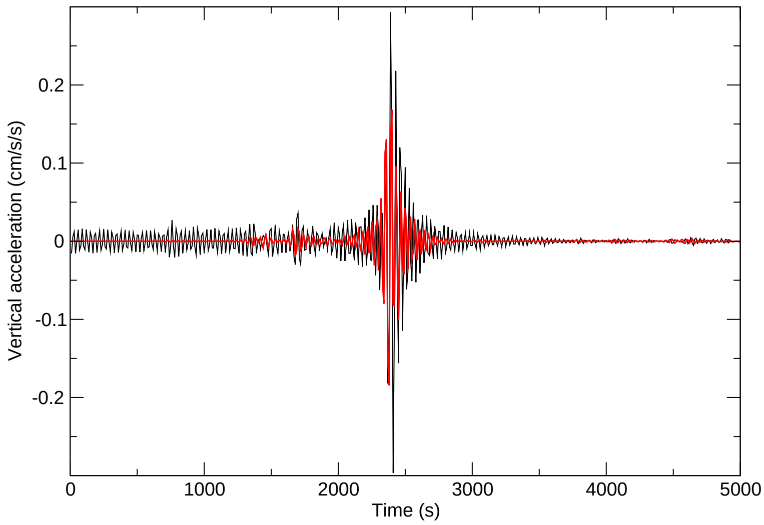
The difference in amplitude is something you would not expect, since the moment tensor components are determined with an inversion technique using similar expressions for the seismogram as the one used in this study. It might have something to do with the specific earthquake chosen. Because the event was very big, it might be that the point source approximation is not valid anymore. Now all energy emerges from one point, it might very well be that the amplitude of the synthetic seismogram would be lower if the earthquake energy was spread over a larger source area by taking for example more point sources in the earthquake region replacing the single point source used in this study.

It is quite certain that the mismatch in amplitude is not a numerical issue, since another program exists (`apsyn`) [33], doing the same calculation, giving the same amplitude. This can be seen in figure 5. The reason the signal obtained from `apsyn` is more noisy and has a bigger amplitude is that the signal obtained from this study has been filtered more. But most importantly, the amplitudes are in the same order of magnitude.

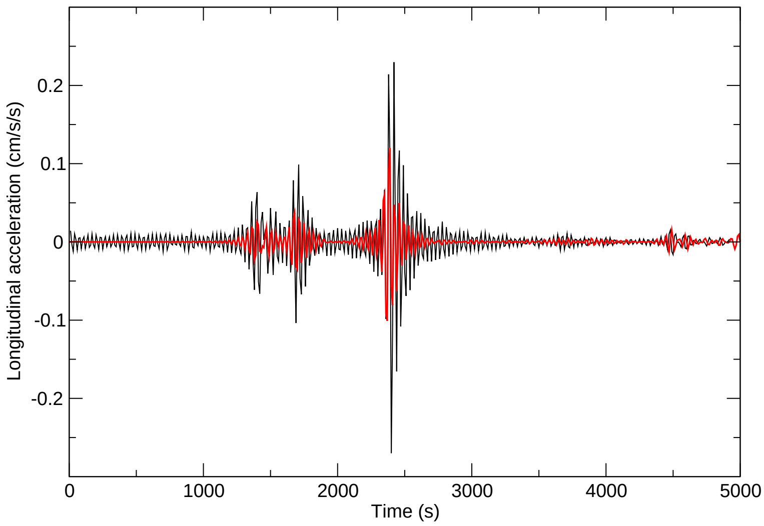
A possible explanation for the difference in P-wave arrival match with the surface wave match is the SNREI model: it is not laterally heterogeneous. Since the surface waves are trapped on the Earth's surface, which is not homogeneous at all, differences with the mean homogeneous earth occur quickly. The surface waves arriving in the Netherlands from Japan mostly travel through continental lithosphere, in which the surface wave speed is lower than in oceanic lithosphere, because of the lower density of the continental lithosphere. The P-waves travel through the deeper mantle, which is less heterogeneous than the crust and therefore the arrival times match better.

A reason for other differences between the real data and the synthetics is the fact that after the Japan earthquake many strong aftershocks took place, which were significantly lower in magnitude, but which could be the explanation for the noise in the data, which lacks in the synthetic seismogram.

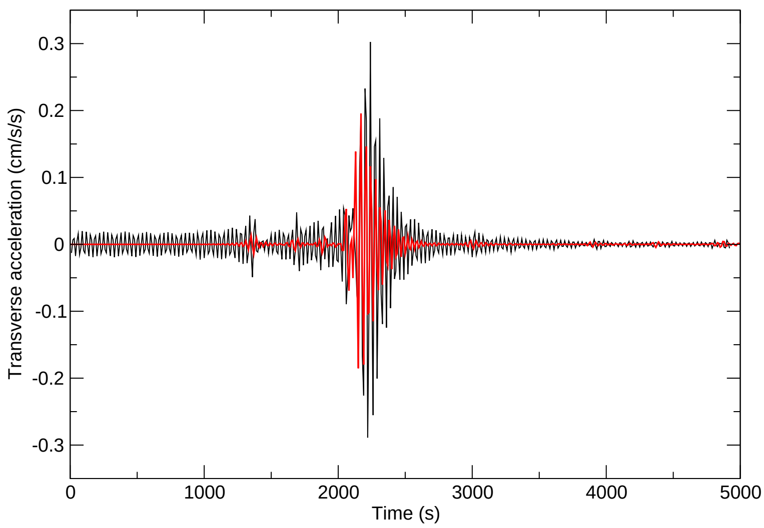
The importance of synthetic seismograms for seismology is its application in the solution of inverse problems. By comparing synthetic seismograms with real data, the misfit between the two originates from deviations of the model and source parameters from the real Earth situation. The misfit can be defined in several ways. One could define the misfit in the frequency domain for the eigenfrequencies of the Earth. Another common way of defining misfit is to take the difference between the synthetic seismogram and the real data, square it and integrate it over the entire seismogram. Often, certain portions of the seismogram are taken. For example, one can take the mismatch of the P-wave arrival times, S-wave arrival times, amplitude or surface waves. As soon as the misfit is defined, the challenge is to find an inverse operation which can be performed on the misfit to obtain the model which minimalizes the misfit. An example of the solution of such a problem can be found in [34].



(a) Vertical displacement



(b) Longitudinal displacement



(c) Transverse displacement

Figure 5: Synthetic acceleration at Fort Hoofddijk, Utrecht in meters. The black signal is from `apsyn`, the red signal is from this study. The time at the origin is the time of the event.

The main direction of further research is the application of normal mode synthetic seismograms in inverse problems, to obtain a more detailed picture of the Earth's interior. Because nowadays more advanced codes exist for calculating synthetic seismograms, this relatively simple way of calculating synthetics for a SNREI Earth can be used as a starting point in the development of an inverse method, since calculations can be done quickly. A more specific research direction is to find out why such a big amplitude difference between data and synthetics exists in this particular study. The same comparison has to be made for a smaller and less powerful source to see whether the cause is indeed the failure of the point source approximation.

Regarding the calculation of synthetic seismograms, of course it is possible to do better than the result presented in this study. Important effects that are neglected such as the rotation and ellipticity of the Earth, which causes splitting in de $2l+1$ modes corresponding to each set of modes (n, l). Lateral heterogeneity and non hydrostatic initial stress will also change the results, as well as anisotropy. There are codes currently available taking this into account, but a discussion of the theory behind these codes was not the aim of this study.

5 Conclusions

The conclusions of this study can be split in two categories: conclusions regarding the theory of the normal modes of oscillation of the Earth and the theoretical expression of a seismogram and conclusions about the application of the expression of a seismogram on the 9.0 MW earthquake on 11-03-2011 near the east coast of Honshu, Japan.

To find the normal modes of oscillation of the Earth, the normal modes of a spherically symmetric, non-rotating, elastic and isotropic Earth model (SNREI) are discussed. A linear differential equation for free oscillations with small displacements $\vec{s}$ is derived which has the form $\mathcal{L}_i(\vec{s}) = \rho^0\left(\vec{x}^0, t\right)\partial_t^2 s_i\left(\vec{x}^0, t\right)$, where $\mathcal{L}_i$ is a linear, self adjoint operator acting on the displacement field $\vec{s}$, depending only on known model parameters. The equilibrium density distribution is given by ρ^0, ∂_t^2 denotes the second partial derivative with respect to time.

This equation is solved in spherical coordinates, with a vector spherical harmonic expansion of the displacement, under boundary conditions of continuous traction on all interfaces of the model and zero traction on the surface. Displacement is continuous everywhere, except at fluid-solid boundaries where displacement tangential to the boundary can be discontinuous.

The differential equations allow two types of normal modes: toroidal oscillations in which the motion is perpendicular to straight lines through the center of the Earth. The other type of oscillations are called spheroical oscillations in which both radial and tangential motion exists. All separate modes are completely decoupled from each other.

If all normal modes of the Earth have been determined, any seismogram can be obtained by performing a normal mode sum in which the coefficient for each mode is determined by the seismic source. The seismic source can be represented by a force f_i and a moment tensor M_{ij} in the case of a point source. For an earthquake, $f_i = 0$ and M_{ij} is symmetric. The expression for a seismogram is:

$$\vec{s}\left(\vec{x}^0, t\right) = \sum_k \frac{1 - e^{\alpha_k t}\cos(\omega_k t)}{\omega_k^2} \sum_m E_{km}\vec{s}_k^m\left(\vec{x}^0\right) \tag{209}$$

where $\vec{s}(\vec{x}^0, t)$ is the displacement as a function of the position of a particle initially at $\vec{x}^0$ at time t. Indices k and m are numbers identifying separate modes, α_k is a constant determining the decay of a mode (k, m). The eigenfrequency of mode (k, m) is given by ω_k. The excitation coefficient for a mode (k, m) is given by E_{km} and is determined by the source. The displacement of a normal mode (k, m) is given by $\vec{s}_k^m$.

The synthetic seismogram calculated for the earthquake in Japan has generally the same waveform as the real data. There are however some remarkable differences. The amplitude of the theoretical seismogram is approximately 6 times bigger than the amplitude of the data. The arrival time of the surface waves is earlier in the theoretical seismogram, but the P-wave arrival times coincide.

Possible explanations for these differences are the scale of the event and the heterogeneity of the lithosphere. Because the Japan earthquake was an event which did not occur at a point, but in a larger region, the point source approximation might not be valid. This is why the equivalent point source might produce such large amplitudes, because all the energy is released at one instant and at one point. The path between Japan and the Netherlands consists mostly of continental lithosphere. In continental lithosphere, the surface wavespeed is relatively slow, which explains the late arrival time of the real surface wave signal relative to the synthetics. Because P-waves travel through the mantle, which is less heterogeneous than the crust, the P-wave arrival time misfit is small.

The normal modes of the Earth can be used to tackle inverse problems regarding the structure of the Earth or the description of an earthquake source. Programs are available to calculate more precise synthetic seismograms which take into account more Earth model features, such as rotation, ellipticity and lateral heterogeneity, but these are impractical if a new inverse technique is developed because of the long calculation time. Synthetic seismograms calculated with the expressions in this study have as main advantage that a relatively accurate synthetic seismogram can be generated in a couple of seconds. Besides, the theory presented in this study is a good introduction in normal mode seismology and prepares for the next step in the theory: the breaking of the spherical symmetry of the Earth.

References

[1] J. H. Woodhouse, Lecture notes, obtained from Jeannot Trampert

[2] A. M. Dziewonski and J. H. Woodhouse, Studies of the seismic source using normal-mode theory. In Kanamori, H. and Boschi, E., editors, Earthquakes: observation, theory, and interpretation: notes from the International School of Physics "Enrico Fermi" (1982: Varenna, Italy), pages 45-137. North-Holland Publ. Co., Amsterdam., 1983.

[3] USGS, FAQs at `http://earthquake.usgs.gov`

[4] S. D. Poisson, 1829 Mémoire sur l'équilibre et le mouvement des corps élastiques, Mém. Acad. Roy. Sci. Inst. France, 8, 357-570.

[5] W. Thomson, 1863 On the rigidity of the Earth, Phil. Trans. Roy. Soc. Lond., 153, 573-582.

[6] W. Thomson, 1863 Dynamical problems regarding elastic spheroidal shells and spheroids of incompressible liquid, Phil. Trans. Roy. Soc. Lond., 153, 583-616.

[7] H. Lamb, 1882, On the vibrations of an elastic sphere, Proc. Lond. Math. Soc., 13, 189-212.

[8] J. W. S. Rayleigh, 1906, On the dilatational stability of the Earth, Proc. Roy. Soc. Lond., Ser. A, 77,486-499.

[9] A. E. H. Love, 1907, The gravitational stability of the Earth, Phil. Trans. Roy. Soc. Lond., Ser. A, 207, 171-241.

[10] A. E. H. Love, 1911, Some Problems of Geodynamics, Cambridge University Press, Cambridge. Reprinted in 1967 by Dover Publications, New York.

[11] L. M. Hoskins, 1920, The strain of a gravitating sphere of variable density and elasticity, Trans. Am. Math. Soc., 21, 1-43.

[12] J. H. Jeans, 1927, The propagation of earthquake waves, Proc. Roy. Soc. Lond., Ser. A, 102, 554-574.

[13] N. Jobert, 1956, Évaluation de la période d'oscillation d'une sphère hétèrogene, par application du principe de Rayleigh, Comptes Rendus Acad. Sci. Paris, 243, 1230-1232.

[14] N. Jobert, 1957, Sur la période propre des oscillations sphéroïdales de la Terre, Comptes Rendus Acad. Sci. Paris, 244, 921-922.

[15] N. Jobert, 1961, Calcul approché de la période des oscillations sphéroïdales de la Terre, Geophys. J. Roy. Astron. Soc., 4, 242-258.

[16] C. L. Pekeris and H. Jarosch, 1958, The free oscillations of the Earth. In Benioff, H., Ewing, M., Howel, Jr., B.F. & Press, F., editors, Contributions in Geophysics in Honor of Beno Gutenberg, pages 171-192, Pergamon, New York.

[17] H. Takeuchi, 1959, Torsional oscillations of the Earth and some related problems, Geophys. J. Roy. Astron. Soc., 2, 89-100.

[18] Z. Alterman, H. Jarosch and C. L. Pekeris, 1959, Oscillations of the Earth, Proc. Roy. Soc. Lond., Ser. A, 252, 80-95.

[19] C. L. Pekeris, Z. Alterman and H. Jarosch, 1961, Comparison of theoretical with observed values of the periods of the free oscillations of the Earth, Proc. Nat. Acad. Sci. USA, 47, 91-98.

[20] F. Gilbert and A. M. Dziewonski, 1975, An application of normal mode theory to the retrieval of structural parameters and source mechanisms from seismic spectra, Phil. Trans. Roy. Soc. Lond., Ser. A, 278, 187-269.

[21] A. M. Dziewonski and D. L. Anderson, 1981, Preliminary reference Earth model, Phys. Earth Planet. Int., 25, 297-356.

[22] F. Gilbert, 1970, Excitation of the normal modes of the Earth by earthquake sources, Geophys. J. Roy. Astron. Soc., 22, 223-226.

[23] B. V. Kostrov, 1970, The theory of the focus for tectonic earthquakes, Izv. Bull. Akad. Sci. USSR, Phys. Solid Earth, 4, 84-101.

[24] A. M. Dziewonski, T.-A. Chou and J. H. Woodhouse, 1981, Determination of earthquake source parameters from waveform data for studies of global and regional seismicity, J. Geophys. Res., 86, 2825-2852.

[25] K. Aki and P. G. Richards, 2002, Quantitative Seismology, Chapter 2.1, University Science Books, California, Second Edition.

[26] W. S. Slaughter, 2002, The linearized theory of elasticity, page 51, Birkhäuser Boston.

[27] F. A. Dahlen and J. Tromp, 1998, Theoretical Global Seismology, Princeton University Press, Princeton, New Jersey.

[28] P. M. Morse and H. Feshbach, 1953, Methods of Theoretical Physics, pp 1898-1901, McGraw-Hill.

[29] G. E. Backus and M. Mulcahy, 1976, Moment tensors and other phenomenological descriptions of seismic sources, Geophys. J. Roy. Astron. Soc., 46, 341-361 and Geophys. J. Roy. Astron. Soc., 47, 301-329.

[30] The earthquake data can be obtained at:
http://earthquake.usgs.gov/earthquakes/eqinthenews/2011/usc0001xgp/

[31] The animations can be obtained at:
http://icb.u-bourgogne.fr/nano/MANAPI/saviot/terre/index.en.html

[32] The program can be obtained at the following internet address:
http://www.geodynamics.org/cig/software/mineos/

[33] A computer program made by John Woodhouse in the seventies. Obtained from Jeannot Trampert at the University of Utrecht.

[34] J. Tromp, C. Tape and Q. Liu, 2005, Seismic Tomography, Adjoint Methods, Time Reversal, and Banana-Doughnut Kernels, Geophysical Journal International, Vol. 160, Issue 1, 195-216.