Topographic Filtering and Reflectionless Transmission of Long Waves

Leo R.M. Maas
Netherlands Institute for Sea Research, P.O. Box 59, 1790 AB Texel, The Netherlands

Abstract The equation governing the passage of monochromatic, long waves over variable topography can be transformed into a Schrödinger equation. There are several transformations accomplishing this. A 'naive' transformation (in which only the horizontal coordinate is stretched) has a 'potential' that is non-vanishing, even if the slope in topography vanishes. A transformation, in which also the surface elevation field is stretched, has a 'potential' vanishing outside the sloping region. This transformation has the property that it displays scattering against a background of adiabatic variations. That is, the plane waves that result if the potential is approximated to vanish identically, display amplitude and wave length variations, in the original frame, as in WKB-theory. For smooth bottom profiles, typical for the continental slope, the potential has a positive lobe, the top of which acts as a 'topographic cut-off frequency'. This lobe is missed by piecewise-linear topographies. The trapped modes of the adjacent negative side lobe are the topographic Rossby wave modes. For a particular smooth bottom-profile, long waves, coming from a specific direction, can be shown to pass reflectionless.

Schrödinger equation

It is often instructive to consider scattering problems in terms of a Schrödinger equation, because it allows one to qualitatively assess the local nature of the wave field under consideration. The relevance of this equation for the long-wave equations is discussed in the next section. The Schrödinger equation,

$$\frac{d^2\Psi}{dx^2} + [E - V(x)]\Psi = 0, \qquad (1)$$

describes the shape of the state variable $\Psi(x)$, related to the wave field, $\psi(x,t) = \Psi(x)e^{-i\sigma t}$, (with t denoting time and σ the frequency), due to inhomogeneities of the 'medium' through which the wave propagates as a function of the coordinate x. The variations of the medium are here represented by the 'potential' $V(x)$. For localized variations of the medium it is natural to expect the potential to vanish outside the x-region for which these variations occur. For values of the 'energy' E greater than the maximum value of the potential, V_{max}, (like E_1 in Figure 1) the quantity in square brackets in (1) is everywhere positive and hence the solution is locally sinusoidal. It is therefore expected that the wave will not be greatly attenuated by the scattering potential. If E drops below this maximum, but is otherwise positive (E_2 in Figure 1), this quantity is negative over some x-interval and hence the wave field will be exponentially decaying over this range leading to great attenuation of an incoming wave field: waves can pass only through 'tunneling'. For negative values of the 'energy', $V_{min} < E < 0$, like E_3 in Figure 1, trapped waves can exist. Finally, for still lower values, like E_4, wave solutions no longer exist. Here we will concentrate on positive values of the 'energy' parameter, E. Actually, this quantity is usually not related to the true energy of the wave field, but rather is to be regarded a metaphor for the frequency of the wave involved. Therefore, for an incoming spectrum of waves, the existence of a maximum in the potential, V_{max}, can directly be interpreted as a (soft) cut-off frequency: for waves with energy (frequency) well above it, waves can pass unimpeded, for waves with energy (frequency) below V_{max}, waves are strongly attenuated. The cut-off frequency is *soft*, however, since no rigorous cut-off (zero transmission) of the incoming wave field below this frequency is implied. A potential like the one shown in Figure 1 thus acts as a high-pass filter.

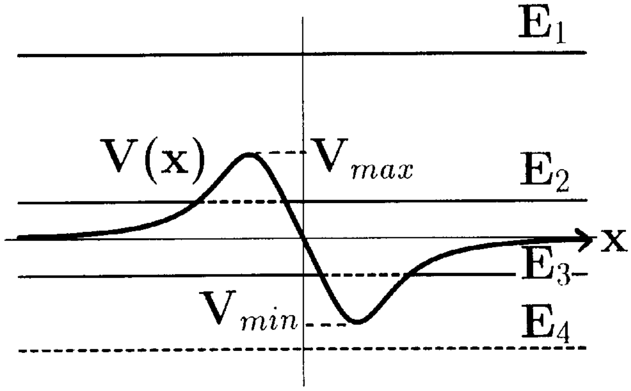

Figure 1. Sketch of a 'typical' potential $V(x)$ and four levels of the energy, E_1 to E_4. Solid (dashed) parts of the energy levels refer to sinusoidal (exponential) behaviour of the wave field.

Topographic filtering

Consider a long, plane wave propagating on an f-plane at the surface of a homogeneous fluid incident on a smoothly and monotonically varying topography $H(x)$. Let the surface elevation take the form $\zeta(x)\exp(ily - i\sigma t)$. Here l indicates the wavenumber in the along-slope direction y. We non-dimensionalize with length (L) and depth (H_0) scales appropriate to the shelf edge and with the inertial frequency, f:

$$\mathbf{x} \to L\mathbf{x},\ l \to l/L,\ H \to H_0 h(x),\ \sigma \to f\sigma,$$

where $h(x)$ is a nondimensional shape-function, modeling the shelf edge. The cross-isobath structure of the elevation field, $\zeta(x)$, is then determined by

$$\frac{d}{dx}\left(h\frac{d\zeta}{dx}\right) + \left[\epsilon^2(\sigma^2-1) - l^2 h - \frac{l}{\sigma}\frac{dh}{dx}\right]\zeta = 0. \quad (2)$$

Here $\epsilon = L/R$ is the ratio of the external scale L and the Rossby deformation scale, $R = \sqrt{gH_0}/f$. The square of ϵ is known as the divergence parameter. In general this is a small quantity. For instance, taking $H_0 = 1$ km, $L = 100$ km, $g = 10m^2s^{-1}$, $f = 10^{-4}s^{-1}$ one obtains $\epsilon = 10^{-1}$. However, since the theory may equally be applied to interfacial waves this quantity may be order one. In this case depth is replaced by equivalent depth, $h_e(x) = h_1 h_2(x)/(h_1 + h_2(x))$ and gravity, g, by reduced gravity g', being equal to g multiplied by the relative density difference of thetwo layers. Typical values of these lead to a phase speed of about one-hundredth of its value in the barotropic case.

Equation (2) can be 'naively' transformed to a Schrödinger equation by multiplying it with h and identifying hd/dx with $d/d\xi$, which amounts to a stretching of the horizontal coordinate,

$$\xi = \int_0^x \frac{1}{h(x')}dx', \quad (3)$$

in inverse proportion to water depth. The equation then takes the form

$$\frac{d^2\zeta}{d\xi^2} + \left[\epsilon^2(\sigma^2-1)h(\xi) - l^2h^2(\xi) - \frac{l}{\sigma}\frac{dh}{d\xi}\right]\zeta = 0, \quad (4)$$

where $h(\xi) = h(x(\xi))$. This equation was employed by *Saint-Guily* (1976), who, for the depth profile

$$h(\xi) = 1 + \lambda \tanh\xi, \quad (5)$$

with $\lambda \in (-1, 1)$, was able to calculate the *trapped* modes — the topographic Rossby waves — exactly. For the case of *free, propagating* waves, however, the potential is not 'physically realistic' as one may observe by concentrating for example on waves of normal incidence ($l = 0$). In this case (4) simplifies to

$$\frac{d^2\zeta}{d\xi^2} + \epsilon^2(\sigma^2-1)h(\xi)\zeta = 0,$$

which, for the tanh-shaped topography considered, leads to a potential that is non-vanishing at infinity and therefore does not satisfy the requirement that the scattering be localized.

For this reason a transformation is employed that stretches not only the horizontal but also the vertical coordinate (the elevation). Discussion is for the sake of simplicity here limited to waves of normal incidence ($l = 0$), for which case (2) takes the form

$$\frac{d}{dx}\left(h\frac{d\zeta}{dx}\right) + \epsilon^2(\sigma^2-1)\zeta = 0. \quad (6)$$

The general case of obliquely incident waves can be treated likewise. If we now adopt the following stretching (*Morse and Feshbach,* 1953, p.730)

$$\xi = \int_0^x \frac{1}{\sqrt{h(x')}}dx', \quad \zeta = \frac{Z}{h^{1/4}}, \quad (7)$$

then (6) takes the form

$$\frac{d^2Z}{d\xi^2} + \left[\epsilon^2(\sigma^2-1) - \frac{1}{h^{1/4}}\frac{d^2h^{1/4}}{d\xi^2}\right]Z = 0,$$

which is a Schrödinger equation once we identify energy and potential as $E = \epsilon^2(\sigma^2-1)$ and $V(\xi) = h^{-1/4}d^2h^{1/4}/d\xi^2$ respectively. This shows that 'energy' E is indeed related to frequency σ. Since the second ξ-derivative of $h^{1/4}$ is related to first and second x-derivatives of the topography $h(x)$ (see below), which vanish away from the sloping region, it is evident that this form of the potential does also vanish outside that region and hence is of the localised form we expect it to have as a scattering potential. Also, the expression for E nicely identifies its positive values with freely propagating, superinertial ($\sigma > 1$) waves and its negative values with trapped, subinertial ($\sigma < 1$) waves.

Without actually solving the resulting equation a number of inferences can be drawn from the form it has.

First, assume that we are dealing with energy-values (frequencies) which are much greater than the maximum value of the potential $E >> V_{max}$. Then, we can approximate the potential by assuming that it vanishes identically, $V(\xi) = 0$. In this case, the solutions in the transformed plane consist, of course, of plane waves of the form $Z = Z_0\ \exp(\pm i\sqrt{E}\xi)$. In the original frame this solution reads

$$\zeta = \frac{Z_0}{h^{1/4}} \exp(\pm i\sqrt{E} \int_0^x \frac{1}{\sqrt{h(x')}} dx' - i\sigma t),$$

which contains wave number variations (the x-derivative of the phase factor) inversely proportional to $\sqrt{h}$ and amplitude variations inversely proportional to $h^{1/4}$. The latter is known as Green's law (*Mei,* 1989). These are consistent with adiabatic variations such as occur in WKB approximation, which would have group velocity c_g and phase velocity proportional and hence wave number variations inversely proportional to $\sqrt{h}$, and which would require conservation of energy flux, proportional to $|\zeta|^2 c_g$. Since amplitude (and wave number) variations associated with adiabatic changes do not form part of the scattering process the physically appropriate frame of reference in which to consider scattering is that based on (7). Any amplitude and wave number variations obtained in that frame are truely associated with scattering.

Second, consider a typical monotonic shelf edge, like $h^{1/4} = 1 + \lambda \tanh \xi$, which is similar to the topography employed by *Saint-Guily* (1976) except that it now applies to the quarter power of the topography. Then, the potential takes the form

$$V(\xi) = -2\lambda \frac{T(1-T^2)}{1+\lambda T},$$

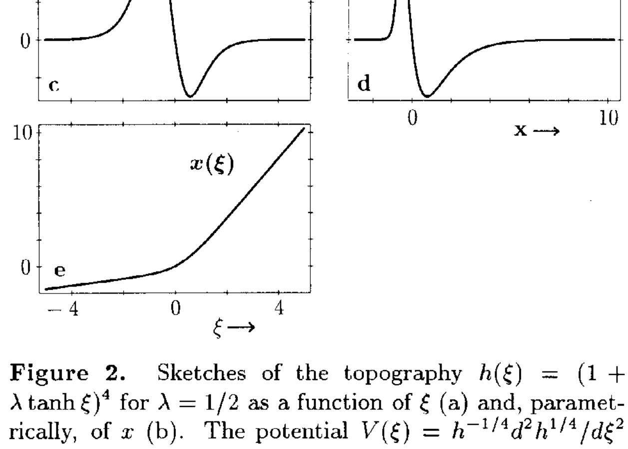

Figure 2. Sketches of the topography $h(\xi) = (1 + \lambda \tanh \xi)^4$ for $\lambda = 1/2$ as a function of ξ (a) and, parametrically, of x (b). The potential $V(\xi) = h^{-1/4} d^2 h^{1/4} / d\xi^2$ can be obtained from these, both in the transformed (c), as well as, again parametrically, in the original frame (d). Here the x-dependence on ξ is given by $x = \int_0^\xi \sqrt{h(\xi)} d\xi = \xi(1+\lambda^2) + 2\lambda \ln[\cosh \xi] - \lambda^2 \tanh \xi$, see (e).

where $T \equiv \tanh \xi$, see Figures 2 and 3a. The potential is observed to have the typical two-lobed shape adopted in the discussion of Figure 1, the positive lobe extending over the top of the shelf edge. The position of the maximum value of the potential, T_{max}, as well as its value at this position, V_{max}, can be determined analytically as a function of the 'depth-contrast parameter' λ. Their expressions are rather cumbersome and are therefore just shown graphically (Figure 3b). Now, since the topology of the problem would not change when we vary the shape of the monotonically sloping topography, we may expect the occurence of a positive lobe ($V(\xi) > 0$ for some range of ξ) to be a generic feature of this scattering problem. Therefore, following the discussion in the introduction, one may infer that there exists a topographic cut-off frequency $\sigma_T \equiv (1 + \epsilon^{-2} V_{max}(\lambda))^{1/2}$, for *any* monotonically-sloping topography, which is a function of the geometrical parameters (the divergence parameter ϵ^2 and the depth-contrast parameter λ). For the parameters considered previously $\epsilon = 10^{-1}$, and for $\lambda = 1/2$, which has $V_{max} \approx 1/2$, we obtain $\sigma_T \approx 7$. For interfacial waves, with $\epsilon \approx 1$, σ_T approaches the inertial frequency even closer ($\sigma_T \downarrow 1$). It is likely that this quantity σ_T must be observable as a dividing frequency when comparing adjacent directional deep-sea and shelf spectra, such that for $\sigma > \sigma_T$, waves can pass the sloping region fairly easy (and vice versa).

Third, expanding the expression of the potential we find

$$V = \frac{1}{4h}\frac{d^2h}{d\xi^2} - \frac{3}{16}\left(\frac{1}{h}\frac{dh}{d\xi}\right)^2 = \frac{1}{4}\frac{d^2h}{dx^2} - \frac{1}{16h}\left(\frac{dh}{dx}\right)^2.$$

Since $h > 0$ for all x, the second term in the last expression at the right is always negative. Hence the potential is positive only because of the existence of the first term in that expression, which is related to the (convex) curvature of the topography at the top of the shelf slope (*Meyer,* 1979). It is clear that

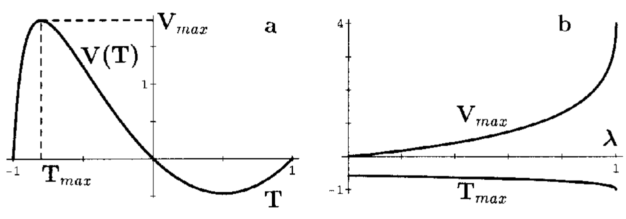

Figure 3. (a) Potential V as a function of T for $\lambda = 0.9$. Since $T \equiv \tanh \xi$ is a monotonic function of ξ this is a compact way of representing the ξ-dependence of the potential. In this figure the position, T_{max}, and value of the peak of the potential, V_{max}, have been indicated, which are shown as a function of λ in (b) and (c).

this term would be absent in piecewise linear topographies, which would therefore exclude the phenomenon of tunneling and, to some extent, topographic filtering.

Reflectionless transmission

Another artifact would actually be introduced when approximating a smooth, monotonic topography by a piecewise-linear topography. This is the spurious phenomenon of reflectionless transmission at certain discrete wave frequencies (which have wave lengths that are multiples of twice the size of the linearly sloping region). It is generally regarded that the occurence of these reflectionless frequencies is not realistic (*Kajiura,* 1963; *Meyer,* 1979; *Mei,* 1989). Indeed, for a *smooth* monotonic profile, *Kajiura* (1963) showed that the transmission coefficient is a steadily decreasing function of frequency. Reflectionless transmission of normally-incident long waves over smooth topographies is found only for (symmetric) ridges. *Fitz-Gerald* (1976) obtained this result semi-analytically without restriction on wave length by an iterative scheme (see also *Rouseau,* 1952).

Surprisingly, as will be shown below, reflectionless transmission may also occur for a smooth, monotonically-increasing depth profile. This occurs for a particular angle of incidence of the long waves. Ironically, this can be demonstrated most easily by employing the naive transformation (3) and using the *Saint-Guily* (1976) topography (5). Eq. (4) then reads

$$\frac{d^2\zeta}{d\xi^2} + \left[E - l^2(1+\lambda^2) + \lambda(E - 2l^2)\tanh\xi\right.$$

$$\left. -l\lambda\left(\frac{1}{\sigma} - l\lambda\right)\mathrm{sech}^2\xi\right]\zeta = 0.$$

This is of the form

$$\frac{d^2\zeta}{d\xi^2} + \left[E' + n(n+1)\mathrm{sech}^2\xi\right]\zeta = 0,$$

with $n \in \mathbb{N}$, for which *Kay and Moses* (1956) showed that the potential is reflectionless for any positive value of the energy E' (see also, *Lamb,* 1980). Identifying coefficients between these equations we find that long waves are able to pass the tanh-shaped shelf edge reflectionless provided

$$E = 2l^2, \tag{8}$$

and

$$l\lambda\left(\frac{1}{\sigma} - l\lambda\right) = -n(n+1). \tag{9}$$

Because $E' = E - l^2(1+\lambda^2)$ we verify from (8) that E' satisfies the positivity constraint: $E' = l^2(1-\lambda^2)$. Since

$$E = \epsilon^2(\sigma^2 - 1), \tag{10}$$

which, from the dispersion relation applied in the far field, equals $h(l^2 + k^2)$, we find, assuming the waves to enter from the deep region, where $h = 1 + \lambda$,

$$E = (1+\lambda)\frac{l^2}{\sin^2\alpha}.$$

Here the absolute value κ of the wavenumber vector $\mathbf{k} = (k, l) = \kappa(\cos\alpha, \sin\alpha)$ that makes an angle α with the x-direction, has been replaced in terms of l and α. Hence, inserting this expression for E in (8) and eliminating l^2, we find that waves coming from directions α_* determined by

$$\sin^2\alpha_* = \frac{1+\lambda}{2} \tag{11}$$

are able to pass this shelf edge reflectionless, provided they satisfy also the second constraint (9). From (10) and (8) we obtain σ as a function of $b \equiv l\lambda$:

$$\sigma = \sqrt{1 + a^2b^2}, \tag{12}$$

where $a^2 = 2/\lambda^2\epsilon^2$. Inserting this into (9) and plotting both the left and right-hand sides of this equation as a function of b (Figure 4), we find at their intersections the wave numbers for which waves coming from directions determined by (11) are able to pass the shelf edge reflectionless. From (12) their frequencies can be obtained. For $a > 1$, these are approximately determined by the two asymptotic (dashed) curves of Figure 4 which lead to

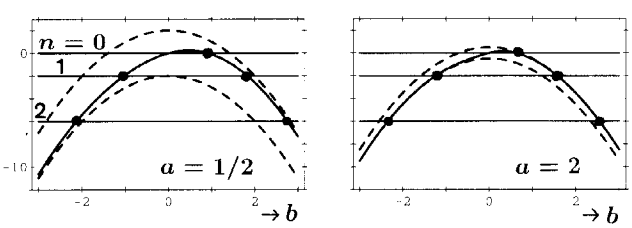

Figure 4. Plot of $b/\sqrt{1+a^2b^2} - b^2$ and $-n(n+1)$ (solid lines) as a function of b for two values of a and integer $n \in \{0, 1, 2\}$. Heavy dots indicate values of b (*i.e.* scaled along-isobath wavenumber l) for which waves, coming from a direction α_* given by (11), are able to pass the tanh-shaped shelf edge reflectionless. Dashed lines indicate the two asymptotes $\pm 1/a - b^2$.

$$\sigma_n^2 = 1 + \frac{l}{|l|}a + a^2 n(n+1), \quad n \in \mathbb{N}.$$

For $n = 0$ only one physically realistic solution is obtained, having the coast at its left (in the Northern Hemisphere), seen from the along-shelf propagation direction ($l > 0$). The expression of the frequency for general values of a is slightly more involved.

Conclusions

It is shown qualitatively that a monotonically-sloping shelf edge generally acts as a topographic filter for incoming long super-inertial waves. This filter can be characterized by a (soft) cut-off frequency above (below) which waves can pass the topography without (with) much attenuation. This frequency is solely dependent on parameters characterizing the geometry of the problem (topographic scales, latitude and earth rotation rate). The filtering properties are crucially dependent on the existence of a convex part of the bottom shape at the top of the shelf edge. It provides the positive lobe of the localized potential in the Schrödinger equation to which the scattering problem can be transformed. It is attractive to view this positive lobe of the potential of a shelf edge as providing a natural shield by which the shelf region is 'protected' against incoming waves. Each shelf edge, however, also has an Achilles' heel. It is transparent for waves of particular discrete frequencies, coming from two directions determined by the 'depth-contrast parameter', that are able to pass the shelf edge without any reflection. These are the two directions for which the shelf edge under consideration is particularly 'vulnerable'. Although no parameter-sensitivity analysis of these results has been made yet it is conjectured that a true shelf edge should show its 'vulnerability' over some range of angles and frequencies around those calculated. It would be useful, therefore, to make a catalogue for shelf edges around the world identifying these reflectionless angles and frequencies at each location.

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