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Lecture series given at the Institute of Meteorology and Oceanography (IMOU), University of Utrecht on the nonlinear dynamics of geophysical fluid flow.

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Lecture series given at
the Institute of Meteorology and Oceanography (IMOU),

University of Utrecht on

The Nonlinear Dynamics of Geophysical Fluid Flow.

by

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Preface

This report contains the material covered in a series of lectures given during the autumn semester of 1982 at the Institute of Meteorology and Oceanography, University of Utrecht. The principal lecturers were Erland Källén and Leo Maas. The course intends to demonstrate some techniques which can be applied to tackle nonlinear problems in geophysical fluid dynamics and it is mainly directed towards research students in meteorology and oceanography. The notes give a fairly detailed account of the topics covered on the course and we have also included some exercises which were discussed during the course. The solution of exercises was very valuable for a deeper understanding of the underlying and are to be seen as an integral part of the course.

These notes were written in a first preliminary version by A. van Delden, M. Scheffers and H. de Swart. As a result of one of the exercises, W. Verkley of the KNMI supplied a very general solution which is given in an appendix. L. Maas gave the lectures corresponding to chapter 3 of the notes, while E. Källén's lectures covered the rest of the material. A careful reading of the notes by G.J.E. van Heijst, H. Oerlemans, J.D. Opsteegh and C. Schuurmans has very much improved the readability of the material.

1. Introduction

One of the basic equations governing the dynamics of fluid flow both in the atmosphere and in the oceans, is Newton's second law of motion. Following a fluid particle with velocity $\overline{v}$ ($\overline{x}$,t) we have

$$\frac{d\overline{v}}{dt} = \sum_i \overline{F}_i \tag{1.1}$$

where $\sum_i \overline{F}_i$ is a summation over all viscous and pressure forces acting on the fluid particle. The $\overline{F}_i$'s are expressed as force per unit mass. The velocity field $\overline{v}$ is a function of space $\overline{x}$ and time t.
As equation (1.1) is written following a fluid particle we have to expand the total time derivative $\frac{d}{dt}$ in its space and time dependent parts to investigate what happens at a certain time and place, i.e.

$$\frac{\partial\overline{v}}{\partial t} + \overline{v} \cdot \nabla \; \overline{v} = \sum_i \overline{F}_i \tag{1.2}$$

When the equation of motion is written in this form, we see explicitly the nonlinear character of the equation. The local acceleration of a fluid particle at a certain point in space is governed not only by the forces at that point but also by the advection of momentum by the fluid itself in an infinitesimal surrounding of the point. It is this implicit nature of the advective nonlinearity which makes it rather complicated to handle in many cases. The nonlinear character of fluid flow is however also the reason why fluid dynamics is such a fascinating field of research. The nonlinearity is responsible for the lack of predictability of atmospheric flow beyond a week or so, but it may also contribute to the appearance of certain stable flow types which persist for a week or more. The onset and maintenance of turbulence in a fluid is certainly due to nonlinear advection while also regularly appearing convection patterns result from nonlinear advection. The advection may thus have a stabilizing effect depending on the particular flow situation.

In this series of lectures we will investigate how nonlinear advective effects may be analysed in simplified models based on Newton's second law of motion. The method of expanding the space dependent part of the velocity field in a series of orthogonal functions and then to limit the expansion to only a few terms will be applied to some

different models of geophysical fluid flow. The emphasis will be on the methods used, although some geophysically relevant examples will be worked out to show the applications of the various models. In order to demonstrate the nonlinear phenomena of breaking waves and to show a simple example of a series expansion of the solution in terms of orthogonal functions, the one dimensional advection equation will be analysed.
Some models including wave dispersion as well as an advective term will be used to discuss breaking of water waves. In this connection the Korteweg-de Vries equation will also be taken up.

In the second part of the course, the atmospherically relevant example of two dimensional, non-divergent fluid flow on a rotating sphere will be studied. The method of expanding the space dependent part of the solution in spherical harmonics and analyzing the advective effects through a coupled set of nonlinearly coupled ordinary differential equations, will be covered in detail. Examples of Rossby-Haurwitz wave instability and instabilities associated with mountain forcing at the lower boundary of a quasi-two dimensional flow will be given. Some particular localized two dimensional flow solutions of a modon type will be discussed.

In the final chapter we will discuss the two types of motion associated with the shallow water equations in a rotating coordinate system namely the Rossby and gravity modes. The modes are only a linear concept, we will discuss how nonlinear advection will effect the transfer of energy between the modes and how the structure of the modes change due to a change of the basic state. The relation between the practical problem of providing initial data for a numerical forecasting model and the concept of normal modes (Rossby and gravity modes) will be discussed.

2. The one-dimensional advection equation

2a. General properties

To study nonlinear effects on fluid flow we should really start by considering the full 3-dimensional Navier-Stokes equations with some relevant boundary conditions. Except for some rather special cases, this would very quickly lead us to a mathematical problem with such a complexity that it would be difficult to overview. Therefore we will instead start with the simplest possible equation which contains the advective nonlinearity in its most rudimentary form. Following Platzmann (1964) and Källén and Wiin-Nielsen (1980) we will discuss some particular properties of this equation which are characteristic for advective models of fluid flow in general. It is difficult, if not impossible, to relate the one-dimensional advection equation to any particular fluid flow situation, but the presence of the advection term gives the equation properties which are related to the properties of more realistic models of fluid flow.

We start by considering the advection equation in one dimension without any external forces:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0 \qquad (2.1)$$

The equation simply states that the velocity of a fluid particle $u(x,t)$ is conserved following the particle motion.

We impose the following boundary conditions

$$u(0,t) = u(L,t) = 0$$

An initial condition which obeys these boundary conditions is

$$u(x,0) = u_o \sin(\frac{2\pi x}{L})$$

which is shown in Fig. 1a. This initial state may be thought of as a wave disturbance in a fluid flow and we will now investigate how the advective process changes the shape of this wave. One way of doing this is to use the method of characteristics, i.e. to determine curves in the x-t plane which fluid particles follow. For eq. (2.1) these curves are simply straight lines where the slope $u = \frac{dx}{dt}$ is given by the initial value of u.

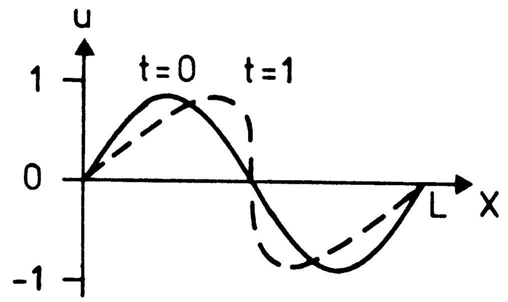

Fig. 1a. Initial distribution of u (t = 0, full line) and distribution just at the onset of breaking (t = 1, dashed line).

Some typical characteristics of equation (2.1) (i.e. lines of constant u in the t-x plane) have been drawn in Fig. 1b.

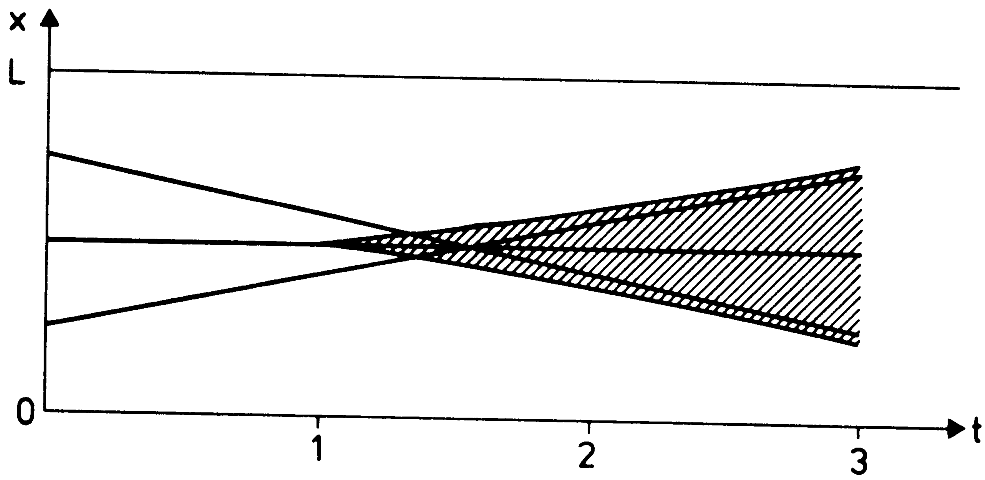

Fig. 1b. Characteristics of eq. (2.1) in the t-x plane. Inside the hatched, cuspshaped region the solution is multivalued, i.e. breaking has occurred.

We see from the figure that after some time the characteristics from different initial points will intersect and thus we have a multivalued velocity at certain points. Also shown in fig. 1b is the envelope of the "cusp-region", inside which the solution is triple-valued (three characteristics through each point). This behaviour is clearly unphysical, but we may interpret it in terms of a wave breaking in the following way. Since each value of u is propagated along x with speed u, it follows that the wave crest($u > 0$) is propagated forward and the trough ($u < 0$) backwards. So the slope $S = \frac{\partial u}{\partial x}$ of the wave profile becomes steeper where S is negative initially and flatter where S is positive initially. This breaking process may be examined quantitatively by computation of the change of slope along a characteristic:

$$\frac{dS}{dt} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) + u\,\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

After changing the order of differentiation in the first term on the right-handside and using (2.1) we obtain

$$\frac{dS}{dt} = -\,S^2$$

Integration yields

$$S = (t + S_o^{-1})^{-1} ,$$

where S_o is the initial value of S. This equation shows that for each value of u where $S_o < 0$, the slope becomes increasingly negative until at a critical time $t_c = -\,S_o^{-1}$ it becomes infinite, after which it is positive and declines to zero. At the critical time t_c we may say that breaking occurs and after this time the model solution is no longer meaningful. The minimum value of t_c occurs for $u = 0$ at $x = L/2$, since there S_o^{-1} has its minimum value of -1. Breaking will therefore occur first at time $t = 1$.

Physically this is not a satisfying model because of the discontinuity forming at the breaking point. We therefore incorporate a simple dissipation term into equation (2.1):

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = -\,\varepsilon\,u \qquad (2.2)$$

The slope equation now becomes

$$\frac{dS}{dt} = - S^2 - \varepsilon S \tag{2.3}$$

If we substitute $y = S^{-1}$ we get

$$\frac{dy}{dt} = 1 + \varepsilon y$$

which has the solution

$$y = -\frac{1}{\varepsilon} + C' e^{\varepsilon t}$$

Thus,

$$S = \frac{\varepsilon}{C e^{\varepsilon t} - 1}$$

where C is a constant which we can relate to the initial slope S_o as follows

$$C = \frac{\varepsilon}{S_o} + 1$$

Breaking occurs if S goed to infinity, in other words if

$$S_o = \frac{\varepsilon}{e^{-\varepsilon t} - 1}$$

Since $-1 < (e^{-\varepsilon t} - 1) < 0$, breaking only occurs if $-\infty < S_o < -\varepsilon$. Thus, not all initial slopes will lead to breaking. This is illustrated in Fig. 2. in which three solutions for three different initial conditions are drawn. Note that the value of parameter ε determines for which initial slope breaking will occur.

This is an example of a typical situation in many nonlinear models. From eq. (2.2) we see that there are two terms governing the time evolution of u. The dissipation term on the right hand side of (2.2) will try to damp the initial value towards zero while the advective term will try to increase the velocity gradients to give a breaking. The two competing effects will either eventually lead to a breaking or the initial velocity profile will just be damped down to zero. Which one of these two possibilities that eventually will occur depends on the initial slope in relation to the dissipation parameter ε. Rewriting the slope

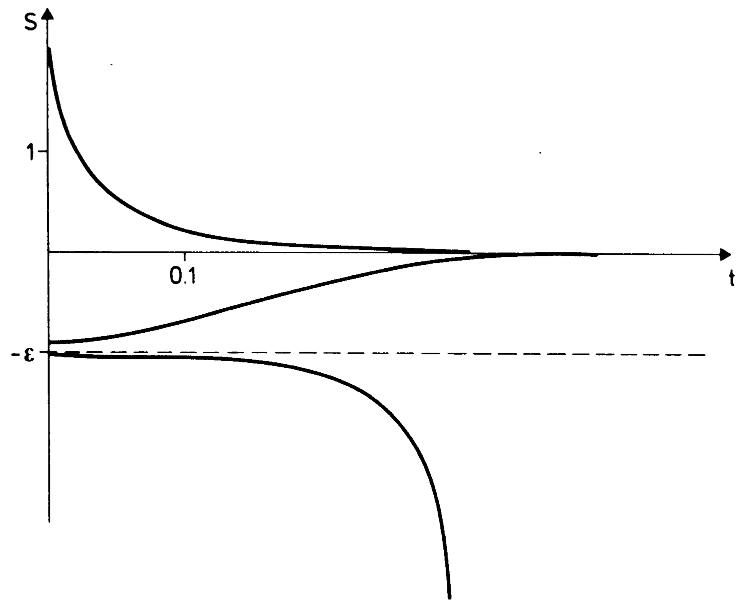

Fig. 2. Time evolution of the slope for some different initial values in the presence of dissipation.

equation (2.3) as

$$\frac{dS}{dt} = - S(S + \varepsilon) \tag{2.4}$$

we see that if $S < -\varepsilon$, $\frac{dS}{dt}$ will be negative and an initially negative slope will become increasingly negative, in other words the wave will break.

If $S > - \varepsilon$, $\frac{dS}{dt} > 0$ for $S < 0$ and $\frac{dS}{dt} < 0$ for $S > 0$. This will give a solution which asymptotically approaches $S = 0$ as $t \to \infty$. We may also see from (2.4) that both $S = 0$ and $S = - \varepsilon$ are steady solutions of the slope equation. A slight perturbation around $S = 0$ will eventually lead to S returning to zero and therefore the steady-state $S = 0$ is said to be stable. The steady-state $S = - \varepsilon$, on the other hand, is unstable to small perturbations as any small deviation from it will either lead to the steady-state $S = 0$ or $S \to - \infty$.

The method of determining the steady-states and then analyzing the stablity of each steady-state to small perturbations will be used extensively in this series of lectures. In this case we see that we have two steady-states, but only one of them is stable to small perturbations. Solutions starting close to the unstable steady-state will either reach the stable steady-state asymptotically or they will be negatively infinite. Solutions starting sufficiently close to the stable steady-state will always approach this steady-state asymptotically, and it is thus this steady state which from a physical point of view is most interesting.

Finally it should be mentioned that we can also parameterize dissipation by replacing the friction term on the righthandside of (2.2) by a diffusion term. This result is the so called Burgers' equation:

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = \nu\,\frac{\partial^2 u}{\partial x^2} \tag{2.5}$$

This equation can be solved analytically by the so-called Cole-Hopf transformation (see Platzman, 1964). This transformation reduces (2.5) to a linear diffusion equation which may be solved by standard methods. The solution does not have any discontinuities in contrast to the solution of (2.2) which also involves a "frictional" process in addition to the advective one. So different parameterizations of the dissipative process may lead to a quite different qualitative behaviour. In Burgers' equation the dissipation term strongly damps the solutions with large gradients and therefore breaking will never occur. In eq. (2.2) the dissipation is independent of the particular scale of motion and therefore the magnitude of the initial slope determines whether breaking will occur or not.

2b. Spectral method

When examining the nonlinear properties of the advection equation in section 1, we assumed an initial condition in the form of a sine wave. Any initial condition satisfying the boundary conditions $u = 0$ at $x = 0$ and $x = L$ may of course be constructed by the addition of sine waves with different wave lengths in the form of a Fourier series. This also applies to the solution for all times as long as it is single valued. When breaking occurs the solution has an infinite derivative at one point and a Fourier series expansion is therefore not valid. It may nevertheless be instructive to express the solution of the equation in terms of a Fourier series, in particular as this method will be applied to more complicated models later on this course. We must, however, keep in mind that a Fourier series expansion is not valid when we get close to a point of breaking.

As in exercise 1, we will consider the one-dimensional advection equation with forcing and a linear drag term

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \varepsilon (u_E - u) \tag{2.6}$$

The boundary conditions will be the same as previously, $u = 0$ at $x = 0$ and $x = L$. The same boundary conditions apply for u_E. For algebraic convenience we will non-dimensionalize (2.6) with the length scale L and the time scale $1/\varepsilon$.

Denoting the nondimensional variables with a star we thus obtain

$$u = L \varepsilon u^*$$

$$u_E = L \varepsilon u_E^*$$

$$t = t^* / \varepsilon$$

$$x = L \xi \text{ with } 0 < \xi < \pi$$

and equation (2.6) takes the form

$$\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial \xi} = u_E^* - u^* \tag{2.7}$$

Taking the boundary conditions into account a spectral expansion of u^* and u_E^* can be made with sine functions only forming an orthogonal system as follows:

$$u^* = \sum_{k=1}^{N} u_k^*(t^*) \sin(k\xi) \tag{2.8}$$

The solution is thus separated into a time and a space dependent part over $0 < \xi < \pi$.

For u_E^* we similarly have

$$U_E^* = \sum_{k=1}^{N} u_{E,k}^* \sin(k\xi)$$

with $u_{E,k}^*$ independent of time.
Note that we have truncated the summation at N terms.

Choosing a single sine component as our initial state we can easily see from fig. 1a that after some time this single component will not be sufficient to describe our solution accurately. If there was no advective term in (2.6) it would have been possible to treat each component in the expansion (2.8) separately, but due to the advective nonlinearity we obtain an energy transfer from one spectral component to another. The more terms we include in our expansion (2.8), the more accurately this process will be described, but as we have limited our expansion to a certain truncation N there is always a limit to the accuracy which we can obtain. In fig. 3 this point is illustrated by showing how well a wave is approximated when it is just before the point of breaking, for different values of N.

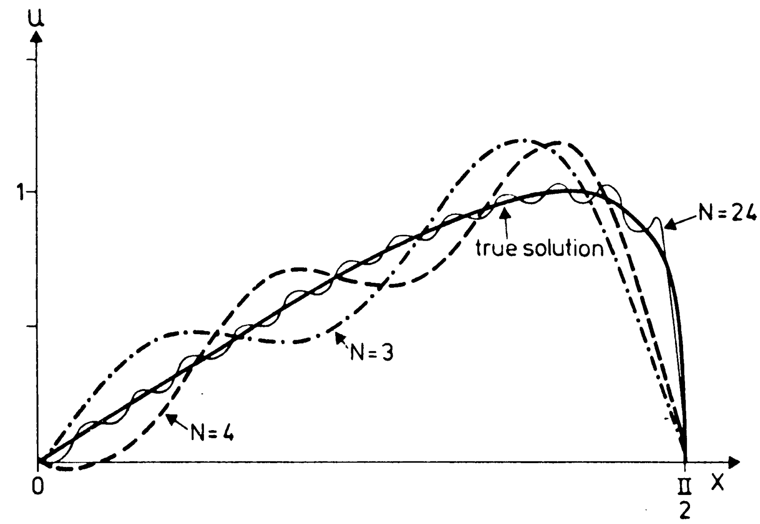

Fig. 3. A wave at the onset of breaking (full line) and its approximate form given by a limited number of spectral components.

To explicitly investigate how energy is transferred between the Fourier components, we can derive equations for the time variation of each spectral component as follows.

Inserting the expansions in (2.7) we have the equation

$$\sum_{k=1}^{N} \frac{du_k^*}{dt} \sin k\xi + \sum_{k=1}^{N} \sum_{l=1}^{N} u_k^* u_l^* \, l \sin k\xi \cos l\xi = \\ = \sum_{k=1}^{N} (u_{E,k}^* - u_k^*) \sin k\xi \tag{2.9}$$

We may now separate eq. (2.9) into a set of nonlinearly coupled ordinary differential equations by making use of the orthogonal properties of the expansion functions. By considering the expansion functions as a set of base vectors we may "project" eq. (2.9) on one base function, sin $n\xi$, by letting the operator

$$\frac{1}{\int_0^\pi \sin^2 n\xi \, d\xi} \int_0^\pi d\xi \sin(n\xi) \tag{2.10}$$

act on equation (2.9).
There is only a contribution from the linear terms when n is equal to k. The nonlinear term, which is more complicated, will require some more detailed derivations,

$$\sum_{k=1}^{N} \sum_{l=1}^{N} u_k^* u_l^* \, l \int_0^\pi \sin n\xi \sin k\xi \cos l\xi \, d\xi / \int_0^\pi \sin^2 n\xi \, d\xi \tag{2.11}$$

We evaluate the equation for every combination of k and l contributing to the component n, on which we are projecting. The integral

$$l \int_0^\pi \sin n\xi \sin k\xi \cos l\xi \, d\xi \tag{2.12}$$

may be written

$$\frac{1}{2} \int_0^\pi \sin n\xi \sin(k+l)\xi \, d\xi + \frac{1}{2} \int_0^\pi \sin \xi \sin(k-l)\xi \, d\xi$$

We can now see that only if $n = k + l$ or $n = |k - l|$ one of the integrals will be non zero. Noting that n is positive (follows from eq.

(2.8)) this can be plotted in a diagram (fig. 4).

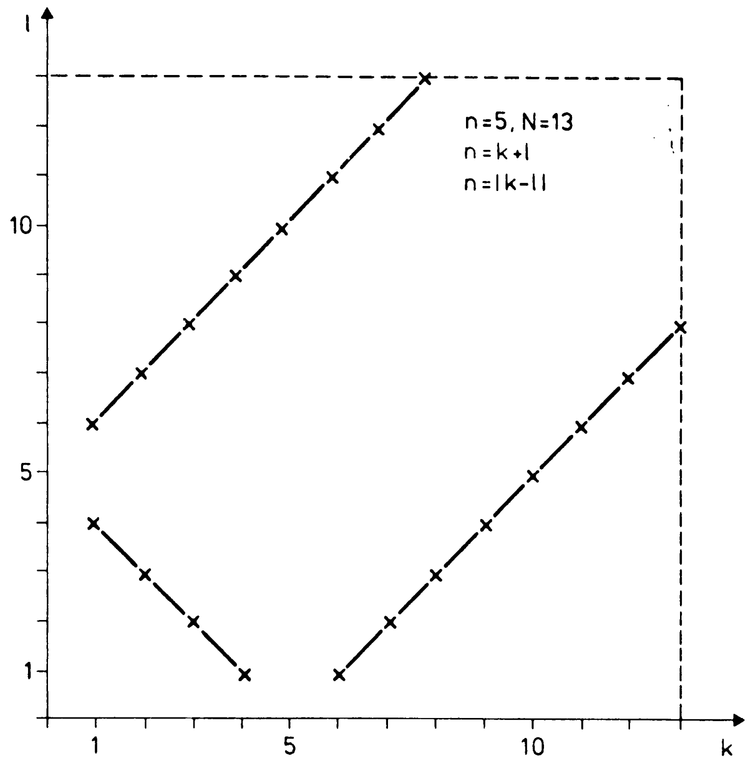

Fig. 4. Wave number combinations giving non zero contributions to the interaction integral.

The time change of the component with wave number n follows from equation (2.9)

$$\frac{du_n^*}{dt} = \frac{n}{2} \left(\sum_{l=1}^{N-n} u_l^* u_{u+l}^* - \frac{1}{2} \sum_{l=1}^{n-1} u_l^* u_{n-l}^* \right) + u_{E,u}^* - u_n^* \qquad (2.13)$$

The derivation of the advective term is given as an exercise.
From the structure of eq. (2.13) we see that all nonlinear terms will be quadratic and that for a certain component (n) terms involving all other components will be present. Note, however, that a particular component will not have any terms with its own amplitude squared. The addition of an extra component beyond the truncation limit (N + 1) will thus affect

the time evolution of all the other components through the nonlinearity. The spectral method may therefore seem to be quite useless, as all components are affected when the truncation limit is slightly changed. In some cases it will however appear that some interactions dominate over the others. The closer we come to the point of breaking, the more energy will be distributed to the small scale components as this is the only way in which we can describe a function with a large derivative.

To investigate more schematically how the nonlinear energy transfer between components takes place, we will here analyse some very severely truncated spectral models. The solutions we obtain are therefore not very accurate in describing the solution of the original equation but they do show some very characteristic types of nonlinear behaviour and we will therefore use them as illustrative examples.

If N = 1 there is no nonlinear interaction, because a component cannot interact with itself to itself.

Now take N = 2 and we define

$$\overset{*}{u}_1 = x \quad \overset{*}{u}_2 = y \quad \overset{*}{u}_{E,1} = x_E \quad \overset{*}{u}_{E,2} = y_E$$

We insert this in equation (2.13).

This gives a system of two ordinary differential equations,

$$\frac{dx}{dt} = \frac{1}{2}xy + x_E - x$$
$$\frac{dy}{dt} = -\frac{1}{2}x^2 + y_E - y \qquad (2.14)$$

We first investigate, the behaviour of the solutions when there is no forcing($x_E = y_E = 0$) and no dissipation.

Multiplying the first equation by x and the second by y and summing gives

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

or alternatively written

$$\frac{d}{dt}\left(\frac{1}{2}(x^2 + y^2)\right) = 0$$

This shows that there is only a redistribution of kinetic energy between the two components and the total energy is conserved. The kinetic energy of each component is given by half its amplitude squared, and due to the orthogonality of the basis functions the total kinetic energy is the sum of the kinetic energies of the individual components.
The solution of (2.14) without forcing and dissipation will therefore lie on a circle as shown in fig. 5. The radius of the circle depends on the initial energy in the system. The steady-state solutions are defined as values of x and y for which

$$\frac{dx}{dt} = 0 \text{ and } \frac{dy}{dt} = 0$$

We thus find the system

$$\begin{cases} \frac{1}{2} x^2 = 0 \\ \frac{1}{2} xy = 0 \end{cases} \rightarrow \quad \begin{array}{l} x = 0 \\ y \in \mathrm{IR} \end{array}$$

Together with the equation for the total energy

$$E = \tfrac{1}{2}(x_o^2 + y_o^2)$$

we find that the steady-state points are $(0, y_o)$ and $(0, -y_o)$.
These points are circled in fig. 5.
The stability of the steady-state solutions can be examined by linearizing the model around a steady-state and examining perturbations in the x-direction.
Defining the perturbations as

$$x = x_o + \delta x$$

$$y = y_o + \delta y$$

we have from eq. (2.14) without forcing and dissipation terms and when neglecting second order quantities in the primed variables,

$$\frac{dx}{dt} = \tfrac{1}{2}\,(x_o \delta y + y_o \delta x)$$

$$\frac{dy}{dt} = -\,x_o \delta x$$

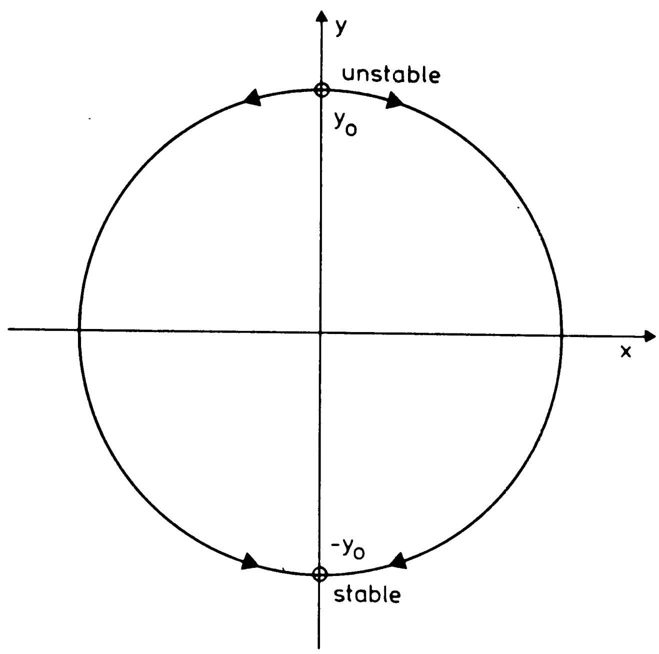

Fig. 5. Steady-states and stability properties of conservative two component system.

Perturbing around the steady-state $(0,y_o)$ we have for the derivative $\frac{dx}{dt}$

$$\delta x > 0 \rightarrow \frac{dx}{dt} > 0$$

$$\delta x < 0 \rightarrow \frac{dx}{dt} < 0$$

so the point $(0,y_o)$ is unstable to small perturbations.
For the steady-state $(0,-y_o)$ we similarly have

$$\delta x > 0 \rightarrow \frac{dx}{dt} < 0$$

$$\delta x < 0 \rightarrow \frac{dx}{dt} > 0$$

so the point $(0,-y_o)$ is stable to small perturbations (see fig. 5.).
The non-linearity thus transports energy from the larger scale component (x) to the smaller scale component (y). This may be demonstrated with an

example. If we start with an initial condition where we only have energy in the x-component and the y-component is equal to zero, (see fig. 6,a) we find after some time that they have changed as shown in fig. 6,b. Because of the non-linear interaction the y component grows at the expense of the x component. Finally the x component is equal to zero and a steady-state solution where $y = -x_o$ is found (see fig. 6,c).

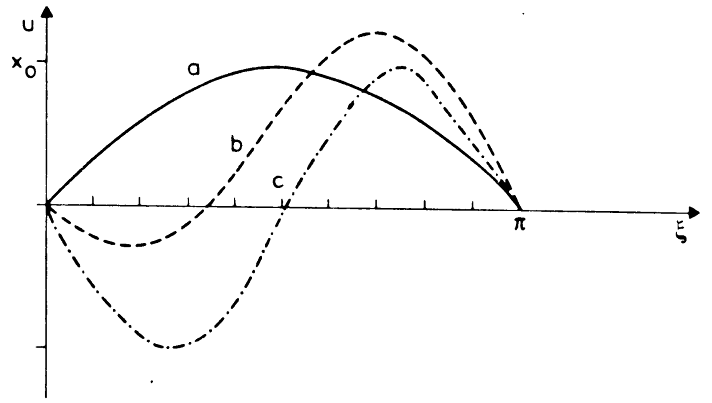

Fig. 6. Time evolution of the velocity field in the two component model. Initial state is given by curve labelled a), b) is an intermediate state while c) is the asymptotic state.

If we change the sign in the initial state we get a similar type of behaviour (see fig. 7,a). After some time we see that all the energy in the x-component goes to the y-component (fig. 7,b and 7,c).

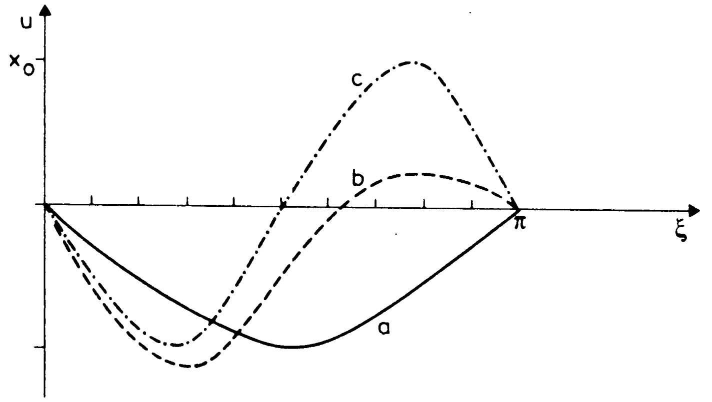

Fig. 7. Same as fig. 6, but for a different initial state.

This is called a cascade process. It always stops in the smallest scale which is taken into account; in this case the y-component. Breaking will never occur because we have truncated the expansion of u and u_E after two terms. If we would have taken more terms into account the process would finally have resembled breaking. Next we will consider a two component system with forcing and dissipation. We first suppose that there is only a forcing in the small scale, i.e. we set $x_E = 0$. We are then able to find the steady states of (2.14) for which

$$\frac{dx}{dt} = \frac{dy}{dt} = 0 \tag{2.15}$$

holds. Combining (2.14) and (2.15) we find

$$\begin{aligned} &\tfrac{1}{2}\, x(y-2) = 0 \\ &-\tfrac{1}{2}\, x^2 + y_E - y = 0 \end{aligned} \tag{2.16}$$

From this it can be seen that

$$x_s = 0 \;\; ; \;\; y_s = y_E \text{ or } x_s = \pm\sqrt{2(y_E-2)}; \; y_s = 2 \tag{2.17}$$

are the steady states. It will be clear that as long as the forcing y_E is less than two there only exists one steady state; in all other cases three steady states are found (see fig. 8).

The sudden change from one to three steady states as the forcing parameter is increased is called a bifurcation. When dealing with the full one dimensional advection equation in exercise 1 we had a similar type of behaviour, only there we went from zero to two steady states as the forcing was increased beyond a critical value.

From fig. 8 we also see that the behaviour of the two component system changes drastically as forcing aand dissipation is included. In the conservative case, with only nonlinear terms coupling the two components, the energy which initially is in the system is exchanged between the two components. In the forced and dissipative system we have a constant energy supply in the small scale component while energy is being dissipated in both components. If the energy supply is low enough (y_E below a certain value) we have one steady-state where the energy is

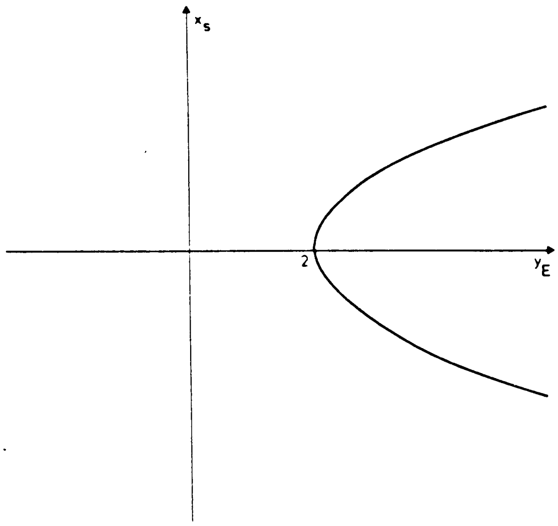

Fig. 8. Bifurcation diagram for the the two component system with forcing only on the smallest scale of motion and dissipation on both components.

just dissipated in the small scale component. If the energy supply exceeds a critical value ($y_E > 2$) the non-linearity can take care of the energy transfer to the larger scale component in more than one way and the system now has three different steady state solutions. All of these steady states cannot be stable to small perturbations and we will now turn to the problem of determining the stability of the linearized equations.

Considering only small perturbations (δx, δy) around a steady-state (x_s, y_s) we have

$$x = x_s + \delta x$$
$$y = y_s + \delta y \tag{2.18}$$

Insertion of (2.18) in (2.14) gives the linearized equation (neglecting terms of second order in δx and δy).

$$\dot{\delta x} = (\tfrac{1}{2}\, y_s - 1)\ \delta x + \tfrac{1}{2}\, x_s\ \delta y$$
$$\dot{\delta y} = -\, x_s\ \delta x - \delta y \qquad (2.19)$$

where a dot indicates time differentiation.
We can try solutions of the form

$$\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \begin{pmatrix}x'\\ y'\end{pmatrix} e^{\lambda t} \qquad (2.20)$$

Substitution in (2.19) leads to

$$\begin{pmatrix}\tfrac{1}{2}\, y_s - 1 - \lambda & \tfrac{1}{2}\, x_s\\ -\, x_s & -1 - \lambda\end{pmatrix}\begin{pmatrix}x'\\ y'\end{pmatrix} = 0 \qquad (2.21)$$

This is an eigenvalue problem: the λ's are called the eigenvalues, and $\binom{x'}{y'}$ the eigenvectors. (2.21) has nontrival solutions if the determinant of the coefficient matrix is equal to zero. From this condition we find the quadratic equation in λ.

$$\lambda^2 + (2 - \tfrac{1}{2}\, y_s)\lambda + \tfrac{1}{2}\, x_s^2 - \tfrac{1}{2}\, y_s + 1 = 0$$

which has solutions

$$\lambda_{1,2} = \tfrac{1}{4}(y_s - 4) \pm \tfrac{1}{4}\sqrt{y_s^2 - 8\, x_s^2} \qquad (2.22)$$

The steady states of the two component system were already found in (2.17). Substituting them in (2.22) we obtain for $x_s = 0$, $y_s = y_E$ the eigenvalues

$$\lambda_{1,2} = \tfrac{1}{4}(y_E - 4) \pm \tfrac{1}{4}\sqrt{\ y_E^2} \ \rightarrow\ \begin{matrix}\lambda_1 = \frac{y_E}{2} - 1\\ \lambda_2 = -\, 1\end{matrix} \qquad (2.23)$$

It can be seen that if $y_E \leq 2$ both λ_1 and λ_2 are nonpositive. From (2.20) it then follows that the perturbations are bounded and thus the steady state is stable. If y_E is greater than two one eigenvalue is positive and we have an unstable steady state. In mathematics this type of a steady state is called a saddle point.For the steady states $x_s = \pm \sqrt{2(y_E - 2)}$, $y_s = 2$ we have from (2.22) the eigenvalues

$$\lambda_{1,2} = - \tfrac{1}{2} \pm i \sqrt{y_E - \tfrac{9}{4}} \tag{2.24}$$

The real part is always negative for $y_E \geq 2$, which means that these steady states are stable. The imaginary part causes on oscillatory behaviour of the solutions (2.20) near the steady states. Such a steady state is called a stable spiral point.

We can now show a rough sketch of the solutions in the x-y plane. In figure 9 this is done for the case $y_E > 2$. Again it may be seen that the behaviour of the solutions depends on the initial conditions. Furthermore the y-axis acts as a dividing ridge separating the (x,y) plane in two regions. All trajectories in the left half plane will reach $(-\sqrt{2(y_E-2)}, 2)$ as an asymptotic state, while the trajectories in the right hand half plane will reach $(\sqrt{2(y_E - 2)}, 2)$.

To look at another type of bifurcation which also often appears in non-linear dynamics we will now extend our equations to a three component system. We retain the forcing in the y-component, now the middle scale component, and include a dissipative term in all components. This can be developed in the same way as (2.14). We find

$$\begin{aligned} \dot{x} &= \tfrac{1}{2} xy + \tfrac{1}{2} yz - x \\ \dot{y} &= -\tfrac{1}{2} x^2 + xz + y_E - y \\ \dot{z} &= -\tfrac{3}{2} xy - z \end{aligned} \tag{2.25}$$

As has been shown in the exercises the nonlinear interactions keep the energy in the system; only forcing and dissipation can change the total energy. The steady states of (2.25) can be found algebraically. From $\dot{z} = 0$ it follows that $z = -\frac{3}{2} x y$. Subsituting this in $\dot{x} = 0$ we obtain

$$\tfrac{1}{2} xy - \tfrac{3}{4} xy^2 - x = -\tfrac{3}{4} x \left[(y - \tfrac{1}{3})^2 + \tfrac{11}{9} \right] = 0$$

so x must be zero, and thus $z = 0$.

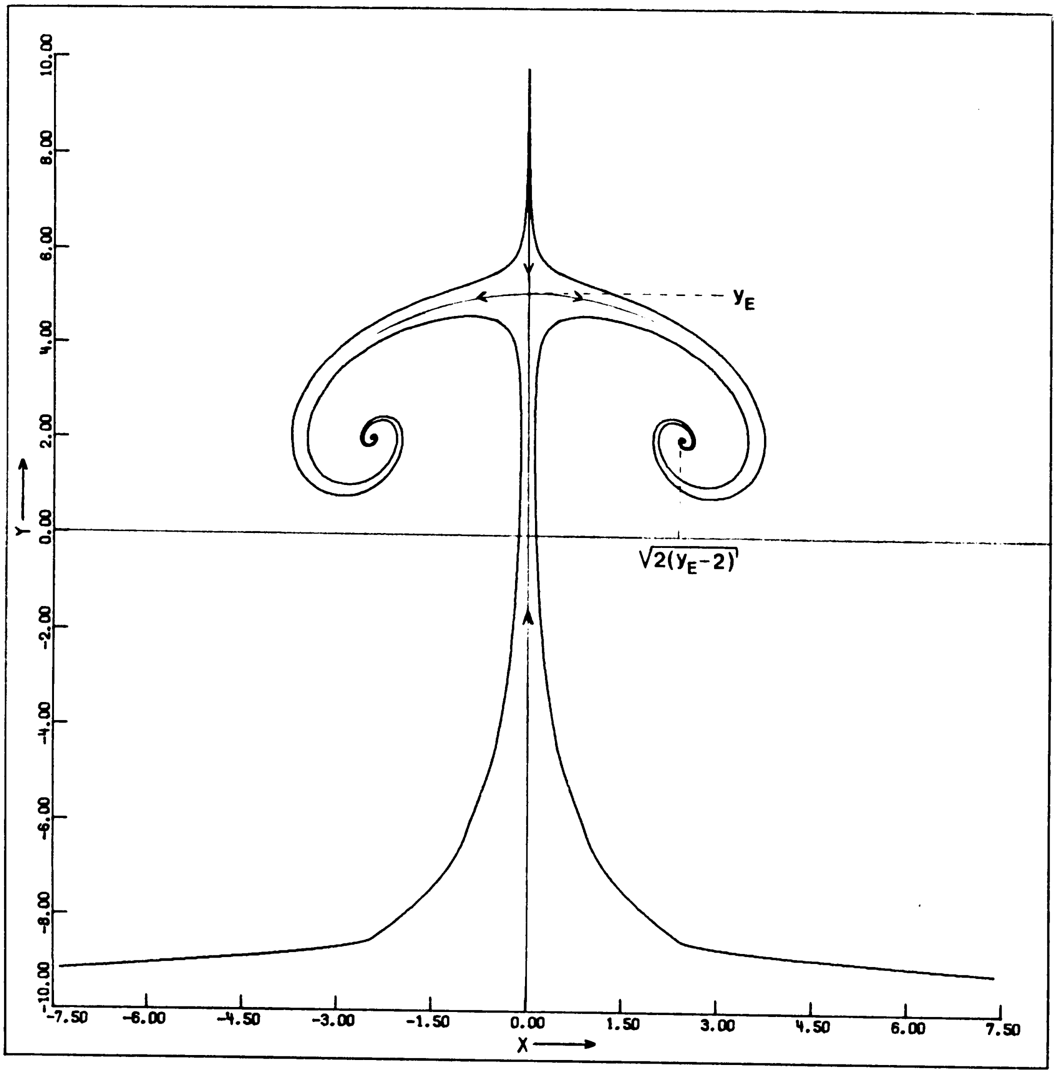

Fig. 9. Phase plane portrait of the characteristic trajectories in case of two components with amplitudes x and y.

Considering finally $\dot{y} = 0$ we find $y = y_E$. There is then a balance between forcing and dissipation. The only possible steady state is thus

$$x_s = 0 \quad ; \quad y_s = y_E \quad ; \quad z_s = 0 \qquad (2.26)$$

This is quite different from the two component system, since in this case no change of the number of steady states can occur. However, there is a bifurcation, because the stability of the steady state strongly depends on the forcing. To show this we linearize (2.25) around $(0,\ y_E,\ 0)$:

$$
\begin{aligned}
x &= \delta x \\
y &= y_E + \delta y \qquad\qquad (2.27)\\
z &= \delta z
\end{aligned}
$$

and inserting this into (2.25) we find (neglecting second order terms in the perturbation quantities)

$$
\begin{pmatrix} \delta\dot{x} \\ \delta\dot{y} \\ \delta\dot{z} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} y_E - 1 & 0 & \frac{1}{2} Y_E \\ 0 & -1 & 0 \\ -\frac{3}{2} y_E & 0 & -1 \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix} \qquad (2.28)
$$

Determining the eigenvalues we obtain the cubic equation

$$
-(1 + \lambda)[(1 + \lambda)^2 - \tfrac{1}{2} y_E (1 + \lambda) + \tfrac{3}{4} y_E^2] = 0
$$

with solutions

$$
\lambda_1 = -1 \quad ; \lambda_{2,3} = \tfrac{1}{4}(y_E - 4) \pm i \frac{\sqrt{11}}{4} y_E \qquad (2.29)
$$

It is clear that λ_2 and λ_3 have positive real parts if the forcing y_E is greater than four. All eigenvalues have negative real parts as long as y_E is less than four. So although there is only one steady state it can change its stability.

When the single steady state becomes unstable we have a new type of situation which has not been treated in the previous examples. When an unstable steady state appeared in the two component system, we had two other stable steady states emerging at the same time and a trajectory starting close to the unstable steady state would reach one of the stable steady states as an asymptotic state. In this case there is no stable steady state and therefore a solution starting close to the unstable steady state has to continue its trajectory in the three dimensional phase space forever. There is the possibility that the trajectory will approach infinity for large times, but we can of course also have some type of limiting behaviour within a certain region of the phase space.

As we have included dissipation terms in the governing equations (2.25) it is not very likely that the trajectory will go to infinity. To prove this, we will do a global stability analysis.
Define

$$\bar{r} = (u_1^*, \ldots, u_N^*) \text{ and } \bar{r} = (u_{E,1}^*, \ldots, u_{E,N}^*) \qquad (2.30)$$

it then follows from (2.13) that we can write an energy equation

$$\bar{r}\frac{d\bar{r}}{dt} = \frac{1}{2}\frac{d}{dt}|\bar{r}|^2 = \bar{r}_E \cdot \bar{r} - |\bar{r}|^2 \qquad (2.31)$$

This shows a balance between the forcing and the dissipation; nonlinear interactions do not contribute to a change in the total amount of energy. It can be seen that if $|\bar{r}|$ is large the energy will always decrease because the quadratic term will dominate the first term on the right side of (2.31). It means that the solutions will always be bounded, a property which often occurs in physical systems.
After this global analysis we return to the local analysis. The eigenvalues in (2.29) can now be used to determine the eigenvectors. For $\lambda_1 = -1$ we get by using (2.28).

$$\begin{pmatrix} \frac{1}{2}y_E & 0 & y_E \\ 0 & 0 & 0 \\ -\frac{3}{2}y_E & 0 & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = 0$$

From which it follows that $x' = z' = 0$ and y' is arbitrary. Thus the eigenvector is (0, 1, 0). Along this direction the solution is always stable. It is much more complicated to calculate the two other eigenvectors along which the solution is unstable. However, it is known from linear algebra that the eigenvectors are linearly independent. So these other two eigenvectors span a plane which only cuts the y-axis at $y = y_E$. In the case of y_E greater than four the solution thus spirals out in this plane if we look sufficiently close to the steady state. See figure 10.

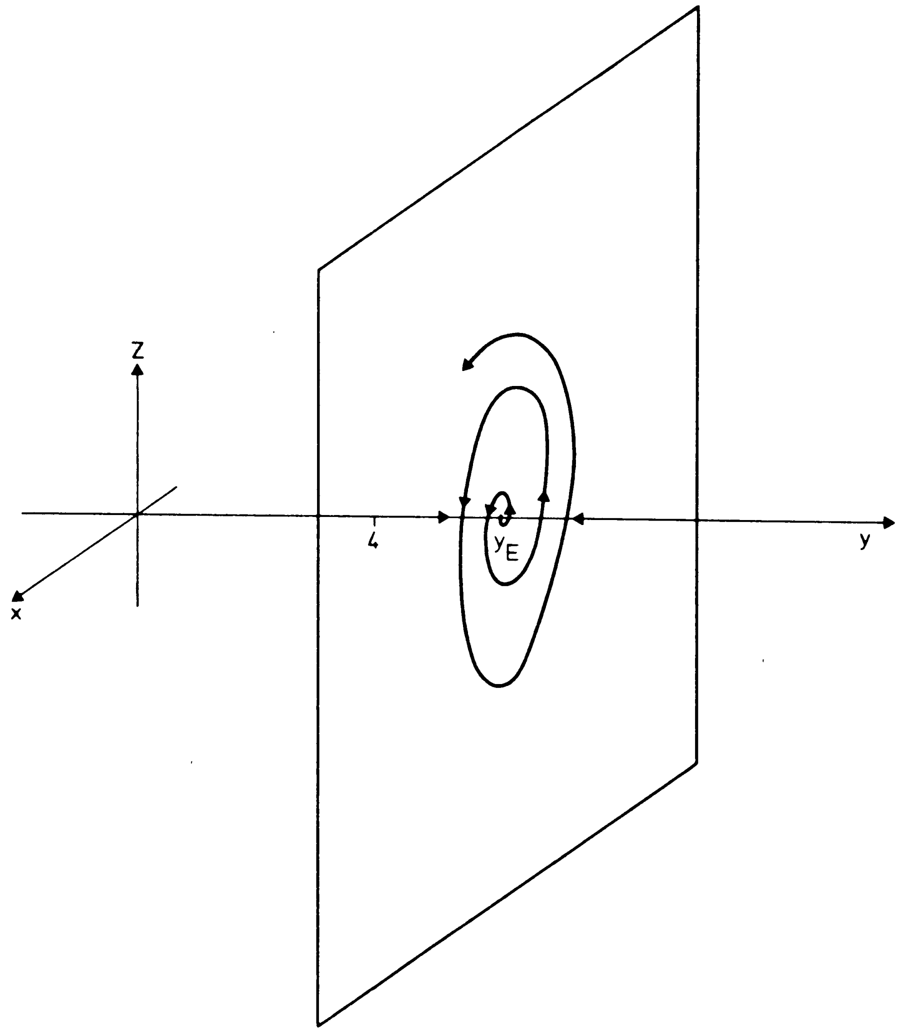

Fig. 10. Trajectory of the solution for $y_E > 4$ in a plane normal to the direction of the stable eigenvector.

The first one to discuss this type of behaviour was the German mathematician Hopf (193). The appearance of a period solution in connection with a change in stability of a steady state is thus called a Hopf bifurcation.

The trajectories spiralling out from the unstable steady state in this example will approach a closed orbit asymptotically (periodic solution). Hopf showed that a necessary condition for the appearance of a periodic orbit is that the imaginary parts of the eigenvalues are nonzero when the real part changes sign. One may distinguish between two cases, namely a supercritical and a subcritical Hopf bifurcation. In the supercritical case a stable periodic orbit develops around an unstable steady state, while in the sub-critical case an unstable periodic orbit, which exists around the stable steady state, collapses around the steady state as it becomes unstable (see fig. 11). In the example treated here we have a supercritical Hopf bifurcation.

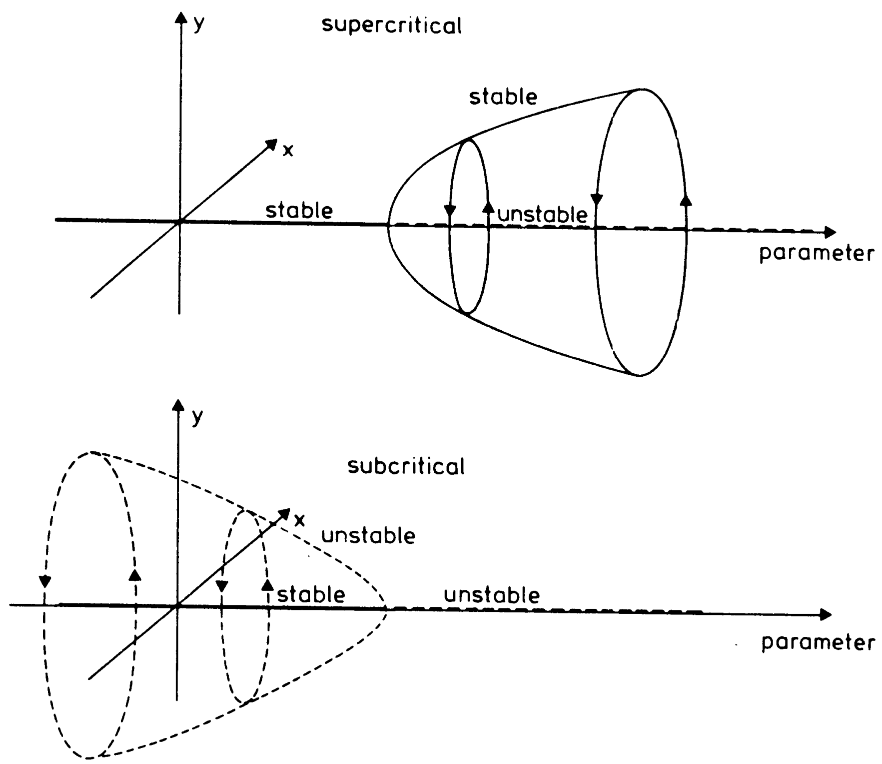

Fig. 11. Super- and Subcritical Hopfbifurcation showing appearance appearance (disappearance) of a stable (unstable) limit cycle for parametervalues greater than a critical value.

The Hopf bifurcation theory is only valid when the forcing parameter is close to its critical value. When increasing y_E beyond the value of 4, numerical experiments reveal that the stable periodic orbit will increase its amplitude up to a certain, second limiting value in

y_E. For y_E above this value we will have a new periodic orbit with twice the period of the first one. According to some recent literature (see Hofstadter, 1981) this period doubling will continue until, at some finite value of the forcing parameter y_E, the solutions will lose their periodicity. The solution then enters a chaotic regime, but because of the global stability it will stay within a confined region of the phase space. As this region still contains infinitely many points the solution may continue to go around forever without returning to a point where it has been before, i.e. to be non-periodic (for an example of such a system see Lorenz, 1963).

Excercises

1. Give the equation for the time dependence of the slope, $\frac{\partial u}{\partial x}$, in the case of a nondimensionalized, forced and dissipative advection equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = u_E - u .$$

Are there any stationary solutions to the slope equation? Assuming an initial slope of $-\frac{1}{2}$ at a point where $u = 0$, give explicitly the time evolution of the slope. (Use the standard solution of a Riccati equation, which is as follows:

Given $\frac{dv}{dx} + r(x)\, v^2 + s(x) = 0$, the solution is

$$v(x) = \frac{1}{r(x)} \frac{u'(x) + C\, t'(x)}{u(x) + C\, t\,(x)}$$

where u and t are linearly independent solutions of

$$\frac{d}{dx} \left(\frac{1}{r(x)} \frac{dz}{dx}\right) + s(x)\, z = 0.$$

2. Show, that the contribution to a component n in a spectral expansion of the advective term $\left(u \frac{\partial u}{\partial x}\right)$ may be written

$$\frac{n}{2} \left(\frac{1}{2} \sum_{\ell=1}^{n-1} u_n\, u_{n-\ell} - \sum_{\ell=1}^{n-1} u_\ell\, u_{n+\ell}\right)$$

3. Show by induction that energy is conserved for any truncated system where components up to a certain wavenumber, N, are included.

4. Find the steady-state solutions to the two component system

$$\dot{x} = \frac{1}{2}\, xy - x + x_E$$

$$\dot{y} = - \frac{1}{2}\, x^2 - y + y_E$$

Sketch the solution surface in (x, x_E, y_E) space.

3. Nonlinear and dispersive effects in one-dimensional models

In this chapter we will derive some simplified models originating from the equations of motion which govern the dynamics of fluid flow. Following the main theme of this lecture series we will examine the nonlinear advective process, but here in an oceanographic relevant model.

The nonlinear advection term as treated in the simple one-dimensional case will be further investigated and we will show how terms related to wave dispersion will affect the wave breaking phenomenon, which is mainly due to advection. The Korteweg-de Vries equation will be derived as an approximation to the equation governing the dynamics of surface waves and its solution will be examined and related to the breaking and peaking of surface waterwaves.

3.1 Dispersion

In linear problems, dispersive waves are characterized by sinusoidal wavetrains

$$\phi(\bar{x},t) = A \exp\,[i(\bar{k}.\bar{x} - \omega)] \tag{3.1}$$

where ϕ measures some quantity related to the wave, as its height, or its velocity etc. A is an amplitude, $\bar{k}$ the wavenumbervector, i.e. the number of waves per unit of time (at a certain position).
The quantity

$$\theta \equiv \bar{k}.\bar{x} - \omega t \tag{3.2}$$

is called the phase of the wave and therefore

$$\bar{k} = \nabla\theta, \qquad \omega = -\frac{\partial\theta}{\partial t} \tag{3.3}$$

if k and ω are slowly varying functions in space and time.
The rate of progression of a surface of constant phase, θ = constant, is found by noting that

$$d\theta = 0$$

or

$$\frac{\partial\theta}{\partial t}\,dt + \nabla\theta \cdot d\overline{x} = [\frac{\partial\theta}{\partial t} + \nabla\theta \cdot \frac{d\overline{x}}{dt}]dt = 0 \qquad (3.4)$$

With the previous expression (3.3) we find

$$-\omega + \overline{k} \cdot \overline{c} = 0$$

where we have adopted the expression $\overline{c} = \frac{d\overline{x}}{dt}$. Since this is the speed with which surfaces of constant phase move, this may appropriately be termed the phase speed: $\overline{c} = \frac{\omega}{k} \cdot \hat{k}$ with $k = |\overline{k}|$ and $\hat{k} = \frac{\overline{k}}{k}$ a unitvector in the $\overline{k}$-direction.

Substituting a linear wave expression (3.1) into a linear advection equation

$$\frac{\partial\phi}{\partial t} + c_o \frac{\partial\phi}{\partial x} = 0 \qquad (3.5)$$

we find

$$-\, i\omega + c_o\, ik = 0 \rightarrow c = \frac{\omega}{k} = c_o \qquad (3.6)$$

Adding to this equation a higher order derivative,

$$\frac{\partial\phi}{\partial t} + c_o \frac{\partial\phi}{\partial x} + \gamma \frac{\partial^3\phi}{\partial x^3} = 0 \qquad (3.7)$$

we find a relation between ω and k

$$-\, i\omega + c_o\, ik + \gamma\, i^3 k^3 = 0 \rightarrow$$

$$c = \frac{\omega}{k} = c_o - \gamma\, k^2 \qquad (3.8)$$

The relation between ω and k is termed the dispersion-relation.

In general it is possible for linear polynomial equations of the form

$$P(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_i}; \gamma_i) = 0 \qquad (3.9)$$

(where γ_i are parameters of the system) to derive a dispersion relation by substituting the linear wavesolution (3.1) which yields a polynomial dispersion relation

$$P(-i\omega, ik; \gamma_i) = 0 \qquad (3.10)$$

More general dispersion relations, beyond the polynomial types arise
i) when wave motion takes place in a limited number of space-coordinates, while showing a more complicated behaviour in the other(s). An example will arise in the linear potential theory of waterwaves.
ii) If we construct a "wave" equation in the following way

$$\frac{\partial}{\partial t}\phi(x,t) + \int_{-\infty}^{\infty} K(x-\xi)\frac{\partial \phi}{\partial \xi}(\xi,t)\, d\xi = 0 \qquad (3.11)$$

with $\phi(x,t) = A \exp[i(kx - \omega t)]$ this yields

$$-i\omega A \exp[i(kx-\omega t)] + ik A \int_{-\infty}^{\infty} K(x-\xi)e^{ik\xi}\, d\xi\, e^{-i\omega t} = 0$$

or

$$c = \frac{\omega}{k} = \int_{-\infty}^{\infty} K(x-\xi)e^{-ik(x-\xi)}\, d\xi = \int_{-\infty}^{\infty} K(\zeta)e^{-ik\zeta}\, d\zeta \qquad (3.12)$$

after a transformation $\zeta = x - \xi$.
This is the Fourier transform for a given function $K(\zeta)$, or conversely

$$K(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} c(k)e^{ikx}\, dk \qquad (3.13)$$

The Kernel function may be chosen as the Fourier-transform of any desired phase-speed $c(k)$, extending beyond the polynomial types.

Now that we have established the existence of a relation between ω and k: $\omega(k)$ in the dispersion relation, we may now want to investigate its effect on the propagation of waves. Over a fixed interval (Δx), we consider the local change in time of the number of waves $(k.\Delta x)$. This can only change, in the absence of sources and sinks of wavenumbers (like that produced by a stone thrown into a water basin), if there is a difference in the frequencies at the boundaries. By calculating the limit when $\Delta x \to 0$, we find

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0 \qquad (3.14)$$

This may also be inferred from the definitions (3.3) of $k = \nabla\theta$ and $\omega = -\frac{\partial \theta}{\partial t}$. This is often called the conservation of crest equation.

Since $\omega = \omega(k)$ this can be rewritten as

$$\frac{\partial k}{\partial t} + \frac{d\omega}{dk} \cdot \frac{\partial k}{\partial x} = 0$$

or, interpreting this as the material derivative

$$\frac{dk}{dt} = 0 \quad \text{if} \quad \frac{dx}{dt} = \frac{d\omega}{dk} \qquad (3.15)$$

we can say that the wavenumber is a conserved quantity when we move in a frame of reference with speed $\frac{dx}{dt} = \frac{d\omega}{dk}$, the group velocity: c_g. This is the velocity with which the energy moves. Therefore if c_g is a function of wavenumber k a disturbance $\Phi(x,t)$, which may be thought of as being built up of a large number of wavenumber components, will disperse since each wavenumber moves with a different group velocity thereby spreading or dispersing the energy in the physical domain (fig. 12). Note that in general the group velocity differs from the phase velocity. The phase velocity is the speed with which individual waves move, while a wave group may be identified with the envelope of a set of waves with different wavelengths.

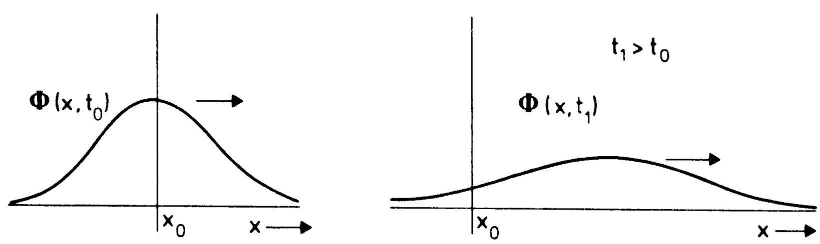

Fig. 12. Illustration of the effect of dispersion: flattening of a disturbance.

3.2 Linear Potential theory

We will now consider the dynamics of small amplitude waterwaves. Because of the high frequencies involved as compared with the rate of rotation of the earth we neglect Coriolis forces. We also assume the fluid to have a constant density and viscous effects to be absent. We take the Euler equations as a starting point.

$$\frac{D\bar{u}}{Dt} = -\frac{1}{\rho}\nabla p - g\hat{k} \tag{3.16}$$

$$\nabla.\bar{u} = 0 \tag{3.17}$$

where the vertical unit vector $\hat{k}$ points upward and $\frac{D}{Dt}$ is the total derivative.

Taking the curl of (3.16) we eliminate the pressuregradient, since $\nabla x \nabla() = 0$ and $g\hat{k}$ since it is a constant. The material derivative leaves us with

$$\frac{\partial\bar{\omega}}{\partial t} + \nabla x\,(\bar{\omega} x \bar{u}) = 0 \qquad (\bar{\omega} \equiv \nabla x \bar{u}) \tag{3.18}$$

or

$$\frac{D\bar{\omega}}{Dt} - (\bar{\omega}.\nabla)\bar{u} = 0 \tag{3.19}$$

If $\bar{\omega} = 0$ initially $\frac{D\bar{\omega}}{Dt} = 0$, so $\bar{\omega}$ will remain zero. Therefore we may assume the motion to be irrotational throughout. This implies that

$$\bar{u} = \nabla\phi \tag{3.20}$$

From the continuity equation (3.17) we find that ϕ has to obey a Laplace equation

$$\nabla^2\phi = 0 \tag{3.21}$$

The boundary conditions are as follows:

The free upper surface can be described by a functional relationship $f(x,y,z,t) = 0$, from which we assume that we can derive an explicit description

$$z = \zeta(x,y,t) \tag{3.22}$$

A physically reasonable boundary condition is that the rate of change of this surface is equal to the vertical component of the velocity of the moving fluid $w = \frac{Dz}{Dt}$.
We thus demand

$$w = \frac{D\zeta}{Dt} = \frac{\partial\zeta}{\partial t} + u\,\frac{\partial\zeta}{\partial x} + v\,\frac{\partial\zeta}{\partial y} \text{ on } z = \zeta \tag{3.23}$$

However, since ζ itself is unknown a priori, we have introduced an unknown quantity which must be related to the flowfield which we are determining. Therefore we also pose a second boundary condition on the free surface which is that

$$p_{ocean} = p_{atmosphere} \text{ at } z = \zeta. \tag{3.24}$$

To use this relation we must determine p_{ocean} from a Bernouilli equation. We can form a Bernouilli-equation by considering that $\overline{u} = \nabla\phi$ and $g\hat{k} = \nabla(gz)$. Therefore from (3.16)

$$\nabla\left(\frac{\partial\phi}{\partial t} + \tfrac{1}{2}(\nabla\phi.\nabla\phi) + \frac{p-p_o}{\rho_o} + gz\right) = 0 \tag{3.25}$$

or

$$\phi_t + \tfrac{1}{2}(\nabla\phi.\nabla\phi) + \frac{p-p_o}{\rho_o} + gz = B(t) \tag{3.26}$$

Absorbing the "constant" B(t), from the integration, in $\phi = \tilde{\phi} - \int B(t)dt$ we get as boundary condition on $z = \zeta$ (dropping ~), where $p = p_o$

$$\phi_t + \tfrac{1}{2}(\nabla\phi.\nabla\phi) + g\zeta = 0 \qquad , z = \zeta$$

Note that this is the place where the time variations of the flow enter explicitly.

As a bottom boundary condition we assume that there is no normal flow at the bottom, or

$$\frac{D}{Dt}(z + H) = 0 \quad , \; z = -H$$

This is equivalent to

$$w = -\frac{DH}{Dt} = -\bar{u}.\nabla_h H \quad , \; z = -H \tag{3.27}$$

If we now assume a flat bottom ($\nabla_h H = 0$) and only one horizontal wave-direction (x), the resulting system of equations describing irrotational waterwaves is

$$\left.\begin{aligned} &\nabla^2\phi = 0 \\ &\zeta_t + \phi_x\zeta_x = \phi_z \\ &\phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_z^2) + g\zeta = 0 \end{aligned}\right\} \; z = \zeta(x,t) \qquad \phi_z = 0 \quad z = -H_o \tag{3.28}$$

Due to the upper boundary condition this system is highly nonlinear. Linearizing around z = 0 by forming Taylor expansions of $\phi(\zeta)$ around z = 0:

$$\phi(\zeta) = \phi(0) + \zeta\,\frac{\partial\phi(0)}{\partial z} + \frac{\zeta^2}{2}\,\frac{\partial^2\phi(0)}{\partial z^2} + \ldots. \tag{3.29}$$

we get for small ζ (retaining only first order terms in ζ):

$$\begin{aligned} &\nabla^2\phi = 0 \\ &\zeta_t = \phi_z \\ &\phi_t + g\zeta = 0 \qquad z = 0 \\ &\phi_z = 0 \qquad\;\; z = -H_o \end{aligned} \tag{3.30}$$

The two surface boundary conditions may be combined as

$$\phi_{tt} + g\phi_z = 0 \quad , \; z = 0 \tag{3.31}$$

Assuming sinusoidal wave solutions for $\zeta = A \exp[i(kx - \omega t]$ and ϕ (in the horizontal direction)

$$\phi = Z(z) \exp[i(kx - \omega t)] \tag{3.32}$$

then the Laplace equation in ϕ yields

$$Z'' - k^2 Z = 0 \tag{3.33}$$

with fundamental solutions e^{kz}, e^{-kz}.
From the lower boundary condition (3.30^{iv}) we find that

$$Z \sim \cosh k(z + H_o)$$

From the 2nd upper boundary condition (3.30^{iii}) we find that

$$- i\omega\, Z(o) = -gA$$

and therefore

$$Z(z) = - \frac{ig}{\omega} A \frac{\cosh k(z + H_o)}{\cosh kH_o} \tag{3.34}$$

We thus have

$$\phi = R\ell\{Z \exp[i(kx - \omega t)]\} =$$

$$= \frac{gA}{\omega} \frac{\cosh k(z + H_o)}{\cosh kH_o} \sin (kx - \omega t) \tag{3.35}$$

and

$$\zeta = A \cos(kx - \omega t)$$

From the combined upper boundary condition (3.31)

$$\phi_{tt} + g\phi_z = 0 \quad , z = 0$$

we then find a dispersion relation:

$$-\omega^2 + gk \tanh kH_o = 0 \rightarrow \omega = \pm \{gk \tanh kH_o\}^{\frac{1}{2}} \tag{3.36}$$

We will now examine some asymptotic limits of this relation.
First the deepwater limit: $H_o >> \lambda$, or $kH_o >> 1$. We can reach this limit either by taking the limit $H_o \to \infty$ or by making the waves very short. We then find the dispersion relation

$$\omega^2 = gk$$

The phase velocity is thus

$$c = \pm \sqrt{g/k} \tag{3.37}$$

which clearly shows wave dispersion since $c = c(k)$. Taking $kH_o << 1$ we have the so-called shallow water limit:
The dispersion relation becomes

$$\omega^2 = gH_o k^2 \tag{3.38}$$

or

$$c = \pm \sqrt{gH_o}$$

These are non-dispersive waves because the phase-velocity equals the group velocity.

3.3 Shallow water equations

Although it would be natural at this point to progress by considering the effects of nonlinearities in potential theory, we will delay this step and pause to study a wellknown system of equations namely the shallow water equations. In doing this we concentrate on the qualitative features as offered by the linear potential theory i.e. dispersive effects (in the deep-water limit), versus those of the shallow water equations, i.e. nonlinear breaking.

In shallow water theory, the waves under consideration are assumed to be long compared to the water depth i.e., $\lambda >> H_o$. Therefore the associated vertical motions are much smaller than the horizontal motions and we may assume a hydrostatic balance in the vertical

$$\frac{\partial p}{\partial z} = - \rho g \tag{3.39}$$

After a vertical integration we get

$$p = p_{atm} + \rho g(\zeta - z) \tag{3.40}$$

where p_{atm} is the atmospheric pressure at the mean height of the surface; it is assumed to be constant in time and space.
Assuming only one horizontal dimension the equation of motion reads:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -g \frac{\partial \zeta}{\partial x} (x,t) \tag{3.41}$$

When the velocity field is independent of height at some instant $u(x,z,t = 0) = u(x)$ then $\frac{\partial u}{\partial z} = 0$ for all t since the right-hand side is z-independent. So (3.41) reduces to

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial \zeta}{\partial x} = 0 \tag{3.42}$$

The continuity equation (3.17)

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

may now be integrated vertically as all horizontal variations are independent of height

$$\int_{-H_o}^{\zeta} [\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}] dz = 0$$

Applying Leibniz' rule for interchanging integration and differentiation we arrive at

$$\frac{\partial}{\partial x}[\int_{-H_o}^{\zeta} u \, dz] - u(\zeta) \frac{\partial \zeta}{\partial x} + u(-H_o) \frac{\partial(-H_o)}{\partial x} + w(\zeta) - w(-H_o) = 0 \tag{3.43}$$

and since

$$w(\zeta) \equiv \frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + u(\zeta) \frac{\partial \zeta}{\partial x}$$

$$w(-H_o) \equiv \frac{d(-H_o)}{dt} = u(-H_o) \frac{\partial(-H_o)}{\partial x}$$

we get

$$\frac{\partial}{\partial x}\,[u(H_o + \zeta)] + \frac{\partial \zeta}{\partial t} = 0 \tag{3.44}$$

From (3.42) and (3.44) we can now write the shallow water equations (with the total depth $H = H_o + \zeta$) (see figure 13).

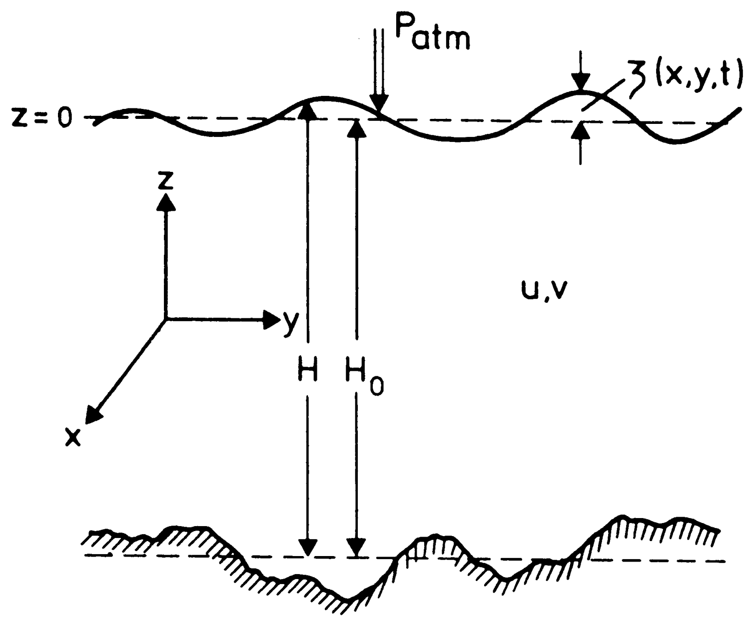

Fig. 13. Definition sketch.

$$\begin{aligned} u_t + uu_x + gH_x &= 0 \\ H_t + Hu_x + uH_x &= 0 \end{aligned} \tag{3.45}$$

Note that for small amplitude waves the same long wave limit as in linear potential theory is obtained:

$$\begin{aligned} u_t + g\zeta_x &= 0 \\ \zeta_t + H_o u_x &= 0 \end{aligned} \tag{3.46}$$

yielding $\zeta_{tt} - c_o^2\, \zeta_{xx} = 0$ or for sinusoidal waves the dispersionless relation

$$\omega = c_o k \tag{3.47}$$

where we introduced the symbol $c_o = \sqrt{gH_o}$, for the phase speed of long waves in shallow water.

If we assume no surface height variations we obtain the nonlinear advection equation

$$u_t + uu_x = 0$$

As shown in chapter 2 we can solve this equation with the method of characteristics. In general we have the material derivative

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} \tag{3.48}$$

Equation (3.48) can be interpreted as the material derivative along curves C, where $\frac{dx}{dt} = u$. See the situation sketch in figure 14.

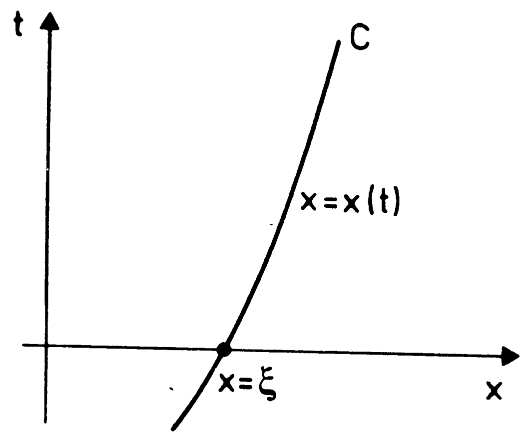

Fig. 14. The characteristic plane with one of the characteristic curves C defined by $\frac{dx}{dt} = u$.

Supplemented with some initial conditions $u = f(x)$ for $t = 0$ over het whole interval $-\infty < x < \infty$
we get a system of equations:

$$\frac{dx}{dt} = u \quad , \quad x(0) = \xi$$
$$\frac{du}{dt} = 0 \quad , \quad u(0) = f(\xi) \tag{3.49}$$

From the second equation (3.49ii) $u(t) = \text{const.}$ on C, therefore $u(t) = f(\xi)$ and from the first (3.49i)

$$x = u.t + x(0) \text{ or } x = \xi + f(\xi).t. \tag{3.50}$$

Changing the emphasis we may regard ξ as a function of x and t: $\xi = \xi(x,t)$ and extract it from this last relation. Inserting it again in $u = f(\xi)$ we then get the solution over the whole domain.
Example $f(x) = x$ would give the following solutions

$$x = \xi + \xi.t \quad \rightarrow \quad \xi = \frac{x}{1 + t}$$

and $u = f(\xi) = \xi = \frac{x}{1 + t}$ solves the equation.
Other examples will be dealt with in the exercises.
The occurrence of breaking in the solution depends on the functional form of $f(\xi)$. If $f'(\xi)$ is less than zero for some x, then breaking will occur. Assume for example an $f(\xi)$ of the form shown in figure 15.
Breaking can be seen as the intersecting point of two characteristics emanating from different values of x, for a certain value of t. For $x = x_o$ there is an intersection at $t = 0$. For a continuous function $f(x)$ the time of first breaking will be finite if for some region $f'(x) < 0$.
The method of characteristics can be extended to more general equations than the one-dimensional advection equation.
If we have an equation of the form

$$u_t + a(u,x,t)u_x = b(u,x,t) \tag{3.51}$$

we may interpret it as having an observer moving with the speed $\frac{dx}{dt} = a(u,x,t)$ and he will "see" u changing as $\frac{du}{dt} = b(u,x,t)$. The second

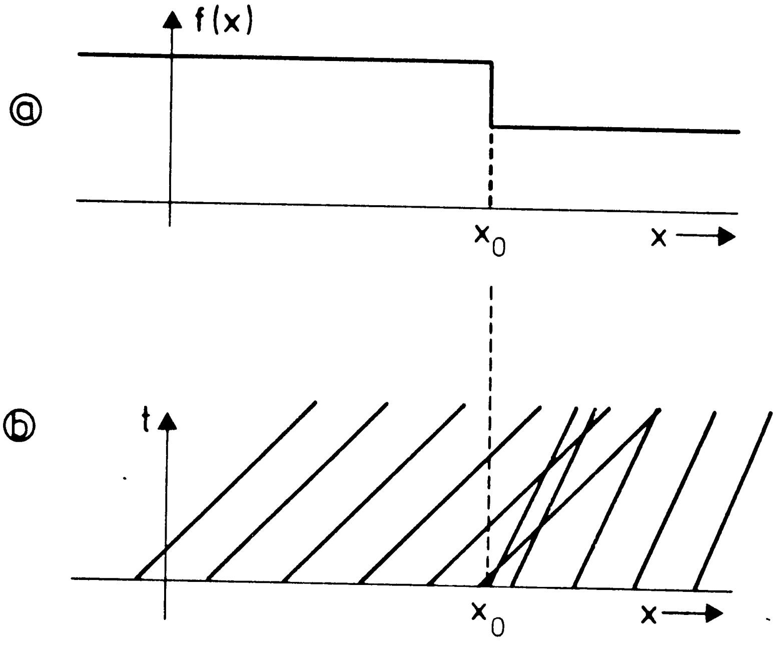

Fig. 15. Intersecting characteristics, indicating breaking, are shown in b) for the initial velocity profile in a).

extension may be applied to the problem of solving the shallow water equations:

$$u_t + uu_x + gH_x = 0 \qquad (3.52)$$

$$H_t + Hu_x + uH_x = 0 \qquad (3.53)$$

with the new variable c defined as $c^2 = gH$ this system can be rewritten as

$$\begin{aligned} u_t + uu_x + 2\,cc_x &= 0 \\ c_t + \tfrac{1}{2}\,cu_x + uc_x &= 0 \end{aligned} \qquad (3.54)$$

Multiplying the second equation by a constant λ and adding it to the first we obtain

$$(u + \lambda c)_t + (u + \frac{\lambda}{2}\,c)u_x + (2\,c + \lambda u)c_x = 0$$

Noting that we have complete freedom in the choice of λ this can be rewritten as

$$(u + \lambda c)_t + (u + \frac{\lambda}{2} c)(u + \lambda c)_x = 0 \qquad (3.55)$$

if

$$2 c + \lambda u = \lambda(u + \frac{\lambda}{2} c) \qquad (3.56)$$

From this condition we find $\lambda = \pm 2$. This may then alternatively be written as

$$\frac{d}{dt} (u + \lambda c) = 0 \text{ on curves with } \frac{dx}{dt} = u + \frac{\lambda}{2} c \qquad (3.57)$$

Or with $\lambda = \pm 2$, $u \pm 2c$ is invariant, the so called Riemann-invariants on curves that move with a speed

$$\frac{dx}{dt} = u \pm c$$

the so called characteristic velocities.

On all C^-curves, defined by $\frac{dx}{dt} = u - c$ we have $u-2c$ = constant. When we assume $u = 0$ at $t = 0$ this constant may be resolved as $-2c_o$, proportional to the phase speed in undisturbed water. So $u - 2c = -2c_o$ over the whole plane, since C^- (as the C^+-curves) span the whole plane. We can therefore use this relation between u and c also on C^+-curves defined as

$$\frac{dx}{dt} = u + c$$

After an integration and substitution $u = 2(c - c_o)$ we are left with

$$x = \xi + (3c - 2c_o)t = \xi + (3\sqrt{gH(\xi)} - 2\sqrt{gH_o}).t \qquad (3.58)$$

Note that the solution is again in the implicit form

$$x(t) = \xi + F(\xi).t \qquad (3.59)$$

as derived for the one-dimensional advection equation. The same conclusion therefore holds: if $F'(\xi) < 0$ somewhere over the interval the wave will ultimately break in a finite time interval. Since for a wave ($F'(\xi) < 0 \leftrightarrow H'(\xi) < 0 \leftrightarrow \frac{\partial \zeta}{\partial \xi} < 0$) by definition has some part where its derivative is negative, we are led to the conclusion that in this description all waves will tend to break.

So we see that the non-linearities in the shallow water equations give rise to breaking phenomena, or from another point of view, form a concentration mechanism for energy in the physical plane since potential energy is accumulating in the top of the wave and at the breaking slope. However, we saw that the associated linear system (3.46) was dispersionless. The neglect of dispersive effects is therefore mainly responsible for the fact that in the shallow water equations all waves will ultimately evolve into breaking, which is in contradiction with the observed state of a sea.

3.4 Combined effects of dispersion and non-linearity

In the potential and shallow water theories either pure dispersive - or pure nonlinear effects were present. It is obvious that we want to combine the dispersivity and the nonlinearity in one set of equations. There are several possible approaches. Using non-linear potential theory we can expand the velocity potential as a series in powers of a small parameter. Each truncation of the expansion enables one to investigate the physical meaning of the constructed system. Another approach is to add a dispersive term to the shallow water equations by expanding the dispersion relation (3.36) for linear water waves one step further. Restricting the waves to propagate in one direction we will end up with one differential equation which can be solved exactly. Finally we can follow the Seliger-Whitham approach which extends the shallow water equations with full dispersive effects.

I) The nonlinear potential theory starts from the general water wave problem (3.28) for an irrotational flow. Assuming that the waves only travel in the x-direction and that the bottom is flat we nondimensionalize the equations as follows:

$$x = \lambda x' \ , \ z = H_o z' \ , \ t = \frac{\lambda}{c_o} t' \ , \ \zeta = A\zeta' \ , \ \phi = \frac{g\lambda}{c_o} A\phi' \qquad (3.60)$$

where $c_o = \sqrt{gH_o}$ and A is the amplitude of the linear waves, we find

$$\begin{aligned}
&\beta\,\phi'_{x'x'} + \phi'_{z'z'} = 0 && -1 < z' < \alpha\zeta' \\
&\phi'_{z'} = 0 && z' = -1 \\
&\zeta'_{t'} + \alpha\phi'_{x}\ \zeta'_{x'} - \frac{1}{\beta}\,\phi'_{z'} = 0 && \Big\} \; z' = \alpha\zeta' \\
&\zeta' + \phi'_{t'} + \tfrac{1}{2}\,\alpha\phi'^{2}_{x'} + \tfrac{1}{2}\,\frac{\alpha}{\beta}\,\phi'^{2}_{z'} = 0 &&
\end{aligned} \tag{3.61}$$

where

$$\alpha = \frac{A}{H_o} \quad , \qquad \beta = \left(\frac{H_o}{\lambda}\right)^2 \tag{3.62}$$

which can be seen as a small amplitude parameter and a long wave parameter, respectively.
To eliminate the nonlinearity, which is due to the freely varying upper boundary condition to be applied at $z' = \alpha\zeta'$. We approximate as before in (3.29),

$$\phi'(\alpha\zeta') = \phi'(0) + \alpha\zeta'\,\frac{\partial\phi'}{\partial z'}(0) + \ldots$$

where the index 0 means that the functions are evaluated at $z' = 0$. This is useful since we know the velocity potential at $z' = 0$ from the linear potential theory. Assuming that β is of order 1 and α is much smaller than 1 we have α as the small parameter and we may expand the velocity potential and the surface elevation in powers of α:

$$(\phi,\zeta) = \sum_{n=0}^{\infty} \alpha^n(\phi^{(n)},\ \zeta^{(n)}) \tag{3.63}$$

To zeroth order in α the linear equations are (in a nondimensional form):

$$\begin{aligned}
&\beta\phi^{(o)}_{xx} + \phi^{(o)}_{zz} = 0 && -1 < z < 0 \\
&\phi^{(o)}_{z} = 0 && z = -1 \\
&\zeta^{(o)} - \frac{1}{\beta}\,\phi^{(o)}_{z} = 0 && \\
&\zeta^{(o)} + \phi^{(o)}_{t} = 0 && z = 0
\end{aligned} \tag{3.64}$$

The solution of this problem has already been determined (see eq. (3.35). In a nondimensional form we have

$$\zeta^{(o)} = \cos\theta \quad , \quad \theta = kx - \omega t$$

$$\phi^{(o)} = \frac{\cosh[k\sqrt{\beta}(z+1)]}{\omega\cosh(k\sqrt{\beta})} \cdot \sin\theta$$

and

$$\omega^2 = \frac{k}{\sqrt{\beta}} \tanh(k\sqrt{\beta}) \qquad (3.65)$$

To order α we have the following linear system of equations

$$\nabla^2\phi^{(1)} = 0 \qquad -1 < z < 0$$

$$\phi_z^{(1)} = 0 \qquad z = -1$$

$$\beta\zeta_t^{(1)} - \phi_z^{(1)} = \zeta^{(o)}\phi_{zz}^{(o)} - \beta\zeta_x^{(o)}\phi_x^{(o)} \qquad z = 0 \qquad (3.66)$$

$$\phi_t^{()} + \zeta^{(1)} = -(\zeta^{(o)}\phi_{tz}^{(o)} + \tfrac{1}{2}\phi_x^{(o)}\phi_x^{(o)} + \frac{1}{2\beta}\phi_z^{(o)}\phi_z^{(o)}) \qquad z = 0$$

Substituting (3.65) and eliminating $\zeta^{(1)}$ we obtain a condition for the velocity potential at $z = 0$, namely

$$\phi_{tt}^{(1)} + \frac{1}{\beta}\phi_z^{(1)} = \frac{-3\,\omega k\sqrt{\beta}}{\sinh(2k\sqrt{\beta})} \cdot \sin 2\theta \qquad z = 0 \qquad (3.67)$$

where $\sin 2\theta$ is due to product terms of $\sin\theta$ and $\cos\theta$. From this it can be seen that $\phi^{(1)}$ must be proportional to $\sin 2\theta$ whereas from (3.66) it follows that $\zeta^{(1)}$ is proportional to $\cos 2\theta$. So the expansion for the surface elevation to first order in α is:

$$\zeta = \cos\theta + \alpha . a(k,\omega)\cos 2\theta + O(\alpha^2)$$

where $a(k,\omega)$ is a known function of the wavenumber k and frequency ω. In figure 16 the result is sketched. Note that the correction to the zeroth order linear wave solution is small, since α is a small parameter. It shows that the throughs of the original sinusoidal wave flatten while the crests become steeper. In a higher order approximation this tendency

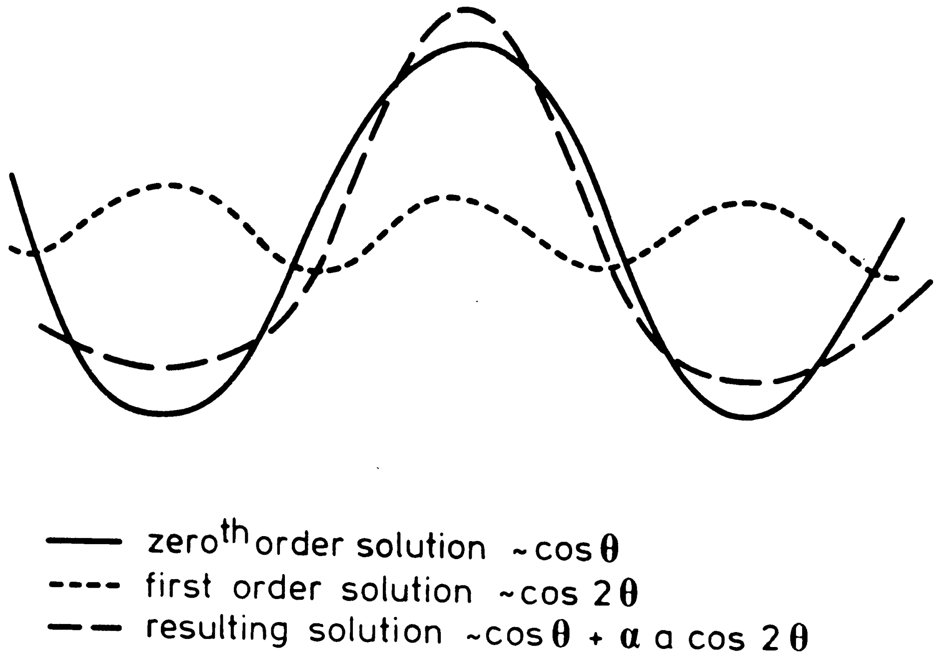

Fig. 16. The peaking of crests and flattening of troughs demonstrated by adding to the linear solution the first nonlinear correction term: its first higher harmonic.

is accentuated and the final result is peaked crests, a phenomenon frequently observed in water waves. The physical interpretation is that dispersion and nonlinearity balance in such a way that peaked waves are formed.

Stokes has given a method for calculating the angle that will ultimately be reached. This so-called Stokes angle is 120°. However, no breaking will occur and the reason for this is that the nonlinearities are weak due to the small value of the parameter α.

Note that in the second order approximations we will derive contributions proportional to sin 3θ and cos 3θ, since there will be interaction between the zeroth- and first order solution. But these interactions also give rise to terms proportional to sin θ and cos 3θ. They appear to be secular, that is the solution becomes unbounded due to these terms. Since this is not physically realistic we also have to expand the dispersion relation in powers of α and choose the coefficients such that secular terms are avoided. This will involve

the amplitude of the wave in the dispersion relation.
The method of expansion for weakly nonlinear waves (the so-called Stokes expansion) is also illustrated in the third exercise and applied to the Korteweg-de Vries equation. We will see that to keep this expansion bounded we have to conclude that the dispersion relation is also affected by the presence of nonlinearities.

II) The next method is the extension of the shallow water equations (3.45) with a dispersive term. As has been shown previously this model describes wave breaking. This is due to the lack of an energy spreading mechanism. The linearised shallow water equations

$$\zeta_t + H_o u_x = 0$$
$$u_t + g\zeta_x = 0 \qquad (3.68)$$

on the other hand, lead to a simple wave equation

$$\zeta_{tt} - c_o^2 \zeta_{xx} = 0 \qquad (3.69)$$

with the dispersion relation

$$\omega^2 = c_o^2 k^2 \qquad (3.70)$$

for sinusoidal wave solutions. In general the solution consists of disturbances $\zeta^L(x+c_o t)$ and $\zeta^R(x - c_o t)$, travelling to the left and right respectively, associated with the simple wave equations

$$\zeta_t^L - c_o \zeta_x^L = 0 \quad \text{and} \quad \zeta_t^R + c_o \zeta_x^R = 0 \qquad (3.71)$$

From potential theory we saw that for general linear water waves the dispersion relation is given by $\omega^2 = gk \tanh kH_o$. We expect that we must extend the shallow water equations in such a way that the linearised version has the same dispersion relation.
An expansion of the dispersion relation (3.36) beyond the dispersionless

case leads to

$$\omega^2 = c_o^2k^2 - \frac{1}{3} c_o^2H_o^2k^4 + O(k^6) \tag{3.72}$$

As we have seen in the introduction this polynomial dispersion relation can be associated with a linear equation, which has periodic solutions with a corresponding wave number relation. In this case the equation corresponding to (3.72) reads

$$\zeta_{tt} - c_o^2\zeta_{xx} - \frac{1}{3} c_o^2H_o^2 \zeta_{xxxx} = 0 \tag{3.73}$$

Comparing with (3.69) we can now reconstruct (3.68) with an extra term coming from the fourth order derivative in the equation above.

$$\begin{aligned} &\zeta_t + H_o u_x = 0 \\ &u_t + g\zeta_x + \frac{1}{3} c_o^2H_o \zeta_{xxx} = 0 \end{aligned} \tag{3.74}$$

Finally adding the nonlinear terms we end up with the extended shallow water equations

$$\begin{aligned} &H_t + (uH)_x = 0 \\ &u_t + uu_x + gH_x + \frac{1}{3} c_o^2H_oH_{xxx} = 0 \end{aligned} \tag{3.75}$$

We see that the continuity equation is unmodified but the momentum equation has one extra term which physically may be interpreted as the curvature correction of the water level due to fluid motion.
When we restrict ourselves again to waves moving to the right we separate (3.72) into

$$\omega = + \{c_o^2k^2 - \frac{1}{3} c_o^2H_o^2k^4 + O(k^6)\}^{\frac{1}{2}} = c_ok(1 - \frac{1}{6} H_o^2k^2) + O(k^5)$$

or

$$\omega = c_ok - \gamma k^3 \text{ with } \gamma = \frac{1}{6} c_oH_o^2 \tag{3.76}$$

We want to add a nonlinear term to this. Therefore we look at the shallow water equations for waves moving to the right. It has been shown in (3.3) that they satisfy the Riemann invariant $u = 2\sqrt{gH} - 2\sqrt{gH_o}$.

Substituting this in the continuity equation and again using $H=H_o + \zeta$ we obtain (see also (3.58))

$$\zeta_t + (3\sqrt{g(H_o + \zeta)} - 2\sqrt{gH_o})\zeta_x = 0 \qquad (3.78)$$

Combining with (3.77) we find

$$\zeta_t + (3\sqrt{g(H_o + \zeta)} - 2\sqrt{gH_o})\zeta_x + \gamma\zeta_{xxx} = 0$$

and for ζ much smaller than the depth H_o this may be approximated as

$$\zeta_t + c_o(1 + \frac{3}{2}\frac{\zeta}{H_o})\zeta_x + \gamma\zeta_{xxx} = 0$$

which is the famous Korteweg-de Vries equation.
The derivation is rather intuitive. However, it may also be done in a formal way by starting from the nonlinear potential theory and expanding the velocity potential in the following way

$$\phi = \sum_{n=o}^{\infty} f_n(x,t).\ z^n$$

In its normalized form this will be an expansion in the assumed small parameter β. Retaining terms proportional to α and β and neglecting higher order contributions we obtain the Korteweg-de Vries equation from the free surface boundary condition. We will however, continue with our investigation of the KdV equation. For the moment it is enough to realise that it is the first expansion of the full nonlinear set of equations describing water waves and combines both dispersion and non-linearity. We will now proceed by investigating the similarity solutions of the KdV-equation.

$$\zeta = \zeta(x - Ut) = \zeta(X) \qquad (3.80)$$

which have a steady shape for an observing moving with velocity U along the x-axis. Introducing this dependence in the Korteweg-de Vries equation we find ($\hat{\zeta} = \zeta/H_o$)

$$-(\frac{U}{c_o} - 1)\hat{\zeta}' + \frac{3}{2}\hat{\zeta}\,\hat{\zeta}' + \frac{1}{6}H_o^2\,\hat{\zeta}''' = 0$$

Integrating once gives

$$-(\frac{U}{c_o} - 1)\hat{\zeta} + \frac{3}{4}\hat{\zeta}^2 + \frac{1}{6}H_o^2\hat{\zeta}'' + B = 0$$

Multiplying by $\hat{\zeta}'$ and integrating once more yields

$$-(\frac{U}{c_o} - 1)\frac{\hat{\zeta}^2}{2} + \tfrac{1}{4}\hat{\zeta}^3 + \frac{1}{12}H_o^2(\hat{\zeta}')^2 + B\hat{\zeta} + A = 0 \qquad (3.81)$$

where A and B are integration constants.
A special solution may be obtained when A = B = 0 and ζ and its derivatives tend to zero at $\pm\infty$. Then

$$\frac{1}{3}H_o^2(\hat{\zeta}')^2 = \hat{\zeta}^2(\alpha - \hat{\zeta}) \text{ with } \alpha = 2(\frac{U}{c_o} - 1) \qquad (3.82)$$

A real solution is found when α is positive, that is U larger then c_o. Coming from $-\infty$, $(\alpha - \hat{\zeta})$ will be positive and $\hat{\zeta}' > 0$. When $\hat{\zeta} = \alpha$ (at X = 0 for instance) ζ' changes sign and the curve symmetrically decreases to zero for $X \to \infty$. The corresponding water level is drawn in figure 17.

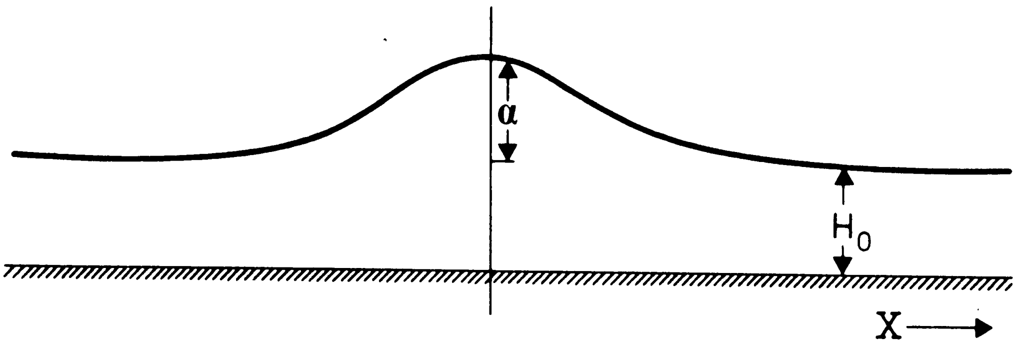

Fig. 17. Soliton.

This identifies α as ζ_o. Actually the solution of (3.82) reads

$$\hat{\zeta} = \alpha \cosh^{-2}[(\frac{3\alpha}{4H_o^2})^{\frac{1}{2}} X]$$

or

$$\zeta = A \cosh^{-2}[(\frac{3A}{4 H_o^3})^{\frac{1}{2}} (x - Ut)] \tag{3.83}$$

This is a so-called solitary wave. Its propagation velocity turns out to be

$$U = c_o(1 + \tfrac{1}{2} \frac{A}{H_o}) \tag{3.84}$$

It is clear that (3.83) is a nonlinear wave due to the fact that its phase speed is a function of the amplitude: the higher the waves are the faster they move. The name solitary wave is due to the fact that it consists of only one single hump. They can be observed in watertanks but also in canals as was first done by Scott Russell in 1844.
In fact (3.83) is the limiting wave form of the special similarity solutions of the Korteweg-de Vries equation. Without restricting ourselves to the case A = B = 0 the general solution may be calculated in terms of elliptic functions, but the derivation goes beyond the scope of these lectures. The result is a train of cnoidal waves, which are plotted in figure 18.

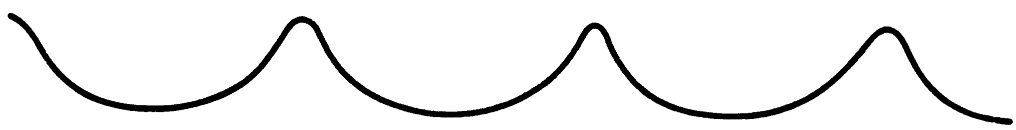

Fig. 18. Cnoidal waves.

They have steep crests and flat troughs, resembling the Stokes wave somewhat. However, no peaking nor breaking occurs for these waves. The reason is that the dispersion relation (3.76) is only valid for small values of kH_o. This means that the introduced dispersion is much too strong for the short waves, and this prevents breaking and peaking. The parameter that controls the behaviour of the solution is

$$s = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}} = \frac{A\lambda^2}{H_o^3} \tag{3.85}$$

which measures the relative importance of nonlinearity versus dispersion. If s reaches zero we have linear dispersive waves and in the limit $s \to 1$ we have solitary waves: there is then an exact balance between nonlinearity and dispersion. A very important feature of these solitary waves is that they preserve their identity when interacting with other solitary waves (apart from a phase shift). This means that they behave like linear waves. This particle-like behaviour is the main reason for the introduction of the name soliton.

III) Seliger-Whitham approach

The Korteweg-De Vries equation, having both nonlinearity and dispersion, yields steady state waves that become solitary waves when s in (3.85) is close to 1. However, from a physical point of view, the dispersion is too severe as there is no wave breaking anymore.

Returning to our goal of constructing an equation which combines non-linear and dipersive effects we may combine the two results arrived at previously. The breaking of waves in shallow water is effectively described by (3.78), where we may introduce the approximation that the sealevel variations ζ are much smaller than the water depth H_o. We then have

$$\zeta_t + \zeta_x + \frac{3}{2}\zeta\,\zeta_x = 0 \tag{3.86}$$

where we have scaled all quantities with a length scale H_o and a velocity scale c_o. On the other hand the most general way of describing dipsersive waves in a linear system can be written in the form (see eq. 3.11)

$$\zeta_t + \int_{-\infty}^{\infty} K(x - \xi)\,\zeta_\xi\,d\xi = 0 \tag{3.87}$$

where the kernel K is the Fouriertransform of the phase velocity:

$$K(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} c(k) e^{ikx} \, dk$$

and c(k) is taken from dispersion relation (3.36) for linear water waves and reads in nondimensional form

$$c(k) = \{ \frac{\tanh k}{k} \}^{\frac{1}{2}} \tag{3.88}$$

Combining (3.87) and (3.13) intuitively, in the same way as has been done by deriving the Korteweg-de-Vries equation the result is

$$\zeta_t + \frac{3}{2} \zeta \, \zeta_x + \int_{-\infty}^{\infty} K(x-\xi) \zeta_\xi \, d\xi = 0$$

Note that the linear term ζ_x is now incorporated in the kernel-term. This equation may also be derived from nonlinear potential theory by taking into account all orders of β and the nonlinear term proportional to α, disregarding cross-products terms as well as higher order terms in α. However, the calculation of all these terms is rather tedious and the result cannot be put in an integral representation. But it is clear that (3.89) describes a very realistic situation in which the nonlinearity is combined with full dispersive effects. Equation (3.89) looks as a straightforward extension of the non-linear advection equation, here in terms of ζ

$$\zeta_t + \zeta \zeta_x = 0$$

(apart from the constant 3/2, which can be absorbed via a simple transformation). However, solving it with the method of characteristics is difficult. Witham (1974, 1979) and Seliger (1968) show however, that equation (3.89) may exhibit both peaking and breaking depending on the initial steepness of the wave. Or in other words the original criterium that the solution at t = 0, should have a negative slope somewhere, in order to evolve into breaking, is now relaxed into a limit, much like in the solution of the model equation treated in chapter one.

Exercises on water waves

5. Find the dispersion relation for the following equations:

$$\eta_t + c_0\, \eta_x = 0$$

$$\eta_t + c_0\, \eta_x = \eta_{xx} \qquad \text{(linearized Burgers eq.)}$$

$$\eta_t + c_0\, \eta_x + \eta_{xxx} = 0 \qquad \text{(linearized Korteweg-de Vries eq.)}$$

Discuss the difference in the resulting behaviour of the solutions. What would the Kernel-function $K(x)$ be when these equations are written in general waveequation form

$$\eta_t + \int_{-\infty}^{\infty} K(x - \xi)\eta_\xi(\xi,t)d\xi = 0$$

(Note that one of the definitions

of $\delta(x-a) = \lim\limits_{L\to\infty} \frac{1}{2\pi} \int_{-L}^{L} \exp i(x-a)t\, dt$)

6. Solve with the method of characteristics:

i) $\phi_t + \exp(-t)\phi_x = 0 \qquad t > 0,\ -\infty < x < \infty$

$\phi = 1/(1 + x^2) \qquad t = 0$

ii) $x^2\phi_t + \phi_x + t\phi = 0 \qquad x > 0,\ -\infty < t < \infty$

$\phi = \Phi(t) \qquad x = 0$

iii) $\phi_t + \phi\phi_x + \alpha\phi = 0 \qquad t > 0,\ -\infty < x < \infty,\ \alpha > 0$

$\phi = F(x) \qquad t = 0$

(treated as example in first lecture).
Discuss the occurrence of breaking.

7. Solve the Korteweg-de Vries equation

$$\zeta_t + c_o(1 + \frac{3}{2}\frac{\zeta}{h_o})\zeta_x + \gamma\zeta_{xxx} = 0 \qquad \gamma = \frac{1}{6}\, c_o h_o^2$$

with a perturbation expansion technique and look for periodic solutions i.e. write

$$\frac{\zeta}{h_0} = \alpha\zeta_1(\theta) + \alpha^2\zeta_2(\theta) + \alpha^3\zeta_3(\theta) + \ldots \quad \theta = kx - \omega t, \alpha = \frac{a}{h_0} << 1$$

i) Insert this expansion into the KdV-eqn. What do the resulting equations in the first three orders in α look like?

ii) Find a solution for the first two equations (taking only one of the two fundamental solutions of the first equation).

iii) The nonlinear contribution in the third order equation has a $\sin\theta$-term. This (secular) term resonates with a $\cos\theta$ solution, which means that $\theta\cos\theta$ would be part of the solution of ζ_3. Since this is unbounded in θ it is argued that this cannot be physically realistic, since we are looking for periodic solutions.
Therefore also expand $\omega(k) = \omega_o(k) + \alpha\omega_1(k) + \alpha^2\omega_2(k) + \ldots.$ and find the value of ω_1 and ω_2 in order to avoid the occurrence of a secular term in the equation for α^3. (Note that we now allow $\omega = \omega(k,\underline{a})$).

4. Two-dimensional, nondivergent flow on a sphere

To investigate the nonlinear dynamics of large scale atmopheric flow we will now turn our attention to two dimensional models on a rotating spherical geometry. We will only concern ourselves with nondivergent flow, thus working with the barotropic vorticity equation. The motion is of a Rossby wave type where the characteristic time scale is the inverse of the rotation rate. The procedure for investigating the nonlinearity will be the same as the one used earlier in chapter one. We will expand the space dependent part in terms of a set of orthogonal functions and we will truncate the expansion at a very low order. Due to the spherical geometry, the expansion functions will be more complex than in the one-dimensional case and the nonlinear interactions between the components will also be more involved. In principle, however, the procedure is similar and we hope that the one dimensional example will serve as an analogue to the more mathematically involved two dimensional case. Through the assumption of non-divergence we will describe a two-dimensional velocity field with one scalar quantity, namely the vorticity. For atmospheric flow conditions this is a very good approximation. The nonlinearity appearing in the governing equation is of the advective type, albeit somewhat different from the advection term in the one-dimensional equation. In the one-dimensional case the velocity field is advecting itself. For non-divergent, two dimensional flow the vorticity, which is a function of the velocity field, is being advected and this gives the nonlinearity some particular properties which we will examine in detail. The rotation of the sphere is another important factor which will make the two-dimensional flow quite different from the one-dimensional one.

We will examine the nonlinear structure of truncated systems in detail and by considering energy and enstrophy conservation principles a very important restriction on any truncated system is established. The principle states that we must have at least three different components in a truncated system to have non-trival nonlinear energy exchanges within the system. This principle has important implications for the atmospheric energy spectrum and it also limits the type of interactions allowed in a given low order system.

4.1 Spectral expansion

For a two-dimensional velocity $\overline{v}$ the equation of motion reads

$$\frac{\partial \overline{v}}{\partial t} + \overline{v} \cdot \nabla \overline{v} + f\, \overline{k} \times \overline{v} = - \nabla \phi \qquad (4.1)$$

where $f = 2\, \Omega \sin \phi$ is the coriolisparameter (Ω is the angular speed of rotation of the sphere), $\overline{k}$ is the unity vector along the radius of the sphere (pointing outwards) and ϕ the geopotential. Furthermore we suppose the flow to be incompressible, so the continuity equation is

$$\nabla \cdot \overline{v} = 0 \qquad (4.2)$$

in other words the flow is non-divergent. This means that a stream-function ψ can be introduced, and the velocity field may be written

$$\overline{v} = \overline{k} \times \nabla \psi \qquad (4.3)$$

Now defining the relative vorticity

$$\zeta = \overline{k} \cdot (\nabla \times \overline{v}) = \nabla^2 \psi \qquad (4.4)$$

we can derive a vorticity equation by operating on (4.1) with $(k \cdot \nabla \times)$. The result is

$$\frac{\partial \zeta}{\partial t} + \overline{v} \cdot \nabla (\zeta + f) = 0 \qquad (4.5)$$

which is independent of the forcing by the geopotential. This is the barotropic vorticity equation. From this equation it is clear that the velocity field is advecting the vorticity field.
Rewriting eq. (4.5) in terms of a total derivative

$$\frac{d}{dt} (\zeta + f) = 0 \qquad (4.6)$$

we see that the equation just expresses the conservation of total vorticity which consists of the local component, ζ, and the component arising from the rotation vector of the sphere projected on the radial

vector, f. Through the definition of the vorticity (4.4) we can associate it with a local rotation and a local shear of the velocity field. If a fluid particle is advected in the latitudinal direction eq. (4.5) states that the local vorticity must change due to the change of f. Integrating eq. (4.5) over the surface of the sphere we can see that the integrated, total vorticity must remain unchanged.
The vorticity equation may als be written as

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta + f) = 0 \tag{4.7}$$

with the jacobian

$$J(a,b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x} = \frac{\partial(a,b)}{\partial(x,y)} \tag{4.8}$$

To arrive at a final equation in a non-dimensional form we define a time scale Ω^{-1} and the radius of the earth as a length scale. Thus

$$t' = \Omega t \qquad \zeta' = \frac{1}{\Omega}\zeta \tag{4.9}$$

where the primed variables denote nondimensional quantities.
As we will be working on a spherical geometry we wish to express the velocity vector $\overline{v} = (u,v)$ in terms of the radius of the earth (a), the longitude (λ) and $\mu = \sin\phi$, where ϕ is latitude. See figure 19. Now introducing a local cartesian coordinate system (x,y) at point P on the surface of the sphere we have from figure 19

$$dx = a\cos\phi\, d\lambda \qquad dy = \frac{a}{\cos\phi} d\mu \tag{4.10}$$

and thus we obtain

$$u = \frac{dx}{dt} = a\cos\phi\frac{d\lambda}{dt} \qquad v = \frac{dy}{dt} = \frac{a}{\cos\phi}\frac{d\mu}{dt} \tag{4.11}$$

Applying these relations to the Jacobian we have

$$J(a,b) = \frac{1}{a^2}\frac{\partial(a,b)}{\partial(\lambda,\mu)} \tag{4.12}$$

and the nondimensional vorticity equation reads

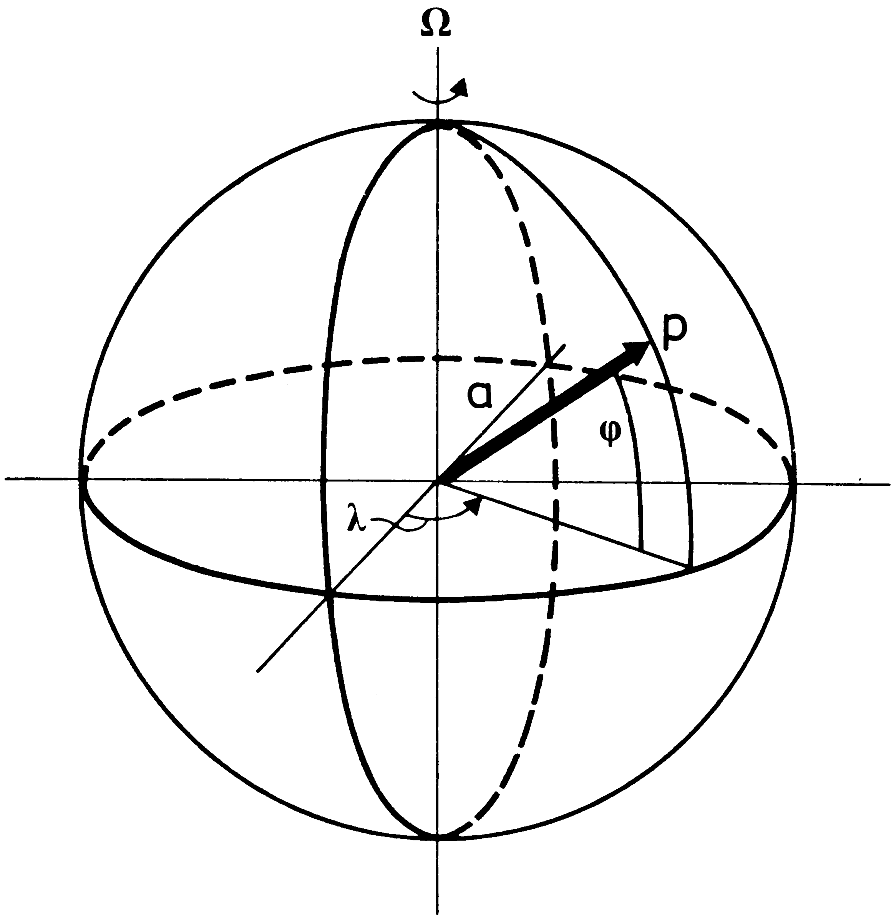

Fig. 19 Geometrical configuration for the two dimensional flow on a rotating sphere.

$$\frac{\partial \zeta'}{\partial t'} + J(\psi',\zeta') + 2\frac{\partial \zeta'}{\partial \lambda} = 0 \qquad (4.13)$$

This states that the total vorticity $\eta = \zeta + f$ is conserved for a particle following the fluid motion. The treatment of (4.13) is made difficult by the second term: it is a nonlinear contribution due to the advection of the vorticity field. In this section we will discuss a method for analysing this nonlinearity. For convenience we will hereafter drop the primes from (4.13).

Using the same method as for the non-dimensional advection equation, we will separate the time and space dependent parts of the solution to

(4.13) and we will expand the space dependent part in a series of orthogonal functions. As we are working on a spherical geometry and the governing equation is expressed in terms of a stream function and a vorticity, it is natural to use Legendre polynomials, or spherical harmonics, as our expansion functions. The relation between the stream-function and the vorticity will now be particularly simple as the spherical harmonics are eigenfunctions of the Laplacian operator in the sphere.

For the vorticity we thus write

$$\zeta = \sum_n \sum_\ell \zeta_{n\ell}(t)\ P_n^\ell(\mu) e^{i\ell\lambda} \tag{4.14}$$

The functions $P_n^\ell(\mu)$ have different definitions in mathematical textbooks. Here we will follow the definition given by Platzmann (1962), which is

$$P_n^\ell(\mu) = \left[(2n+1)\ \frac{(n-\ell)!}{(n+\ell)!}\right]^{\frac{1}{2}} \frac{(1-\mu^2)^{\ell/2}}{2^n n!} \left(\frac{d}{d\mu}\right)^{n+\ell} (\mu^2-1)^n \tag{4.15}$$

with $\ell \geq 0$ and $n \geq 1$. If ℓ is negative we define

$$P_n^{-\ell}(\mu) = P_n^\ell(\mu) \tag{4.16}$$

Some more information about Legendre polynomials and their properties can be found in appendix A.

Now n determines the order of the polynomial, and ℓ corresponds to the longitudinal wave number. By defining a complex wave number

$$\gamma = n + i\ell \tag{4.17}$$

the double summation in (4.14) can be reduced to a single one:

$$\zeta = \sum_\gamma \zeta_\gamma(t)\ P_n^\ell(\mu) e^{i\ell\lambda} = \sum_\gamma \zeta_\gamma Y_\gamma(\mu,\lambda) \tag{4.18}$$

In this summation ℓ can also have negative values, since there are sine- and cosine parts. Thus we sum over all complex values of γ.

From the definition of Y_γ in (4.18) we see that

$$Y_{\bar{\gamma}} = \overline{Y}_\gamma, \tag{4.19}$$

where the bar denotes complex conjugation.
The polynomials are normalized in the following way

$$\frac{1}{2} \int_{-1}^{1} [P_n^\ell(\mu)]^2 \, d\mu = 1 \tag{4.20}$$

and from (4.18) it may be seen that

$$\frac{1}{4\pi} \int_S \overline{Y}_\alpha \, Y_\beta \, dS = 1 \quad \text{if } \alpha = \beta \tag{4.21}$$

with S the surface of the sphere.
From (4.4) we find that there is a simple relation between the amplitudes of the relative vorticity and the stream function. To obtain this we expand

$$\psi = \sum_\gamma \psi_\gamma(t) \, Y_\gamma \tag{4.22}$$

and calculate

$$\nabla^2 \psi = \sum_\gamma \psi_\gamma(t) \, \nabla^2 Y_\gamma = - \sum_\gamma n(n+1) \psi_\gamma(t) Y_\gamma = \sum_\gamma \zeta_\gamma \, Y_\gamma \tag{4.23}$$

Here a basic property of the Legendre functions is used (see appendix A). Thus,

$$\psi_\gamma = - \frac{1}{n(n+1)} \zeta_\gamma = - c_\gamma \, \zeta_\gamma \tag{4.24}$$

This relation makes it easy to alternate between the vorticity and the stream function. Now inserting the expansion (4.14) in the barotropic vorticity equation (4.13) we have

$$\frac{\partial}{\partial t} \sum_\gamma \zeta_\gamma \, Y_\gamma + J(\sum_\alpha \psi_\alpha \, Y_\alpha, \sum_\beta \zeta_\beta \, Y_\beta) + 2 \sum_\gamma \psi_\gamma \frac{\partial Y_\gamma}{\partial \lambda} = 0 \tag{4.25}$$

where α and β are complex wave numbers. The first and third term are linear and can therefore be written as follows:

$$\frac{\partial}{\partial t} \sum_\gamma \zeta_\gamma Y_\gamma = \sum_\gamma \frac{d\zeta_\gamma}{dt} Y_\gamma \tag{4.26}$$

$$2 \sum_\gamma \psi_\gamma \frac{\partial Y_\gamma}{\partial \lambda} = -2\, i \sum_\gamma \ell_\gamma c_\gamma \zeta_\gamma Y_\gamma \tag{4.27}$$

Now we turn to the nonlinear term, and using some of the properties derived above we find

$$\begin{aligned} J(\sum_\alpha \psi_\alpha Y_\alpha, \sum_\beta \zeta_\beta Y_\beta) &= - J(\sum_\alpha c_\alpha \zeta_\alpha Y_\alpha, \sum_\beta \zeta_\beta Y_\beta) = \\ &= \sum_{\alpha,\beta} c_\alpha \zeta_\alpha \zeta_\beta \left(\frac{\partial Y_\alpha}{\partial \mu}\frac{\partial Y_\beta}{\partial \lambda} - \frac{\partial Y_\alpha}{\partial \lambda}\frac{\partial Y_\beta}{\partial \mu}\right) = \\ &= \sum_{\alpha,\beta} \zeta_\alpha \zeta_\beta\, i\, c_\alpha\, e^{i(\ell_\alpha+\ell_\beta)\lambda} \left(\ell_\beta P_\beta \frac{dP_\alpha}{d\mu} - \ell_\alpha P_\alpha \frac{dP_\beta}{d\mu}\right) \end{aligned} \tag{4.28}$$

As the expansion functions are orthogonal we want to project te resulting vorticity equation on a specific wavenumber, γ, which is the same thing as multiplying the equation by $Y_{\bar\gamma}$ and integrating over the surface S of the sphere. For the first term (4.26) this results in:

$$\int Y_{\bar\gamma}\{\sum_\alpha \frac{d\zeta_\alpha}{dt} Y_\alpha\} dS = 4\pi \frac{d\zeta_\gamma}{dt}$$

because of the orthogonality of Y_γ and Y_α if $\gamma \neq \alpha$. Operating in the same way on the other two terms the spectral form of the vorticity equation becomes,

$$\begin{aligned} 4\pi\left[\frac{d\zeta_\gamma}{dt} - 2\, i\, \ell_\gamma c_\gamma \zeta_\gamma\right] &+ \sum_{\alpha,\beta} \{ i\, \zeta_\alpha \zeta_\beta c_\alpha \cdot \\ \int P_\gamma\left(\ell_\beta P_\beta \frac{dP_\alpha}{d\mu} - \ell_\alpha P_\alpha \frac{dP_\beta}{d\mu}\right) & e^{i(\ell_\alpha+\ell_\beta-\ell_\gamma)\lambda} dS\} = 0 \end{aligned} \tag{4.29}$$

because $Y_{\bar\gamma}(\mu,\lambda) = P_{n_\gamma}^{-\ell_\gamma}(\mu) e^{-i\ell_\gamma\lambda} = P_{n_\gamma}^{\ell_\gamma}(\mu) e^{-i\ell_\gamma\lambda} = P_\gamma e^{-i\ell_\gamma\lambda}$.

If $\ell_\alpha + \ell_\beta - \ell_\gamma \neq 0$ then the integral in (4.29) is zero because the exponent is the only longitude-dependent term. Integrated over longitude (around a latitude circle) this term gives zero.

If $\ell_\alpha + \ell_\beta - \ell_\gamma = 0$ we may have the integral nonzero, but to determine this we have to examine the latitudinally dependent part of the integral. We can now rewrite the integral, which is only dependent on latitude.

As

$$\int (\quad) dS = \int_{-1}^{1} \int_{0}^{2\pi} (\quad)\, d\lambda\, d\mu$$

equation (4.29) becomes,

$$\frac{d\zeta_\gamma}{dt} = 2i\ell_\gamma c_\gamma \zeta_\gamma + i \sum_{\alpha,\beta} \{\zeta_\alpha \zeta_\beta (c_\beta - c_\alpha)\tfrac{1}{2} \int_{-1}^{1} P_\gamma (\ell_\beta P_\beta \frac{dP_\alpha}{d\mu} - \ell_\alpha P_\alpha \frac{dP_\beta}{d\mu}) d\mu\}$$

where the summation in the second term on the right is non-redundant; that is, it only includes all distinct combinations (without permutation) of the pair of vectors α and β. The above can also be written in the following form:

$$\frac{d\zeta_\gamma}{dt} = 2i\, \ell_\gamma\, c_\gamma\, \zeta_\gamma + i \sum_{\alpha,\beta} I_{\gamma,\alpha,\beta}\, \zeta_\alpha\, \zeta_\beta \tag{4.30}$$

where $I_{\gamma,\alpha,\beta}$ are interaction coefficients, determined as follows:

$$I_{\gamma,\alpha,\beta} = \tfrac{1}{2}(c_\beta - c_\alpha) \int_{-1}^{1} P_\gamma (\ell_\beta P_\beta \frac{dP_\alpha}{d\mu} - \ell_\alpha P_\alpha \frac{dP_\beta}{d\mu}) d\mu \tag{4.31}$$

We can also define so called coupling integrals

$$K_{\gamma,\beta,\alpha} = \int_{-1}^{1} P_\gamma (\ell_\beta P_\beta \frac{dP_\alpha}{d\mu} - \ell_\alpha P_\alpha \frac{dP_\beta}{d\mu}) d\mu \tag{4.32}$$

It is clear that I is symmetric and K is antisymmetric:

$$I_{\gamma,\alpha,\beta} = I_{\gamma,\beta,\alpha} \qquad K_{\gamma,\beta,\alpha} = -\, K_{\gamma,\alpha,\beta} \tag{4.33}$$

We will now set up some selection rules so that large classes of interaction coefficients can be excluded from consideration. Due to various properties of the Legendre functions, $I_{\gamma,\alpha,\beta}$ must vanish even when $\ell_\alpha + \ell_\beta - \ell_\gamma = 0$ unless the following scalar selection rules are satisfied:

$$\ell_\alpha^2 + \ell_\beta^2 \neq 0 \tag{4.34}$$

$$|n_\beta - n_\alpha| < n_\gamma < n_\beta + n_\alpha \tag{4.35}$$

$$n_\gamma + n_\beta + n_\alpha \text{ is odd} \tag{4.36}$$

$$n_\beta \neq n_\alpha \tag{4.37}$$

The last rule (4.37) comes from the factor $(c_\beta - c_\alpha)$ and therefore does not hold for the coupling integrals, all the others are true for both types of integrals. Rule (4.34) is merely a reflection of the vanishing of $K_{\gamma,\beta,\alpha}$ when both ℓ_α and ℓ_β are zero; in other words the interaction of two zonal components is nugatory. The "triangle" rule (4.35) and the "parity" rule (4.36) are consequences of properties of the associated Legendre functions. Another way of stating these two rules is that the three n's must form the sides of a triangle of odd perimeter. An elementary proof of (4.36) is easily given from the fact that a Legendre polynomial is an even function of μ if $1 + n$ is even and an odd function of μ if $1 + n$ is odd (see the definition (4.15)). From reference to (4.32), the integrand of $K_{\gamma,\beta,\alpha}$ must be either an even or an odd function of μ and, in the latter case, K must vanish. To make the integrand of $K(= P_\gamma(\ell_\beta P_\beta \frac{dP_\alpha}{d\mu} - \ell_\alpha P_\alpha \frac{dP_\beta}{d\mu}))$ even we can have the following combinations of P_α, P_β and P_γ (note: if P. is even, $\frac{dP.}{d\mu}$ is odd and vice versa):

Combination number	1	2	3	4
P_β:	odd	even	odd	even
P_α:	even	odd	odd	even
P_γ:	even	even	odd	odd

Combination (1) means that

$\ell_\beta + n_\beta$ is odd

$\ell_\alpha + n_\alpha$ is even

$\ell_\gamma + n_\gamma$ is even

which leads to

$\ell_\beta + \ell_\alpha + \ell_\gamma + n_\beta + n_\alpha + n_\gamma$ being odd.

This relation in fact holds for all 4 combinations of P_α, P_β and P_γ. If use is made of the fact that $\ell_\alpha + \ell_\beta = \ell_\gamma$, then we find that $n_\alpha + n_\beta + n_\gamma$ must be odd.

The proof of the "triangle" rule (4.35) is not so straight forward. It is given by Silberman (1954). $I_{\gamma,\alpha,\beta}$ must also vanish when $\ell_\alpha + \ell_\beta - \ell_\gamma = 0$ unless the following vector selection rules are satisfied:

$$\beta \neq \alpha \tag{4.38}$$

$$\beta \neq \overline{\gamma} \text{ and } \alpha \neq \overline{\gamma} \tag{4.39}$$

Selection rule (4.38) is a consequence of the antisymmetry of K and selection rule (4.39) is a consequence of the redundancy relations,

$$K_{\gamma,\beta,\alpha} = K_{\alpha,\overline{\beta},\gamma} \qquad K_{\gamma,\beta,\alpha} = K_{\beta,\gamma,\overline{\alpha}} \tag{4.40}$$

(see exercise).

Corresponding to each expansion coefficient there is a wave vector $\gamma = (\ell,n)$. The infinite set of all wave vectors (for spherical harmonics) can be presented geometrically in the ℓ,n plane by the integral lattice points which lie within the semi infinite triangle $|\ell| < n$, or on the boundary of this region (see fig. 20). From this set, we select a finite subset S which is symmetric with respect to the axis $\ell = 0$.

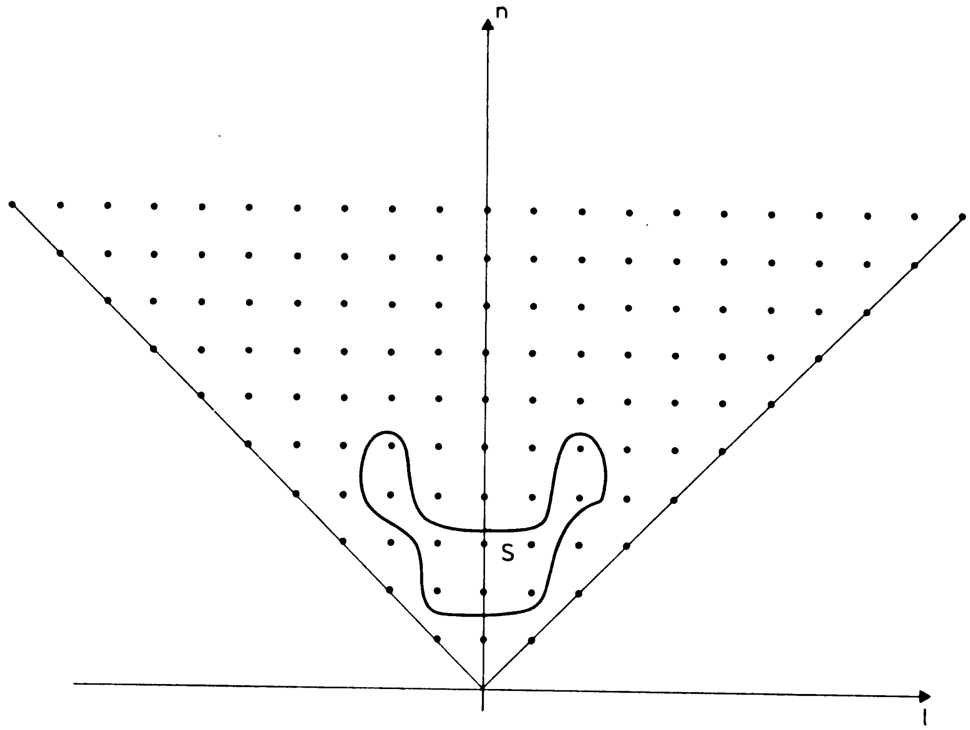

Fig. 20. Representation in the (l,n) wavenumber plane of the nonzero Legendre functions and a certain subset, S.

In other words for each vector (ℓ,n) in S we include the image $(-\ell,n)$. This is necessary if the truncated spectrum is to represent a real function. All equations in the system (4.30) for which ζ_γ corresponds to a wave vector outside of S will be ignored, so we are left with a finite number of equations. We further ignore all nonlinear interactions which involve expansion coefficients corresponding to wave vectors outside of S. This means, then, that all three vectors α,β,γ in (4.30) must be members of S. The set of equations thus obtained evidently involves only the expansion coefficients corresponding to the wave vectors in S and is a closed system with a solution $\zeta_\gamma(t)$ determined uniquely by the initial conditions $\zeta_\gamma(0)$.
If we look at the wave vector corresponding to $n = 0$ and $\ell = 0$ ($\ell = 0$ corresponds to zonal flow) we find that due to the fact that P_0^0 is constant ψ must also be constant so that there is no flow.
The amplitude of this wave vector can thus arbitrarily be set to zero.

The next zonal wave component ($n = 1$, $\ell = 0$) corresponds to a solid body rotation of the whole atmosphere, because $P_1^o = \sqrt{3}\mu$.
We can include this in the Coriolis force. Take $\alpha = 1$ in the spectral vorticity equation (4.30). Due to selection rule (4.35) and the fact that $\ell_\alpha + \ell_\beta - \ell_\gamma = 0$ and $\ell_\alpha = 0$, β must be equal to γ. Substituting $\alpha = 1$ and $\beta = \gamma$ in equation (4.30) gives for each $\gamma \neq 1$

$$\frac{d\zeta_\gamma}{dt} = 2i\ell_\gamma\, c_\gamma\, \zeta_\gamma + i\, \zeta_1\, \zeta_\gamma (c_\gamma - \tfrac{1}{2})\tfrac{1}{2} \int_{-1}^{1} (P_\gamma)^2 \ell_\gamma \sqrt{3}\, d\mu =$$

$$= 2\, i\, \ell_\gamma\, c_\gamma\, \zeta_\gamma + i\, \zeta_1\, \zeta_\gamma (c_\gamma - \tfrac{1}{2})\sqrt{3}\, \ell_\gamma$$

This can be written as

$$\frac{d\zeta_\gamma}{dt} = i\, \ell_\gamma\, \omega_\gamma\, \zeta_\gamma \tag{4.41}$$

where

$$\omega_\gamma = 2\, c_\gamma + \sqrt{3}\, \zeta_1 (c_\gamma - \tfrac{1}{2}) \tag{4.42}$$

For $\gamma = 1$ we find that

$$\frac{d\zeta_1}{dt} = 0 \tag{4.43}$$

All nonlinear contributions to this equation must vanish due to selection rules (4.35) and (4.37).
Let us first consider a one component system i.e. eq. (4.41). In this equation there is no nonlinear term. ζ_γ ($\ell_\gamma > 0$) is an element which, together with its conjugate $\zeta_{\bar{\gamma}}$, forms a one-component system. The solution of this linear system is

$$\zeta_\gamma = \zeta_\gamma(0)\; e^{i\ell_\gamma \omega_\gamma t} \tag{4.44}$$

This is a "simple" planetary (Rossby-Haurwitz) wave with constant amplitude $|\zeta_\gamma|$ and phase speed ω_γ. If $\omega_\gamma = 0$ the wave is stationary. From (4.42) and remembering that $c = 1/n(n + 1)$ it follows that, for a stationary wave, we must have

$$\zeta_1 = \frac{4}{\sqrt{3}(n_\gamma(n_\gamma+1)-2)} \qquad (4.45)$$

In other words the higher the total wavenumber, n_γ, the smaller the solid body rotation (ζ_1) needed to keep the wave stationary. Given a wave component with $n_\gamma \neq 1$ we can thus always find an amplitude of the zonal flow which will make this wave stationary. This result is analogous to Rossby's equation for the phase speed of a wave on a β-plane. Due to the finiteness of the spherical geometry we can only have integer values of the total wavenumber n_γ and the vorticity of the solid body rotation required to keep the wave stationary thus has an upper limit which in nondimensional units is $1/\sqrt{3}$.

Exercises

8. Show that for the coupling integrals defined as

$$K_{\gamma,\beta,\alpha} = \int_{-1}^{1} P_\gamma \left(\ell_\beta P_\beta \frac{\partial P_\alpha}{\partial \mu} - \ell_\alpha P_\alpha \frac{\partial P_\beta}{\partial \mu}\right) d\mu$$

the following redundancy relations are satisfied.

$$K_{\gamma,\beta,\alpha} = K_{\alpha,\bar{\beta},\gamma} \qquad K_{\gamma,\beta,\alpha} = K_{\beta,\gamma,\bar{\alpha}}$$

9. What is the (dimensional) zonal windspeed required at 45° latitude to make a Rossby-Haurwitz wave with total wavenumber n = 3 stationary?

4.2 Energy and enstrophy conservation

In this section we will prove that kinetic energy and enstrophy are conserved in general for the barotropic vorticity equation and also in any truncated system.
Since $\overline{v}.\overline{v} = (\nabla\psi)^2$, we have

$$\overline{v}.\overline{v} = \nabla.(\psi\nabla\psi) - \psi\zeta \tag{4.46}$$

The mean square velocity may therefore be expressed as

$$V^2 = \frac{1}{4\pi} \int \overline{v}.\overline{v} \; dS = -\frac{1}{4\pi} \int \psi\zeta \; dS \tag{4.47}$$

Since $\psi_\gamma = - c_\gamma \, \zeta_\gamma$ and $\frac{1}{4\pi} \int \overline{Y}_\alpha \, Y_\beta \; dS = \delta_{\alpha,\beta}$, we have

$$V^2 = \sum_\gamma v_\gamma \, |\zeta_\gamma|^2 \tag{4.48}$$

where $|\zeta_\gamma|^2 = \zeta_\gamma \, \overline{\zeta}_\gamma$. The sum in (4.48) is over all $\gamma(\gamma$ and $\overline{\gamma})$. So components with $\ell \neq 0$ will contribute twice to the summation.
Since absolute vorticity $\zeta + 2\,\mu$ is conserved, the mean-square integral of absolute vorticity over the sphere must be invariant; but it is also true that the mean-square integral of relative vorticity is invariant. To show this, we note that

$$\frac{1}{4\pi} \int (\zeta + 2\mu)^2 dS = Z^2 + 4M + \frac{4}{3} \tag{4.49}$$

where

$$Z^2 = \frac{1}{4\pi} \int \zeta^2 \; dS \tag{4.50}$$

and

$$M = \frac{1}{4\pi} \int \zeta\mu \; dS \tag{4.51}$$

M is proportional to the projection of the vorticity on the solid body rotation component. In other words M represents the total angular momentum, which must be conserved. Considering the integral on the left of (4.49) as invariant, Z must be invariant too. The spectral form of the mean square vorticity Z^2 is

$$Z^2 = \sum_{\gamma} |\zeta_\gamma|^2 \tag{4.52}$$

which we obtain in the same way as (4.48).

A general proof that kinetic energy (or mean square velocity) and enstrophy for any expansion in orthogonal functions are conserved is given in appendix B.

We will here show that kinetic energy and enstrophy are conserved in any truncated system where the expansion functions are Legendre functions (finite γ). First consider enstrophy. We start with the idendity

$$\frac{d}{dt} |\zeta_\gamma|^2 = \zeta_\gamma \frac{d\bar{\zeta}_\gamma}{dt} + \bar{\zeta}_\gamma \frac{d\zeta_\gamma}{dt} \tag{4.53}$$

On the righthand side we introduce the expression for $\frac{d\zeta_\gamma}{dt}$ and $\frac{d\bar{\zeta}_\gamma}{dt}$ from (4.30); this yields

$$\frac{d}{dt} |\zeta_\gamma|^2 = \sum_{\alpha,\beta} i(\bar{\zeta}_\gamma \zeta_\alpha \zeta_\beta \, I_{\gamma,\alpha,\beta} - \zeta_\gamma \bar{\zeta}_\alpha \bar{\zeta}_\beta \, I_{\gamma,\alpha,\beta}) \tag{4.54}$$

where the linear Coriolis terms have canceled out. In the righthandside of (4.54) we can substitute the coupling integrals K for the interaction coefficients I which leaves us with

$$\frac{d}{dt} |\zeta_\gamma|^2 = i\,\bar{\zeta}_\gamma \sum_{\alpha,\beta} (-c_\alpha \zeta_\alpha \zeta_\beta K_{\gamma,\beta,\alpha}) + i\zeta_\gamma \sum_{\alpha,\beta} c_\alpha \bar{\zeta}_\alpha \bar{\zeta}_\beta K_{\gamma,\beta,\alpha} \tag{4.55}$$

where now the summation is over all combinations (with permutation) of α and β. Equation (4.55) can also be written as

$$\frac{d}{dt} |\zeta_\gamma|^2 = \bar{\zeta}_\gamma \sum_{\beta} \zeta_\beta i \sum_{\alpha} (-c_\alpha \zeta_\alpha) K_{\gamma,\beta,\alpha} - \zeta_\gamma \sum_{\beta} \bar{\zeta}_\beta i \sum_{\alpha} (-c_\alpha \bar{\zeta}_\alpha) K_{\gamma,\beta,\alpha} \tag{4.56}$$

We define a coupling matrix M as follows

$$M_{\gamma,\beta} = i \sum_{\alpha} - c_\alpha \, \zeta_\alpha \, K_{\gamma,\beta,\alpha} \tag{4.57}$$

$M_{\gamma,\beta}$ is antisymmetric in a Hermitian sense because using (4.40) we have

$$M_{\gamma,\beta} = i \sum_{\alpha}(-c_\alpha \zeta_\alpha K_{\beta,\gamma,\overline{\alpha}}) = i \overline{\sum_{\alpha} c_\alpha \overline{\zeta}_\alpha K_{\beta,\gamma,\overline{\alpha}}} = - \overline{M_{\beta,\gamma}} \tag{4.58}$$

So we can rewrite (4.56) as

$$\frac{d}{dt} |\zeta_\gamma|^2 = \sum_{\beta}(\overline{\zeta}_\gamma \zeta_\beta M_{\gamma,\beta} + \zeta_\gamma \overline{\zeta}_\beta \overline{M}_{\gamma,\beta})$$

Summing over all components γ we get

$$\sum_{\gamma} \frac{d}{dt} |\zeta_\gamma|^2 = \sum_{\gamma} \sum_{\beta} D_{\gamma,\beta} \tag{4.59}$$

where

$$D_{\gamma,\beta} = \overline{\zeta}_\gamma \zeta_\beta M_{\gamma,\beta} + \zeta_\gamma \overline{\zeta}_\beta \overline{M}_{\gamma,\beta} = - \zeta_\beta \overline{\zeta}_\gamma \overline{M}_{\beta,\gamma} - \overline{\zeta}_\beta \zeta_\gamma M_{\beta,\gamma} = - D_{\beta,\gamma} \tag{4.60}$$

So D is antisymmetric. Therefore from (4.59) it follows

$$\sum_{\gamma} \frac{d}{dt} |\zeta_\gamma|^2 = 0 \tag{4.61}$$

because in the double sum of $D_{\gamma,\beta}$ each pair of wave vectors must occur twice, the second pair being a permutation of the first. We therefore have the result that (4.52) is invariant. Clearly, the foregoing proof does not depend upon the nature of the set S of which the wave vectors γ and β are members; this set may be quite arbitrary in configuration, and it may have a finite or an infinite number of elements.
The corresponding proof for conservation of kinetic energy in a truncated system is analogous. Equation (4.55) can also be written as

$$\frac{d}{dt} |\zeta_\gamma|^2 = \overline{\zeta}_\gamma \sum_{\alpha} c_\alpha \zeta_\alpha i \sum_{\beta} \zeta_\beta K_{\gamma,\alpha,\beta} - \zeta_\gamma \sum_{\alpha} c_\alpha \overline{\zeta}_\alpha i \overline{\sum_{\beta} \zeta_\beta K_{\gamma,\alpha,\beta}} \tag{4.62}$$

where we have used the fact that the coupling integrals are antisymmetric. Introducing the coupling matrix M we get

$$\frac{d}{dt} |\zeta_\gamma|^2 = \sum_{\alpha}(c_\alpha \overline{\zeta}_\gamma \zeta_\alpha M_{\gamma,\alpha} + c_\alpha \zeta_\gamma \zeta_\alpha \overline{M}_{\gamma,\alpha}) = \sum_{\alpha} c_\alpha D_{\gamma,\alpha} \tag{4.63}$$

Multiplication by c_γ and summation yields

$$\sum_{\gamma} \frac{d}{dt} c_\gamma |\zeta_\gamma|^2 = \sum_{\gamma} \sum_{\alpha} c_\gamma c_\alpha D_{\gamma,\alpha} \qquad (4.64)$$

and the double summation vanishes again because of the antisymmetry of D.

4.3 Truncated systems: systems with one or two components

At the end of the section (4.1) we investigated a one component system. We found that it described a so called Rossby-Haurwitz wave with an invariant amplitude and a phase-speed dependent on the total wavenumber. There was no nonlinearity in the equation as one component cannot interact with itself, to itself.

To consider a two component system, let α and β denote the relevant wave vectors, ordered such that $0 \leq \ell_\alpha \leq \ell_\beta$ and $0 < n_\alpha < n_\beta$.
This involves no loss of generality. It is helpful to think of the interaction coefficients $I_{\gamma,\beta,\alpha}$ arranged in the form of an interaction matrix I_γ. The numerical values of its elements are fixed by (4.31), and its structure is governed largely by the locations of nonzero elements. It is symmetric ($I_{\gamma,\beta,\alpha} = I_{\gamma,\alpha,\beta}$).
First we construct the interaction matrixes I_α and I_β. For this purpose it is convenient to classify the possible ℓ-configurations of α and β in three groups as in table I (see also figure 21)

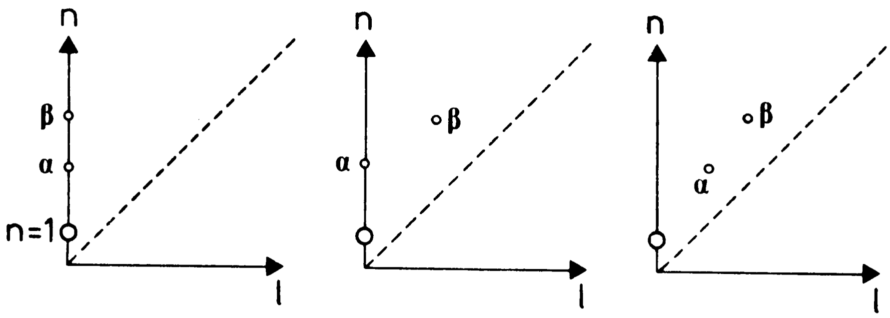

Fig. 21. Rank classification of two-component system. The half plane (1 < 0 is omitted because the configuration is symmetric).

Table 1. The rank classification of a two-component system.

Class	ℓ_α	ℓ_β
L1	$= 0$	$= 0$
L2	$= 0$	> 0
L3	> 0	$\leq \ell_\alpha$

In class L1, both interaction matrices must be zero because the flow is purely zonal. Formally this is imposed by selection rule (4.43). Similarly, both I_α and I_β are zero in class L3 because the condition $\ell_\alpha + \ell_\beta = \ell_\gamma$ cannot be satisfied.

In class L2 the matrices showing the possible values of the interaction coefficients are displayed below (Fig. 22). Since the interaction matrix is symmetric in general, only one side of the diagonal need be considered. Each is a 3 x 3 (rather than a 4 x 4) matrix in this case because $\ell_\alpha = 0$.

Note first that the diagonal elements of all interaction matrices are zero owing to rule (4.37) or (4.38)

I_α	α	β	$\bar{\beta}$
α	o	o	o
β		o	o
$\bar{\beta}$			o

I_β	α	β	$\bar{\beta}$
α	o	–	o
β		o	o
$\bar{\beta}$			o

Fig. 22. Interaction matrices for a two component system.

The former rule also excludes the interaction of any element with its conjugate, such as β, $\bar{\beta}$. This leaves only α, β and α, $\bar{\beta}$ in the two matrices of Fig. 22. In I_α these interactions are zero because the condition $\ell_\alpha + \ell_\beta = \ell_\gamma$ cannot be satisfied in class L2; hence $I_\alpha = 0$. In I_β, rule (4.39) excludes the interaction α, $\bar{\beta}$ so that only α, β

remains to be considered.

One finds, indeed, that the interaction coefficient $I_{\beta,\alpha,\beta}$ (= $I_{\beta,\beta,\alpha}$) is the only one in I_β of class L2 which is not excluded by any of the selection rules. Therefore, in class L2 the dynamical equations reduce to

$$\frac{d\zeta_\alpha}{dt} = 0 \qquad (4.65)$$

$$\frac{d\zeta_\beta}{dt} = i[\ell_\beta\ \omega_\beta\ \zeta_\beta + \zeta_\alpha\ \zeta_\beta\ I_{\beta,\alpha,\beta}] \qquad (4.66)$$

where $\ell_\alpha = 0$. Hence the wavecomponent does not interact back to the zonal component. This is why a linearization around the basic state zonal flow is permissible. It is clear that there are no interesting nonlinearities in the above system, not for that matter in any two component system. This can also be seen as a consequence of the conservation of energy and enstrophy. Energy conservation requires the solution to move on the circumference of an ellipse in the $|\zeta_\alpha|^2 - |\zeta_\beta|^2$ plane (see Fig. 23), while enstrophy conservation requires the solution to stay on the circumference of a circle. Both the ellipse and the circle have their centre in the origin. Because of these two constraints and the fact that the amplitudes can of course only be positive the solution is restricted to the intersection of the ellips and the circle in the first quadrant. In other words the solution is "locked" to a point in the amplitude plane. It can only change its phase. In three dimensions (three component system) the solution can move on the intersectionline of a sphere and an ellipsoid in the first quadrant. This makes the behaviour of a three component system much more interesting. We will therefore consider a three component system in the following section.

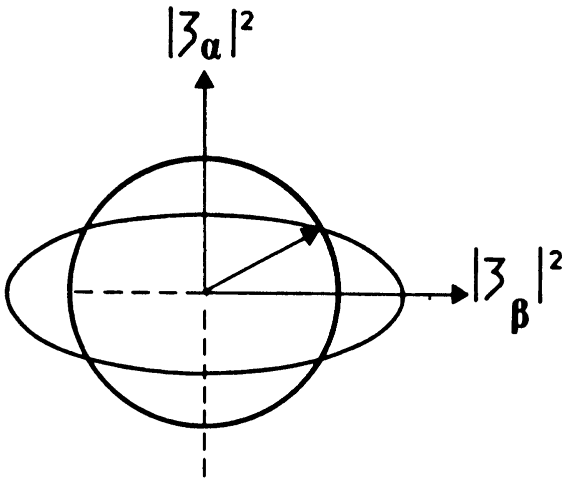

Fig. 23. The solution of a two component system is restricted to a point in the amplitude-plane because of energy and enstrophy conservation.

4.4 A three component system

The simplest three component system having active non-linear interactions is one in which one component is purely zonal, i.e. $\ell = 0$, and the two other components have the same non-zero wavenumber (Platzman, 1962). This is illustrated in Fig. 24.
The wave components do not interact directly with each other. If they did interact, we could drop the zonal component and we would be left with a two component systems with active nonlinear interactions. In the preceding section we showed that this is not possible. Thus the wave-components can only interact via the zonal component. This is illustrated in Fig. 24, by the arrows.
Let us first of all look at energy and enstrophy transfer within this system. In any three component system the invariance of enstrophy and kinetic energy may be stated as

$$\frac{d}{dt}(x_1^2 + x_2^2 + x_3^2) = 0 \qquad (4.67a)$$

$$\frac{d}{dt}(c_1 x_1^2 + c_2 x_2^2 + c_3 x_3^2) = 0 \qquad (4.67b)$$

where $x_i \equiv \zeta_i\bar{\zeta}_i$ for a wave component ($\ell \neq 0$) and $x_i = \frac{1}{2}\zeta_i^2$ for a zonal component ($\ell = 0$). From these it is a simple matter to establish the following symmetric relations:

$$p_1 \frac{dx_1^2}{dt} = p_2 \frac{dx_2^2}{dt} = p_3 \frac{dx_3^2}{dt}$$

$$p_1 \equiv (c_1 - c_2)(c_1 - c_3)$$

$$p_2 \equiv (c_2 - c_3)(c_2 - c_1) \qquad (4.68)$$

$$p_3 \equiv (c_3 - c_1)(c_3 - c_2)$$

It is clear from (4.68) that the signs of the p's are decisive in determining directions of energy exchange between three components.

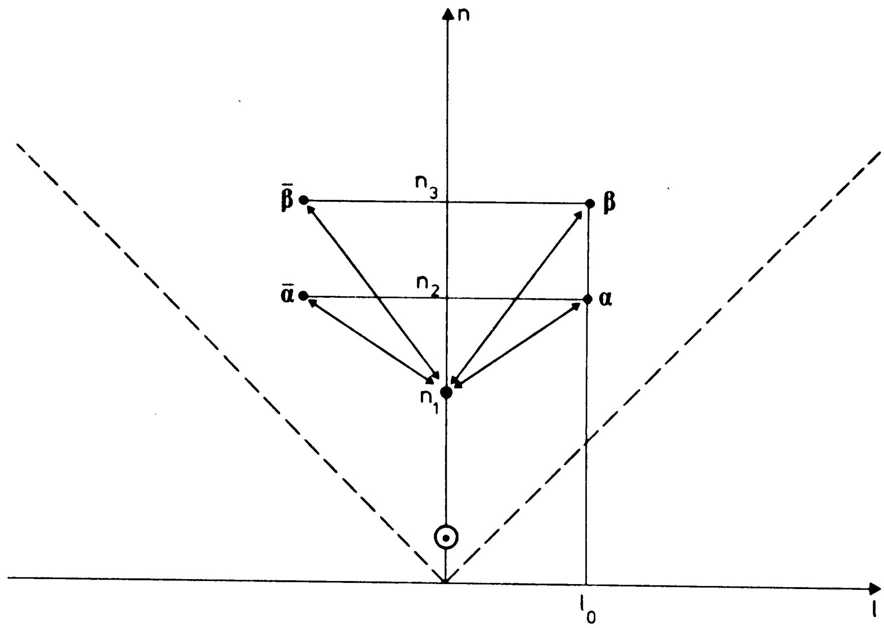

Fig. 24. Representation in wavenumber plane of the components of the simplest three component system having active nonlinear interactions. Arrows indicate direct interactions.

Now if one adopts the ordering in Fig. 24, the numbers $c \equiv \frac{1}{n(n+1)}$ must be in the sequence $c_1 > c_2 > c_3$. Therefore $P_1 > 0$, $P_2 < 0$ and $P_3 > 0$This means that concurrent enstrophy changes in the components of smallest and largest degree are of the same sign, and are opposite in sign to enstrophy changes of the component of intermediate degree. In other words, three components of unequal scales interact in such a way that the component of intermediate scale blocks the transfer of enstrophy from large to small scales (or, indeed, from small to large scales). The existence of such a spectral "blocking" is a direct consequence of the existence of two quadratic invariants, and is a crucial aspect of the exchange processes. It is also evident from(4.68) that in a two-component system energy or enstrophy flow between the components is impossible. We will now use a three component system to discuss the barotropic instability of large scale waves in the atmosphere.

4.5 Stability of the Rossby-Haurwitz wave

In this section we shall consider one kind of barotropic instability, of which there are three types. The first two types are concerned with the transformation of zonal kinetic energy (K_Z) to eddy kinetic energy (K_E) or, in other words, the conditions which have to be fulfilled for the amplification of wave disturbances on a given zonal flow. The first of these is inertial instability. The condition for inertial instability is,

$$f - \frac{du}{dy} < 0$$

where u is the zonal windspeed and the y-axis is directed in south-north direction. Since $f - \frac{du}{dy}$ is the absolute vorticity of the basic flow, the inertial stability condition is simply that the absolute vorticity be positive.

Observations indicate that on the synoptic scale the absolute vorticity is nearly always positive. The occurrence of a negative absolute vorticity over any large area would be expected to trigger immediately inertially unstable motions which would mix the fluid laterally and reduce the shear until the absolute vorticity was again positive.

The other type of instability is the so called "Kuo" instability which can be stated as follows:

$$\beta - \frac{d^2u}{dy^2} = 0$$

somewhere in the fluid is a necessary but not sufficient condition for instability.

Since the positive northward gradient of the earth's vorticity dominates mostly, the condition is that there must exist negative meridional gradients in zonal absolute vorticity. Because they occur with such small time and space scales these two types of instability are not very important for the total energetics of the atmosphere. It is thought that the Kuo type of instability is important for the initiation of tropical disturbances. Once a tropical disturbance has been formed, it is maintained mainly by latent heat release.

The third kind of barotropic instability is concerned with the reverse energy transformation, $K_E \rightarrow K_Z$, which is actually observed in the mid latitude atmosphere. This problem was first investigated by Lorenz (1972). He studied the barotropic stability of Rossby waves of infinite meridional extent on a β-plane and found that if the waves are short enough and of sufficient amplitude, then they are unstable, in other words, they are destroyed. This indicated one reason why the classic Rossby waves are not regularly observed in the atmosphere and a mechanism which may be responsible for the breakdown of baroclinic disturbances when they have obtained a large amplitude. We shall now investigate in more detail the stability of the Rossby-Haurwitz wave (Rossby wave on the sphere) of finite amplitude, following closely an article by Hoskins (1973).

We shall examine what happens in a three component system with one zonal component and two wave components with the same zonal wavenumber. We will initially insert all the energy in one wavecomponent and we will determine under which circumstances it becomes unstable such that energy is transferred to the other components. In fig. 24 the rank classification of the three component system we will consider is shown. The dynamical equations (see 4.30) for this system are: For the zonal component

$$\frac{d\zeta_\gamma}{dt} = i[\zeta_{\bar{\alpha}}\ \zeta_\beta\ I_{\gamma,\bar{\alpha},\beta} + \zeta_\alpha\ \zeta_{\bar{\beta}}\ I_{\gamma,\alpha,\bar{\beta}})$$

which because of (4.31) can also be written as,

$$\frac{d\zeta_\gamma}{dt} = i[\zeta_{\bar{\alpha}}\ \zeta_\beta - \zeta_\alpha\ \zeta_{\bar{\beta}}]\ I_{\gamma,\bar{\alpha},\beta} \tag{4.69}$$

and for the wave components:

$$\frac{d\zeta_\alpha}{dt} = im\ \omega_\alpha\ \zeta_\alpha + i[\zeta_\beta\ \zeta_\gamma\ I_{\alpha,\beta,\gamma} + \zeta_\alpha\ \zeta_\gamma\ I_{\alpha,\alpha,\gamma}] \tag{4.70}$$

$$\frac{d\zeta_\beta}{dt} = im\ \omega_\beta\ \zeta_\beta + i[\zeta_\alpha\ \zeta_\gamma\ I_{\beta,\alpha,\gamma} + \zeta_\beta\ \zeta_\gamma\ I_{\beta,\beta,\gamma}] \tag{4.71}$$

We will rewrite these equations in terms of amplitudes and phases of the waves, such that:

$$\zeta_\alpha = \tfrac{1}{2}\ A\ e^{i\theta_\alpha}$$

$$\zeta_\beta = \tfrac{1}{2}\ B\ e^{i\theta_\beta}$$

$$\zeta_\gamma = C$$

Here C, A and B are real numbers. The factor $\frac{1}{2}$ has been introduced to make things simpler later. If we substitute these expressions in (4.69), (4.70) and (4.71) we find that

$$\dot{C} = \tfrac{1}{2}\ I_{\gamma,\bar{\alpha},\beta}\ A\ B\ \sin\ (\theta_\alpha - \theta_\beta) \tag{4.72}$$

$$\dot{A} = I_{\alpha,\beta,\gamma}\ C\ B\ \sin\ (\theta_\alpha - \theta_\beta) \tag{4.73}$$

$$\dot{B} = -\ I_{\beta,\gamma,\alpha}\ C\ A\ \sin(\theta_\alpha - \theta_\beta) \tag{4.74}$$

$$\dot{\theta}_\alpha = m\ \omega_\alpha + I_{\alpha,\alpha,\gamma}\ C + I_{\alpha,\beta,\gamma}\frac{CB}{A}\ \cos(\theta_\alpha - \theta_\beta) \tag{4.75}$$

$$\dot{\theta}_\beta = m\ \omega_\beta + I_{\beta,\beta,\gamma}\ C + I_{\beta,\gamma,\alpha}\frac{CA}{B}\ \cos(\theta_\alpha - \theta_\beta) \tag{4.76}$$

where the dot stands for a time derivative.

We will now consider the β and γ components as perturbations to the α

wave, and by linearizing about this wave determine its stability to the perturbation. Hence we are testing the barotropic stability of a wave governed by the equation

$$\frac{d\zeta_\alpha}{dt} = i\, m\, \omega_\alpha\, \zeta_\alpha \tag{4.77}$$

with the solution

$$\zeta_\alpha = A\, e^{i m \omega_\alpha t}, \tag{4.78}$$

to a perturbation composed of a wave of the same zonal wavelength, but shorter scale in the meridional direction, and a zonal flow. From (4.78) it is clear that

$$\dot{\theta}_\alpha = m\, \omega_\alpha$$

Because B and C are assumed small initially, the perturbation equation for the phase difference becomes,

$$\dot{\theta} = \dot{\theta}_\alpha - \dot{\theta}_\beta = m(\omega_\alpha - \omega_\beta) - \frac{AC}{B} I_{\beta,\alpha,\gamma} \cos\theta \tag{4.79}$$

The other two perturbation equations for the amplitudes become,

$$\dot{C} = \tfrac{1}{2} AB\, I_{\gamma,\bar{\alpha},\beta} \sin\theta \tag{4.80}$$

$$\dot{B} = -\, AC\, I_{\beta,\alpha,\gamma} \sin\theta \tag{4.81}$$

Equations (4.80) and (4.81) can also be written in the form

$$\dot{C}^2 = I_{\gamma,\bar{\alpha},\beta}\, ABC \sin\theta \tag{4.82}$$

$$\dot{B}^2 = -\, 2\, I_{\beta,\alpha,\gamma}\, ABC \sin\theta \tag{4.83}$$

Therefore it follows that

$$\dot{C}^2 = a\, \dot{B}^2 \tag{4.84}$$

where $a = \frac{I_{\gamma,\bar{\alpha},\beta}}{2\, I_{\beta,\alpha,\gamma}} = \frac{1}{2}\, \frac{c_\alpha - c_\beta}{c_\gamma - c_\alpha}$

The solution of (4.84) is,

$$C = \pm \sqrt{a}\, B \tag{4.85}$$

If there is to be growth of the zonal flow and the wave, then clearly we must have $a > 0$. Since $n_\gamma < n_\alpha < n_\beta$, a is actually positive.
If we would have chosen the other wave component as the energy containing component, a would have been negative, and the perturbation components would never have grown. This is again just a reflection of the fact that energy can flow only from the intermediate wavelength to the longer and shorter wavelengths or vice versa.
We note that the phase speed equation (4.79) contains two terms. The first one gives a constant growth of θ because the two waves are Rossby waves with different total wavenumbers. The second term, which may be of either sign, is an interaction effect proportional to the amplitude of the main wave. This suggests the possibility of the interaction effect cancelling the Rossby effect to give a solution with constant phase difference. Substitution of (4.85) into (4.79) gives

$$\dot{\theta} = m(\omega_\alpha - \omega_\beta) \mp \sqrt{a}\, A\, I_{\beta,\alpha,\gamma} \cos\theta.$$

For exponentially growing solutions of (4.80) and (4.81) we must require, $\dot{\theta} = 0$. The perturbation on the basic state wave must have the same phase to pick up energy from the basic state wave with maximum efficiency. Hence,

$$\frac{m\, \omega_\alpha - \omega_\beta}{a\, A\, I_{\beta,\alpha,\gamma}} \leq 1$$

or

$$A \geq \frac{m\, \omega_\alpha - \omega_\beta}{\sqrt{a}\; I_{\beta,\alpha,\gamma}} \tag{4.86}$$

This, together with the condition that $a > 0$, is the instability criterium for the Rossby-Haurwitz wave for this severely truncated system. Another way of obtaining the same result is by transforming the equations (4.69), (4.70) and (4.71) to the real domain through a suitable transformation and subsequently finding the eigenvalues of the equations (4.69), (4.70) and (4.71) to the real domain through a suitable transformation and subsequently finding the eigenvalues of the system of equations in matrixform (see exercise).

The system of equation (4.72)-(4.74) and (4.79) was solved exactly by Platzman (1962). He obtained a periodic solution, but the instability criterium could not easily be deduced from it.

It is possible to include more zonal components, but there will not be any qualitative change of behavior. The problem becomes intractable when we include more wave components. We can then resort to a numerical model.

Hoskins computed theoretically e-folding times for the growth of perturbations to Rossby-Haurwitz waves with a wave vector $\alpha = (m, m + 1)$. The result is shown in Fig. 25. The r.m.s. vorticity divided by the earth's angular velocity is used as a measure of the amplitude of the main wave. Waves with wavenumber less than 6 are stable to perturbations, while shorter waves with sufficient amplitude are unstable.

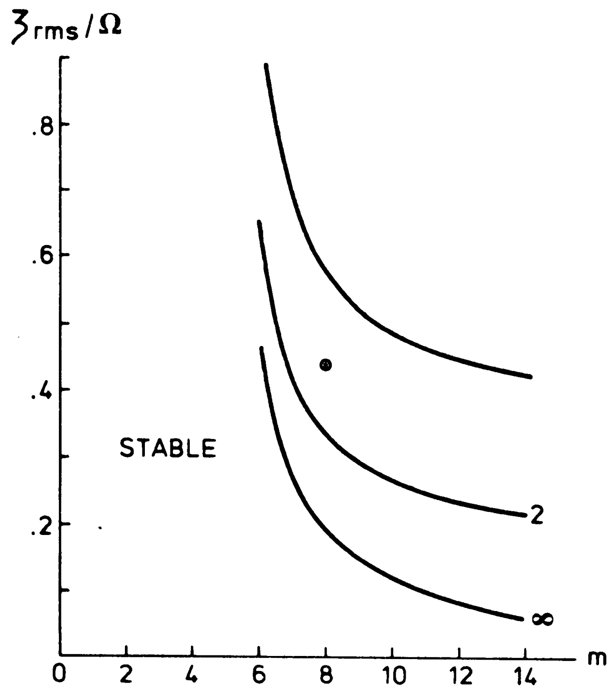

Fig. 25. Amplitude of wave given by its r.m.s. vorticity as a function of the wavenumber. Numbers on curves refer to e-folding times, to the left of the curve marked ∞ all waves are stable (From Hoskins, 1973).

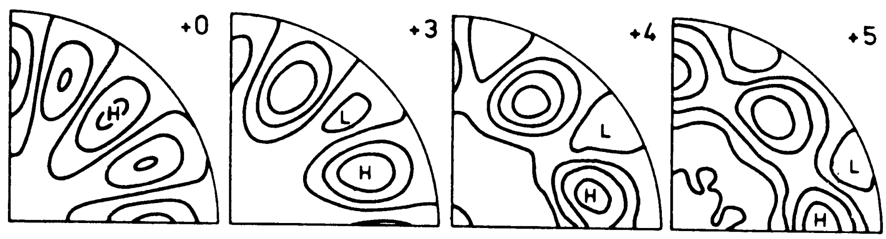

Fig. 26. Example of flow configurations from a numerical experiment on Rossby-Haurwitz wave instability. One quadrant of a hemisphere is shown on a polar stereographic projection, full lines are isolines for the streamfunction. The initial state is to the far left the other flows are taken from days 3, 4 and 5 respectively. The breakdown of the initial wave into zonally sheared flow agrees well with the theory. (From Hoskins, 1973).

Hoskins also carried out a series of numerical experiments using the barotropic vorticity eqaution on the sphere. One of these integrations is shown in Fig. 26. All the energy is initially inserted in wave (8,9). It can be seen that by day 5 this wave has lost a great deal of its energy to the zonal flow and smaller latitudinal scale waves. When wave (4,5) was chosen as main wave, the energy content of this wave only decreased by 3½% in 5 days. We can conclude this section by saying that eddy kinetic energy, which is created from eddy potential energy by baroclinic instability, is destroyed again and converted to zonal kinetic energy by barotropic instability.

According to the above theory long waves are stable. In the next section we will investigate a possible source of instability for long waves due to boundary forcings.

4.6 Orographic effects on barotropic flow

In the previous section we have studied two-dimensional, nonviscous flow on a rotating sphere. This is a reasonably good approximation for the flow in the "free" atmosphere, but to study boundary effects such as horizontal variations of the surface elevation (orography) and dissipation we have to take a third, vertical dimension into consideration. This may be done parametrically, i.e. we can still work with a two-dimensional model which takes the vertical variations into account in a vertically integrated sense.
Variations in the surface elevation (orography) may influence the flow in essentially two different ways. The first is just an "obstacle" effect which means that the flow tries to avoid the obstacle by flowing around it. The second effect arises through the conservation of potential vorticity. When a fluid parcel is advected across an orographic rdige its vertical dimension shrinks and to conserve its potential vorticity the fluid parcel has to gain anticyclonic vorticity in the horizontal plane. The trajectory of the parcel will thus be deflected and from fundamental dynamic meteorology it is well known that together with the β-effect this may set up a standing wave pattern downwind of an orographic ridge. For large scale atmospheric flow the second type of effect is dominating while the "obstacle" effect is more relevant for small scale mountains. We will therefore disregard the "obstacle" effect in this section while the vorticity effect will be introduced through a forced vertical velocity at the lower boundary. The forced vertical velocity is assumed to be dependent on the mountain slope and the intensity of flow. The effect of orography is thus flow dependent ant it is this feature in combination with the advective nonlinearity which will lead to long wave instability and a bifurcation.

We first consider a three dimensional model where only the vertical component of the flow vorticity is taken into account. By taking a vertical average and making a few assumptions about the vertical variation of the flow we will reduce this to a two dimensional model (equivalent barotropic assumption, see Haltiner, 1971).
Figure 27 gives a vertical cross section where the upper and lower boundaries are indicated. The upper boundary is taken as the "top" of the atmosphere, where $p = 0$, and the vertical motions must be zero at

this level. The orographic height is denoted by h, which is a function of the spherical coordinates λ and μ.

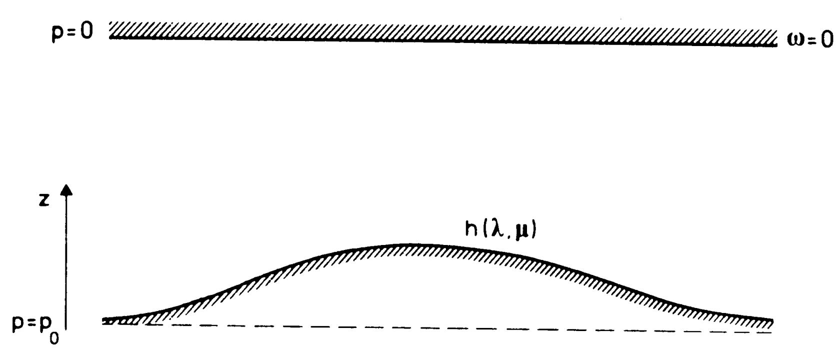

Fig. 27. Vertical cross-section of the equivalent barotropic atmosphere.

A sloping bottom boundary will give rise to vertical motions with a vertical velocity given by

$$w = \frac{dz}{dt} = \overline{v}_o . \nabla\ h \tag{4.87}$$

Using pressure as a vertical coordinate we wish to express w in terms of the individual pressure changes at the lower boundary,

$$\omega_{p_o} = \frac{dp_o}{dt} \tag{4.88}$$

Using the hydrostatic equation this may be approximated by

$$\omega_{p_o} \simeq - \rho_o\ gw = - \rho_o\ g\ \overline{v}_o . \nabla h \tag{4.89}$$

Since the vertical velocities in the atmosphere are an order of magnitude smaller than the horizontal velocities we may consider the flow as being quasi-two dimensional. The next step is to find a vorticity equation for the flow. The equation for the vertical component of the flow vorticity reads

$$\frac{d}{dt}(\zeta + f) = -(\zeta + f)\nabla.\overline{v} \qquad (4.90)$$

Eliminating the divergence from the continuity equation

$$\nabla.\overline{v} = -\frac{\partial\omega}{\partial p} \qquad (4.91)$$

we obtain

$$\frac{\partial\zeta}{\partial t} + \overline{v}.\nabla(\zeta + f) = f_o \frac{\partial\omega}{\partial p} \qquad (4.92)$$

On the right hand side we have set $\zeta = 0$ and the Coriolisparameter equal to a constant value f_o. This is necessary for energy consistency. A full discussion of this derivation can be found in Holton (1979). The final result (4.92) is valid in midlatitudes, but it fails near the equator since the Coriolisparameter becomes zero in this case. To apply (4.98) to a strictly two dimensional flow we will now average it vertically. We assume that the relative vorticity and the horizontal velocities, which are functions of pressure, can be written as

$$\zeta = A(p)\,\overline{\zeta} \text{ and } \overline{v} = A(p)\overline{\overline{v}} \qquad (4.93)$$

where

$$\overline{(\quad)} = \frac{1}{p_o}\int_{p_o}^{p} (\quad)dp \qquad (4.94)$$

A(p) is a weighting function; its shape for average atmospheric conditions is drawn in figure 28. We furthermore define the equivalent barotropic level to be the pressure, p_{EB} at which

$$A(p_{EB}) = \overline{A^2(p)} = A^* \qquad (4.95)$$

From figure 28 it is clear that p_{EB} is close to 500 mb.

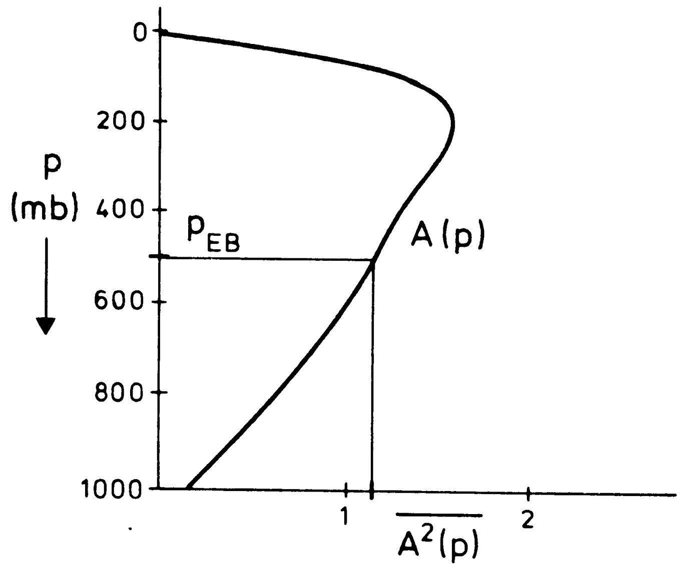

Fig. 28. An example of the function governing the vertical variation of the variables in an equivalent barotropic model. Also indicated is the equivalent barotropic level.

We first average the vorticity equation (4.92) using (4.93) and (4.94) and we obtain

$$\frac{\partial \bar{\zeta}}{\partial t} + A^* \, J(\bar{\psi}, \bar{\zeta} + \frac{f}{A^*}) = \frac{f_o}{p_o} (\omega_{p_o} - \omega_o) \tag{4.96}$$

where we have introduced a stream function for the vertically averaged flow; the vorticity $\bar{\zeta}$ can be expressed as $\bar{\zeta} = \nabla^2 \psi$.
Using the boundary conditions (4.89) for ω_{p_o} and $\omega_o = 0$ and introducing a stream function ψ_o for the lower boundary, we find

$$\frac{\partial \bar{\zeta}}{\partial t} + A^* \, J(\bar{\psi}, \bar{\zeta} + \frac{f}{A^*}) = -\frac{f_o}{p_o} \, g \, \rho_o \, J(\psi_o, h) \tag{4.97}$$

We next express all quantities in the terms of the values at the equivalent barotropic level by means of the relations

$$\bar{\zeta} = \frac{\zeta^*}{A^*} \quad , \quad \bar{\psi} = \frac{\psi^*}{A^*} \quad , \quad \psi_o = A_o \bar{\psi} = \frac{A_o}{A^*} \, \psi^* \tag{4.98}$$

and finally we obtain the vorticity equation applied at the equivalent barotropic level

$$\frac{\partial \zeta^*}{\partial t} + J(\psi^*, \zeta^* + f) = - f_o A_o \frac{g\rho_o}{p_o} J(\psi^*, h) \tag{4.99}$$

This result can be used for our purposes; it describes the vorticity balance of a two dimensional flow at the equivalent barotropic level, where the orographic effects are taken into account parametrically. To simplify the algebra we introduce nondimensional quantities. They read

$$\zeta = \frac{\zeta^*}{\Omega} \quad , \quad \psi = \frac{\psi^*}{a^2\Omega} \quad , \quad \tau = \Omega t \tag{4.100}$$

and $h' = 2 \sin \phi_o \; A_o \frac{h}{H}$

where

$$H = \frac{p_o}{\rho_o g} = \frac{RT_o}{g} \tag{4.101}$$

is a scale height. The rewriting is possible by using the ideal gaslaw. Note that due to the approximations inherent in eq. (4.89) the orography is assumed to be much smaller than the actual scale height ($h' < 1$). The most important restrictive assumption in (4.89) is that we assume a windspeed, representative of the surface layer, to blow across the orography. Introducing these expressions in (4.99) we find

$$\frac{\partial \zeta}{\partial \tau} + J(\psi, \zeta + h') + 2 \frac{\partial \psi}{\partial \lambda} = 0 \tag{4.102}$$

From this it can be seen that the orographic effects enter in the Jacobian. It can be shown that if there exists a westerly flow over an orographic ridge a wave train will be formed on the leeward side. Again we develop the relative vorticity and the streamfunction in a series of Legendre functions. Writing the nonlinear Jacobian term in (4.102) as

$$J(\psi, \zeta + h') = J(\psi, \zeta) + J(\psi, h')$$

we see that $J(\psi, h')$ is the only new term when comparing with the standard barotropic vorticity equation. This term gives rise to "interactions" between the flow and the orographic field and it can be written

$$J(\psi,h') = J(\sum_\alpha \psi_\alpha Y_\alpha, \sum_\beta h_\beta Y_\beta) =$$

$$= - J(\sum_\alpha c_\alpha \zeta_\alpha Y_\alpha, \sum_\beta h_\beta Y_\beta) =$$

$$= \sum_{\alpha,\beta} c_\alpha \zeta_\alpha h_\beta [\frac{\partial Y_\beta}{\partial \mu} \frac{\partial Y_\alpha}{\partial \lambda} - \frac{\partial Y_\alpha}{\partial \mu} \frac{\partial Y_\beta}{\partial \lambda}] =$$

$$= \sum_{\alpha,\beta} i\, c_\alpha \zeta_\alpha h_\beta [\ell_\alpha P_\alpha \frac{dP_\beta}{d\mu} - \ell_\beta P_\beta \frac{dP_\alpha}{d\mu}] e^{i(\ell_\alpha + \ell_\beta)\lambda}$$

Projecting this sum on a certain component γ we obtain

$$\tfrac{1}{2}\, i \sum_{\alpha,\beta} c_\alpha \zeta_\alpha h_\beta K_{\gamma,\alpha,\beta}$$

Restricting the forcing to one component (q) we have

$$\tfrac{1}{2}\, i \sum_\alpha c_\alpha \zeta_\alpha [K_{\gamma,\alpha,q}\, h_q + K_{\gamma,\alpha,\bar{q}}\, h_{\bar{q}}] \qquad (4.103)$$

and from this it can be seen that we end up with the selection rules

$$\ell_\gamma = \ell_\alpha + \ell_q \quad \text{or} \quad \ell_\gamma = \ell_\alpha - \ell_q \qquad (4.104)$$

Due to the orography a wave can interact with the solid body rotation, since ℓ_γ can be zero while ℓ_α and ℓ_q are nonzero. This is just an expression of the fact that a zonal flow over orography will generate wave energy.

The orography thus acts to transfer energy between components of different scales. The generation of wave energy is taken out of the zonal flow and this is why we will have terms involving the orography in the equation for the amplitude of the solid body rotation. Because of this energy transfer, we can form a low order system only containing two flow components which has interesting nonlinear properties. One component is a solid body rotation zonal flow while the other is a wave-component. The orographic forcing is introduced in the same component as the wave. Fig. 29 shows the choice of components in a wavenumberplane.

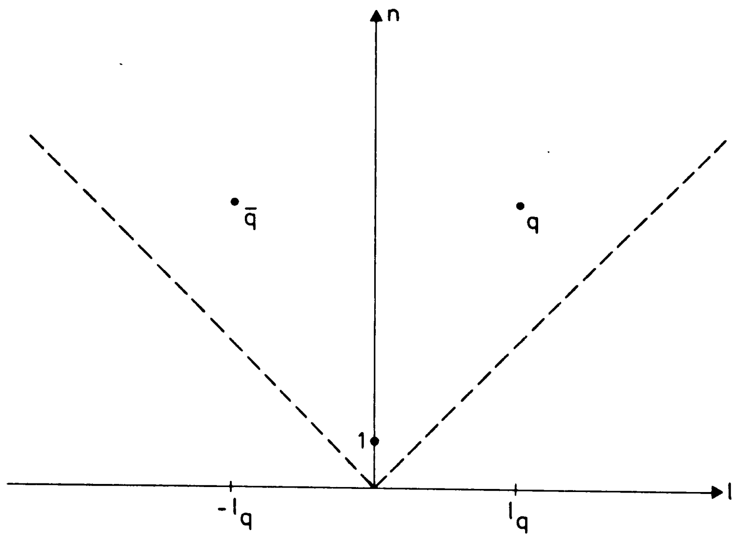

Fig. 29. Wavenumber configuration for the low order system used to investigate the orographic instability.

For $\gamma = 1$ (solid body rotation) we have from eq. (4.104)

$$\frac{d\zeta_1}{dt} = \frac{i}{2} \sum_{\alpha} c_\alpha \zeta_\alpha (K_{1,\alpha,q} h_q + K_{1,\alpha,\bar{q}} h_{\bar{q}}) \qquad (4.105)$$

As we have seen earlier the right hand side of (4.105) vanishes unless the following selection rules are satisfied for α and q: $\ell_\alpha = -\ell_q$ and $|n_q - n_\alpha| < 1 < n_q + n_\alpha$ from which it follows that $n_q = n_\alpha$. We thus have to include orography in the same component as the wave to obtain a nonzero energy transfer to/from the zonal component. We can simplify (4.105) somewhat by using the defintion of $K_{1,\alpha.q}$ and $K_{1,\alpha,\bar{q}}$: $K_{1,\bar{q},q} = -K_{1,q,\bar{q}} = 2\sqrt{3}\,\ell_q$.
We thus have

$$\frac{d\zeta_1}{dt} = i\sqrt{3}\,\ell_q\, c_q (h_q \zeta_{\bar{q}} - h_{\bar{q}} \zeta_q) \qquad (4.106)$$

The equation for the wave component is

$$\frac{d\zeta_q}{dt} = i\,\ell_q\,\omega_q\,\zeta_q + \frac{i}{2}\zeta_1 c_1 K_{q,1,q} h_q \qquad (4.107)$$

where

$$K_{q,1,q} = -2\sqrt{3}\ \ell_q, \quad \omega_q = 2\ c_q + \sqrt{3}(c_q - \tfrac{1}{2})\zeta_1$$

The ω_q also includes the amplitude of the solid body rotation, and it is the advective nonlinearity together with the orographic forcing, which gives the equations interesting properties.
In order to find the steady states of the system eq. (4.106) and (4.107) and their stability properties more easily, we convert the system into three equations in the real domain.
We define

$$\zeta_1 = u$$

$$\zeta_q = x + iy$$

$$h_q = h_{\bar{q}} = h$$

We have fixed the phase of the orography and look at the response of the other components. Substituting this in the eq. (4.106) and (4.107) we obtain

$$\frac{du}{dt} = \underline{\delta_1\ h\ y}$$

$$\frac{dx}{dt} = -(\beta - \alpha\ u)y$$

$$\frac{dy}{dt} = (\beta - \alpha\ u)x - \underline{\delta_2\ h\ u}$$

with

$$\delta_1 = 2\sqrt{3}\ \ell_q\ c_q$$

$$\delta_2 = \frac{\sqrt{3}}{2}\ \ell_q$$

$$\alpha = \sqrt{3}\, \ell_q (\tfrac{1}{2} - c_q) \qquad (> 0)$$

$$\beta = 2\, \ell_q\, c_q$$

The system above consists of three nonlinearly coupled differential equations. The underlined terms are responsible for the exchange of energy between the zonal flow and the wave component due to the orography. The $\beta - \alpha u$ terms can be compared with the ω_q in eq. (4.107) and thus give the phase speed of a free wave.

The combination of a zonal flow and a wave component gives rise to a flow which is sketched in figure 30. The orographic forcing is indicated in terms of "land" areas, where the height above its mean value, and "ocean" areas where the height is below its mean value. The system of equations (4.108) do not conserve enstrophy but there is energy conser-vation. Instead of enstrophy another quantity involving the orography is conserved.

For the total energy, E, and the other quantity, which will be called F and is related to the enstrophy, we have

$$\frac{dE}{dt} = \frac{d}{dt} \left(\frac{u^2}{4} + c_q(x^2 + y^2)\right) = 0$$

and

$$\frac{dF}{dt} = \frac{d}{dt} \left[(\beta - \alpha u)^2 - 2\, \alpha\, \delta_1\, hu\right] = 0$$

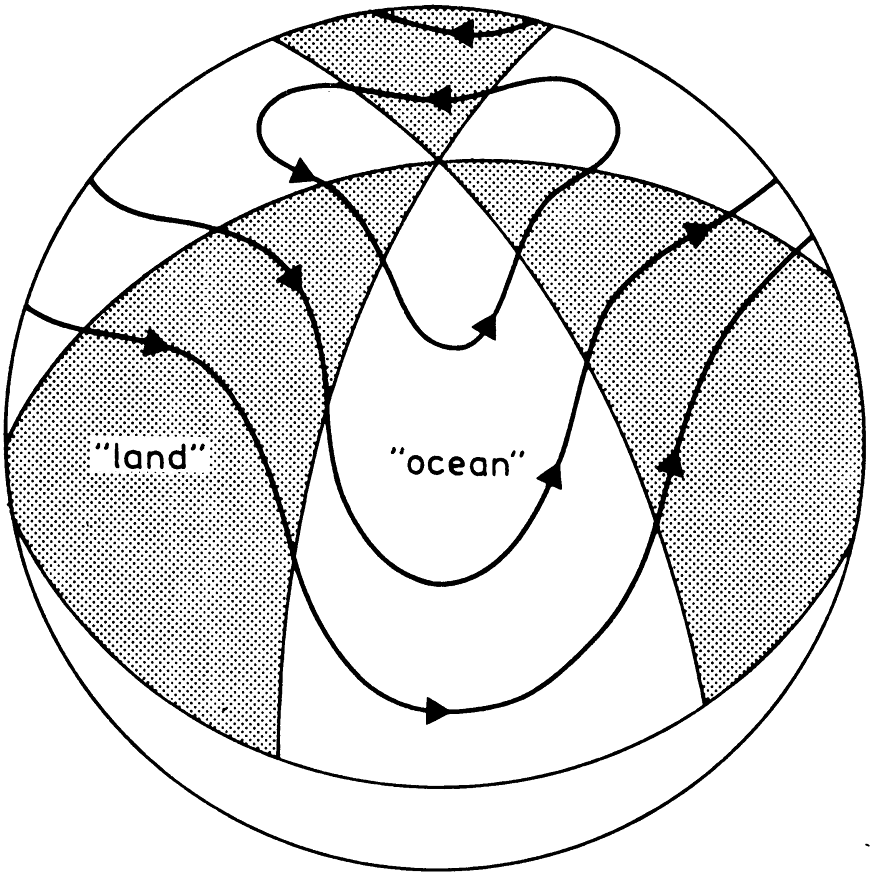

Fig. 30. Schematic picture of the flow described by the low order model. The structure of the orographic forcing is indicated with a shading where the orography is above its mean value. Full lines are isolines for the streamfunction, flow direction is indicated with arrows.

The steady states of (4.108) can be found by setting the l.h.s. equal to zero. We obtain

$$y = 0$$

$$x = \frac{\delta_2 hu}{\beta - \alpha u} \tag{4.109}$$

We here have one degree of freedom in determining a steady state, and we will thus have steady state curves instead of points. As $y = 0$ at a steady state we only have to look in the x-u plane. Along the full lines in fig. 31 $\frac{dy}{dt}$ changes sign, and we thus have three regions in the x-u plane inside which $\frac{dy}{dt}$ is of equal sign. There will be periodic solutions in time, which circle around the curves $\frac{dy}{dt} = 0$ in the x, y, u space. The periodic solutions have to remain on the intersection between surfaces of constant E and F and this constraint together with the initial conditions completely determines the path of the periodic solutions. Unless the initial state is on a steady curve, we will always have a solution which periodically exchanges energy between the zonal flow and the waves due to the effect of the orography.

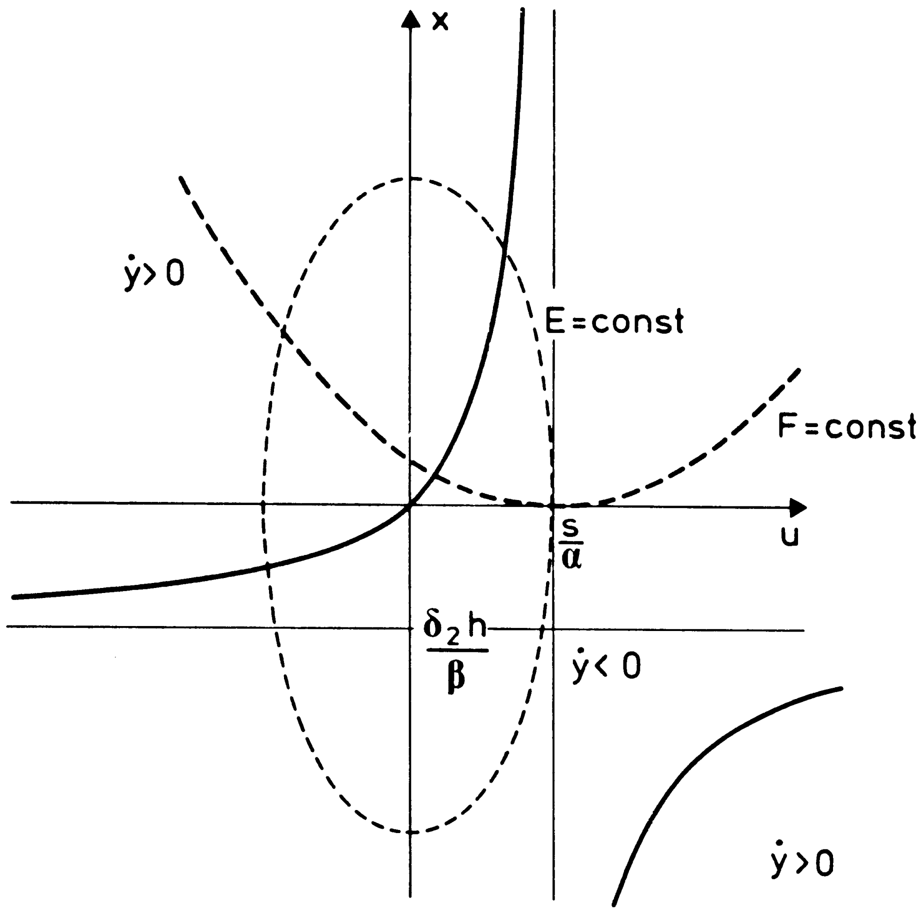

Fig. 31. Diagram to describe the dynamics of an orographically forced low order system. For explanations see text.

In the atmosphere we know that the orography generates waves which can easily be identified on long-term average maps of the atmospheric mean flow. A strictly periodic energy exchange between the zonal flow and the waves is, however, not observed although we do find periods when the long waves are particularly intense and the zonal flow is rather weak and other time periods when the situation is reversed. The transition from one type of circulation to the other is not periodic, it rather occurs at irregular intervals and it is short compared to the characteristic persistence time scale of the circulation types. These observed features suggest a bifurcation mechanism where we have two different stable flow types and where the transition occurs due to a change in the external flow parameters.

To model this type of behavior we will now go one step further with the orographically forced barotropic model, by including the effects of dissipation and a momentum forcing. In the vorticity equation dissipation can be included linearly due to the effect of Ekman-pumping which acts directly on the vorticity field. To balance the dissipation we have to include a vorticity forcing term which will act as a source of kinetic energy. We will restrict this vorticity forcing to the zonal flow. Writing the effects of forcing and dissipation in a Newtonian form we thus have

$$\frac{d}{dt}(\zeta + f + h) = e(\zeta^* - \zeta) \qquad (4.110)$$

The parameter e is the dimensional dissipation rate, which has the dimensions s^{-1} and is given by the intensity of the Ekman pumping. Looking at the total energetics of the model we may also interpret e as a characteristic residence time of the kinetic energy. The energy input to the model is given by $e\langle -\zeta^* \psi \rangle$ where $\langle \quad \rangle$ denotes an area integrated value. The energy output, or the total dissipation, is given by $e\langle -\zeta\psi \rangle$. As a long term time average these two terms must balance, and in particular if we have a steady state they must balance exactly as the total kinetic energy K_E then remains constant (see fig. 32). The input and output of kinetic energy may thus be interpreted as fluxes in and out of a reservoir containing a certain amount of kinetic energy, K_E. The characteristic flux rate is thus governed by e and 1/e can be interpreted as a residence time for the energy. Observations from the

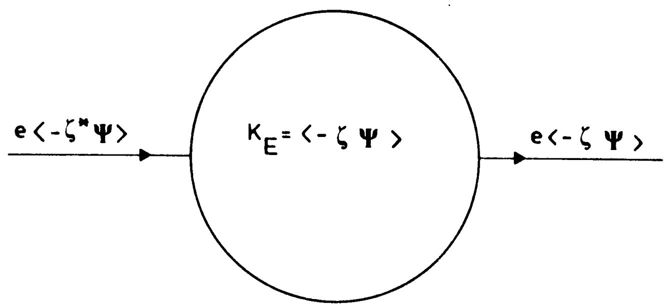

Fig. 32. Total energetics of the barotropic model.

atmosphere show that a reasonable value for $1/e$ is around 5 days. Non dimensionalizing e with the rotation rate of the earth we find a nondimensional dissipation rate $\varepsilon = \frac{e}{\Omega} \sim 0.03$. Returning to the vorticity equation (4.110) we can now nondimensionalize and insert an orthogonal expansion for the space dependent variables just as we have done earlier in this chapter. The forcing and dissipation terms are linear and may thus just be added on to eq. (4.108) which now will be

$$\frac{du}{dt} = \delta_1 hy + \varepsilon(u^* - u)$$

$$\frac{dx}{dt} = -(\beta - \alpha u)y + \varepsilon(x^* - x) \qquad (4.111)$$

$$\frac{dy}{dt} = (\beta - \alpha u)y - \delta_2 hu + \varepsilon(y^* - y)$$

The forcing terms, denoted with a star, have been nondimensionalized with Ω, just as the vorticity.
Assuming a forcing only in the zonal flow ($x^* \equiv y^* \equiv 0$) we have a nonlinear system which will give us the desired bifurcations. The steady states of the system follow from $\dot{x} = 0$ and $\dot{y} = 0$.

$$\bar{x} = \frac{\delta_2 h\bar{u}(\beta - \alpha\bar{u})}{\varepsilon^2 + (\beta - \alpha\bar{u})^2} \qquad (4.112a)$$

$$\bar{y} = \frac{-\varepsilon\, \delta_2\, h\bar{u}}{\varepsilon^2 + (\beta - \alpha\bar{u})^2} \qquad (4.112b)$$

and from $\dot{u} = 0$

$$u^* = -\frac{\delta_1 hy}{\varepsilon} + \overline{u} = [\frac{\delta_1 \delta_2 h^2}{\varepsilon^2 + (\beta - \alpha u)^2} + 1]\overline{u} \qquad (4.113)$$

The last equation gives the forcing as a function of the response. It is a cubic equation in $\overline{u}$, so we may thus have three steady state solutions. Note that if there is no orography eq. (4.113) is linear in u.
We will now investigate eq. (4.113) graphically to see how the number of steady states varies with the forcing parameters u^* and h. The dissipation rate, ε, will be kept constant and we will choose the wavenumber dependent parameters to represent a large scale wave where $\ell = \zeta$, $n_1 = 4$ and $\varepsilon = 0.03$. Fig. 33 gives a plot in the $u^* - \overline{u}$ plane of eq. (4.113) for some values of h.

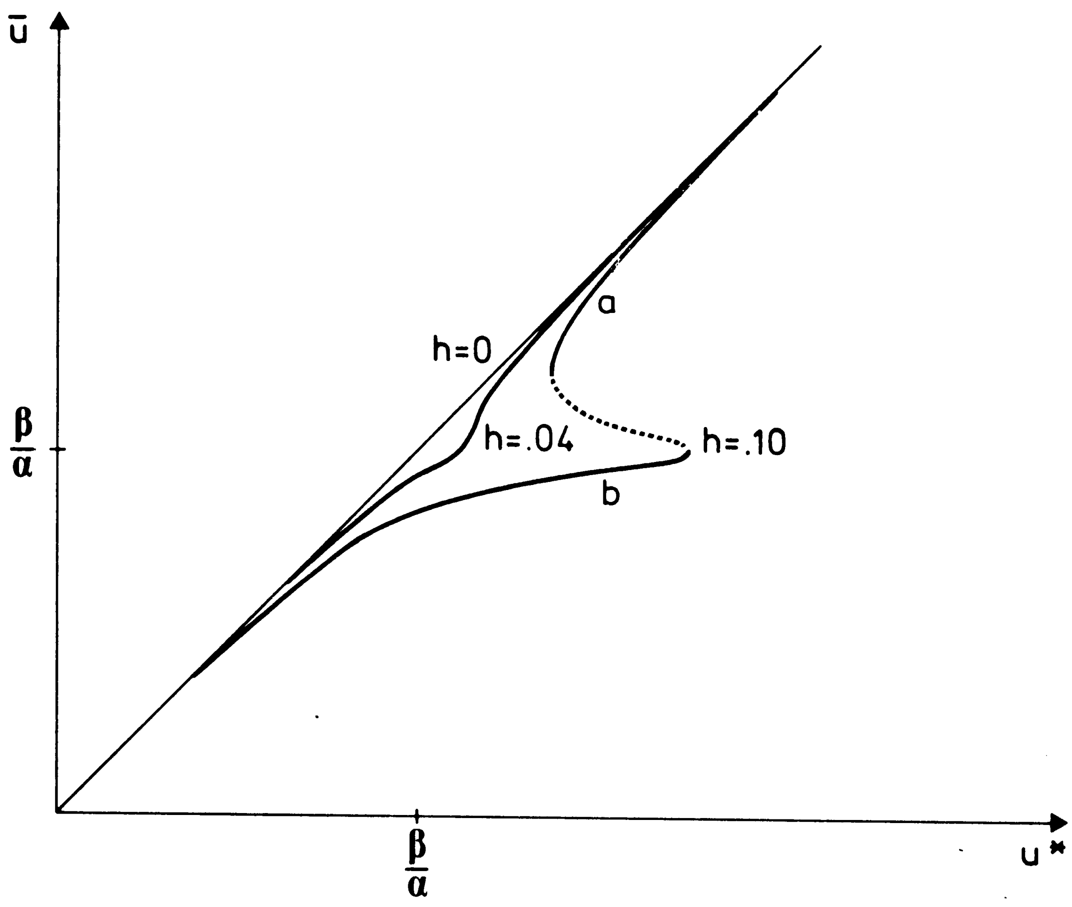

Fig. 33. Steady states in a $\overline{u} - u^*$ diagram for different values of the orographic parameter.

We see from fig. 33 that when the orographic forcing is small, there is only one value of $\overline{u}$ for each value of u^* and the steady state solution is thus unique. When h exceeds a certain critical value we have a bifurcation and we can now find three values of $\overline{u}$ for each u^* within a certain range of u^* values. This means that for a given forcing the system can have three different steady solutions.
An eigenvalue analysis of eq. (4.111), linearized around each of these steady states, reveals that one of them is unstable (the middle one) while the two others are stable. In one of the stable states (the upper solution branches marked a in fig. 33) the response of the zonal flow ($\overline{u}$) is very close to the forcing (u^*) while in the other stable state (lower solution branches marked b in fig. 33) the response $\overline{u}$ is much lower than the forcing u^*.

Returning to the energetics, we may divide the total kinetic energy into a zonally averaged part, $K_Z = \frac{1}{2} u^2$ and a wave part, $K_W = c_q(\overline{x}^2 + \overline{y}^2)$. In a diagram of the energetics (fig. 34) we can now interpret the effect of the orography. We know that the orographic term is the only one which can transfer energy between K_Z and K_W. The size of this term is determined by the steady state values of $\overline{u}$ and $\overline{y}$.

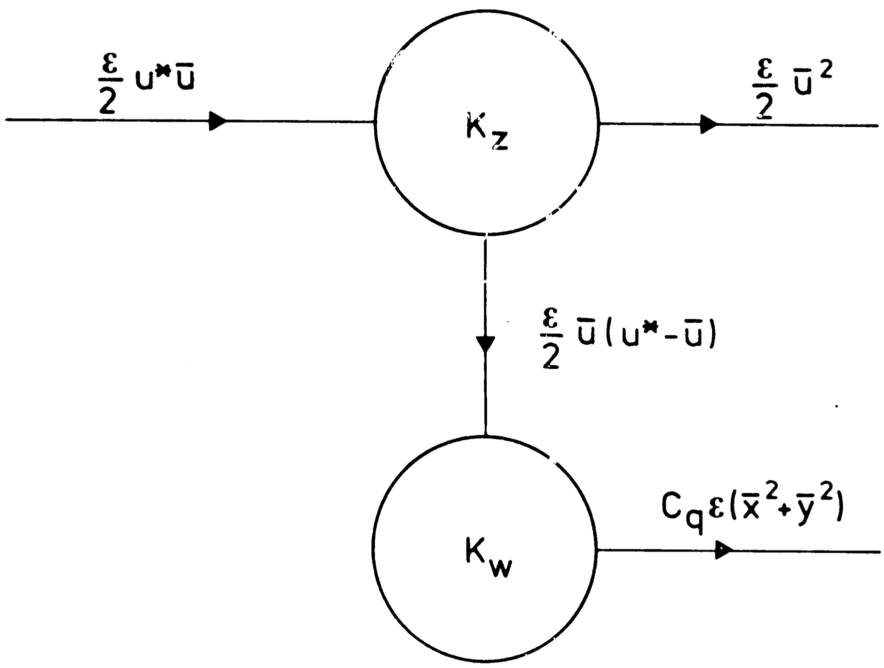

Fig. 34. Energetics for the low order model separated into contributions from the zonal component (K_Z) and the wave components (K_W).

From (4.108) it follows from the underlined terms that the energy transfer is given by $-\sqrt{3} \cdot \ell_q c_q h \bar{u} \bar{y}$ and using (4.113) this may be written

$$\frac{\varepsilon}{2} \bar{u}(u^* - \bar{u})$$

which is just the difference between the generation and dissipation of zonal kinetic energy. When $(u^* - \bar{u})$ is large we have a strong energy transfer from the zonal flow to the waves and thus high dissipation in the waves which implies a large wave amplitude.
If $\bar{u}$ and u^* are close together we have a strong zonal flow but the orographic energy transfer is weak and we thus expect low wave amplitudes. The solution branches denoted a in fig. 33 correspond to this latter situation while those marked b are characterized by a large amplitude wave flow and a weak zonal flow.

The nonlinearity in the system giving rise to the bifurcation is the advection of the wave by the zonal flow given by the terms $(\beta-\alpha u)x$ and $(\beta-\alpha u)y$ (see eq. 4.111). If $\beta = \alpha u$ we have a cancellation of the β-effect and the zonal advection which gives us a type of resonance in this model. From the steady state equations for the waves (eq. 4.112) it can be seen that the wave amplitude is near a maximum when $\beta - \alpha u = 0$. The large amplitude wave branches (b in fig. 33) are thus situated at a value of $\bar{u}$ which is close to resonance.

A simple wave-zonal mean flow interaction through the effect of the orography may thus give rise to a bifurcation and the stable steady states qualitatively agree well with observed quasi-stationary circulation types in the atmosphere. The model property that a fairly strong zonal forcing is needed for the existence of multiple steady-states agrees well with the observed fact that quasi-stationary flow patterns such as "blocking" occur most frequently during winter or early spring and in the Northern hemisphere. Whether the interesting nonlinear behaviour found in simple models like the one treated here also exists in more complicated and realistic models of the atmosphere is a subject which presently is under intense research. If this is indeed a general property which can be identified in a general circulation model this opens up new prospects for long range weather forecasting.

Exercises

10. Consider the equations for the perturbation components in the Rossby-Haurwitz wave instability problem,

$$\frac{d\zeta_\gamma}{dt} = i\,(\zeta_{\bar{\alpha}}\,\zeta_\beta - \zeta_\alpha\,\zeta_{\bar{\beta}})\,I_{\gamma,\bar{\alpha},\beta}$$

$$\frac{d\zeta_\beta}{dt} = i\,(\ell\omega_\beta\,\zeta_\beta + \zeta_\gamma\,(\zeta_\alpha\,I_{\beta,\alpha,\gamma} + \zeta_\beta\,I_{\beta,\beta,\gamma}))$$

Convert these equations to the real domain through the transformation

$$\zeta_\gamma = z, \quad \zeta_\beta = x + iy, \quad \zeta_\alpha = A \qquad (z,x,y,A) \in \mathrm{IR}$$

a. Determine the linearized local stability of the steady-state

$$\zeta_\gamma = \zeta_\beta = 0$$

b. What happens with the stability properties if we add a dispersion term $-\varepsilon\,\zeta_{(\gamma,\beta)}$ to the right hand sides of the above equations?

11. Given the orographically forced two-component system with momentum forcing and dissipation

$$\frac{d\zeta_1}{dt} = i\,\sqrt{3}\,\ell_\alpha\,c_\alpha\,(h_\alpha\,\zeta_{\bar{\alpha}} - h_{\bar{\alpha}}\,\zeta_\alpha) + \varepsilon\,(\zeta_1^* - \zeta_1)$$

$$\frac{d\zeta_\alpha}{dt} = i\,[\ell_\alpha\,\omega_\alpha\,\zeta_\alpha - \frac{\sqrt{3}}{2}\,\ell_\alpha\,\zeta_1\,h_\alpha] - \varepsilon\,\zeta_\alpha$$

$$(\omega_a = 2\,c_\alpha + \sqrt{3}\,(c_\alpha - \tfrac{1}{2})\zeta_1)$$

Show how the transformation below converts it to a system in the real domain

$$\zeta_1 = u \qquad \zeta_\alpha = x + iy \qquad h_\alpha = h \qquad (u,x,y,h) \in \mathrm{IR}$$

a. Determine the eigenvalue equation for the system when linearized around a certain steady-state.

b. What is the condition for a Hopf-bifurcation in this case?

5. Localized solutions to the barotropic vorticity equation-modons

When looking for solutions to the barotropic vorticity equation we have up to now concerned ourselves with functions having a global character. The wavelike structure of the solutions implies that motions take place over the whole domain, but as the wave solutions are eigenfunctions of the Laplace operator we found it a convenient method. To represent a solution which is more localized in space we can superimpose waves of many different scales, but as we have seen earlier this will give us a rather complicated and lengthy set of nonlinear equations. Another way of representing a localized solution is to choose another set of basis functions which are localized in space and preferably also eigenfunctions of the Laplace operator. We may here draw an analogy with the KdV equation (chapter 3) where we also had an advective nonlinearity but where we managed to find a localized solution, a so called soliton. Here we have a two-dimensional flow but in principle we are looking for the same type of solution, i.e. a localized structure where nonlinear and dispersive effects balance. To distinguish these from the one-dimensional solitons, they will be called modons.

Our strategy here will be to construct a modon type of solution step by step and we will closely follow the approach given by Leith (1981). Because of our interest in a localized solution we will work with the barotropic equation on a β-plane and thus the geometry will be slightly simpler than the spherical one used in chapter 4. We will have a Cartesian coordinate system and the dispersive effects will be retained by allowing the Coriolis parameter to be a linear function of the north-south coordinate. We first seek solutions which satisfy $\nabla^2\psi = \mu^2\psi$, thus having squared eigenvalues of opposite sign as compared with the normal Rossby-wave type of solutions on the β-plane. Assuming a cylindrical symmetry we find such solutions to be modified Bessel functions of the second kind. These functions have a localized character in that they monotonically approach zero as the distance from the origin goes to infinity, but they also have a singularity at the origin. To avoid this singularity we introduce a second solution inside a certain region which encloses the singularity. The second, or "inner", solution has a dipole structure and by matching the inner and outer solutions at the boundary we can determine some constants of integration.

Additionally we also demand that the propagation velocities of the inner and outer solutions must be the same.

We can thus construct a localized dipole type of solution and in an exercise it will be shown that a monopole of arbitrary amplitude may be added on to the dipole. With this procedure we can construct a localized solution with a shape and structure that looks rather similar to some characteristic atmospheric flow patterns, i.e. blocking highs. To determine whether these type of solutions are likely to be found in the atmosphere an investigation of their stability has to be made. In the literature only numerical investigations of the stability of dipole modons have been reported, and these modons appear to be remarkably stable to various types of perturbations. We must of course remember that the modon solutions as described here are essentially linear phenomena, the advective nonlinearity only contributes to give the advective phase speed necessary to "hold" the modon together. A perturbation analysis, through which the nonlinear stability of the modon could be investi-gated, is still lacking in the theory of modons.

5.1 Modon structure

We shall describe a localized modon solution for the equivalent barotropic vorticity equation

$$\frac{\partial}{\partial t}(\nabla^2 - \alpha^2)\psi + \beta\frac{\partial\psi}{\partial x} + J(\psi,\nabla^2\psi) = 0 \qquad (5.1)$$

which determines the evolution of the stream function ψ for the equivalent barotropic flow of mean depth H on a β-plane with Coriolis coefficient $f = f_0 + \beta y$. Here α is the deformation wavenumber with

$$\alpha^2 = \frac{f_o^2}{gH} \qquad (5.2)$$

and g is an equivalent gravitational acceleration such that $(gH)^{\frac{1}{2}}$ is the speed of gravity waves.

The deformation wavenumber α is mainly introduced to give a lower bound on the phase velocities of free Rossby waves (see below). Through the equivalent barotropic assumption this term may be associated with a large scale divergence which is added to slow down the long waves. The Jacobian here is defined in the usual way as

$$J(\psi,\phi) = \psi_x\phi_y - \psi_y\phi_x \tag{5.3}$$

There exists in this case a potential vorticity

$$Z = f + \nabla^2\psi - \alpha^2\psi \tag{5.4}$$

in terms of which Eq. (5.1) may be rewritten

$$Z_t + J(\psi,Z) = 0 \tag{5.5}$$

displaying Z as conserved following the flow.
The linear Rossby wave solutions of Eq. (5.1) are given by eigenfunctions of ∇^2 such that

$$\nabla^2\psi = -\lambda^2\psi \tag{5.6}$$

For these the Jacobian term vanishes and Eq. (5.1) reduces to the linear equation

$$-(\alpha^2 + \lambda^2)\psi_t + \beta\psi_x = 0 \tag{5.7}$$

describing waves propagating in the x-direction with velocity

$$c = -\frac{\beta}{\alpha^2 + \lambda^2} \tag{5.8}$$

Since $0 < \lambda^2 < \infty$, c is bounded with $-\beta/\alpha^2 < c < 0$. Rossby waves are oscillatory in space like $\sin \lambda x$ and are not therefore localized solutions.

A localized solution must drop off rapidly away from some central region. As an outer solution with this property we take another eigenfunction of ∇^2 but one such that

$$\nabla^2\psi = \mu^2\psi \tag{5.9}$$

In particular we choose

$$\psi = A\,K_1(\mu r)\sin\theta \tag{5.10}$$

where K_1 is the modified Bessel function of the second kind of order 1. Again in the outer region the Jacobian vanishes and Eq. (5.1) reduces to Eq. (5.7) but with λ^2 replaced by $-\mu^2$. The outer solution (5.10) propagates therefore in the x-direction with velocity

$$c = -\frac{\beta}{\alpha^2 - \mu^2} \tag{5.11}$$

Since $0 < \mu^2 < \infty$, the range of possible c values for localized solutions is $-\infty < c < -\beta/\alpha^2$ and $0 < c < \infty$, disjoint from the possible Rossby wave velocities of Eq. (5.8). In Eq. (5.10) r and θ are polar coordinates in a moving frame with, say,

$$\begin{aligned} r^2 &= (x-ct)^2 + y^2 \\ \sin\theta &= y/r \end{aligned} \tag{5.12}$$

To avoid the singularity in ψ at $r = 0$ given by Eq. (5.10), we introduce a smooth inner solution which we let join the outer one at a circle of radius $r = a$. We take as the inner solution for $r \leq a$

$$\psi = BJ_1(\lambda r)\sin\theta - C\,r\sin\theta \tag{5.13}$$

where J_1 is the Bessel function of order 1.

The first term is again an eigenfunction of ∇^2 satisfying Eq. (5.6) and would by itself propagate in the x-direction with a velocity given by Eq. (5.8). The second term, however, introduces a constant advecting velocity C. In order that the inner and outer propagation velocities be the same we must impose a velocity constraint

$$C = \beta\left[\frac{1}{\lambda^2} - \frac{\alpha^2 + \lambda^2}{\lambda^2(\alpha^2 - \mu^2)}\right] \tag{5.14}$$

that determines the coefficient C for any choice of inner and outer wavenumbers, λ and μ.

We match the inner and outer solutions at $r = a$ by imposing as many continuity conditions as possible. From the continuity of ψ and ψ_r at $r = a$ we have

$$A\ K_1(\mu a) = B\ J_1(\lambda a) - Ca\ , \tag{5.15}$$

and

$$A\ \mu a\ K_1'(\mu a) = B\lambda a\ J_1'(\lambda a) - Ca \tag{5.16}$$

Let $\lambda a = \bar{\lambda}$, $\mu a = \bar{\mu}$. By subtraction we may eliminate the term Ca and find

$$A\ [K_1(\bar{\mu}) - \bar{\mu}K_1'(\bar{\mu})] = B[J_1(\bar{\lambda}) - \bar{\lambda}J_1'(\bar{\lambda})] \tag{5.17}$$

Recursion relations for Bessel functions permit Eq. (5.17) to be written in simpler form as

$$A\ [\bar{\mu}K_2(\bar{\mu})] = B[\bar{\lambda}J_2(\bar{\lambda})] \tag{5.18}$$

whence

$$\begin{aligned} A &= S[\bar{\mu}K_2(\bar{\mu})]^{-1} \\ B &= S[\bar{\lambda}J_2(\bar{\lambda})]^{-1} \end{aligned} \tag{5.19}$$

The coefficient S is determined by Eq. (5.15) to be

$$S = Ca\left[\frac{J_1(\bar{\lambda})}{\bar{\lambda}J_2(\bar{\lambda})} - \frac{K_1(\bar{\mu})}{\bar{\mu}K_2(\bar{\mu})}\right]^{-1} \tag{5.20}$$

The conditions imposed so far suffice to determine the amplitude coefficients A, B, and C for any choice of radius a and wavenumbers λ and μ. By Eq. (5.11) a choice of μ is equivalent to a choice of the overall propagation velocity c. Then the choice of λ determines C by Eq. (5.14). If we next choose a radius a then $\bar{\lambda}$ and $\bar{\mu}$ are defined, S is determined by Eq. (5.20) and finally A and B by eqs. (5.19).

The most important continuity conditions have been satisfied, but we still have the freedom to choose λ for a given value of μ in such a way that the vorticity $\zeta = \nabla^2\psi$ is also continuous at $r = a$. The continuity condition for ζ at $r = a$ is

$$A\,\bar{\mu}^2\,K_1(\bar{\mu}) = -B\,\bar{\lambda}^2\,J_1(\bar{\lambda}) \tag{5.21}$$

which may be combined with Eqs. (5.19) to give

$$\frac{\bar{\lambda}J_1(\bar{\lambda})}{J_2(\bar{\lambda})} = -\frac{\bar{\mu}K_1(\bar{\mu})}{K_2(\bar{\mu})} \tag{5.22}$$

For any value of $\bar{\mu}$ the expression on the right is well defined and negative. Thus λ must be in those intervals of the $\bar{\lambda}$ range where J_1 and J_2 have opposite sign. We shall consider only the gravest such interval $(j_1^{(1)}, j_2^{(1)})$ where $\bar{\lambda}$ is smallest and the inner solution has the smoothest structure. The solid curve in Fig. 35 shows the mapping $\bar{\mu} \rightarrow \bar{\lambda}$ into this interval given by Eq. (5.22).

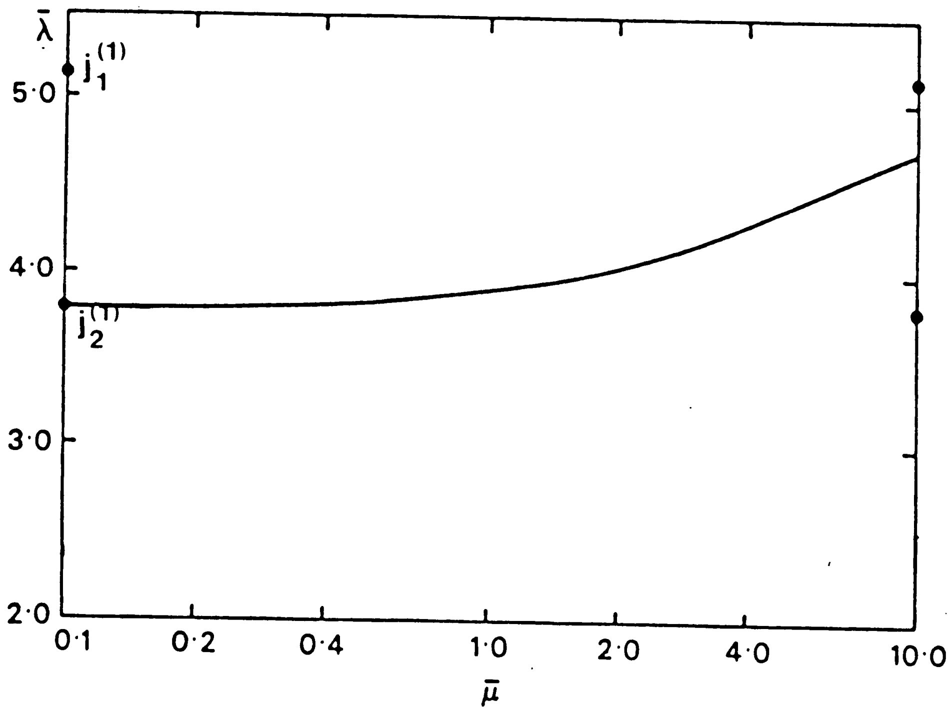

Fig. 35 Inner wavenumber $\bar{\lambda}$ vs. outer wavenumber $\bar{\mu}$ satisfying vorticity continuity conditions for a modon. Dots on $\bar{\lambda}$-axis delimit solution interval (From Leith).

The modon so constructed is a localized vorticity dipole with an amplitude determined by its radius a and its velocity c. The required form of the second term in Eq. (5.13) imposes the dipole structure on the first term and on Eq. (5.10).

Excersises

12. Show how the amplitudes of the inner and outer modon solutions are determined through the use of recursive relations for the Bessel functions.

13. Once a modon has been constructed, a so called rider can be added to it. The stream function of a rider has the following structure

$$\psi_R = D\ K_o\ (\mu r)\ \text{for}\ r > a$$

$$\psi_R = E\ J_o\ (\lambda r) + F\ \text{for}\ r < a$$

D, E and F are amplitudes to be determined while the wavenumbers μ and λ are the same as for the modon.

a. Show that an addition of a rider on top of a modon will not affect the velocity constraint arrived at for the modon.

b. Determine the coefficients D and E as functions of F through continuity conditions on ψ_R and $\frac{\partial \psi_R}{\partial r}$ at $r = a$.

c. Sketch the structure of a rider.

6. Rossby and Gravity waves

6.1 Geostrophic adjustment theory

To model the dynamics of the long waves in the atmosphere we have so far derived our results from the barotropic vorticity equation or an equivalent barotropic model where we have assumed that the quasi-geostrophic approximation is valid. In deriving the barotropic vorticity equation we assume that the divergence is zero. For midlatitude, large scale motions this is a very reasonable assumption. The horizontal divergence is small and the vertical motions are at least an order of magnitude smaller than the horizontal ones. The small divergence and vertical wind fields are, however, the essential driving forces which change the weather pattern from day to day. Therefore a model in which the divergence is a prognostic rather than a diagnostic variable is a better model for forecasting the weather than a model which is based on the barotropic vorticity equation or a model which involves a quasi-geostrophic balance. Such a model, however, not only permits the existence of the meteorologically important, slow moving Rossby waves, but also gravity waves. The latter wave-type is probably not of any direct importance for large scale numerical weather prediction because amplitudes are very small, which means that they play an insignificant role in the energetics of the atmosphere. Nevertheless gravity waves have an important function in that they adjust imbalances between the pressure and the velocity fields which then tend to a quasi-geostrophic state. This process is usually referred to as geostrophic adjustment. In the following introductory section we will describe Rossby's original ideas from 1937 and 1938 on this matter. Rossby found the lengthscale related to the geostrophic adjustment problem by finding an answer to the question, what is the distance upto which the pressure field (or mass field or height field) is influenced by an initial perturbation in the velocity field or vice versa?

Consider a rotating cylinder filled up to a certain height H with a fluid (see Fig.36). If H is small compared to the radius of the cylinder (in fact we assume that the radius is infinite) we can use the so-called shallow water equations to describe the flow of the fluid:

$$\frac{d\overline{v}}{dt} + 2\,\overline{\Omega} \times \overline{v} = -\nabla\Phi \tag{6.1}$$

$$\frac{d\Phi}{dt} + \Phi\,\nabla\cdot\overline{v} = 0 \tag{6.2}$$

The first equation expresses conservation of momentum and the second equation expresses conservation of mass or continuity of the upper surface. $\overline{V}$ is the horizontal wind vector, $\overline{\Omega}$ is the angular velocity of the cylinder and Φ is the geopotential.

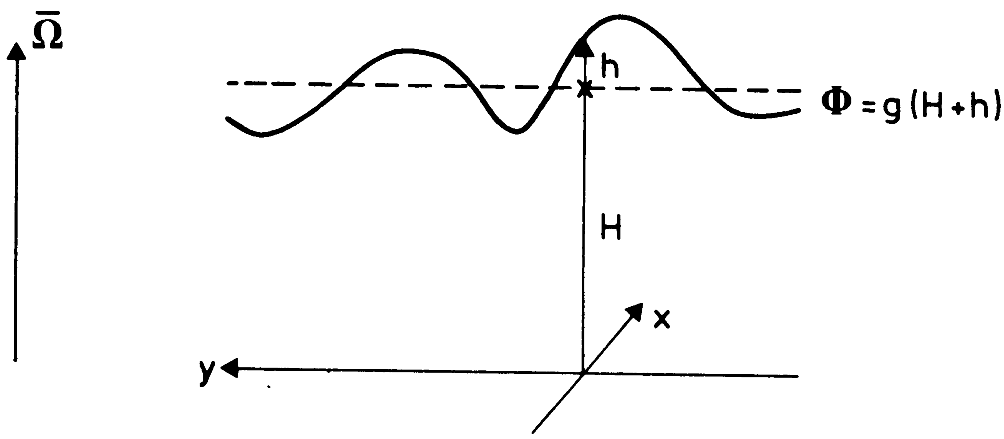

Fig. 36. Geometry of rotating fluid.

Considering the flow of a fluid in a rotating cylinder is analoguous to considering the fluid flow on the f-plane with $f = 2\,|\overline{\Omega}|$. We have thus neglected the β-effect. We will avoid an explicit treatment of the time dependent problem by employing a Lagrangian technique. By specifying the initially perturbed state and applying a geostrophic constraint on the final, asymptotic state we will be able to deduce the structure of the height and wind fields. The Lagrangian technique takes the advective nonlinearity into account, but we avoid the explicit treatment necessary in an Eulerian formulation.
We define the velocity components as follows:

$$u = \frac{dx}{dt} \text{ and } v = \frac{dy}{dt} \tag{6.3}$$

Let us initially assume a state of rest where we impose a velocity perturbation in the x-direction. The mass field will then try to adjust to the new velocity field such that geostrophic balance is attained. Initially equation (6.1) becomes

$$\frac{du}{dt} - fv = \frac{d\Phi}{dx} = 0$$

or

$$\frac{d}{dt}(u - fy) = 0 \tag{6.4}$$

which means that (u - fy) is conserved. The final state is assumed to be in a geostrophic balance, and therefore,

$$u = -\frac{1}{f}\frac{d\Phi}{dy} \tag{6.5}$$

The Lagrangian technique implies that we consider the flow as a mapping of the initial position of the fluid particles onto the positions of the fluid particles in the final state. To simplify the problem we only consider displacements in the direction of the y-axis, the flow is assumed to be homogeneous in the x-direction. We thus define two y-axes, y_0 and y, representing the initial state and the final state respectively (see fig. 37). All variables with index $_0$ refer to the initial state while unindexed variables refer to the final state.

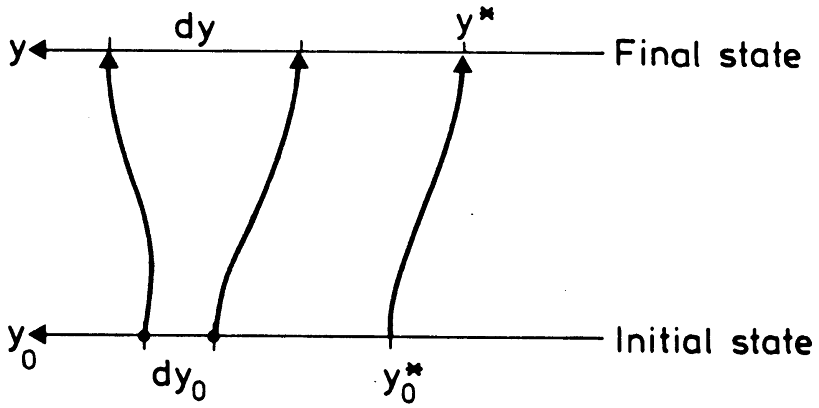

Fig. 37. Displacement of fluid particles.

Conservation of mass implies that

$$\Phi_o \, dy_o = \Phi \, dy$$

or

$$\Phi = \Phi_o \frac{dy_o}{dy} \qquad (6.6)$$

Combining (6.5) and (6.6) yields

$$u = -\frac{\Phi_o}{f} \frac{d^2 y_o}{dy^o} \qquad (6.7)$$

From equation (6.4), which expresses the conservation of (u - fy), we find

$$u_o - fy_o = u - fy \qquad (6.8)$$

This expresses vorticity conservation in an integrated form. From (6.7) and (6.8) it can be deduced that

$$\frac{d^2}{dy^2}(y - y_o) - \frac{y - y_o}{\lambda^2} = \frac{u_o}{f\lambda^2} \qquad (6.9)$$

where

$$\lambda^2 = \frac{\Phi_o}{f^2} = \frac{gH}{f^2} \; [m^2]$$

Assume now that we give the fluid a "push" in the x-direction with a velocity distribution of the form,

$$u_o(y_o) = \delta(y_o - y_o^*) \qquad (6.10)$$

We can now solve (6.9) with the boundary conditions that the displacement $(y - y_o)$ should vanish as y goes to $\pm\,\infty$. The homogeneous solution is, $y - y_o = a' \exp(\pm\, y/\lambda)$. Using the boundary conditions we have

$$y - y_o = a \exp\left(\frac{y - y^*}{\lambda}\right) \quad \text{for} \quad y < y^*$$

$$y - y_o = a \exp\left(\frac{y^* - y}{\lambda}\right) \quad \text{for} \quad y > y^* \qquad (6.11)$$

where a and a' are constants. This displacement field has been drawn in Fig. 38.

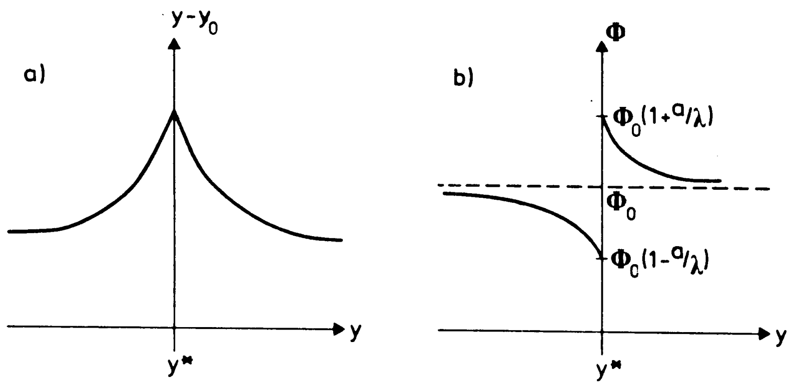

Fig. 38. Displacement and geopotential as functions of the space coordinate in the final state.

The continuity conditions (6.6) can also be written

$$\Phi = \Phi_o(1 - \frac{d}{dy}(y - y_o))$$

Using (6.11) we find that

$$\Phi = \Phi_o(1 - \frac{a}{\lambda}\exp(\frac{y - y^*}{\lambda})) \quad \text{for} \quad y < y^*$$

$$\Phi = \Phi_o\ (1 + \frac{a}{\lambda}\exp(\frac{y^* - y}{\lambda})) \quad \text{for} \quad y > y^* \qquad (6.12)$$

This gives us the geopotential of the final state as a function of y. It has been drawn in Fig. 38. The discontinuity in Φ arises because we assume that the initial velocity perturbation is given by a delta-function. The parameter λ can be interpreted as being the e-folding distance up to which the geopotential field or pressure field is influenced by the initial momentum perturbation due to the adjustment process; λ is therefore the lengthscale related to the geostrophic adjustment problem and is usually called the "Rossby radius of deformation". For the atmosphere in midlatitudes we have $\lambda \sim 10^6$ m. In this section we have only considered the structure of a final, balanced state as a function of the structure of an initial state. We have not found how this adjustment takes place, although we know from linear theory that an unbalanced flow will generate gravity waves. How these gravity waves are formed and how they may be avoided will be the subject of the next section.

6.2 Initialization techniques

Using a Lagrangian technique, we have determined the height field corresponding to a certain velocity perturbation in a shallow water model. We assumed that there exists a geostrophic balance in the asymptotic state, but we never had to consider the type of motions which led to the geostrophically balanced state. It is not even certain that the geostrophically balanced state is the asymptotic one, there may be instabilities developing which will lead the solution to some other type of balance or we may have an oscillating solution. (It can in fact be shown that the asymptotic solution must oscillate, but the time averaged motion is in a geostrophic balance).

The practical problem which is connected with geostrophic adjustment theory, is the problem of providing initial data for a primitive equation model. From atmospheric observations, wind and mass field data are available. These data are sufficient to define an initial state, but due to unavoidable measuring errors in the data it appears that they are not accurate enough. To avoid spurious oscillations in a primitive equation model, there must exist a balance between the wind and the mass field and even if the atmosphere is in a balanced state the accuracy of the data is not sufficient to describe this balance. We must therefore find a procedure by which the data can be changed so as to achieve this balance. The data changes must of course be done in such a way that the meteorological information is not lost.

From linear theory we know that the primitive equations basically can describe two types of wave motion, Rossby waves and gravity waves. They also permit sound waves, but these are not relevant in the present context. The Rossby waves are slowly varying waves, the time scale being governed by the rate of rotation. The gravity waves have much shorter time scale essentially determined by the mass of a vertical column and

the vertical stability of the system.
The large-scale, atmospheric flow is dominated by Rossby waves while the gravity waves from an energetical point of view play an insignificant role. It would therefore seem logical to filter the data in such a way that only Rossby wave type of motions are retained. The simplest filter of this type is a geostrophic balance. Given a certain mass field we can determine the geostrophic wind field or vice versa. Insertion of a geostrophically balanced state in a primitive equation model will, however, not result in a time evolution which is free from spurious oscillations. Gravity waves will be generated and they may reach such an amplitude that they dominate the solution.
Another drawback of a geostrophic initialization is that it does not make optimal use of the observations. Since either the mass or the windfield is redundant.

A slightly more sophisticated version of the geostrophic balance is the so-called nonlinear balance equation. It essentially describes a balance between pressure gradient, Coriolis and local centrifugal forces. Given a certain mass field it is possible to solve for a wind field except for those regions in which a gradient wind balance cannot be found. The existence of such regions is due to the nonlinearity inherent in the centrifugal forces. These regions tend to be found in subtropical areas, particularly in regeions where the curvature and/or shear of the jetstream is anticyclonic. Insertion of gradient wind balanced data into a primitive equation model gives better results than purely geostrophically balanced data, but it is still not satisfactory. In this chapter we will outline a third method, which has proven to be very useful in practice. It is the so called nonlinear normal mode initialization method, first proposed by Machenhauer (1977). The basic idea of the method is to first separate the data into contributions from Rossby wave type of motions and gravity wave type of motions. This is done by projection onto modes which are found by solving for the eigenvalues and eigenvectors of a linearized primitive equation model. Secondly, the amplitudes of the gravity wave part of the data are adjusted so that the time evolution of the gravity models is "smooth". This may be done by ensuring that the first order time derivative of the gravity modes is zero initially.
We will describe this method by considering a particularly simple model,

a rotating shallow water system (following Tribbia, 1981). The model is nonlinear, but caution has to be taken in interpreting the effect of the nonlinearity. The dominating nonlinearity of the model is the centrifugal force and this is not a dominating effect in the dynamics of the atmosphere. The nonlinear properties found in this example may, however, be used as an instructive prototype to understand the nonlinear properties of the normal mode initialization technique.
It will be shown how the slow Rossby modes (in this case stationary) and the fast gravity modes can be separated and how the Machenhauer initialization method may be applied to the model.
We start by writing the model equations in cylindrical coordinates (r,θ), and define the velocity components as follows

$$u = r\frac{d\theta}{dt}, \quad v = \frac{dr}{dt} \tag{6.13}$$

We assume all variables to be independent of θ. The shallow water equations now become

$$\frac{du}{dt} = - fv \tag{6.14a}$$

$$\frac{dv}{dt} = fu - \frac{\partial\phi}{\partial v} + \frac{u^2}{r} \tag{6.14b}$$

$$\frac{d\phi}{dt} = - \frac{(gH + \phi)}{r}\frac{\partial}{\partial r}(rv) \tag{6.14c}$$

The nonlinear centrifugal force appears as the last term on the r.h.s. of Eq. (6.14b). The local steady states $(\overline{u},\overline{v},\overline{\phi})$ of system (6.14) are found by putting the total time derivatives equal to zero. Therefore

$$\overline{v} = 0 \tag{6.15a}$$

$$\frac{\partial\overline{\phi}}{\partial r} = f\overline{u} + \frac{\overline{u}^2}{r} \tag{6.15b}$$

Equation (6.15b) is actually the gradient wind equation, (see Holton, 1979, p. 63). It is clear from this equation that given $\frac{\partial\overline{\phi}}{\partial r}$, there are two balanced states possible. We will now investigate the stability of these steady states to perturbations δu in the u-field. Therefore we set

$$u = \overline{u} + \delta u$$
$$\phi = \overline{\phi} \qquad (6.16)$$

Combining equations (6.14a) and (6.14b) we obtain

$$\frac{d^2 u}{dt^2} = - f(fu - \frac{\partial \phi}{\partial r} + \frac{u^2}{r}) \qquad (6.17)$$

Subsituting (6.16) into (6.17) we have

$$\frac{d^2 \delta u}{dt^2} = - f^2 \overline{u} - f \frac{\overline{u}^2}{r} + f \frac{\partial \overline{\phi}}{\partial r} - f^2 \delta u - 2f \frac{\overline{u}}{r} \delta u$$

Using equation (6.15b) for the steady state we get

$$\frac{d^2 \delta u}{dt^2} = - \delta u \, f \, (f + \frac{2\overline{u}}{r}) \qquad (6.18)$$

We see from this equation that the stability of a locally balanced state to perturbations δu is determined by the sign of $(f + \frac{2\overline{u}}{r})$. Therefore the balanced state is stable when

$$\overline{u} > \frac{-fr}{2} \qquad (6.19)$$

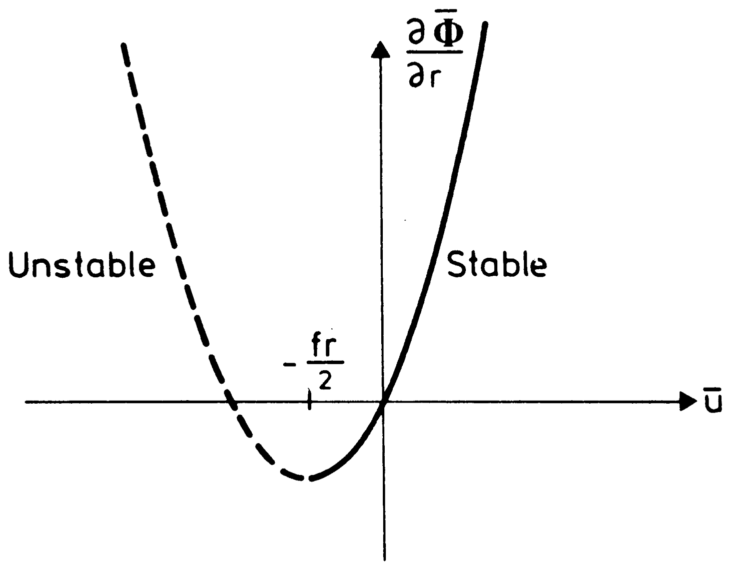

Fig. 39. Balance between wind and geopotential field for a local steady-state.

In Fig. 39 $\frac{\partial \Phi}{\partial r}$ is drawn as a function of $\bar{u}$ as given by equation (6.15b). From condition (6.19) we find that one balanced state (left part of the curve in Fig. 39 is unstable. It gives "supergeostrophic" winds. Therefore we conclude that, given observed height data, there is only one corresponding stable balanced wind field possible.

To determine the eigenmodes an the eigenvalues we will expand the space dependent parts of the governing equations for the fluid in a rotating cylinder in orthogonal functions. Before doing this we nondimensionalize the equations with characteristic length, time, velocity and height scales defined respectively as,

$$r' = r/a \quad (a = \text{radius of the cylinder}). \tag{6.20a}$$

$$t' = tf \tag{6.20b}$$

$$(u',v') = (u,v)/U \tag{6.20c}$$

$$\phi' = \frac{\phi}{afU} \tag{6.20d}$$

The inertial velocity scale U is the same for the u- and v-component. In fact we have two velocity scales, namely U and a velocity scale imposed by the geometry. The ratio of these two velocity scales is defined as the Rossby number,

$$R_o = \frac{U}{fa} \tag{6.21}$$

The Rossby number expresses the relative importance of the nonlinear terms. If it is small, the advective nonlinearities are also small. The imposed velocity scale fa is then dominant.

Substituting the new variables (6.20) in the governing equations and dropping the primes we obtain the following system.

$$\frac{\partial u}{\partial t} + v = - R_o \left(v \frac{\partial u}{\partial r} + \frac{uv}{r}\right) \tag{6.22}$$

$$\frac{\partial v}{\partial t} - u = - \frac{\partial \phi}{\partial r} - R_o\left(v \frac{\partial v}{\partial r} - \frac{u^2}{r}\right) \tag{6.23}$$

$$\frac{\partial \phi}{\partial t} + F \frac{1}{r} \frac{\partial}{\partial r} (rv) = - R_o \frac{1}{r} \frac{\partial}{\partial r} (rv\phi) \tag{6.24}$$

where the Froude number F is defined as

$$F = \frac{gH}{f^2 a^2} = \frac{\lambda^2}{a^2} \tag{6.25}$$

Note that the Rossby number appears in front of all nonlinear advection terms and that the Froude number appears in front of the divergence term in equation (6.24). If F is small ($a^2 > \lambda^2$) the geostrophic adjustment can freely take place inside the container. If F is of ther order 1 or larger the adjustment is forced to take place inside the container, and the laterel boundaries of the container will influence the adjustment process. Due to the cylindrical geometry, the most appropriate orthogonal functions to expand the space dependent parts in are Bessel functions. Therefore we set

$$\binom{u}{v} = \sum_{n=1}^{\infty} \binom{u_n}{v_n} J_1(\lambda_n r) \tag{6.26a}$$

$$\phi = \sum_{n=1}^{\infty} \phi_n J_o (\lambda_n r) \tag{6.26b}$$

The Bessel functions J_o and J_1 are drawn as functions of r in fig. 40.

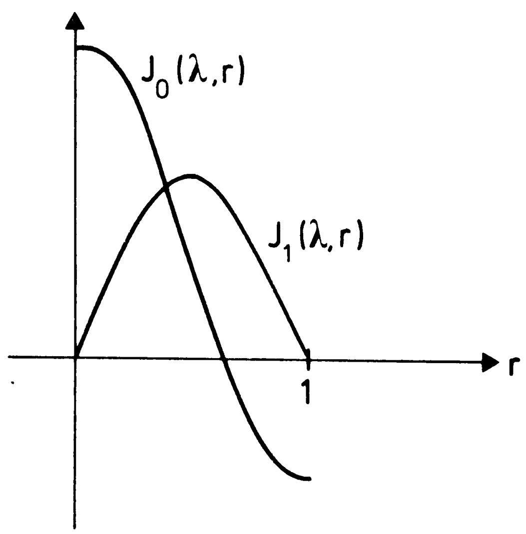

Fig. 40. Structure of the eigenmodes for the height (J_o) and the wind (J_1) fields.

The J_1 function is chosen for the wind field as this satisfies the boundary conditions for v at r = 0 and r = a. For the height field we then must choose J_o as can be seen through the fact that the height and wind fields are related via a horizontal deviative (eq. 6.23 and 6.24). Tha parameter λ_n can be seen as a wave number determining the number of maxima and minima.
Subsituting (6.26a), (6.26b) into (6.22), (6.23) and (6.24) and restricitng our attention to only the gravest mode λ, (see fig. 40) we obtain the following nonlinear system:

$$\dot{u}_1 + v_1 = - R_o(I_1 + I_2)\, u_1 v_1 \tag{6.27}$$

$$\dot{v}_1 - u_1 = \lambda_1 \phi_1 - R_o(I_1 v_1^2 - I_2 u_1^2) \tag{6.28}$$

$$\dot{\phi}_1 + \lambda_1 F v_1 = - R_o\, I_3\, \phi_1\, v_1 \tag{6.29}$$

The interaction integrals are defined as

$$I_1 = \frac{2}{[J_1'(\lambda_1)]^2} \int \lambda_1\, r\, J_1^2\, J_1'\, dr \tag{6.30a}$$

$$I_2 = \frac{2}{[J_1'(\lambda_1)]^2} \int J_1^3\, dr \tag{6.30b}$$

$$I_3 = \frac{2}{J_o^2(\lambda_1)} \int J_o \frac{d}{dr} (r\, J_1\, J_o) dr \tag{6.30c}$$

We will now look at the steady state of the above system where $\overline{v}_1 = 0$ (overbars denote steady-state values). All other steady states are unstable (see exercise). We then have

$$\overline{u}_1 + R_o\, I_2\, \overline{u}_1^2 + \lambda_1\, \overline{\phi}_1 = 0 \tag{6.31}$$

This function is drawn in fig. 41. The dynamics of this system is the same as for the nontruncated system (compare fig. 39 with fig. 41).

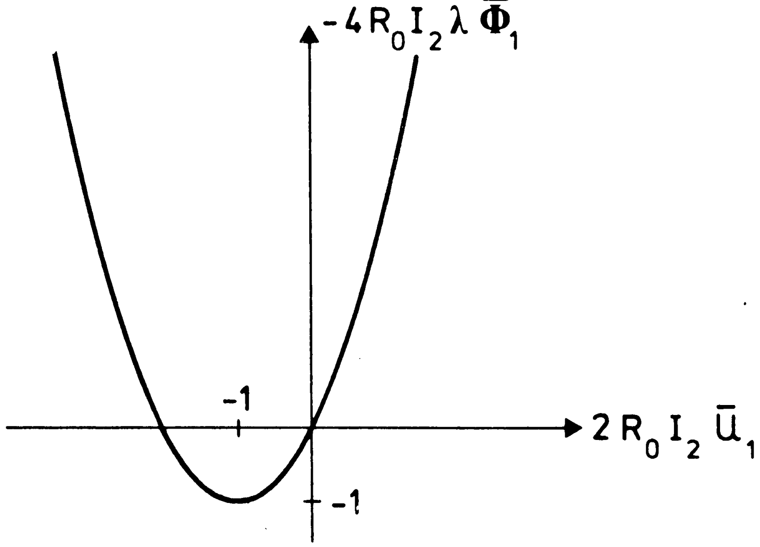

Fig. 41. Steady-states of the truncated system.

There are two possible values of $\overline{u}_1$ for each $\overline{\phi}_1$. We again investigate the stability of this steady state to perturbations δu_1, δv_1 and $\delta\phi_1$ by substituting

$$u_1 = \overline{u}_1 + \delta u_1 \quad v_1 = \delta v_1 \text{ and } \phi_1 = \overline{\phi}_1 + \delta\phi_1 \tag{6.32}$$

into equations (6.27), (6.28) and (6.29)

$$\begin{aligned} \delta\dot{u}_1 &= - [R_o(I_1 + I_2)\overline{u}_1 + 1]\delta v_1 \\ \delta\dot{v}_1 &= (2R_o\ I_2\ \overline{U}_1 + 1)\delta u_1 + \lambda_1\ \delta\phi_1 \\ \delta\dot{\phi}_1 &= - (R_o\ I_3\ \overline{\phi}_1 + \lambda_1 F)\delta v_1 \end{aligned} \tag{6.33}$$

The eigenvalue (ω) for this system is,

$$\omega\{\omega^2 + \lambda_1(R_o\ I_3\ \overline{\phi}_1 + \lambda_1 F) + (R_o(I_1 + I_2)\overline{U}_1 + 1).$$
$$(2\ R_o\ I_2\ \overline{U}_1 + 1)\} = 0 \tag{6.34}$$

One of the eigenvalues (ω_R) is equal to zero. We identify this solution with the slowly moving Rossby mode. In fact in this case the Rossby mode is stationary. The other two eigenvalues ($\pm\ \omega_g$) correspond to the gravity mode frequencies. They are given by

$$\omega_g = i\ (1 + \lambda_1^2\ F + R_o\ (....))^{\frac{1}{2}} \tag{6.35}$$

For small values of R_o, which is mostly the case in the atmosphere, we may neglect the last term under the square root sign. The fact that ω_g is imaginary shows that we are dealing with oscillating solutions. The eigenvectors corresponding to these eigenvalues can also be found. We will do this for the state of rest i.e. when $\overline{u}_1 = 0$ and $\overline{\phi}_1 = 0$. System (6.33) in matrix form then becomes

$$\delta \begin{vmatrix} \dot{u}_1 \\ \dot{v}_1 \\ \dot{\phi}_1 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & \lambda_1 \\ 0 & -\lambda_1 F & 0 \end{vmatrix} \delta \begin{vmatrix} u_1 \\ v_1 \\ \phi_1 \end{vmatrix} \tag{6.36}$$

The eigenvectors are:

$$\text{For } \omega = \omega_R = 0 \ : \ \underset{\sim}{e}_R = \begin{vmatrix} 1 \\ 0 \\ -\lambda_1{}^{-1} \end{vmatrix}$$

$$\text{For } \omega = \pm\ \omega_g \ : \ \underset{\sim}{e}_G = \pm \begin{pmatrix} \omega_g^{-1} \\ 1 \\ \omega_g^{-1}\lambda_1 F \end{pmatrix} \tag{6.37}$$

The eigenvectors corresponding to the gravity modes just oppose each other and are therefore linearly dependent. This is due to the fact that the matrix in eq. (6.36) is singular in that the top and bottom rows are linearly dependent. Physically, this may be interpreted as a constraint on the system; the u_1 and ϕ_1 time evolutions are directly coupled and although we have three dependent variables the model essentially only has two degrees of freedom. If we had include a β-effect in the model, the gravity wave frequencies and eigenvectors would have differed and the Rossby mode would not be stationary. We will return to this case later, but first we want to use the f-plane model to illustrate some basic concepts in nonlinear normal mode initialization.

In a steady state we have a coupling between $\overline{u}_1$ and $\overline{\phi}_1$ given by eq. (6.31) while $\overline{v}_1 = 0$. If we upset this steady-state balance, the system will generally respond with gravity mode oscillations and the frequency is given by (6.35). If, however, we perturb the system in such a way that (6.31) is satisfied and $v_1 = 0$ we will not create any gravity wave motions and the system will just smoothly adjust to a new steady-state balance. When the large scale wind and pressure fields charge in the atmosphere, it is this type of adjustment which takes place. The gravity mode oscillations are insignificant compared to the changes in the Rossby wodes. In this simple model we identify the slowly evolving solution with a gradient wind balance, in the nonlinear primitive equations governing the atmosphere it is impossible to find such a relation, but we may conceptually think of it as a "slow manifold" (Leith, 1981). The slow manifold is thus a subspace of the space spanned by the complete statevector of an atmospheric model. A realistic solution of the model will always keep close to this slow manifold and the initialization problem may be thought of as a way of adjusting the data to ensure that the initial state is on the slow manifold. To find the slow manifold we first have to determine the Rossby and gravity mode eigenvectors for the model linearized around a given state, usually the state of rest. Assuming that the data describes a state which is only a small pertubation from the state of rest, we wish the initial time evolution of the model state vector to be in the direction of the Rossby mode. If the nonlinear effects are small, this can be accomplished by projecting the data onto the respective eigenmodes and subtract that part of the data which projects onto the gravity-modes. This should only cause a slight change in the data. In practice it turns out that this is not a satisfactory method, the nonlinearities are large enough to create spurious gravity mode oscillations which is due to the fact that the eigenmodes are determined for a state of rest, which is too far away from the actual state. It is, however, impossible to linearize around the actually observed state as this is the one we wish to determine.

To illustrate the nonlinear features, let us return to the f-plane model. On the slow manifold we have a relation between ϕ_1 and u_1, and from (6.37) we have the eigenmodes for a state of rest. Due to the linear dependence of the gravity modes, we can define one gravity mode amplitude, G, and one Rossby mode amplitude, R. Any combination of ϕ_1 and u_1 may now be written in terms of these eigenmodes as

$$\begin{pmatrix} u_1 \\ \phi_1 \end{pmatrix} = R.\underset{\sim}{e}_R + G.\underset{\sim}{e}_G \tag{6.38}$$

This defines a nonsingular linear transformation which may be inverted to give

$$R = \frac{\lambda_1}{1 + \lambda_1^2 F} (\lambda_1 F u_1 - \phi_1)$$

$$G = \frac{\omega_g}{1 + \lambda_1^2 F} (u_1 + \lambda_1 \phi_1) \tag{6.39}$$

Using this transformation we can determine the relation between R and G for any state which is in a gradient wind balance (note that we have disregarded the condition $v_1 = 0$ in the gradient wind balanced state; this we are forced to do because of the singularity in the eigenmode representation).

The steady state gradient wind balance curve in the R-G plane has been drawn in fig. 42, and this is the "slow manifold" of an model. The problem of initialization is now how to change the data so as to minimize the spurious gravity mode oscillations and find a solution which evolves smoothly in time on the slow manifold.

A slowly evolving solution may still contain gravity mode contributions, as the separation into the eigenmodes is only valid close to a state of rest. Given a data point (see fig. 7) we can now proceed in two different ways:

1. Linear mode initialization.

here we project the data on the Rossby mode. Therefore G = 0, but this is not on the gradient wind balance curve. Gravity waves can develop during the integration and the solution will display large amplitude oscillations.

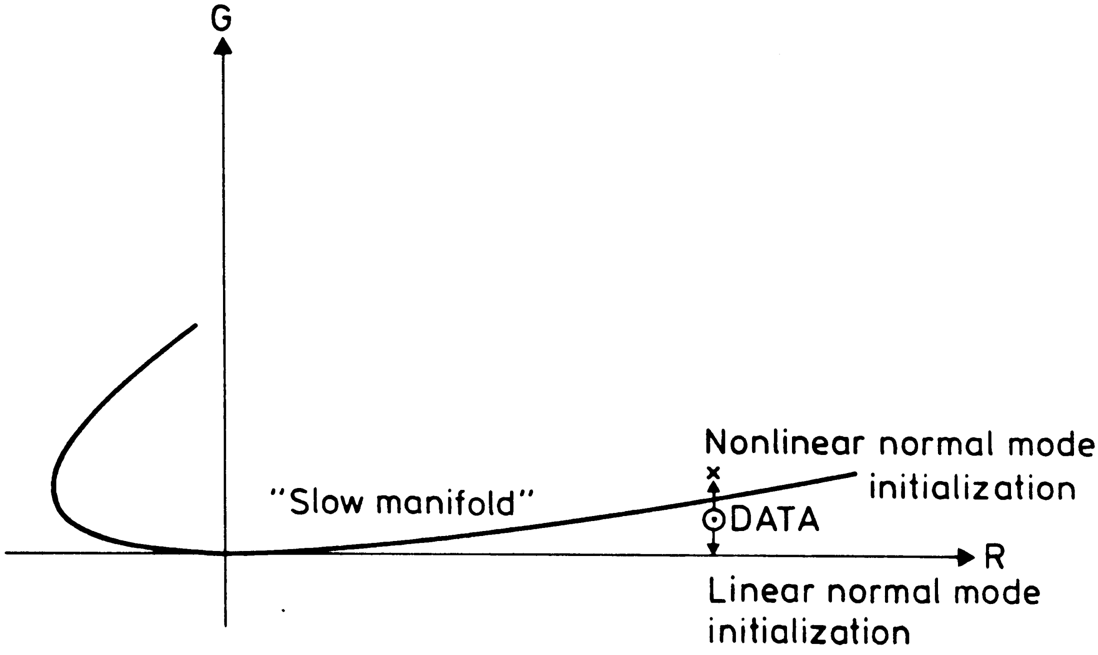

Fig. 42. Gradient wind balance curve in the R-G plane ("slow manifold") and a sketch to demonstrate different initialization methods (see text).

2. Nonlinear normal mode initialization (Machenhauer, 1977).
Here we keep the Rossby mode constant and choose the gravity modes in such a way that $\dot{G} = 0$. (see fig. 41).
From eqs. (6.39) and (6.27-6.29) we find

$$\dot{G} = -(\omega_g + \frac{R_o}{1 + \lambda_1^2 F}(I_3 \phi_1 + (I_1 + I_2)u_1))v_1 \quad (6.40)$$

Setting $\dot{G} = 0$ in this model thus implies $v_1 = 0$ or a linear relation between u_1 an ϕ_1. The first condition implies a gradient wind balance, but as we are considering relations between u_1, and ϕ_1 (disregarding v_1) only the second condition is relevant. Through the transformation (6.39) we can rewrite the linear relation between u_1 and ϕ_1 in terms of R and G. This is a straight line in the R-G diagram of fig. 7 and for the data point shown in fig. 42 we find the initialized state close to, but not exactly on, the slow manifold. This is a general property of the Machen-

hauer initialization scheme, but it gives an initial state which is sufficiently balanced to give a smoothly evolving solution in a primitive equation model of the atmosphere.
Leith (1982) showed that in a more general model this method, to first order, leads to a quasi-geostrophic balance.

By the Machenhauer initialization method we can thus find an initial state which is almost balanced, and only some small spurious gravity wave oscillation will occur. If we include a dissipation term in the model the gravity wave oscillations will damp out after some time. With a more rigourous analysis it is possible to show that the gravity mode oscillations can be eliminated to any desired degree of accuracy (Baer and Tribbia, 1977). Through expanding the solutions in terms of a small parameter, the Rossby number, and furthermore by defining a slow and a fast time scale, Baer and Tribbia (1977) demonstrate how the fast time scale variations can be eliminated to an accuracy given by the order of the terms included in the series expansion. In practice, it has however, been found that the Machenhauer method is sufficient to balance the data in midlatitudes. The Machenhauer method is equivalent to an expansion of the Baer and Tribbia type, but truncated at the first order.

Rossby and gravity modes on a β-plane .

On the β-plane, the arithmetic becomes much more complicated and it becomes difficult to illustrate the slow manifold concept in a simple way. In principle the same conclusions may be reached concerning the nonlinear effect but another interesting problem is to see how well the Rossby mode type of solutions describe quasi-geostrophic motion. This problem is covered in detail in the ECMWF Lecture Note no. 1 by A. Wiin-Nielsen (1979). The final part of this lecture series addresses this question and as the lectures closely follow the Lecture Note by Wiin-Nielsen the interested reader is referred to that publication.

Excercises

14) Determine the steady-states where $V_1 \neq 0$ for the low order equation system derived from the shallow water equations

$$\dot{U}_1 = - V_1 - R_o (I_1 + I_2) U_1 V_1$$

$$\dot{V}_1 = U_1 + \lambda\Phi_1 - R_o (I_1 V_1^2 - I_2 U_1^2)$$

$$\dot{\Phi}_1 = - \lambda F V_1 - R_o I_3 \Phi_1 V_1$$

Wat is the stability of this (these) steady-state(s)?

Appendix A

Conservation of energy and enstrophy in truncated spectral models of the barotropic vorticity equation.

by Wim Verkley

We work in the complex Hilbert space of quadratically integrable functions ψ on $[0,2\pi] \times [-1,1]$ into the complex numbers, satisfying the following periodicity conditions:

> ψ and all its derivatives are the same at the points $(0,\mu)$ and $(2\pi,\mu)$ for every μ in $[-1,1]$. Additionally $\frac{\partial\psi}{\partial\lambda} = 0$ on the boundaries at $\mu = 1$ and $\mu = -1$.

In H we define the Hermitian product

$$< \psi,\phi > \equiv \frac{1}{4\pi} \int \overline{\psi}\phi dS = \frac{1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} \overline{\psi}(\lambda,\mu)\phi(\lambda,\mu)d\lambda d\mu \tag{A.1}$$

The overbar denotes complex conjugation.
The barotropic vorticity equation can be written as

$$\nabla^2 \dot{\psi} = f(\psi) \qquad \text{where} \tag{A.2}$$

$$f(\psi) = -J(\psi,\nabla^2\psi) - 2 \frac{\partial\psi}{\partial\lambda} \tag{A.3}$$

The partial derivative with respect to time is denoted by a dot. The expression for the Laplacian ∇^2 reads

$$\nabla^2 = (1-\mu^2)^{-1} \frac{\partial^2}{\partial\lambda^2} - 2\mu \frac{\partial}{\partial\mu} + (1 - \mu^2) \frac{\partial^2}{\partial\mu^2} \tag{A.4}$$

and the Jacobian operator is defined by

$$J(\psi,\phi) \equiv \frac{\partial\psi}{\partial\lambda}\frac{\partial\phi}{\partial\mu} - \frac{\partial\psi}{\partial\mu}\frac{\partial\phi}{\partial\lambda} \tag{A.5}$$

Because of J the operator f is nonlinear.

A particular reduced system can be constructed by decomposing H into two mutually orthogonal subspaces M and $M^{\perp}$ i.e., by writing

$$H = M \oplus M^{\perp} \tag{A.6}$$

This implies that every ψ in H can be written in a unique way as

$$\psi = \chi + \eta \tag{A.7}$$

where

χ is in M, η is in $M^{\perp}$ and $\langle \chi,\eta \rangle = 0$.

The projection operator that projects into M will be denoted by P. This means that, when ψ is written as in (A.7), we have

$$P\psi \equiv \chi \tag{A.8}$$

From the definition of P we conclude that $P\chi = \chi$ as χ is in M. Furthermore, it can be seen that P is self adjoint, i.e.,

$$\langle P\psi,\phi \rangle = \langle \psi,P\phi \rangle \tag{A.9}$$

for ψ and ϕ in H.

Using partial integration and making use of the periodicity conditions one can show that for every ψ and ϕ in H we have

$$\langle \nabla^2\psi,\phi \rangle = \langle \psi,\nabla^2\phi \rangle \tag{A.10}$$

$$\langle J(\psi,\phi),\phi \rangle = - \langle J(\psi,\overline{\phi}),\overline{\phi} \rangle \tag{A.11}$$

$$\langle \frac{\partial\psi}{\partial\lambda} , \phi \rangle = - \langle \psi, \frac{\partial\phi}{\partial\lambda} \rangle \tag{A.12}$$

Equation (A.10) says that ∇^2 is also a self adjoint operator. We now assume that M is chosen in such a way that the projection operator P commutes with ∇^2, i.e.,

$$\nabla^2 P = P\nabla^2 \tag{A.13}$$

which is equivalent to the assumption that ∇^2 leaves the spaces M and $M^{\perp}$ invariant.

Furthermore, we will assume that our functions are real. For real functions ψ and ϕ in H we have

$$< \phi,\psi > = < \psi,\phi > \tag{A.14}$$

$$< J(\psi,\phi),\psi > = 0 \tag{A.15}$$

From equation (A.12) and (A.15) we deduce that for every real function ψ in H we have

$$< \psi, f(\psi) > = 0 \tag{A.16}$$

$$< \nabla^2\psi, f(\psi) > = 0 \tag{A.17}$$

The truncated form of the barotropic vorticity equation reads

$$\dot{\nabla^2}\chi = Pf(\chi) \tag{A.18}$$

where χ is an element of M. The energy and enstrophy associated with this system are (E_M and $Z_M \geq 0$).

$$E_M \equiv -\tfrac{1}{2} < \chi,\nabla^2\chi > \tag{A.19}$$

$$Z_M \equiv < \nabla^2\chi,\nabla^2\chi > \tag{A.20}$$

and for the time derivative of E_M and Z_M we have

$$\dot{E}_M = - < \chi,\dot{\nabla^2}\chi > \tag{A.21}$$

$$\dot{Z}_M = 2 < \nabla^2\chi,\dot{\nabla^2}\chi > \tag{A.22}$$

We can now easily see why the energy and enstrophy are conserved in our truncated system. Indeed,

$$\dot{E}_M = - < \chi, \nabla^2 \dot{\chi} > = - < \chi, Pf(\chi) > = - < P\chi, f(\chi) > = \\ = - < \chi, f(\chi) > = 0. \tag{A.23}$$

and

$$\dot{Z}_M = 2 < \nabla^2 \chi, \nabla^2 \dot{\chi} > = 2 < \nabla^2 \chi, Pf(\chi) > = \\ = 2 < P\nabla^2 \chi, f(\chi) > = 2 < \nabla^2 P\chi, f(\chi) > = \\ = 2 < \nabla^2 \chi, f(\chi) > = 0. \tag{A.24}$$

Appendix B

Legendre polynomials

There are many different approaches to the derivation of Legendre polynomials. We shall here introduce them by solving an eigenvalue problem of the form:

$$\nabla^2 y = ky \tag{B.1}$$

on a spherical gecmetry. The Laplacian in spherical coordinates is given by

$$\nabla^2 = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial}{\partial\phi}\right) + \frac{1}{\cos^2\phi}\frac{\partial^2}{\partial\lambda^2}\right] \tag{B.2}$$

We solve the problem in two dimensions on a sphere with unit radius. Consequently the first term on the r.h.s. of eq. (B.2) disappears. In order to separate the variables we assume a particular solution of the form

$$y = G(\phi)H(\lambda)$$

and obtain

$$\frac{H}{\cos\phi}\frac{d}{d\phi}\left(\cos\phi\frac{dG}{d\phi}\right) + \frac{G}{\cos^2\phi}\frac{d^2H}{d\lambda^2} - k\,G\,H = 0$$

Multiplying by $\cos^2\phi$ and dividing by GH we find

$$\frac{\cos\phi}{G}\frac{d}{d\phi}\left(\cos\phi\frac{dG}{d\phi}\right) + \frac{1}{H}\frac{d^2H}{d\lambda^2} - k\cos^2\phi = 0$$

We have one part of the equation depending on ϕ and one part depending on λ. Therefore the equation may be split up into

$$\frac{1}{H}\frac{d^2H}{d\lambda^2} = -\ell^2 \tag{B3}$$

$$\frac{\cos\phi}{G}\frac{d}{d\phi}\left(\cos\phi\frac{dG}{d\phi}\right) - k\cos^2\phi = \ell^2 \tag{B4}$$

where ℓ is a separation variable. Eq. (B.3) is easily solved

$$\frac{d^2H}{d\lambda^2} + \ell^2 H = 0 \quad \leftrightarrow \quad H = A\, e^{\pm i\lambda\ell}$$

As $H(\lambda)$ must satisfy the condition $H(o) = H(2\pi)$, ℓ must be an integer. Equation (B.4) can be written

$$\frac{\cos^2\phi}{G}\frac{d^2G}{d\phi^2} + \frac{\sin\phi\cos\phi}{G}\frac{dG}{d\phi} - k\cos^2\phi = \ell^2$$

We introduce $\mu = \sin\phi$ in the equation and obtain

$$(1-\mu^2)\frac{d^2G}{d\mu^2} - 2\mu\frac{dG}{d\mu} + [-k - \frac{\ell^2}{1-\mu^2}]G = 0$$

or with the eigenvalue k chosen as $k = -n(n+1)$

$$(1-\mu^2)\frac{d^2G}{d\mu^2} - 2\mu\frac{dG}{d\mu} + [n(n+1) - \frac{\ell^2}{1-\mu^2}]G = 0 \qquad \text{(B.5)}$$

This is the so called associated Legendre equation. With $\ell = 0$ the equation reduces to

$$(1-\mu^2)\frac{d^2G}{d\mu^2} - 2\mu\frac{dG}{d\mu} + n(n+1)G = 0 \qquad \text{(B.6)}$$

This is the Legendre equation. We shall solve this equation first and later we shall derive a relation between the solution of this equation and the solutions of the associated Legendre equation. We assume a solution $P_n(\mu)$ in the form of a power series

$$G = P_n(\mu) = \sum_{k=o}^{\infty} a_k\,\mu^k$$

with

$$\frac{dG}{d\mu} = \sum_{k=1}^{\infty} k\,a_k\,\mu^{k-1}$$

$$\frac{d^2G}{d\mu^2} = \sum_{k=2}^{\infty} k(k-1)\,a_k\,\mu^{k-2}$$

Inserting this into equation (B.6) we obtain

$$\sum_{k=2}^{\infty} k(k-1)a_k\,\mu^{k-2} - \sum_{k=2}^{\infty} k(k-1)a_k\,\mu^k - 2\sum_{k=1}^{\infty} k\,a_k\,\mu^k + n(n+1)\sum_{k=o}^{\infty} a_k\,\mu^k = 0$$

or rewritten

$$\sum_{k=o}^{\infty}[(k+2)(k-1)a_{k+2}\ \mu^{k-2} - k(k-1)a_k\ \mu^k - 2k\ a_k\ \mu^k + n(n+1)a_k\ \mu^k] = 0$$

Equating equal powers of k we have

$$(k+2)(k+1)a_{k+2} - k(k-1)a_k - 2k\ a_k + n(n+1)a_k = 0.$$

and we can obtain a recursive relation for the a_k's

$$a_{k+2} = \frac{k(k+1)\ -n(n+1)}{(k+2)(k+1)}\ a_k$$

Now you can see why we have chosen $n(n+1)$ as the eigenvalue because the coefficients a_k will be zero for $k = n+2$ and thus also for $k = n+4$, etc. The series is completely determined when we have chosen $a_o = 0$ and $a_1 = 1$ (if n is odd) or $a_o = 1$ and $a_1 = 0$ (if n is even) and the series will be polynomials with a finite number of terms.

By using the recursive relation inductively we can write the solution of the Legendre equation as

$$G(\mu) = P_n(\mu) = \sum_{j=0}^{n} \frac{(-1)^j(2n-2j)!}{2^n\ j!(n-2j)!(n-j)!}\ \mu^{n-2j}$$

Another form for the coefficients may be obtained by developing the derivative

$$\frac{d^n}{d\mu^n}(\mu^2-1)^n = \frac{d^n}{d\mu^n}\sum_{j=o}^{\infty}\frac{n!}{j!(n-j)!}\ \mu^{2n-2j}$$

$$= \sum_{j=o}^{n}(-1)^j\frac{n!(2n-2j)!}{j!(n-j)!(n-2j)!}\ \mu^{n-2j}$$

We see then that $P_n(\mu)$ can be written as

$$P_n(\mu) = \frac{1}{2^n n!}\frac{d^n}{d\mu^n}(\mu^2-1)^n$$

If we take the longitudinal dependence of the equations into account we have to take $\ell \neq 0$.

We shall now show, that a particular solution of this equation is the polynomial

$$P_n^\ell(\mu) = (1-\mu^2)^{\frac{\ell}{2}} \frac{d^\ell}{d\mu^\ell} P_n(\mu) \tag{B.7}$$

We introduce the variable transformation

$$y = (1-\mu^2)^{\frac{\ell}{2}} \; z \tag{B.8}$$

and arrive at the equation

$$(1-\mu^2) \frac{d^2 z}{d\mu^2} - 2(\ell+1) \frac{dz}{d\mu} + [n(n+1) - \ell(\ell+1)]z = 0 \tag{B.9}$$

The Legendre equation is

$$(1-\mu^2) \frac{d^2 P_n}{d\mu^2} - 2\mu \frac{d\,P_n}{d\mu} + n(n+1)P_n = 0$$

differentiating this equation ℓ times with respect to μ, we obtain

$$(1-\mu^2) \frac{d^2}{d\mu^2} \left(\frac{d^\ell P_n}{d\mu^\ell}\right) - 2(\ell+1) \frac{d}{d\mu} \left(\frac{d^\ell P_n}{d\mu^\ell}\right) + [n(n+1)-\ell(\ell+1)] \frac{d^\ell P_n}{d\mu^\ell} = 0 \tag{B.10}$$

Comparing (B.9) and (B.10) we see

$$z = \frac{d^2 P_n}{d\mu^2}$$

From (B.8) it follows that

$$P_n^\ell(\mu) = (1-\mu^2)^{\frac{\ell}{2}} \frac{d^\ell P_n}{d\mu^\ell}$$

or

$$P_n^\ell = \frac{(1-\mu^2)^{\frac{\ell}{2}}}{2^n \; n!} \frac{d^{\ell+n}}{d\mu^{\ell+n}} (\mu^2-1)^n$$

We will now prove the orthogonality of the associated Legendre polynomials. If the zonal wavenumbers (ℓ) differ, the functions $Y_{n_1}^{\ell_1}$ and $Y_{n_2}^{\ell_2}$ will be orthogonal due to the longitudinally dependent part. We thus only have to consider the case of associated Legendre polynomials of equal ℓ's but different orders n.

Let the Legendre polynomials Y_n^ℓ and Y_k^ℓ be solutions of the associated

Legendre equations

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\,Y_n^\ell}{d\mu}\right] + [n(n+1) - \ell(\ell+1)]Y_n^\ell = 0$$

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\,Y_k^\ell}{d\mu}\right] + [k(k+1) - \ell(\ell+1)]Y_k^\ell = 0$$

Multiplying the first equation by Y_k^ℓ, the second by Y_n^ℓ and taking the difference, we obtain

$$\frac{d}{d\mu}\left[(1-\mu^2)\left(Y_k^\ell \frac{d\,Y_n^\ell}{d\mu} - Y_n^\ell \frac{d\,Y_k^\ell}{d\mu}\right)\right] = [n(n+1)-k(k+1)]Y_n^\ell\, Y_k^\ell \qquad \text{(B.11)}$$

Integration of this relation between -1 and +1 yields

$$[n(n+1)-k(k+1)] \int_{-1}^{+1} Y_n^\ell\, Y_k^\ell\, d\mu = 0,$$

since the integral on the left hand side of (B.11) vanishes for $\mu = \pm 1$. If $n \neq k$ the integral thus has to be zero and orthogonality is proven.

The normalisation factor N for the associated Legendre polynomials can be found by solving

$$\int_{-1}^{+1} [P_n^\ell(\mu)]^2\, d\mu = N^{-2}$$

Defining $1-\mu^2 = X$ the integral can be written

$$\frac{(-1)^\ell}{2^{n+k} n!\; k!} \int_{-1}^{+1} X^\ell \frac{d^{\ell+n}}{d\mu^{\ell+n}} X^n \frac{d^{\ell+k}}{d\mu^{\ell+k}} X^k\, d\mu$$

We integrate $(k+\ell)$ times by partial integration to obtain

$$\int_{-1}^{+1} P_n^\ell(\mu)P_k^\ell(\mu)d\mu = \frac{(-1)^\ell(-1)^{k+\ell}}{2^{n+k}\; n!\; k!} \int_{-1}^{+1} \frac{d^{k+\ell}}{d\mu^{k+\ell}}\left(X^\ell \frac{d^{n+\ell}}{d\mu^{n+\ell}} X^n\right)X^k\, d\mu$$

The integral on the r.h.s. is now expanded by Leibnitz' formula to give

$$X^k \frac{d^{k+\ell}}{d\mu^{k+\ell}}\left(X^\ell \frac{d^{n+\ell}}{d\mu^{n+\ell}} X^n\right) = X^k \sum_{i=0}^{k+\ell} \frac{(k+\ell)!}{i!(k+\ell-i)!} \frac{d^{k+\ell-i}}{d\mu^{k+\ell-i}} X^\ell \frac{d^{n+\ell+i}}{d\mu^{n+\ell+i}} X^n$$

Since the sum X^ℓ contains no power of μ greater than $\mu^{2\ell}$ we must have

$k+\ell-i \leq 2\ell$

or the derivative will vanish.

Similarly $n+\ell+i \leq 2n$.

In the solution of these equation for the index i, the conditions for a non zero result are

$$i \geq k-\ell \quad , \quad i \leq n-\ell$$

We obtain

$$\int_{-1}^{+1} [P_k^\ell(\mu)]^2 \, d\mu = \frac{(-1)^{k+2\ell}(k+\ell)!}{2^{2k}k!k!(2\ell)!(k-\ell)!} \int_{-1}^{+1} X^k \left(\frac{d^{2\ell}}{d\mu^{2\ell}} X^\ell\right)\left(\frac{d^{2k}}{d\mu^{2k}} X^\ell\right) d\mu$$

Since

$$X^\ell = (\mu^2-1)^\ell = \mu^{2\ell} - \ell\mu^{2\ell-2} + \ldots.$$

$$\frac{d^{2\ell}}{d\mu^{2\ell}} X^\ell = (2\ell)!$$

the equation reduces to

$$\int_{-1}^{+1} [P_k^\ell(\mu)^2 d\mu = \frac{(-1)^{k+2\ell}(2k)!(k+\ell)!}{2^{2k}k!k!(k-\ell)!} \int_{-1}^{+1} X^k \, d\mu$$

with

$$(-1)^k \int_{-1}^{+1} X^k \, d\mu = \frac{(-1)^k 2^{2k+1} k!k!}{(2\ell+1)!}$$

the equation above becomes

$$\int_{-1}^{+1} [P_k^\ell(\mu)]^2 d\mu = \frac{2}{2k+1} \frac{(k+\ell)!}{(k-\ell)!}$$

The normalized associated Legendre polynomials are

$$\overline{P_n^\ell} (\mu) = [(2n+1) \frac{(n-\ell)!}{(n+\ell)!}]^{\frac{1}{2}} \frac{(1-\mu^2)^{\frac{\ell}{2}}}{2^n \; n!} \frac{d^{n+\ell}}{d\mu^{n+\ell}} (\mu^2-1)^n$$

and

$$\frac{1}{2} \int_{-1}^{+1} [\overline{P}_n^{\ell}(\mu)]^2 \, d\mu = 1$$

From the normalized associated Legendre polynomials we find

$$\overline{P}_n^{-\ell}(\mu) = (-1)^{\ell} \, \overline{P}_n^{\ell}(\mu)$$

The factor $(-1)^{\ell}$ disturbs symmetry and therefore we restrict the above formula to $\ell > 0$ and define

$$\overline{P}_n^{\ell}(\mu) \equiv \overline{P}_n^{-\ell}(\mu)$$

The normalization of the longitudinally dependent solution of the eigenvalue problem is given by

$$\int_0^{2\pi} (HH^*) d\lambda = \int_0^{2\pi} e^{i\ell\lambda} \, e^{-i\ell\lambda} \, d\lambda = 2\pi$$

with $H = e^{i\ell\lambda}$.

The total solution of the eigenvalue problem is

$$Y_{\gamma} = P_n^{\ell}(\mu) e^{i\ell\lambda} = P_{\gamma}(\mu) e^{i\ell\lambda}$$

with $\gamma = n + i\ell$ and we have $\frac{1}{4\pi} \int Y_{\gamma} \, Y_{\overline{\gamma}} \, ds = 1$

Some examples of the associated Legendre functions are given below (they are unnormalized)

$$P_1^1(\mu) = (1-\mu^2)^{\frac{1}{2}}$$

$$P_2^1(\mu) = 3\mu(1-\mu^2)^{\frac{1}{2}}$$

$$P_2^2(\mu) = 3(1-\mu^2)$$

$$P_3^1(\mu) = \frac{3}{2}(5\mu^2-1)(1-\mu^2)^{\frac{1}{2}}$$

$$P_3^2(\mu) = 15\mu(1-\mu^2)$$

$$P_3^3(\mu) = 15(1-\mu^2)^{3/2}$$

etc.

They have been plotted in figure 43.
In fig. 44 we also show the two dimensional structure of the spherical harmonic function

$$Y_{\gamma}(\lambda,\mu) = \exp(i\ell\lambda)P_n^{\ell}(\mu)$$

for $n = 5$ and $0 < \ell < 5$.

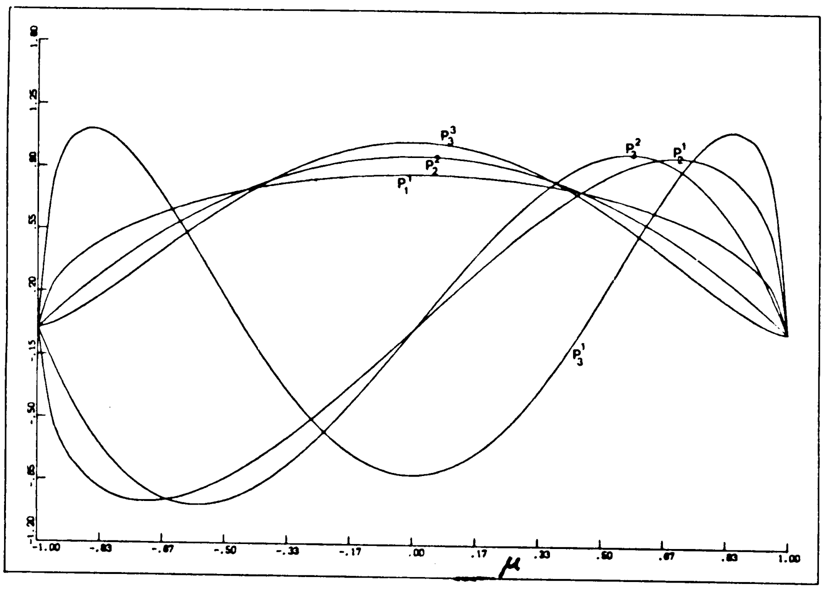

Fig. 43. Normalized associated Legendre functions of 1st kind.

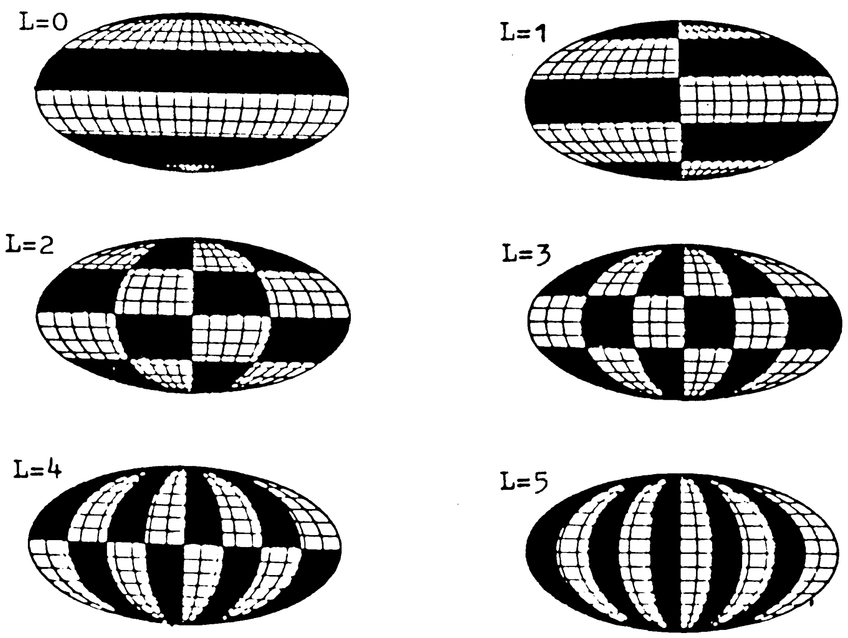

Fig. 44. Example of different cell configurations all having the same two-dimensional index, in this case n = 5. The cells are defined by their nodal lines and are presented on a Mollweide type projection (From Baer, 1972).

References

- Ingemar Holmströn, Partial differential equations with special application to meteorological problems. Sveriges Meteorologiska och Hydrologiska Institut, Stockholm, 1970.
- Enclin and Baer, J.A.S. 29, 1972, p. 649-664.
- George Arfken, Mathematical methods for physicists, 2nd edition, Miami University, Oxford, Ohio.

Solutions to exercises

1. Defining $S = \frac{\partial u}{\partial x}$ and $S_E = \frac{\partial u_E}{\partial x}$ the slope equation is

$$\frac{dS}{dt} = - S^2 - S + S_E$$

Note that this slope equation may only be applied at a point where both $u = 0$ and $u_E = 0$.
With the transformation $v = S + \frac{1}{2}$ and $r = S_E + \frac{1}{4}$ it reduces to

$$\frac{dv}{dt} + v^2 - r = 0$$

which is the required Riccati equation.
The stationary solutions are $v = \pm \sqrt{r}$ or

$$S = - \tfrac{1}{2} \pm \sqrt{S_E + \tfrac{1}{4}}$$

For the time evolution we distinguish between two cases:
1) $r > 0$

$$S = - \tfrac{1}{2} + \sqrt{r}\, \frac{e^{\sqrt{r}t} - C\, e^{-\sqrt{r}t}}{e^{\sqrt{r}t} + C\, e^{-\sqrt{r}t}}$$

2) $r < 0$

$$S = - \tfrac{1}{2} + i\, \sqrt{-r}\, \frac{e^{i\sqrt{-r}t} - C\, e^{-i\sqrt{-r}t}}{e^{i\sqrt{-r}t} + C\, e^{-i\sqrt{-r}t}}$$

For $S(0) = - \frac{1}{2}$ the integration constant $C = 1$.
We thus have for case

1) $S\,(t) = - \frac{1}{2} + \sqrt{r}\, \tanh(t)$

and 2) $S\,(t) = - \frac{1}{2} - \sqrt{r}\, \tan\,(t)$

In case 1) the solution asymptotically approaches $- \frac{1}{2} + \sqrt{r}$ as $t \to \infty$ while in case 2) $S \to - \infty$ as $t \to \frac{\pi}{2}$!

2. Assuming

$$u = \sum_{k=1}^{N} u_k \sin kx$$

we have

$$u \frac{\partial u}{\partial x} = \sum_{k=1}^{N} u_k \sin kx \sum_{\ell=1}^{N} \ell u_\ell \cos kx$$

Projecting on a component n this may be written

$$\sum_{k,\ell=1}^{N} u_k u_\ell \frac{\ell}{2} \frac{\int_0^\pi \sin n\,\xi(\sin(k+\ell)\xi + \sin(k-\ell)\xi)d\xi}{\int_0^\pi \sin^2 n\xi \, d\xi}$$

The integral only gives contributions if $n = k + \ell$ or $n = |k-\ell|$. We thus find (see fig. 4)

$$\tfrac{1}{2} \sum_{\ell=1}^{n-1} \ell \, u_\ell \, u_{n-\ell} + \tfrac{1}{2} \sum_{\ell=1}^{N-n} u_\ell \, u_{n+\ell} - \tfrac{1}{2} \sum_{\ell=n+1}^{N} u_\ell \, u_{\ell-n} =$$

$$= \frac{n}{2} \left(\tfrac{1}{2} \sum_{\ell=1}^{n-1} u_\ell \, u_{n-\ell} - \sum_{\ell=1}^{N-n} u_\ell \, u_{n+\ell}\right)$$

3. Assume energy is conserved up to wavenmber M (< N). For wavenumber M + 1 we have

$$\frac{d\,E_{M+1}}{dt} = \frac{dE_M}{dt} + \sum_{k=1}^{M} u_{M+1-k} \, u_M \, u_k \cdot \frac{k}{2} - \frac{M+1}{4} \sum_{k=1}^{M} u_k u_M u_{M+1-k} =$$

$$= 0 + \frac{u_M}{2} \left(\frac{M+1}{2} \sum_{k=1}^{M} u_{M+1-k} \, u_M \, u_k - \frac{M+1}{2} \sum_{k=1}^{M} u_k \, u_M \, u_{M+1-k}\right) = 0.$$

As we know that energy is conserved for a two component system, it follows by the induction principle that energy is conserved for all truncations.

4. From $\dot{y} = 0$ we have

$$y = y_E - \tfrac{1}{2} x^2$$

Insertion into $\dot{x} = 0$ gives

$$x^3 - 2(y_E - 2)x - 4x_E = 0.$$

Rewriting this as

$$x_E = \tfrac{1}{4}(x^3 - 2(y_E - 2)x)$$

We can plot it in (x, x_E, y_E) - space, see figure.

5. i a) $\omega = c_o K$ no dispersion

b) $\omega = c_o K - iK^2$ dissipative waves $\sim \exp[iK(x-c_o t)]e^{-K^2}$: damping

c) $\omega = c_o K - K^3$ dispersive waves $\sim \exp[iK(x - (c_o - K^2)t)]$: spreading

ii a) $K(x) = c_o\ \delta(x)$

b) $K(x) = c_o\delta(x) + \delta'(x)$ where $\int_{-\infty}^{\infty} \delta'(x-\xi)\eta_\xi d\xi \overset{P.I}{=} \delta\ \eta_\xi - \int_{-\infty}^{\infty} \delta(x-\xi)\eta_{\xi\xi}d\xi$

$$= -\eta_{xx}$$

c) $K(x) = c_o\ \delta(x) + \delta''(x)$

6. i $\frac{d\phi}{dt} = 0 \qquad \phi(o) = \frac{1}{1+x^2} \qquad (1)$

$\frac{dx}{dt} = 0 \qquad x(o) = \xi \qquad (2)$

From (1) $\phi(t) = \phi(o) = \frac{1}{1+\xi^2} = f(\xi)$

From (2) $x(t) = x(o) + 1 - e^{-t} = \xi + 1 - e^{-t}$, or inverting

$\xi = x - 1 + e^{-t}$.

Substition in $\phi = f(\xi) = \frac{1}{1+\xi^2}$ yields

$$\phi(x,t) = \frac{1}{1 + (x - 1 + e^{-t})^2}$$

ii $\phi(x,t) = \Phi(t-1/3x^3)\exp[-x(t-\frac{1}{4}x^3)]$ analogous to i).

iii $\frac{d\phi}{dt} = -\alpha\phi \quad \phi(0) = F(x) = F(\xi) \quad (1)$

$\frac{dx}{dt} = \phi \quad x(o) = \xi \quad (2)$

From (1) $\phi(t) = \phi(o)e^{-\alpha t} = F(\xi)e^{-\alpha t}$

From (2) $x(t) = x(o) + \int \phi \, dt = \xi - \frac{1}{\alpha} F(\xi) \, (e^{-\alpha t} - 1)$

For given profiles $F(\xi)$ we may find $\xi = \xi(x,t)$ and insert in $\phi = F(\xi)e^{-\alpha t}$

Breaking occurs for slopes $\phi_x \to \infty$ (see figure)

Fig. 45. Onset of breaking.

$\phi_x = e^{-\alpha t} \, F'(\xi)\xi_x$

ξ_x may be found
from $x = \xi - \frac{1}{\alpha} F(\xi) \, . \, [e^{-\alpha t} - 1]$, differentiating

to x yields $1 = \xi_x[1 - \frac{1}{\alpha} F'(\xi)(e^{-\alpha t} - 1)]$

So $\phi_x = \dfrac{F'(\xi)e^{-\alpha t}}{1 - \frac{F'(\xi)}{\alpha}(e^{-\alpha t} - 1)}$ $\quad \phi_x \to \infty$ if denominator $\to 0$.
or if $e^{-\alpha t} - 1 = \frac{\alpha}{F'}$ so $t = -\frac{1}{\alpha} \ln(\dfrac{1}{1 + \frac{\alpha}{F'}})$

For $F' > 0$, $1 + \frac{\alpha}{F'} > 1$ so $\ln(\dfrac{1}{1 + \frac{\alpha}{F'}}) < 0$.

Therefore there will be no breaking solution for a positive instant. If $F' < 0$ there will also not occur breaking when $1 + \frac{\alpha}{F'} < 0$ since then the breaking time will be complex.

So no breaking if $1 - \frac{\alpha}{(-F')} < 0$, or $(-F') < \alpha$: This means that breaking will occur if initially at some place

$$-F' > \alpha$$

i.e if the slope is strong enough.

7. The equation of order

$\alpha^1: (-\omega + c_o K)\zeta_1' + \gamma K^3 \zeta_1''' = 0$ (1)

$\alpha^2: (-\omega + c_o K)\zeta_2' + \gamma K^3 \zeta_2''' = -\frac{3}{2} c_o K \zeta_1 \zeta_1'$ (2)

$\alpha^3: (-\omega + c_o K)\zeta_3' + \gamma K^3 \zeta_3''' = -\frac{3}{2} c_o K(\zeta_1 \zeta_2)'$ (3)

From (1) we have $\zeta_1 = A_1 \cos\theta$ with $\omega = c_o K - \gamma K^3 (\equiv \omega_o(K))$.
The right hand side (RHS) of (2) $\sim \sin 2\theta$ is suggesting

$\zeta_2 = A_2 \cos 2\theta.$

This yields $A_2 = \frac{1}{8} \frac{c_o}{\gamma K^2} A_1^2$. Inserting ζ_1 and ζ_2 in the RHS of (3) gives

$$- \frac{3}{4} c_o K A_2 A_1 (\cos 3\theta + \cos \theta)'$$

The sin 3θ term is balanced by a cos 3θ solution, but the sin θ-term is not balanced by a cos θ solution, at least not when we assume $\omega = \omega_o$, since this is a fundamental solution of the first equation. Therefore we have to expand

$$\omega = \omega_o(K) + \omega_2(K)$$

Now assuming $\zeta_3 = A_3 \cos 3\theta + \cos \theta$ we obtain:

$$A_3[3(\omega_o - c_o K) + 27 \gamma K^3 + 3\omega_2] \sin 3\theta +$$

$$[-\omega_o + c_o K + \gamma K^3] \sin \theta + \omega_2 \sin \theta = \frac{3}{4} c_o K A_2 A_1 (3 \sin 3\theta + \sin \theta)$$

With the definition of ω_o we see that the first coefficient in front of sin θ on the LHS vanishes. Therefore the RHS is balanced if we choose

$$\omega_2 = \frac{3}{4} c_o K A_2 A_1 = \frac{3}{32} \frac{c_o^2}{\gamma K} A_1^3$$

The amplitude for A_3 then follows to be

$$A_3 = \frac{1}{1 + \frac{248 \gamma^2 K^4}{3 A_1^3 c_o^2}}$$

8. By partial integration we find

$$K_{\gamma,\beta,\alpha} = \int_{-1}^{1} P_\gamma (\ell_\beta P_\beta P'_\alpha - \ell_\alpha P_\alpha P'_\beta) d\mu = \int_{-1}^{1} - P'_\alpha \ell_\beta (P_\gamma P_\beta)' -$$

$$(\ell_\gamma - \ell_\beta) P_\gamma P_\alpha P'_\beta \, d\mu = \int_{-1}^{1} P_\alpha (- \ell_\beta P_\beta P'_\gamma - \ell_\gamma P_\gamma P'_\beta) d\mu = K_{\alpha,\overline{\beta},\gamma}$$

We also have

$$K_{\gamma,\beta,\alpha} = \int_{-1}^{1} P_\beta(\ell_\gamma - \ell_\alpha)P_\gamma P'_\alpha - P_\beta \ell_\alpha (P_\gamma P_\alpha)' d\mu =$$

$$= \int_{-1}^{1} P_\beta(\ell_\gamma P_\gamma P'_\alpha - (-\ell_\alpha)P_\alpha P'_\gamma) d\mu = K_{\beta,\gamma,\overline{\alpha}}$$

9. From the definition of the streamfunction (taking nondimensionalization into account) we have

$$u = - a\ \Omega \cos\phi \frac{\partial\psi}{\partial\mu} \text{ where } \mu = \sin\phi$$

For stationarity we have from eq. (4.45)

$$\zeta_1 = \frac{4}{\sqrt{3(n(n+1)-2)}} = \frac{1}{\sqrt{3}\ 4.5}$$

We also know that $\zeta_1 = -3\ \psi_1$, and $P_1 = \sqrt{3}\mu$.

We thus find $u = 24.3\ ms^{-1}$.

10. The transformation gives

$$\frac{dz}{dt} = - 2\ AI_{\gamma,\overline{\alpha},\beta}\ y$$

$$\frac{dx}{dt} = -\ell\omega_\beta\ y - zy\ I_{\beta,\beta,\gamma}$$

$$\frac{dy}{dt} = \ell\omega_\beta\ x + Az\ I_{\beta,\alpha,\gamma} + zx\ I_{\beta,\beta,\gamma}$$

Linearized around the steady state z = x = y = 0 we have

$$\delta \begin{pmatrix} \dot{z} \\ x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 0 & -2A\ I_{\gamma,\overline{\alpha},\beta} \\ 0 & 0 & -\ell\omega_\beta \\ A\ I_{\beta,\alpha,\gamma} & \ell\omega_\beta & 0 \end{pmatrix} \cdot \delta \begin{pmatrix} z \\ x \\ y \end{pmatrix}$$

The eigenvalue equation is (eigenvalue λ)

$$\lambda(\lambda^2 + \ell^2\omega_\beta^2 + 2A^2\, I_{\gamma,\bar{\alpha},\beta}\, I_{\beta,\alpha,\gamma}) = 0.$$

For one eigenvalue to have a positive real part we must thus have

$$\ell^2\omega_\beta^2 + 2A^2\, I_{\gamma,\bar{\alpha},\beta}\, I_{\beta,\alpha,\gamma} < 0.$$

We also know that

$$I_{\gamma,\bar{\alpha},\beta} = -\frac{(c_\alpha - c_\beta)}{(c_\gamma - c_\alpha)}\, I_{\beta,\alpha,\gamma}$$

a. The instability condition may thus be written

$$A > \frac{\ell\,|\omega_\beta|}{|I_{\beta,\alpha,\gamma}|}\sqrt{\frac{(c_\gamma - c_\alpha)}{2(c_\alpha - c_\beta)}}$$

b. The addition of dissipative terms on the right hand sides of the governing equations implies an eigenvalue transformation $\lambda \to \lambda + \varepsilon$. The instability condition is thus

$$-\varepsilon + \sqrt{2A^2\frac{(c_\alpha - c_\beta)}{(c_\gamma - c_\alpha)}\, I^2_{\beta,\alpha,\gamma} - \ell^2\omega_\beta^2} > 0 \qquad \Longleftrightarrow$$

$$A > \frac{1}{|I_{\beta,\alpha,\gamma}|}\sqrt{\frac{(c_\gamma - c_\alpha)}{2(c_\alpha - c_\beta)}\,(\varepsilon^2 + \ell^2\omega_\beta^2)}$$

11. The transformed system is given by eq. (4.108).

a. The eigenvalue equation is

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_o = 0$$

where (overbar denotes steady-state)

$$a_2 = 3\,\varepsilon$$

$$a_1 = 3\varepsilon^2 + (\beta - \alpha\bar{u})^2 + \frac{u^* - \bar{u}}{\bar{u}}\,(\varepsilon^2 + \beta^2 - \alpha\beta\,\bar{u})$$

$$a_o = \varepsilon(\varepsilon^3 + (\beta - \alpha\bar{u})^2 + \frac{u^* - \bar{u}}{\bar{u}}(\varepsilon^2 + \beta^2 - \alpha^2 \bar{u}^2)$$

b. The condition for a Hopf-bifurcation is that the real part of an eigenvalue, with a nonzero imaginary part, changes sign. We thus want to find purely imaginary eigenvalues $\lambda = \pm(is)$. From the theory of cubic equations we know that

$$\lambda_1 + \lambda_2 + \lambda_3 = - a_2$$

Assuming $\lambda_{2,3} = \pm(is)$ we thus must have $\lambda_1 = - a_2$. Dividing this root out of the cubic eigenvalue equation we obtain

$$(\lambda + a_2)(\lambda^2 + a_1) + a_o - a_1 a_2 = 0$$

The condition must thus be that

$$a_o - a_1 a_2 = 0 \text{ and } a_1 > 0.$$

We then have $\lambda_1 = - a_2$, $\lambda_{2,3} = \pm i \sqrt{a_1}$.

Using the expressions for the coefficients a_o, a_1 and a_2 it may be shown that this condition cannot be fulfilled when $u^* > u$. The latter relation follows from the steady-state equation (4.113).

12. Both the ordinary Bessel functions $J_p(x)$ and the modified Bessel functions of the second kind $K_p(x)$ satisfy the recursive relation

$$Z'_p(x) = \frac{p}{x} Z_p(x) - Z_{p+1}(x)$$

A simple application of this relation will lead to eq. (5.18).

13. We separate the total solution into a rider (R) and modon (M) part,

$$\psi = \psi_M + \psi_R.$$

a. For the outer region (r > a) we must have the same condition as for the modon, because K_o is also an eigensolution of the Bessel equation. For the inner region we may write

$$\psi = \psi_M + \psi_R = AJ_1(\lambda r)\sin\theta - Cr\sin\theta + EJ_o(\lambda r) + F = \psi_1 + \psi_2 + \psi_3 + \psi_4$$

$$\nabla^2\psi = -\lambda^2(\psi_1+\psi_3)$$

For the Jacobian we thus have

$$J(\psi,\nabla^2\psi) = J((\psi_1+\psi_3) + (\psi_2+\psi_4), -\lambda^2(\psi_1+\psi_3)) =$$

$$-\lambda^2 J((\psi_2+\psi_4), -(\psi_1+\psi_3)) = -\lambda^2 J(\psi_2,(\psi_1+\psi_3)) =$$

$$= -\lambda^2 C\frac{\partial}{\partial x}(\psi_1+\psi_3) = -\lambda^2 C\frac{\partial\psi}{\partial x}$$

The dispersion relation therefore remains unchanged.

b. The continuity conditions are

$$D\,K_o(\mu a) = E\,J_o(\lambda a) + F$$

$$D\,\mu K_o'(\mu a) = E\lambda\,J_o'(\lambda a)$$

Using the relations $K_o' = -K_1$ and $J_o' = -J_1$

we find

$$D = \frac{F}{\mu K_1(\mu a)}\left[\frac{K_o(\mu a)}{\mu K_1(\mu a)} - \frac{J_o(\lambda a)}{\lambda J_1(\lambda a)}\right]$$

$$E = D\,\frac{\mu K_1(\mu a)}{\lambda J_1(\lambda a)}$$

c. For r = 0 we have $\psi_R = E + F$ and for $r \to \infty$, $\psi_R \to 0$. The structure is independent of θ and we thus have a monopole structure with possibly some maxima and minima for r < a (depends on the inner wavenumber). For r > a the function declines to zero.

14. For $V_1 \neq 0$ we have

$$U_1 = \frac{1}{R_o(I_1 + I_2)}$$

$$\phi_1 = -\frac{\lambda F}{R_o I_3}$$

$$V_1 = \pm \sqrt{-\frac{1}{R_o^2}\left(\frac{1}{(I_1 + I_2)^2} + \frac{\lambda^2 F}{I_1 I_3}\right)}$$

at a steady-state. We see that we must require that I_1 and I_3 have opposite signs to make V_1 real.
The eigenvalue equation is (ω is the eigenvalue)

$$-(R_o(I_1 + I_2)V_1 + \omega_1)(2\, R_o\, I_1 V_1 + \omega_2)(R_o I_3 V_1 + \omega_3) = 0.$$

If V_1 is real it can be written $V_1 = \pm\, A$ and either I_1 or I_3 must be positive while the other is negative. This implies that either ω_2 or ω_3 will be positive at the steady-states and they are therefore both unstable.

References

Baer, F. and Tribbia, J.J., 1977. On complete filtering of gravity modes through nonlinear initialization. Mon. Wea. Rev., 105, 1536-1539.

Haltiner, G.J., 1971. Numerical weather prediction. John Wiley and Sons, New York, pp. 317.

Holton, J.R., 1979. An introduction to dynamic meteorology. Academic Press, 2nd edition, pp. 391.

Hofstadter, D.R., 1981. Strange attractors: mathematical patterns delicately poised between order and chaos. Scientific American, November 1981, 16-29.

Hopf, E., 1942. Abzweigung einer periodischen Lösung von einer stationären Lösung eines Differential Systems. Ber. Math. Phys., Kon. Sachs. Akad. Wiss. Leipzig, 94, 1-22.

Hoskins, B.J., 1973. Stability of the Rossby-Haurwitz wave. Quart. J.R. Met. Soc., 99, 723-745.

Källén, E. and Wiin-Nielsen, A.C., 1980. Non-linear, low order interactions. Tellus, 32, 393-409.

Källén, E., 1981. The nonlinear effects of orographic and momentum forcing in a low-order, barotropic model. J. Atmos. Sci., 38, 2150-2163.

Leith, C.E., 1980. Nonlinear normal mode initialization and quasi-geostrophic theory. J. Atmos. Sci., 37, 958-968.

Leith, C.E., 1981. Dynamically stable nonlinear structures. ECMWF seminar 1981 on problems and prospects in long and medium range weather forecasting, published by European Centre for Medium Range Weather Forecasts, Reading, England, pp. 361-370.

Lorenz, E.N., 1963. Deterministic, nonperiodic flow. J. Atmos. Sci., 20, 130-141.

Lorenz, E.N., 1972. Barotropic instability of Rossby wave motion. J. Atmos. Sci., 29, 258-264.

Machenhauer, B., 1977. On the dynamics of gravity oscillations in a shallow water model, with application to normal mode initialization. Beitr. Phys. Atmos., 50, 253-271.

Platzman, G.W., 1960. The spectral form of the vorticity equation. J. Meteor., 17, 635-644.

Platzman, G.W., 1962. the analytical dynamics of the spectral vorticity equation. J. Atmos. Sci., 19, 313-328.

Platzman, G.W., 1964. An exact integral of complete spectral equations for unsteady one-dimensional flow. Tellus, 16, 422-431.

Rossby, C.G., 1937. On the mutual adjustment of pressure and velocity distributions in certain simple current systems, I. J. Mar. Res. 1, 15-28.

Rossby, C.G., 1938. On the mutual adjustment of pressure and velocity distributions in certain simple current systems, II. J. Mar. Res., 1, 239-263.

Seliger, R.L., 1968. A note on the breaking of waves. Proc. Roy Soc. A303, 493-496.

Silberman, I., 1954. Planetary waves in the atmosphere. J. Meteor. 11, 27-34.

Tribbia, J.J., 1981. Nonlinear normal-mode balancing and the ellipticity condition. Mon. Wea. Rev., 109, 1751-1761.

Whitham, G.B., 1974. Linear and Non-linear waves (Ch. 1, 2, 5, 11, 13). Wiley Interscience.

Whitham, G.B., 1979. Lectures on wave propagation (ch. 1). Tata Institute of Fundamental Research.

Wiin-Nielsen, A.C., 1979. Normal mode initialization - a comparative study. Lecture note no. 1 from European Centre for Medium Range Weather Forecasts.