Renormalized Green's function for nonlinear circulation on the β plane

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The nonlinear response of a barotropic quasigeostrophic fluid to a δ forcing by the rotation of the wind stress is discussed in terms of the symmetry properties related to the multipole structure of the response's relative vorticity field and the gradient of planetary vorticity. It is shown that by a global and local renormalization of the β plane, introducing an effective absolute vorticity gradient by means of the global derivatives of the relative vorticity distribution, the three basic symmetry properties, as they are known from numerical simulations, can be explained analytically. This is achieved by means of a renormalized perturbation series which gives the factors that govern the symmetry properties as a function of the parameter measuring the strength of the nonlinearity. These properties are (1) the turning of the symmetry axis for increasing nonlinearity in the direction of the rotation of the wind stress, (2) the concurrent weakening of the symmetry breaking for increasing nonlinearity, and (3) the maximum strength of symmetry breaking around the vorticity dipole axis for intermediate nonlinearity. The first two properties are related to the vorticity distribution's dipole character and the third one to its quadrupole character. The shape of the induced circulation is shown to vary from an oval-shaped pattern with an east-west symmetry axis for weak nonlinearity over a completely asymmetrical swirl for intermediate nonlinearity to a butterfly pattern for the almost-free, strongly nonlinear, inertial mode that is again asymptotically symmetric around the north-south axis.

I. INTRODUCTION

This paper addresses the symmetry properties of a vorticity equation that stands for the simplest nonlinear model of wind-driven mid-ocean gyres. These gyres are characterized by an intense "western intensification" in that current velocities in a small boundary layer on the western side are much higher than in most of the interior.[1] The first successful linear model[2] explaining this property, has shown that this is an internal effect of the dynamics of a fluid forced to circulate horizontally on a rotating sphere; i.e., even for a meridionally symmetric forcing, the fluid reacts with an asymmetrical circulation pattern in a medium in which the horizontal gradient of the local vertical component of the planetary vorticity breaks the symmetry, such that the center of the circulation, being located at the line of maximum forcing (i.e., maximum rotation of the wind stress), is shifted to the west relative to the forcing center (i.e., opposite the direction of the Earth's rotation). Actually observations show that the circulation center is not positioned at the line of maximum rotation of the wind stress. In reality the circulation centers are displaced in the downstream direction, giving rise to a mere "northwestern" intensification. This is generally thought to be a genuine nonlinear effect of gyre dynamics. Interestingly, one of the simplest fully nonlinear models of oceanic circulation, viz., the free (unforced and undamped) mode of a water mass recirculating horizontally on the surface of a rotating sphere exhibits "northern intensification."[3] Rather than having a symmetry axis that runs east-west with strong symmetry breaking round the north-south axis of the circulation, as in the linear model, this mode has a symmetry axis that runs north-south, the symmetry now being broken around the east-west axis of the circulation.

The basic equation that encompasses both asymptotes of the circulation regime is the quasigeostrophic vorticity equation on the β plane:[1]

$$k\Delta\psi+\Im(\psi,\Delta\psi)+\beta_*\frac{\partial\psi}{\partial x_1}=-T\tau(\mathbf{x}) . \quad (1.1)$$

Here ψ is the quasigeostrophic stream function, $\Im$ the Jacobian operator

$$[\Im(a,b)\equiv(\partial a/\partial x_1)(\partial b/\partial x_2)-(\partial a/\partial x_2)(\partial b/\partial x_1)] ,$$

β_* the local planetary vorticity gradient in a plane tangent to the rotating sphere at some arbitrary central latitude, $\mathbf{x}=(x_1,x_2)$ the position vector in that plane, k a damping coefficient to be associated with bottom friction, T the forcing amplitude, and $\tau(\mathbf{x})$ a suitably normalized function representing the shape of the forcing. (Note that in spite of the notation the forcing actually is the rotation of the wind stress at the surface divided by an effective depth of the fluid column.) The minus sign in the right-hand side of (1.1) gives an anticyclonic rotation of the wind stress, as is the case for the gyres on the northern hemisphere.

The two models discussed before describe different dominant balances between the various terms in (1.1). The linear model[2] with its east-west symmetry axis assumes a balance between the forcing in the right-hand side and the first and third terms of the left-hand side, thus a balance between input of relative vorticity ($\Delta\psi$) by the rotation of the wind stress, conservation of planetary vorticity, and damping of relative vorticity by bottom friction. The free fully nonlinear mode[3] with its north-south symmetry axis assumes a balance between the second and third terms in the right-hand side of (1.1), expressing the conservation of absolute vorticity ($\beta_* x_2 + \Delta\psi$) in the absence of forcing and damping [$\Im(\psi, \beta_* x_2 + \Delta\psi) = 0$].

In spite of the simplicity of both asymptotic solutions to Eq. (1) there is no analytical solution known to the full equation, if only in an approximate sense, such that it would unify the linear and "almost free" fully nonlinear regimes and that it would cover the important intermediate regime of "moderate" nonlinearity. There are, however, numerical solutions[4] known, subject to the boundary condition $\psi=0$ at the perimeter of oceanic basins of simple geometry. These solutions show interesting behavior of the symmetry properties of mid-ocean gyres with respect to the degree of nonlinearity of the circulation. These properties are the following.

(a) In a closed basin the broken symmetry manifests itself by the occurrence of a boundary layer with high velocities relative to the interior. In the linear regime this boundary layer occurs on the west side of the basin. For increasing nonlinearity—loosely speaking increasing wind forcing—the boundary layer gradually turns northward; thus the symmetry axis turns in the direction of the rotation of the wind stress, from east-west for very weak forcing to eventually north-south for infinitely large forcing.

(b) Concurrently with the turning of the symmetry axis for increasing nonlinearity, the boundary layer increases in thickness; thus the symmetry breaking decreases in strength.

(c) In the regime of intermediate nonlinearity, when the symmetry axis runs more or less northwest-southeast, the symmetry around the axis actually is also broken. This effect decreases both towards the linear asymptote, when the symmetry axis runs east-west, as towards the fully nonlinear asymptote, when the symmetry axis runs north-south.

In the absence of any analytical theory explaining at least all the mentioned symmetry properties, it seems worthwhile to look for the simplest possible setting that gives an approximate solution to Eq. (1.1) for any degree of nonlinearity. Here we present such a solution, guided by the following statements.

(i) The symmetry properties are more fundamental than the boundary-layer character of the circulation. The latter is just a manifestation of the actual strength of the symmetry breaking which shows up whenever an internal dynamical length scale in the problem—the ratio k/β_* in the linear regime—is much smaller than the external length scale, being the basin width or the length scale of the forcing function. The introduction of an external length scale, however, is not fundamental to the dynamics of forced circulation on the β plane.

(ii) The later is clearly demonstrated by looking at the Green's function for forced circulation on the β plane, as it contains actually *all* symmetry properties of the full nonlinear vorticity equation (1.1). This has already been demonstrated for the linear regime,[5] where the response of the stream function to a δ forcing by the rotation of the wind stress on an unbounded β plane is strikingly asymmetric around the north-south axis and symmetric around the east-west axis. The introduction of a reflecting wall far to the west of the forcing center then produces a boundary layer near the wall, but this boundary layer is just a consequence of the properties of the Green's function for the unbounded plane.

(iii) All the symmetry properties are reflected in the multipole character of the Green's function's relative vorticity field, in which the dynamics of the dipole governs (a) the turning of the symmetry axis for increasing forcing and (b) the weakening of the symmetry breaking for increasing forcing, whereas the dynamics of the quadrupole governs (c) the symmetry breaking around the dipole axis, being strongest for intermediate nonlinearity.

(iv) These properties can be explained by (a) a global renormalization of the β plane, i.e., the introduction of a uniform "effective absolute vorticity gradient" using the circulation's dipole character and (b) a local renormalization of the β plane, by introducing an "effective absolute vorticity field" with hyperbolic contours, associated with the circulation's quadrupole character.

We shall substantiate these statements in the following sections. In Sec. II we define the Green's function in the present nonlinear context, after nondimensionalizing the vorticity equation, laying bare the basic parameter measuring the strength of the nonlinearity; and we introduce a near-field expansion for the induced relative vorticity field. In Sec. III we reconsider the results for weak nonlinearity,[5] which we extend one order deeper in a primitive perturbation series. From these results some inferences are drawn for setting up a renormalized perturbation series which is the subject of Sec. IV. A solution in dipole approximation for the Green's function on an unbounded β plane for an effectively infinitely extended forcing field, to zeroth order in the renormalized perturbation series, is given in Sec. V, which is expanded to a quadrupole approximation in Sec. VI. It is shown that the symmetry properties of the solution behave in the same sense as those of the numerical simulations mentioned before. These results are further discussed in Sec. VII, where it is also shown that the global renormalization of the β plane by means of the relative vorticity dipole in fact leads to a rescaling of the internal dynamical length scale in the problem, such that after rescaling of Eq. (1.1) the familiar form of the vorticity equation for an "almost free" inertial mode is recovered.

II. THE EQUATION FOR THE GREEN'S FUNCTION

Before defining the Green's function in the present nonlinear context, we first nondimensionalize Eq. (1.1). To that end we scale the coordinates with $2k/\beta_*$ and the stream function with $4Tk/\beta_*^2$. The basic equation then reads

$$\Delta\psi+2\epsilon\Im(\psi,\Delta\psi)+2\frac{\partial\psi}{\partial x_1}=-\tau(\mathbf{x}),\quad \epsilon=\frac{T}{2k^2}, \tag{2.1}$$

where the parameter ϵ, sometimes called the Reynolds number, measures the strength of the nonlinear interactions. Note that this parameter is independent of the planetary vorticity gradient β_*, a crucial fact to which we return in the discussion.

Next we define the Green's function for forced circulation on the unbounded β plane as the kernel of the convolution

$$\psi(\mathbf{x};\epsilon,\tau)=\int\int G(\mathbf{x},\mathbf{y};\epsilon,\tau(\mathbf{y}))\tau(\mathbf{y})d\mathbf{y}\ . \tag{2.2}$$

Due to the nonlinear character of (2.1), G depends parametrically on the Reynolds number ϵ, and functionally on the shape of the forcing function τ. In the same sense we define the Green's function g, for the relative vorticity field by

$$\Delta\psi\equiv\zeta(\mathbf{x};\epsilon,\tau)=\int\int g(\mathbf{x},\mathbf{y};\epsilon,\tau(\mathbf{y}))\tau(\mathbf{y})d\mathbf{y}\ , \tag{2.3}$$

where $g\equiv\Delta G$. Substituting (2.3) in (2.1), we get

$$\Delta\psi+2\epsilon\Im\left[\psi,\int\int g(\mathbf{x},\mathbf{y};\epsilon,\tau)\tau(\mathbf{y})d\mathbf{y}\right]+2\frac{\partial\psi}{\partial x_1}=-\tau(\mathbf{x})\ . \tag{2.4}$$

Hence the Green's function should obey

$$\Delta G+2\epsilon\Im\left[G,\int\int g(\mathbf{x},\mathbf{y};\epsilon,\tau)\tau(\mathbf{y})d\mathbf{y}\right]+2\frac{\partial G}{\partial x_1}=-\delta(\mathbf{x}-\mathbf{z})\ . \tag{2.5}$$

We now make a near-field expansion around the forcing position $\mathbf{z}$ of the relative vorticity field as it occurs in (2.5):

$$\zeta(\mathbf{x};\epsilon,\tau)=A(\mathbf{z};g,\tau)+(\mathbf{x}-\mathbf{z})\cdot\mathbf{B}(\mathbf{z};g,\tau)+\tfrac{1}{2}(x_i-z_i)(x_j-z_j):\mathrm{C}_{ij}(\mathbf{z};g,\tau)+\cdots\ , \tag{2.6}$$

where

$$A(\mathbf{z};g,\tau)=\int\int g(\mathbf{z},\mathbf{y};\epsilon,\tau)\tau(\mathbf{y})d\mathbf{y}\cong\tau(\mathbf{z})\int\int g(\mathbf{z},\mathbf{y};\epsilon,\tau)d\mathbf{y}\ , \tag{2.7}$$

$$\mathbf{B}(\mathbf{z};g,\tau)=\left[\int\int\nabla g(\mathbf{x},\mathbf{y};\epsilon,\tau)\tau(\mathbf{y})d\mathbf{y}\right]_{\mathbf{x}=\mathbf{z}}\cong\tau(\mathbf{z})\left[\int\int\nabla g(\mathbf{x},\mathbf{y};\epsilon,\tau)d\mathbf{y}\right]_{\mathbf{x}=\mathbf{z}}, \tag{2.8}$$

$$\mathrm{C}_{ij}(\mathbf{z};g,\tau)=\left[\int\int\frac{\partial^2}{\partial x_i\partial x_j}g(\mathbf{x},\mathbf{y};\epsilon,\tau)\tau(\mathbf{y})d\mathbf{y}\right]_{\mathbf{x}=\mathbf{z}}\cong\tau(\mathbf{z})\left[\int\int\frac{\partial^2}{\partial x_i\partial x_j}g(\mathbf{x},\mathbf{y};\epsilon,\tau)d\mathbf{y}\right]_{\mathbf{x}=\mathbf{z}}. \tag{2.9}$$

In (2.7)–(2.9) the $\cong$ sign applies whenever the external length scale over which the rotation of the wind stress varies is much larger than the internal length scale with which the vorticity distribution and its derivatives drop to zero at infinity. Inserting now expansion (2.6) and (2.5), noticing that the derivatives in the Jacobian operator are taken with respect to $\mathbf{x}$ so that $A(\mathbf{z};g,\tau)$ drops out, the final equation for the Green's function becomes

$$\Delta G+2\epsilon\Im(G,(\mathbf{x}-\mathbf{z})\cdot\mathbf{B}(\mathbf{z};g,\tau)+\tfrac{1}{2}(x_i-z_i)(x_j-z_j):\mathrm{C}_{ij}(\mathbf{z};g,\tau)+\cdots)+2\frac{\partial}{\partial x_1}G=-\delta(\mathbf{x}-\mathbf{z})\ . \tag{2.10}$$

As the coefficients $\mathbf{B}(\mathbf{z};g,\tau)$ and $\mathrm{C}_{ij}(\mathbf{z};g,\tau)$ depend functionally on g, and thus on G, Eq. (2.10) is as yet a fully nonlinear equation. Approximate solutions can be obtained for $\epsilon\ll1$ by means of a primitive perturbation series. For more general ϵ we shall present an iterative solution by means of a renormalized perturbation series. Depending on whether the expansion (2.6) is truncated after the term linear or quadratic in $(\mathbf{x}-\mathbf{z})$, we shall refer to "dipole" or "quadrupole" approximations for reasons that will become apparent in Sec. III.

III. SOME INFERENCES FROM THE WEAKLY NONLINEAR LIMIT

The Green's function corresponding to the linearized version of the quasigeostrophic vorticity equation (1.1) is evidently independent of the shape of the forcing and, of course, translational invariant, i.e., only depending on the difference vector $\mathbf{x}-\mathbf{z}$. Here we extend the linear solution[5] with a first-order nonlinear correction by means of a primitive perturbation series in the Reynolds number. We shall use Eq. (2.10) in dipole approximation, i.e., we truncate the series (2.6) after the second term. Moreover we shall assume that we are dealing with a forcing field varying over a length scale that is much larger than any internal dynamical length scale of the Green's function itself. Then locally we may set $\tau=1$. In view of translational invariance, we just choose $\mathbf{z}=\mathbf{0}$ as the only forcing position to be considered. The relevant equation then reads

$$\Delta G+2\epsilon\Im(G,\mathbf{x}\cdot\mathbf{B}(g;\epsilon,1))+2\frac{\partial G}{\partial x_1}=-\delta(\mathbf{x})\ . \tag{3.1}$$

For $\epsilon\ll1$ we now expand both G and the vector $\mathbf{B}$, which depends functionally on G, in a perturbation series in ϵ:

$$G=G^{(0)}+\epsilon G^{(1)}+\cdots,\quad \mathbf{B}=\mathbf{B}^{(0)}+\epsilon\mathbf{B}^{(1)}+\cdots\ . \tag{3.2}$$

Substitution of (3.2) in (3.1) yields to zeroth order

$$\Delta G^{(0)}+2\frac{\partial G^{(0)}}{\partial x_1}=-\delta(\mathbf{x})\ , \tag{3.3}$$

the solution of which reads[5]

$$G^{(0)}=(2\pi)^{-1}\exp(-r\cos\theta)K_0(r)\ , \tag{3.4}$$

where K_0 is the zeroth-order modified Bessel function of the second kind. [From hereon we shall switch at will between the orthogonal coordinate system (x_1,x_2) and polar coordinates (r,θ).] The corresponding relative vorticity distribution is given by

$$g^{(0)}=\Delta G^{(0)}=-2\frac{\partial G^{(0)}}{\partial x_1}$$
$$=\pi^{-1}\exp(-r\cos\theta)[K_0(r)+(\cos\theta)K_1(r)]\ . \tag{3.5}$$

Both $G^{(0)}$ and $g^{(0)}$ are shown by means of contour plots in Figs. 1(a) and 1(b). Note the striking asymmetry of the stream function with the symmetry axis running east-west (x_1 direction) and in consequence the outstanding dipole character of the relative vorticity distribution. The latter induces an effective global relative vorticity gradient around the forcing position, $\mathbf{z}=\mathbf{0}$, which is represented by $\mathbf{B}^{(0)}$. The latter is calculated by means of Green's theorem (see the beginning of Appendix B), using definition (2.8), as

$$\mathbf{B}^{(0)}=\int\int\nabla g^{(0)}(\mathbf{x})d\mathbf{x}=\lim_{r\to 0}\oint\mathbf{n}g^{(0)}(r,\theta)ds$$
$$=(1,0)\ , \tag{3.6}$$

where $\mathbf{n}$ is the unit vector in the radial direction $(\cos\theta,\sin\theta)$ and ds the arclength of the integration contour. As could be expected from the symmetry of $G^{(0)}$, $\mathbf{B}^{(0)}$ has a component in the east-west direction only.

Although Eq. (3.1) treats the nonlinear interactions in dipole approximation, such does not mean that the relative vorticity distribution $g^{(0)}$ does not have a quadrupole character as well, which is also revealed by closer inspection of Fig. 1(b). For the subsequent discussion it is worthwhile to look at the effective second derivatives connected with the quadrupole structure of the relative vorticity distribution in zeroth order, as represented by the tensor C_{ij}, defined in (2.9):

$$\mathrm{C}_{ij}^{(0)}=\int\int\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}g^{(0)}(\mathbf{x})d\mathbf{x}$$
$$=\lim_{r\to 0}\oint n_i\frac{\partial}{\partial x_j}g^{(0)}(r,\theta)ds$$
$$=\begin{bmatrix}-\frac{3}{2} & 0\\ 0 & -\frac{1}{2}\end{bmatrix}\ . \tag{3.7}$$

Now any vorticity quadrupole around the forcing position can always be thought of as the sum of two contributions, one with its symmetry axes in the east-west (x_1) and north-south (x_2) directions and one with its symmetry axes rotated over an angle of 45° with respect to the former. Both are related to the effective global derivatives C_{ij}. That is, if we define

$$C_{+}^{(0)}=\tfrac{1}{2}(\mathrm{C}_{11}^{(0)}-\mathrm{C}_{22}^{(0)})=-\tfrac{1}{2}\ ,$$
$$C_{\times}^{(0)}=\tfrac{1}{2}(\mathrm{C}_{12}^{(0)}+\mathrm{C}_{21}^{(0)})=0\ , \tag{3.8}$$
$$C_{\circ}^{(0)}=\tfrac{1}{2}(\mathrm{C}_{11}^{(0)}+\mathrm{C}_{22}^{(0)})=-1\ ,$$

$C_{+}^{(0)}$ is related to the quadrupole component with one of its symmetry axes in line with the symmetry axis of the dipole. Evidently this component does not vanish in zeroth order. On the other hand, the quadrupole component that has its symmetry axes rotated with respect to the x_1,x_2 axes, is related to $C_{\times}^{(0)}$. It is this component that would break the remaining symmetry of the dipole, but in accordance with the symmetric character of the zeroth-order solution around the east-west axis, this component vanishes at this order in the expansion. Finally, $C_{\circ}^{(0)}$ is connected with the global second derivatives of the vorticity monopole, which we shall not consider any further.

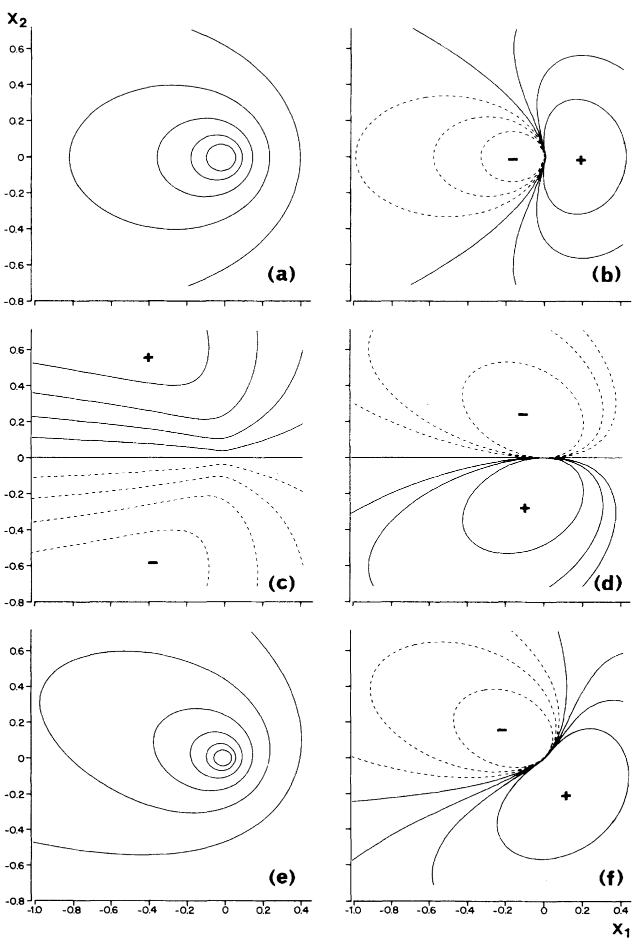

FIG. 1. Contour plots of the Green's function for the stream function (to the left) and for the relative vorticity (to the right). From top to bottom: (a) and (b) the zeroth-order solution, (c) and (d) the first-order solution, and (e) and (f) the sum of zeroth- and first-order solutions for $\epsilon=0.5$. All axes have dimensionless units, the distances being scaled by $2k/\beta_*$.

We now proceed with the first-order equation. From

(3.1) and (3.2) we get to first order in ϵ,

$$\Delta G^{(1)}+2\frac{\partial G^{(1)}}{\partial x_1}=-2\Im(G^{(0)},\mathbf{x}\cdot\mathbf{B}^{(0)})$$
$$=-\pi^{-1}(\sin\theta)\exp(-r\cos\theta)K_1(r)\ , \tag{3.9}$$

the solution of which reads

$$G^{(1)}=x_2G^{(0)}$$
$$=(2\pi)^{-1}r(\sin\theta)\exp(-r\cos\theta)K_0(r)\ , \tag{3.10}$$

which can easily be verified by substitution of the middle expression in (3.10) into (3.9), using (3.3). The corresponding first-order vorticity distribution is then given by

$$g^{(1)}=\Delta G^{(1)}=x_2g^{(0)}+2\frac{\partial G^{(0)}}{\partial x_2}\ , \tag{3.11}$$

where $g^{(0)}$ and $G^{(0)}$ are given by (3.5) and (3.4), respectively. The stream function and vorticity distributions $G^{(1)}$ and $g^{(1)}$ are shown by contour plots in Figs. 1(c) and 1(d). Evidently both the stream function and the vorticity distribution are antisymmetric around the symmetry axis (x_1) of the zeroth-order solution. Hence there is no contribution to the vorticity monopole at this order, whereas the vorticity dipole axis now runs in the north-south direction (x_2). The latter is of course again reflected in the global vorticity gradient at first order:

$$\mathbf{B}^{(1)}=\lim_{r\to 0}\oint \mathbf{n}g^{(1)}ds=(0,-1)\ . \tag{3.12}$$

In the same way the vorticity distribution's quadrupole structure is reflected in the components of $C_{ij}^{(1)}$:

$$C_{ij}^{(1)}=\lim_{r\to 0}\oint n_i\frac{\partial}{\partial x_j}g^{(1)}ds=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\ , \tag{3.13}$$

whence

$$C_0^{(1)}=0,\quad C_+^{(1)}=0,\quad C_\times^{(1)}=1\ . \tag{3.14}$$

This result is in accordance with the antisymmetric character of the first-order solution around the x_1 axis, by which only the quadrupole component with symmetry axes rotated over 45° with respect to the coordinate axes does not vanish.

Finally we may now sum the zeroth- and first-order contributions to the stream function and the relative vorticity field for specific values of ϵ. This is shown by the contour plots in Figs. 1(e) and 1(f). Evidently the whole pattern has now lost all symmetry due to the antisymmetric character of the first-order correction, reflected in $\mathbf{B}^{(1)}$ and $C_\times^{(1)}$. This is in accordance with analytical results for circulation in closed basins, using a primitive perturbation series for weak nonlinearity.[6] This approach thus explains, for small but increasing ϵ, both the initial turning of the symmetry axis, reflected here in the turning of the vector sum $\mathbf{B}^{(0)}+\epsilon\mathbf{B}^{(1)}$, as well as the initial breaking of symmetry around that axis by the sum of the zeroth- and first-order quadrupole. However, there is at this stage, for increasing ϵ, no indication of a weakening of the symmetry breaking around the axis perpendicular to the total dipole axis, as the numerical results[4] show. If this were the case, an increase of the internal length scale of the Green's function should have to be the result, but this length scale is not altered by higher-order terms in a primitive perturbation series. It remains at its zeroth-order value of 1 (nondimensionally) once and for all. Also there is no indication of an ultimate decrease of the symmetry breaking around the dipole axis by the quadrupole structure. Obviously the results for weak nonlinearity are not valid for the regime $\epsilon\geq O(1)$. Moreover, by using a primitive perturbation series together with a dipole approximation, the result (3.10) shows that in fact the first-order correction is only valid in the near field. For, regarding the ratio $G^{(1)}/G^{(0)}=x_2\epsilon$, it appears that only for $|x_2|\ll\epsilon^{-1}$ do we have $\epsilon G^{(1)}\ll G^{(0)}$, as it should be for this type of perturbation series. All these deficiencies of the primitive series ask for a renormalized expansion, the scheme of which shall be discussed in Sec. IV.

IV. THE RENORMALIZATION SCHEME

The formal equivalence of planetary and relative vorticity in the quasigeostrophic vorticity equation (1.1) suggests that they may be combined in a single "effective absolute vorticity field." In its most simple form the effective field is just a uniform gradient of effective absolute vorticity, being the vector sum of the planetary vorticity gradient and an effective gradient of relative vorticity, yet to be defined. The mean value of absolute vorticity on which this uniform gradient is superimposed is evidently unimportant, as the Jacobian operator in Eq. (1.1) is invariant to a change of the vorticity gauge. Such an effective field, consisting of a uniform gradient of absolute vorticity, is in fact the familiar β plane, albeit with arbitrary magnitude and direction of the gradient. The introduction of this field, which changes the β term in the equation in the same way for all positions in the field, is here called a *global* renormalization of the β plane. Of course, the exact distribution of absolute vorticity creates an absolute vorticity gradient that is dependent on position. Any introduction of an effective absolute vorticity gradient that accounts for spatial dependence of the absolute vorticity gradient is called here a *local* renormalization of the β plane. One of our aims is to show that global renormalization of the β plane can be achieved by the dipole character of the relative vorticity distribution and the most simple local renormalization by its quadrupole character.

The results of Sec. III suggest that the planetary vorticity gradient and the self-induced relative vorticity gradients should be treated on an equal footing if any relevant results for moderate to strong nonlinear interactions are to be expected. To that end we introduce an effective absolute vorticity field ζ^*, again in the form of a near-field expansion

$$\zeta^*=1+(\mathbf{x}-\mathbf{z})\cdot\boldsymbol{\beta}+\tfrac{1}{2}(x_i-z_i)(x_j-z_j):\gamma_{ij}\ . \tag{4.1}$$

Here $\boldsymbol{\beta}$ is a vector with as yet undetermined components. The components of the tensor γ_{ij} are undetermined as well, but constrained by the condition that the integral of the third term over all space is zero; i.e., there are no contributions to the vorticity monopole in the expansion (4.1), except for the dynamically unimportant first term. This yields

$$\gamma_{ij} \equiv \begin{bmatrix} \gamma_+ & \gamma_\times \\ \gamma_\times & -\gamma_+ \end{bmatrix} , \quad (4.2)$$

whence $\Delta\zeta^* = 0$.

Next we set up a renormalized perturbation series for the Green's function and the undetermined components of $\boldsymbol{\beta}$ and γ in (4.1). To that end we introduce a formal expansion parameter λ, and we use expression (4.1) as counterterms to the nonlinear term in Eq. (2.10), which we recast as follows:

$$\Delta G + 2\lambda\epsilon\Im(G,(\mathbf{x}-\mathbf{z})\cdot\mathbf{B}(\mathbf{z};g,\tau) + \tfrac{1}{2}(x_i - z_i)(x_j - z_j):\mathrm{C}_{ij}(\mathbf{z};g,\tau))$$
$$+ 2\Im(G,(\mathbf{x}-\mathbf{z})\cdot\boldsymbol{\beta} + \gamma_\times(x_1-z_1)(x_2-z_2) + \tfrac{1}{2}\gamma_+[(x_1-z_1)^2-(x_2-z_2)^2]) = -\delta(\mathbf{x}-\mathbf{z}) , \quad (4.3)$$

where

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \gamma_\times \\ \gamma_+ \end{bmatrix} = \begin{bmatrix} \beta_1^{(0)} \\ \beta_2^{(0)} \\ \gamma_\times^{(0)} \\ \gamma_+^{(0)} \end{bmatrix} + \lambda \begin{bmatrix} \beta_1^{(1)} \\ \beta_2^{(1)} \\ \gamma_\times^{(1)} \\ \gamma_+^{(1)} \end{bmatrix} + \lambda^2 \cdots = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad (4.4)$$

and

$$G = G^{(0)} + \lambda G^{(1)} + \lambda^2 \cdots . \quad (4.5)$$

It can easily be seen, by equality (4.4), that setting $\lambda = 0$ reduces (4.3) to the linear equation (3.3), whereas summing over all powers of λ, for λ reset to 1, reduces (4.3) to (2.10).

As $\mathbf{B}(\mathbf{z};g,\tau)$ and $\mathrm{C}_{ij}(\mathbf{z};g,\tau)$ depend functionally on G, we expand them too:

$$\mathbf{B}(\mathbf{z};g,\tau) = \left[\int\int \nabla[g^{(0)}(\mathbf{x},\mathbf{y}) + \lambda g^{(1)}(\mathbf{x},\mathbf{y}) + \cdots]\tau(\mathbf{y})d\mathbf{y} \right]_{\mathbf{x}=\mathbf{z}}$$
$$= \mathbf{B}^{(0)}(\mathbf{z};g,\tau) + \lambda\mathbf{B}^{(1)}(\mathbf{z};g,\tau) + \lambda^2 \cdots , \quad (4.6)$$

$$\mathrm{C}_{ij}(\mathbf{z};g,\tau) = \left[\int\int \frac{\partial^2}{\partial x_i \partial x_j}[g^{(0)}(\mathbf{x},\mathbf{y}) + \lambda g^{(1)}(\mathbf{x},\mathbf{y}) + \lambda^2 \cdots]\tau(\mathbf{y})d\mathbf{y} \right]_{\mathbf{x}=\mathbf{z}}$$
$$= \mathrm{C}_{ij}^{(0)}(\mathbf{z};g,\tau) + \lambda\mathrm{C}_{ij}^{(1)}(\mathbf{z};g,\tau) + \lambda^2 \cdots . \quad (4.7)$$

Inserting (4.4)–(4.7) in (4.3) gives to zeroth order in λ,

$$\Delta G^{(0)} + 2\Im\{G^{(0)},(\mathbf{x}-\mathbf{z})\cdot\boldsymbol{\beta}^{(0)} + \gamma_\times^{(0)}(x_1-z_1)(x_2-z_2) + \tfrac{1}{2}\gamma_+^{(0)}[(x_1-z_1)^2-(x_2-z_2)^2]\} = -\delta(\mathbf{x}-\mathbf{z}) , \quad (4.8)$$

the solution of which gives $G^{(0)}(\mathbf{x},\mathbf{z};\boldsymbol{\beta}^{(0)},\gamma_\times^{(0)},\gamma_+^{(0)})$ in terms of the undetermined components of $\boldsymbol{\beta}$ and γ. Next, to first order in λ we have

$$\Delta G^{(1)} + 2\Im\{G^{(1)},(\mathbf{x}-\mathbf{z})\cdot\boldsymbol{\beta}^{(0)} + \gamma_\times^{(0)}(x_1-z_1)(x_2-z_2) + \gamma_+^{(0)}[(x_1-z_1)^2-(x_2-z_2)^2]\}$$
$$= -2\epsilon\Im[G^{(0)},(\mathbf{x}-\mathbf{z})\cdot\mathbf{B}^{(0)} + \tfrac{1}{2}(x_i-z_i)(x_j-z_j):\mathrm{C}_{ij}^{(0)}]$$
$$-2\Im\{G^{(0)},(\mathbf{x}-\mathbf{z})\cdot\boldsymbol{\beta}^{(1)} + \gamma_\times^{(1)}(x_1-z_1)(x_2-z_2) + \tfrac{1}{2}\gamma_+^{(1)}[(x_1-z_1)^2-(x_2-z_2)^2]\} . \quad (4.9)$$

If we now truncate the series in λ after the first-order terms and regard the remaining terms as an iterative series for G, then, resetting $\lambda = 1$, we get from (4.4)

$$\begin{bmatrix} \beta_1^{(1)} \\ \beta_2^{(1)} \\ \gamma_\times^{(1)} \\ \gamma_+^{(1)} \end{bmatrix} = \begin{bmatrix} -\beta_1^{(0)} \\ 1-\beta_2^{(0)} \\ -\gamma_\times^{(0)} \\ -\gamma_+^{(0)} \end{bmatrix} . \quad (4.10)$$

Moreover, we require the vanishing of the dipole and quadrupole contributions in the right-hand side of (4.9) in order that the dipole and quadrupole structure of the first iteration $G^{(0)}$ is already a good approximation to the exact solution. Substituting (4.10) in (4.9), this gives the renormalization conditions

$$\begin{bmatrix} 0 \\ -1 \end{bmatrix} + \boldsymbol{\beta}^{(0)} = \epsilon\mathbf{B}^{(0)} = \epsilon \left[\int\int \nabla g^{(0)}(\mathbf{x},\mathbf{y};\boldsymbol{\beta}^{(0)},\gamma_\times^{(0)},\gamma_+^{(0)})\tau(\mathbf{y})d\mathbf{y} \right]_{\mathbf{x}=\mathbf{z}}$$
$$= \epsilon\mathbf{f}(\mathbf{z};\boldsymbol{\beta}^{(0)},\gamma_\times^{(0)},\gamma_+^{(0)}) , \quad (4.11)$$

$$\gamma_{\times}^{(0)}=\epsilon C_{\times}^{(0)}=\epsilon\left[\int\int\frac{\partial^2}{\partial x_1\partial x_2}g^{(0)}(\mathbf{x},\mathbf{y};\boldsymbol{\beta}^{(0)},\gamma_{\times}^{(0)},\gamma_{+}^{(0)})\tau(\mathbf{y})d\mathbf{y}\right]_{\mathbf{x}=\mathbf{z}}$$
$$=\epsilon F_{\times}(\mathbf{z};\boldsymbol{\beta}^{(0)},\gamma_{\times}^{(0)},\gamma_{+}^{(0)})\ , \tag{4.12}$$
$$\gamma_{+}^{(0)}=\epsilon C_{+}^{(0)}=\tfrac{1}{2}\epsilon\left[\int\int\left[\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_2^2}\right]g^{(0)}(\mathbf{x},\mathbf{y};\boldsymbol{\beta}^{(0)},\gamma_{\times}^{(0)},\gamma_{+}^{(0)})\tau(\mathbf{y})d\mathbf{y}\right]_{\mathbf{x}=\mathbf{z}}$$
$$=\tfrac{1}{2}\epsilon F_{+}(\mathbf{z};\boldsymbol{\beta}^{(0)},\gamma_{\times}^{(0)},\gamma_{+}^{(0)})\ . \tag{4.13}$$

The four equations (4.11)–(4.13) permit in principle the determination of the four unknown coefficients $(\beta_1^{(0)},\beta_2^{(0)},\gamma_{\times}^{(0)},\gamma_{+}^{(0)})$ as functions of ϵ. For a general shape of the forcing function these coefficients still depend on $\mathbf{z}$, but if we again restrict ourselves to the case of an effectively infinitely far extended forcing field, then $\tau\simeq 1$, and $G^{(0)}$ will depend on $(\mathbf{x}-\mathbf{z})$ only, whereas the components of β and γ become constants. This is the case we shall deal with in the following sections.

V. THE RENORMALIZED SOLUTION IN DIPOLE APPROXIMATION

Before discussing the solution to the renormalized Green's function in quadrupole approximation we shall first briefly discuss the much more simple problem in dipole approximation which only leads to global renormalization of the β plane. In that case the zeroth-order equation for the Green's function, (4.8), reduces to

$$\Delta G^{(0)}+2\mathfrak{J}(G^{(0)},\mathbf{x}\cdot\boldsymbol{\beta}^{(0)})=-\delta(\mathbf{x})\ , \tag{5.1}$$

where again an effectively infinitely far extended forcing field is assumed. The effective uniform gradient of absolute vorticity, $\boldsymbol{\beta}^{(0)}$, is defined by

$$\boldsymbol{\beta}^{(0)}-\begin{bmatrix}0\\1\end{bmatrix}=\epsilon\mathbf{B}^{(0)}=\epsilon\int\int\nabla g^{(0)}(\mathbf{x};\boldsymbol{\beta}^{(0)})d\mathbf{x}$$
$$=\lim_{r\to 0}\epsilon\oint\mathbf{n}g^{(0)}(r,\theta;\boldsymbol{\beta}^{(0)})ds\ , \tag{5.2}$$

as follows from (4.11). The solution to (5.1) is, analogously to that of (3.3),

$$G^{(0)}=(2\pi)^{-1}\exp(-\beta_2^{(0)}r\cos\theta+\beta_1^{(0)}r\sin\theta)K_0(\hat{\beta}r)\ , \tag{5.3}$$

where $\hat{\beta}=|\boldsymbol{\beta}^{(0)}|$. The corresponding relative vorticity distribution is given by

$$g^{(0)}=-2\beta_2^{(0)}\frac{\partial G^{(0)}}{\partial x_1}+2\beta_1^{(0)}\frac{\partial G^{(0)}}{\partial x_2}\ . \tag{5.4}$$

Inserting (5.3) and (5.4) in the right-hand side of (5.2) gives

$$B_1^{(0)}=\beta_2^{(0)},\quad B_2^{(0)}=-\beta_1^{(0)}\ , \tag{5.5}$$

whence, from the left-hand equality of (5.2),

$$\begin{bmatrix}1 & -\epsilon\\ \epsilon & 1\end{bmatrix}\begin{bmatrix}\beta_1^{(0)}\\ \beta_2^{(0)}\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}\ . \tag{5.6}$$

The solution of (5.6) reads

$$\beta_1^{(0)}=\frac{\epsilon}{1+\epsilon^2}=\hat{\beta}\cos\beta,\quad \beta_2^{(0)}=\frac{1}{1+\epsilon^2}=\hat{\beta}\sin\beta\ , \tag{5.7}$$

in which

$$\hat{\beta}=\frac{1}{(1+\epsilon^2)^{1/2}},\quad \beta=\arctan\epsilon^{-1}\ . \tag{5.8}$$

For $\epsilon\to 0$, these results are in accordance with those of the primitive perturbation series discussed in Sec. III. There we found, (3.6) and (3.12), up to order ϵ, $\mathbf{B}=(1,-\epsilon)$.

Due to the invariance of both the Laplacian and Jacobian operators in (5.1) to a rotation of the coordinate system, the solution (5.3), (5.7), and (5.8) is equivalent to the solution of

$$\Delta G^{(0)}+2\mathfrak{J}(G^{(0)},y_2(1+\epsilon^2)^{-1/2})=-\delta(\mathbf{y})\ , \tag{5.9}$$

which reads

$$G^{(0)}=(2\pi)^{-1}\exp(-y_1\hat{\beta})K_0(\hat{\beta}r)\ , \tag{5.10}$$

where the (y_1,y_2) system is related to the original east-west–north-south system (x_1,x_2) by

$$\begin{bmatrix}y_1\\y_2\end{bmatrix}=\underline{R}_\beta(\epsilon)\begin{bmatrix}x_1\\x_2\end{bmatrix}\ , \tag{5.11}$$

the rotation matrix being given by

$$\underline{R}_\beta(\epsilon)=\hat{\beta}^{-1}\begin{bmatrix}\beta_2^{(0)} & -\beta_1^{(0)}\\ \beta_1^{(0)} & \beta_2^{(0)}\end{bmatrix}$$
$$=(1+\epsilon^2)^{-1/2}\begin{bmatrix}1 & -\epsilon\\ \epsilon & 1\end{bmatrix}\ . \tag{5.12}$$

The structure of the renormalized solution in dipole approximation as given by (5.9) and (5.12) immediately reveals its basic properties. Equation (5.9) is equivalent to the linearized equation of the Green's function, (3.3), provided one introduces an effective uniform absolute vorticity gradient that is turned in the direction of the applied rotation of the wind stress, with an angle dependent on the degree of nonlinearity as given by $\arctan\epsilon^{-1}$. At the same time, the magnitude of the effective gradient is weakened by a factor $(1+\epsilon^2)^{-1/2}$. Both properties are a direct consequence of having an effective absolute vorticity gradient $\boldsymbol{\beta}_e$ that is the vector sum of the planetary vorticity gradient $\boldsymbol{\beta}_p$ and the self-induced global relative vorticity gradient due to the dipole $\boldsymbol{\beta}_d$, as defined by (5.2).

In the $\mathbf{y}$ coordinate system the latter coincides with $\beta_1^{(0)}$, whereas $\boldsymbol{\beta}_e$ coincides with $\beta_2^{(0)}$. The direction of $\boldsymbol{\beta}_d$ is in the positive y_1 direction, which coincides with the symmetry axis to which the effective absolute vorticity gradient $\boldsymbol{\beta}_e$ is constrained to be perpendicular. Thus we have in the $\mathbf{y}$ coordinate system

$$\beta_1^{(0)}=\beta_d,\ \ \beta_2^{(0)}=\beta_e,\ \ \beta_e^2+\beta_d^2=\beta_p^2=1\ ,$$

or (5.13)

$$\beta_e=(1+\epsilon^2)^{-1/2},\ \ \beta_d=\epsilon(1+\epsilon^2)^{-1/2}\ .$$

Global renormalization of the β plane thus already explains two of the three symmetry properties of forced nonlinear circulation: the turning of the symmetry axis with increasing Reynolds number, which is equivalent to the turning of the dipole axis as represented by the rotation matrix $\underline{R}_\beta(\epsilon)$, defined by (5.12); and secondly the concurrent weakening of the symmetry breaking as represented by the absolute value of the effective vorticity gradient β, given by (5.8). This weakening with increasing ϵ is due to a gradual turning of the self-induced relative vorticity gradient together with an increase in strength, such that ultimately it becomes equal in magnitude but opposite to the planetary vorticity gradient, whence the effective absolute vorticity gradient vanishes asymptotically in the fully nonlinear regime. However, for the final property yet to be explained, the breaking of symmetry around the dipole axis for intermediate nonlinearity, we need the mathematically much more intricate local renormalization of the β plane by means of the vorticity's quadrupole structure. This is the subject of Sec. VI.

VI. THE RENORMALIZED SOLUTION IN QUADRUPOLE APPROXIMATION

For an effectively infinitely far extended forcing field, the zeroth-order equation for the renormalized Green's function in quadrupole approximation reads

$$\Delta G^{(0)}+2\mathfrak{J}(G^{(0)},\mathbf{x}\cdot\boldsymbol{\beta}^{(0)}+\gamma_\times^{(0)}x_1x_2+\tfrac{1}{2}\gamma_+^{(0)}(x_1^2-x_2^2)) =-\delta(\mathbf{x})\ . \quad (6.1)$$

The as yet undetermined constants $(\beta_1^{(0)},\beta_2^{(0)},\gamma_\times^{(0)},\gamma_+^{(0)})$ are defined by (5.2) and

$$\begin{aligned}\gamma_\times^{(0)}&=\epsilon C_\times^{(0)}\\ &=\epsilon\int\int\frac{\partial^2}{\partial x_1\partial x_2}g^{(0)}(\mathbf{x};\boldsymbol{\beta}^{(0)},\gamma_\times^{(0)},\gamma_+^{(0)})d\mathbf{x}\\ &=\lim_{r\to 0}\epsilon\oint n_1\frac{\partial}{\partial x_2}g^{(0)}(r,\theta;\boldsymbol{\beta}^{(0)},\gamma_\times^{(0)},\gamma_+^{(0)})ds\ ,\end{aligned} \quad (6.2)$$

$$\begin{aligned}\gamma_+^{(0)}&=\epsilon C_+^{(0)}\\ &=\tfrac{1}{2}\epsilon\int\int\left[\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_2^2}\right]g^{(0)}(\mathbf{x};\boldsymbol{\beta}^{(0)},\gamma_\times^{(0)},\gamma_+^{(0)})d\mathbf{x}\\ &=\lim_{r\to 0}\tfrac{1}{2}\epsilon\oint\left[n_1\frac{\partial}{\partial x_1}-n_2\frac{\partial}{\partial x_2}\right]\\ &\qquad\times g^{(0)}(r,\theta;\boldsymbol{\beta}^{(0)},\gamma_\times^{(0)},\gamma_+^{(0)})ds\ .\end{aligned} \quad (6.3)$$

Equation (6.1) is formally analogous to an advection-diffusion equation of the concentration field $G^{(0)}$ in the presence of a point source at $\mathbf{x}$, the "β term" in the Jacobian standing for a uniform advective velocity field and the "γ terms" for a stretching and straining field. The full details of the solution are given elsewhere.[7] Solving (6.1) involves three steps: (i) a coordinate transformation, (ii) reduction to a Bessel equation and expression of the solution in terms of a Fourier-Bessel series, and (iii) summing of the series to an integral representation of an incomplete modified Bessel function of the second kind. A brief outline of the second step is given in Appendix A. The first step runs as follows. We define

$$\begin{aligned}&\boldsymbol{\beta}^{(0)}=\hat{\beta}(\cos\beta,\sin\beta)\ ,\\ &\hat{\beta}=|\boldsymbol{\beta}^{(0)}|\ , \qquad (6.4)\\ &\arctan\beta=\beta_2^{(0)}/\beta_1^{(0)}\ ,\end{aligned}$$

$$\begin{aligned}&(\gamma_+^{(0)},\gamma_\times^{(0)})=\hat{\gamma}(\cos\gamma,\sin\gamma)\ ,\\ &\hat{\gamma}=[(\gamma_\times^{(0)})^2+(\gamma_+^{(0)})^2]^{1/2}\ , \qquad (6.5)\\ &\arctan=\frac{\gamma_\times^{(0)}}{\gamma_+^{(0)}}\ .\end{aligned}$$

Next we apply a translation and rotation of the (x_1,x_2) coordinate system to a (z_1,z_2) system by

$$\mathbf{z}=\underline{R}_\phi\cdot\mathbf{x}+\mathbf{a}\ , \quad (6.6)$$

where

$$\underline{R}_\phi=\begin{bmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{bmatrix},\ \ \phi=\pi/4-\gamma/2\ , \quad (6.7)$$

$$\begin{bmatrix}a_1\\ a_2\end{bmatrix}\equiv\hat{a}\begin{bmatrix}\cos\alpha\\ \sin\alpha\end{bmatrix},\ \ \hat{a}=\hat{\beta}/\hat{\gamma},\ \ \alpha=\pi/2-\beta-\phi\ . \quad (6.8)$$

In view of the translational and rotational invariance of both the Laplacian and Jacobian operators, Eq. (6.1) in the (z_1,z_2) coordinate system reads

$$\Delta G^{(0)}+2\mathfrak{J}(G^{(0)},\hat{\gamma}z_1z_2)=-\delta(\mathbf{z}-\mathbf{a})\ , \quad (6.9)$$

which is formally analogous to an advection-diffusion equation in a pure straining field with a point source positioned at $\mathbf{a}$ from the center of the hyperbolic streamlines. The solution of Eq. (6.9) reads (see Appendix A)

$$G^{(0)}(\mathbf{z};\hat{\gamma})=(4\pi)^{-1}\exp\left[\frac{\hat{\gamma}}{2}(z_2^2-a_2^2-z_1^2+a_1^2)\right]\times\mathcal{K}_0\left[\frac{\hat{\gamma}}{2}R(\mathbf{z});\eta(\mathbf{z})\right]\ , \quad (6.10)$$

where the zeroth-order incomplete modified Bessel function of the second kind, $\mathcal{K}_0$, is defined in Appendix A as well, and where

$$R(\mathbf{z})=|\mathbf{z}-\mathbf{a}|\ |\mathbf{z}+\mathbf{a}|\ , \quad (6.11)$$

$$\eta(\mathbf{z})=\ln\frac{|\mathbf{z}-\mathbf{a}|}{|\mathbf{z}+\mathbf{a}|}\ . \quad (6.12)$$

The global derivatives of (6.10), necessary for the evaluation of the components of the renormalized β plane, as given by (5.2), (6.2), and (6.3), are derived in Appendix B, in the rotated coordinate system **z**. Transforming back to the original **x** system (west-east-south-north) we get first for the components of $\boldsymbol{\beta}^{(0)}$, using (5.2) and (B13),

$$\boldsymbol{\beta}^{(0)}-\begin{bmatrix}0\\1\end{bmatrix}=\epsilon\mathbf{B}^{(0)}=\epsilon\hat{\beta}\underline{R}_{\phi}^{-1}\begin{bmatrix}\sin(\beta+\phi)\\-\cos(\beta+\phi)\end{bmatrix}=\epsilon\begin{bmatrix}\beta_2^{(0)}\\-\beta_1^{(0)}\end{bmatrix}, \quad (6.13)$$

which is equivalent to (5.6) and thus has the same solution, (5.7) and (5.8). So, the global renormalization of the β plane appears to be independent of whether one works with a dipole or quadrupole approximation, a fact of practical important, as one may limit oneself to the much simpler dipole approximation whenever the details that are added by local renormalization in quadrupole approximation are not needed. Nonetheless, these details are of course of interest in itself, particularly for the symmetry of the solution around the dipole axis. This is shown by solving for the undetermined factors $\gamma_+^{(0)}$ and $\gamma_\times^{(0)}$. Using the result (B18) and (B19), together with (6.2) and (6.3), reminding that $\gamma_+^{(0)}$ and $\gamma_\times^{(0)}$ are composed of components of the tensor $\mathbf{C}_{ij}$, so that tensor transformation rules have to be applied when transforming back from the rotated coordinate system **z** to the original system **x**, we get

$$\begin{bmatrix}\gamma_+^{(0)}\\\gamma_\times^{(0)}\end{bmatrix}=\epsilon\underline{R}_{2\phi}^{-1}\begin{bmatrix}\frac{1}{2}\hat{\beta}^2\cos2(\beta+\phi)+\hat{\gamma}\\\frac{1}{2}\hat{\beta}^2\sin2(\beta+\phi)\end{bmatrix}, \quad (6.14)$$

where

$$\underline{R}_{2\phi}=\begin{bmatrix}\cos2\phi & -\sin2\phi\\\sin2\phi & \cos2\phi\end{bmatrix}=\begin{bmatrix}\sin\gamma & -\cos\gamma\\\cos\gamma & \sin\gamma\end{bmatrix}. \quad (6.15)$$

This appears to be equivalent to

$$\begin{bmatrix}\gamma_+^{(0)}\\\gamma_\times^{(0)}\end{bmatrix}=\frac{\epsilon}{(1+\epsilon^2)^{1/2}}\underline{R}_{\beta}^{-1}\begin{bmatrix}\frac{1}{2}[(\beta_1^{(0)})^2-(\beta_2^{(0)})^2]\\\beta_1^{(0)}\beta_2^{(0)}\end{bmatrix}=\frac{\epsilon}{2(1+\epsilon^2)^3}\begin{bmatrix}3\epsilon^2-1\\\epsilon(3-\epsilon^2)\end{bmatrix}, \quad (6.16)$$

whence

$$\hat{\gamma}=\frac{\epsilon}{2(1+\epsilon^2)^{3/2}},\quad \gamma=\arctan\left[\frac{\epsilon(3-\epsilon^2)}{3\epsilon^2-1}\right], \quad (6.17)$$

such that for $\epsilon=0$, $\gamma=\pi$. For $\epsilon\rightarrow0$ we recover again the result (3.8) for $C_+=\gamma_+^{(0)}/\epsilon$. In the same limit we get from (6.16) $C_\times=\gamma_\times^{(0)}/\epsilon=\frac{3}{2}\epsilon$, whereas (3.14) gives ϵ. The sign is correct, but the difference in magnitude is due to the difference in quadrupole and dipole approximation.

Introducing now the complex constants

$$\tilde{\beta}\equiv\hat{\beta}(\cos\beta+i\sin\beta),\quad \tilde{\gamma}\equiv\hat{\gamma}(\cos\gamma+i\sin\gamma), \quad (6.18)$$

it appears that the whole procedure of global and local renormalization of the β plane in quadrupole approximation can conveniently be summarized in the following set of equations:

$$\tilde{R}\tilde{\beta}=e^{i\pi/2},\quad \tilde{R}\tilde{\gamma}=\frac{\epsilon}{2}\tilde{\beta}^2,$$
$$\tilde{R}=\hat{R}e^{i\rho},\quad \hat{R}=(1+\epsilon^2)^{1/2},\quad \rho=\arctan\epsilon. \quad (6.19)$$

This set can now be used to prove finally that local renormalization of the β plane in quadrupole approximation indeed explains the behavior of symmetry breaking around the dipole axis. To that end we reintroduce the coordinate system **y**, which is the **x** system rotated such that the y_2 direction coincides with the direction of the effective absolute vorticity gradient and the y_1 axis with the direction of the self-induced relative vorticity gradient of the dipole [see (5.11) and (5.12)]. In that system the complex effective absolute vorticity gradient $\tilde{\beta}$, as defined by (6.18), reads

$$\tilde{\beta}=\hat{\beta}e^{i\pi/2}. \quad (6.20)$$

Substituting this in the equation for $\tilde{\gamma}$, (6.19), we get for the components of γ in the **y** coordinate system

$$\tilde{\gamma}=\gamma_++i\gamma_\times=\frac{\epsilon}{2(1+\epsilon^2)^2}(-1+i\epsilon). \quad (6.21)$$

Here the $\gamma_\times$ component is indicative for the symmetry breaking around the dipole axis, whereas γ_+ reinforces the breaking of symmetry by the vorticity dipole around the axis in the direction of the effective absolute vorticity gradient. It is therefore more illuminating to consider the γ components relative to the strength of the gradient of relative vorticity induced by the dipole, as given by (5.13). Then we have

$$\frac{\gamma_+}{\beta_d}=\frac{-1}{2(1+\epsilon^2)^{3/2}},\quad \frac{\gamma_\times}{\beta_d}=\frac{\epsilon}{2(1+\epsilon^2)^{3/2}}. \quad (6.22)$$

This clearly demonstrates that for increasing ϵ the primary symmetry breaking around the axis in the direction of the effective absolute vorticity gradient continuously weakens, in accordance with the results of dipole approximation in Sec. V, whereas the secondary symmetry breaking around the dipole axis obviously has a maximum for maximum $\gamma_\times$, i.e., for $\epsilon\simeq O(1)$.

The shape of the Green's function, for increasing values of ϵ, is finally shown in Fig. 2. One recognizes immediately all the symmetry properties discussed hitherto. For small values of ϵ, in the weakly nonlinear regime, we recover the S mode with its strong symmetry breaking around the axis in the direction of the planetary vorticity gradient and its, asymptotically, perfect symmetry around the dipole axis. For intermediate nonlinearity, the symmetry is also broken around the dipole axis, resulting in a completely asymmetrical swirl, the rudiment of the primary symmetry axis being rotated in the direction of the applied rotation of the wind stress. Ultimately the fully nonlinear regime is characterized by weakening of both symmetry-breaking phenomena; i.e., the symmetry around the dipole axis returns, with the primary symmetry axis now being north-south, whereas concurrently

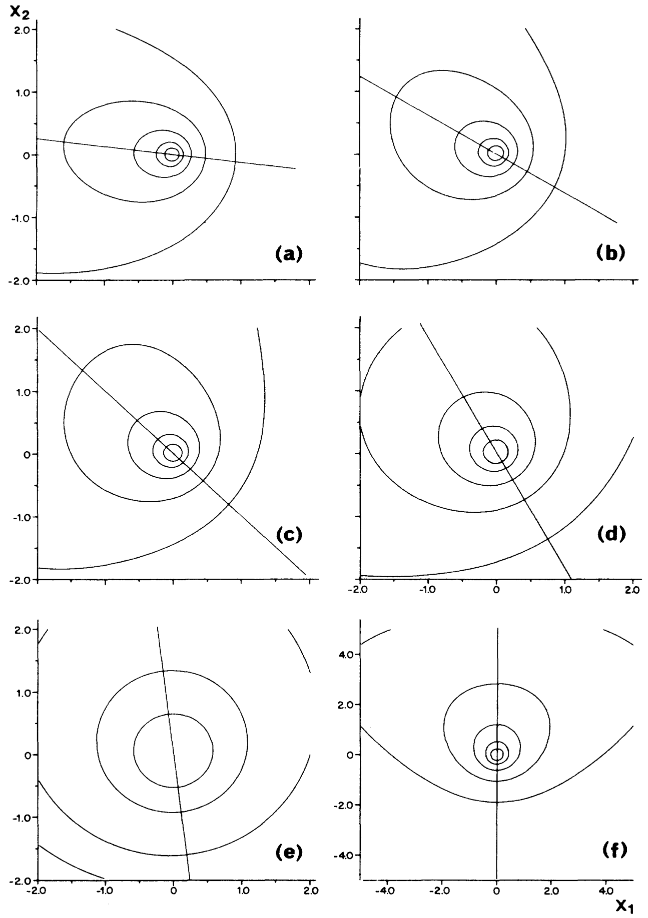

FIG. 2. Contour plots of the solution (6.10) for different values of ϵ, being (a) $\frac{1}{9}$, (b) $1/\sqrt{3}$, (c) 1, (d) $\sqrt{3}$, and (e) 9, respectively. In (a)–(e) the coordinate axes are dimensionless, being scaled by $2k/\beta_*$, the positive x_2 direction being the "north" direction. Also shown is the direction of the dipole axis given by β as defined in (5.8), being (a) 83°, (b) 59°, (c) 45°, (d) 29°, and (e) 6°. In (f) the picture (e) has coordinate axes scaled by an arbitrary external length scale L, such that the dimensionless internal length scale $\delta=5$ [see Eq. (7.9) and the discussion thereafter] for $\epsilon \rightarrow \infty$.

the remaining primary symmetry breaking, asymptotically around the east-west axis, weakens as well. The result is, *in coordinates scaled by* $2k/\beta_*$, a circulation pattern that becomes more and more circularly symmetric in the near field [see Fig. 2(e)]. However, rescaling in this limit, the coordinates with an arbitrary external length scale, shows that in the far field the pattern is still asymmetric around the east-west axis and asymptotically symmetric around the dipole axis. But in contrast to the oval shaped linear regime, the fully nonlinear regime has a butterfly pattern with a broadening to the north of the forcing position [see Fig. 2(f)]. We shall discuss these properties further in Sec. VII.

VII. DISCUSSION AND CONCLUSIONS

Renormalization of the β plane in quadrupole approximation has reduced the basic quasigeostrophic vorticity equation to an advection-diffusion equation for the Green's function in a strained velocity field. This makes it possible to interpret the ultimate results for the symmetry properties of the solution, as shown in Fig. 2, in terms of a trajectory $\mathbf{a}(\epsilon)$ in the effective absolute vorticity field ζ^*, which has hyperbolic contour lines. Here, in terms of the advection-diffusion equation, $\mathbf{a}(\epsilon)$, as given by (6.8), is the distance of the source position from the origin of the hyperbolic streamline pattern, represented by $\hat{\gamma} z_1 z_2$ in (6.9). We shall now discuss this trajectory in two, physically different, ways. First we discuss the trajectory for increasing wind-forcing amplitude T, everything else being constant. Secondly, we discuss the trajectory for decreasing wind forcing, such that the ratio of forcing and damping is constant.

In reality, for a specific area, we deal with the situation that the planetary vorticity gradient β_* is a well-known constant, that the damping coefficient k is rather uncertain, but, as a guess, probably not too dependent on the state of the circulation and thus a constant too, and that the wind-forcing amplitude is the only parameter that may vary over some range. At least, this is the parameter setting that has been explored in the numerical simulations mentioned before,[4] the symmetry properties of which we want to explain. Thus we start our discussion with the internal length scale $2k/\beta_*=\text{const}$ and for the forcing amplitude with $0<T<\infty$. Then, $0<\epsilon=T/2k^2<\infty$. Table I shows all the relevant dimensionless parameters that have been introduced in former

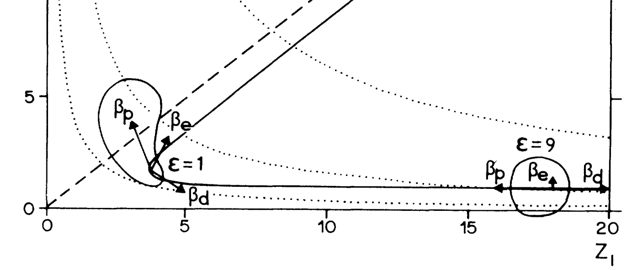

FIG. 3. The trajectory of $\mathbf{a}(\epsilon)$ (——) in the z_1,z_2 coordinate system. The coordinates are scaled by $2k/\beta_*$. Contour lines of effective absolute vorticity are shown by dotted lines. At the positions of $\epsilon=\frac{1}{9}$, 1, and 9 are depicted the direction of the planetary vorticity gradient ($\boldsymbol{\beta}_p$), which coincides with the local north direction in the z_1,z_2 coordinate system for the local value of ϵ, as well as the direction of the effective uniform gradient of absolute velocity ($\boldsymbol{\beta}_e$) and the self-induced global relative vorticity gradient due to the dipole ($\boldsymbol{\beta}_d$). The local shape of the circulation pattern for each value of ϵ is drawn schematically.

chapters, for three different values of ϵ and for the two asymptotes $\epsilon \to 0$ and $\epsilon \to \infty$. Figure 3 then shows in a concise way all the symmetry properties of the solution by depicting the trajectory $\mathbf{a}(\epsilon)$ of the "source position" in the translated and rotated coordinate system (z_1, z_2) for the whole range of ϵ. Also shown are the hyperbolic contours of the effective absolute vorticity distribution (or the streamlines in terms of an advection-diffusion equation) and, for specific values of ϵ, the local "north" direction ($\boldsymbol{\beta}_{np}$) and the directions of the effective uniform absolute vorticity gradient ($\boldsymbol{\beta}_e$) and of the self-induced global relative vorticity gradient of the dipole ($\boldsymbol{\beta}_d$), which coincides with the direction of the dipole axis. The trajectory $\mathbf{a}(\epsilon)$ is parametrically given by

$$\mathbf{a}(\epsilon)=2\left[\epsilon+\frac{1}{\epsilon}\right]\begin{bmatrix}\sin(\beta+\phi)\\ \cos(\beta+\phi)\end{bmatrix}, \tag{7.1}$$

where

$$\beta=\arctan\epsilon^{-1}$$

and

$$\phi=\frac{\pi}{4}-\frac{\gamma}{2},\quad \gamma=\arctan\left[\frac{\epsilon(3-\epsilon^2)}{3\epsilon^2-1}\right],$$

as follows from (6.8), (5.8), and (6.17). The asymptotes of $\mathbf{a}(\epsilon)$ read

$$\epsilon\to 0,\quad a_1\to\sqrt{2}\left[\frac{1}{2}+\frac{1}{\epsilon}\right],\quad a_2\to\sqrt{2}\left[-\frac{1}{2}+\frac{1}{\epsilon}\right], \tag{7.2}$$

whence asymptotically for $\epsilon \to 0$,

$$a_2=a_1-\sqrt{2} \tag{7.3}$$

and

$$\epsilon\to\infty,\quad a_1\to 2\epsilon\to\infty,\quad a_2\to 1\;. \tag{7.4}$$

Thus the trajectory first runs parallel to the bisectrix of the first quadrant of the (z_1, z_2) system, at a distance 1, and then turns and runs parallel to the z_1 axis, at a distance 1. As $\epsilon \to 0$ we see that $\mathbf{a}(\epsilon)$ is infinitely far from the origin in an area where the contour lines of effective absolute vorticity are nearly parallel and in the east-west direction, coinciding with those of the planetary vorticity. The asymptote $\epsilon \to 0$ is of course the linear S mode, with strong symmetry breaking around the north-south axis and perfect symmetry around the east-west axis, in accordance with the contour lines of a concentration field due to a continuous source in a uniform velocity field directed east-west. For increasing ϵ the trajectory approaches the origin along the bisectrix. Then the contour lines become more and more bended, which results in a breaking of symmetry around the dipole axis of the circulation pattern. Concurrently the north direction turns

TABLE I. Values of basic parameters for various values of ϵ. Quantities marked with an asterisk are evaluated in the $\mathbf{y}$ coordinate system with the y_1 axis in the direction of the dipole axis and the y_2 axis in the direction of the effective absolute vorticity gradient β_e.

	Eq.	$\epsilon\to 0$	$\epsilon=1/\sqrt{3}$	$\epsilon=1$	$\epsilon=\sqrt{3}$	$\epsilon\to\infty$
$\hat{\beta}=\beta_e{}^*$	(5.8)	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	ϵ^{-1}
β	(5.8)	$\pi/2$	$\pi/3$	$\pi/4$	$\pi/6$	ϵ^{-1}
$\beta_d{}^*$	(5.13)	ϵ	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	1
$\gamma_+/\beta_d{}^*$	(6.22)	$-\frac{1}{2}$	$-\frac{3}{16}\sqrt{3}$	$-\frac{1}{8}\sqrt{2}$	$-\frac{1}{16}$	$-\frac{1}{2}\epsilon^{-3}$
$\gamma_\times/\beta_d{}^*$	(6.22)	ϵ	$\frac{3}{16}$	$\frac{1}{8}\sqrt{2}$	$\frac{1}{16}\sqrt{3}$	$\frac{1}{2}\epsilon^{-2}$
$\hat{\gamma}$	(6.17)	$\frac{1}{2}\epsilon$	$\frac{3}{16}$	$\frac{1}{8}\sqrt{2}$	$\frac{1}{16}\sqrt{3}$	$\frac{1}{2}\epsilon^{-3}$
γ	(6.17)	π	$\pi/2$	$\pi/4$	0	$-\frac{1}{2}\pi+3\epsilon^{-1}$
ϕ	(6.7)	$-\pi/4$	0	$\pi/8$	$\pi/4$	$\frac{1}{2}\pi-\frac{3}{2}\epsilon^{-1}$
a_1	(6.8)	$\sqrt{2}\left[\frac{1}{2}+\frac{1}{\epsilon}\right]$	4	$\simeq 3.7$	$\simeq 4.5$	2ϵ
a_2	(6.8)	$\sqrt{2}\left[-\frac{1}{2}+\frac{1}{\epsilon}\right]$	$\frac{4}{3}\sqrt{3}$	$\simeq 1.5$	$\simeq 1.2$	$1+\epsilon^{-2}$
$\hat{a}=\hat{\beta}/\hat{\gamma}$	(6.8)	$2\epsilon^{-1}$	$\frac{8}{3}\sqrt{3}$	4	$\frac{8}{3}\sqrt{3}$	$2\epsilon^2$
α	(6.8)	$\frac{1}{4}\pi-\frac{1}{2}\epsilon$	$\pi/6$	$\pi/8$	$\pi/12$	$\frac{1}{2}\epsilon$

anticlockwise in the $\mathbf{z}$ system. When ultimately ϵ reaches a value of $O(1)$ the secondary symmetry breaking is strongest. The trajectory then reaches its apex in the corner of the first octant of the $\mathbf{z}$ system and the resulting circulation pattern is a completely asymmetrical swirl. Increasing ϵ further, the trajectory becomes more and more parallel to the z_1 axis and the secondary symmetry breaking becomes gradually weaker. When finally $\epsilon \to \infty$, all effective vorticity gradients weaken as the amplitude $\hat{\gamma}$ of the effective absolute vorticity field decreases. Ultimately the circulation pattern becomes more and more circularly symmetric for increasing ϵ. All these symmetry properties are in accordance with the behavior of numerical simulations in closed basins.[4] We stress here that the symmetry properties of the circulation, as they evolve along the trajectory $\mathbf{a}(\epsilon)$, are independent of the actual value of the planetary vorticity gradient β_*. Indeed, the components of vector $\mathbf{a}$ are scaled by $2k/\beta_*$, but so are z_1 and z_2, so that in the scaled coordinate system the shape of the trajectory is the same for all values of β_*. Thus the planetary vorticity gradient is a real *catalyst* for the broken symmetry of the circulation pattern. In its absence, for circularly symmetric forcing, the circulation pattern is circularly symmetric for all values of ϵ. Once β_* is present, no matter how small, the symmetry is broken; although, in dimensional coordinates, this manifests itself only at very large distances from the forcing center for very small β_*. However, the behavior of the symmetry properties as a function of ϵ, as represented by $\hat{\beta}$, β, $\hat{\gamma}$, and γ remains independent of β_*.

Clearly, in the limit $T \to 0$, for which $\epsilon \to 0$ if k is constant, we recover the linear "S mode." However, in the other limit, $T \to \infty$, and thus $\epsilon \to \infty$ for k is constant, the resulting circulation pattern is not the sought-after "almost free" fully nonlinear inertial "F mode." For, this mode is characterized by forcing and damping going to zero together, such that at the same time the circulation becomes strongly inertial. Evidently this happens when $T \to 0$ and $k \to 0$, such that T/k is constant, whence $\epsilon = T/2k^2 \to \infty$. The physical interpretation of this limit is as follows. The ratio k/T is the rotation time scale of the vortex. For, if $L = k/\beta_*$ is the vortex length scale and Tk/β_*^2 the scale of the stream function, then the relevant velocity scale $U = \psi/L = T/\beta_*$, whence the rotation time scale $L/U = k/T$. Thus the Reynolds number in this context, $\epsilon = T/k^2$, is the ratio of the damping time scale and the rotation time scale. In consequence, in the present limit the damping time scale goes to infinity for a given constant rotation time scale, so that the circulation becomes strongly inertial. This then is our definition of the F mode, which should be perfectly symmetric around the north-south axis and strongly asymmetric around the east-west axis, asymptotically. In order now to discuss the trajectory $\mathbf{a}(\epsilon)$ so that we may locate the F mode on it, we have to rescale our coordinates, as in their present dimensionless form they contain the now varying damping coefficient k. Besides, this gives us an opportunity to discuss what happens to the circulation pattern as seen by an observer located at the *fixed* dimensional distance L from the forcing center. The introduction of L gives another dimensionless number (Ekman number)

$$E = \frac{2k}{\beta_* L} . \tag{7.5}$$

Now assume T and k to vary over the interval $0-\infty$, such that T/k is constant. Rescaling our trajectory $\mathbf{a}(\epsilon)$ by L, it becomes

$$\mathbf{a}(\epsilon) = 2E \left[\epsilon + \frac{1}{\epsilon} \right] \begin{bmatrix} \sin(\beta+\phi) \\ \cos(\beta+\phi) \end{bmatrix} , \tag{7.6}$$

the asymptotes of which read

$$\epsilon \to 0, \quad a_1 \to \frac{E\sqrt{2}}{\epsilon} + \tfrac{1}{4}E\sqrt{2}, \quad a_2 \to \frac{E\sqrt{2}}{\epsilon} - \tfrac{1}{4}E\sqrt{2} , \tag{7.7}$$

$$\epsilon \to \infty, \quad a_1 \to 2\epsilon E, \quad a_2 \to E . \tag{7.8}$$

Figure 4 shows the trajectory of $\mathbf{a}(\epsilon)$ in the rescaled coordinate system. Obviously now, for $\epsilon \to 0$ and $T/k = \text{const}$, such that $k \to \infty$ and therefore also $E \to \infty$, the linear S mode in the z coordinate system is located infinitely far away from the origin where the effective absolute vorticty contours are nearly parallel. However, the F mode, asymptotically, is located at a fixed and *finite* distance from the origin, exactly at the z_1 axis. This distance is given by

$$\delta \equiv 2\epsilon E = \frac{2T}{k\beta_* L} . \tag{7.9}$$

To an observer at a fixed distance L from the source position the rescaled trajectory of $\mathbf{a}(\epsilon)$ now means that, for $T \to 0$ and $k \to 0$, such that T/k is constant, he sees an *increase* of the primary symmetry breaking around the direction of the effective absolute vorticity gradient, to-

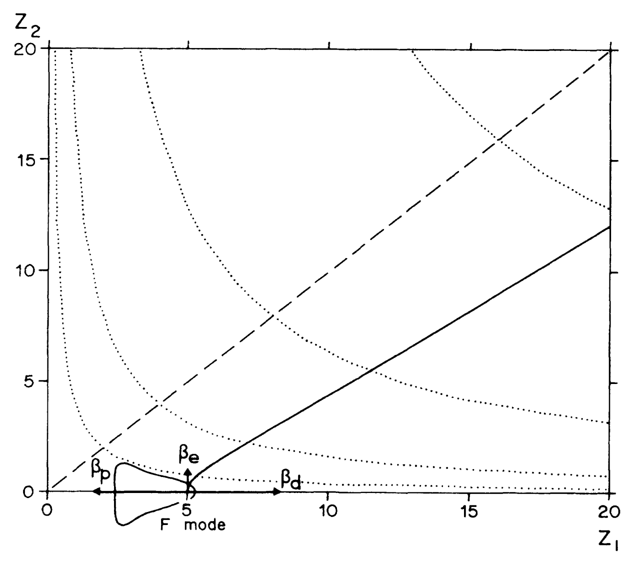

FIG. 4. As Fig. 3, but now with the coordinate axes scaled by an arbitrary external length scale L. The drawn contour plot of the inertial pattern coincides with Fig. 2(f).

gether with a turning of the dipole axis from east-west to north-south, and finally the establishment of a purely symmetric butterfly pattern around the dipole axis, which is along the z_1 axis in this limit, such that the induced relative vorticity gradient again annihilates the planetary vorticity gradient [see Figs. 4 and 2(f)]. That the primary symmetry breaking keeps a finite strength in this limit, as does the spreading of the contour lines giving rise to the butterfly pattern, whereas the secondary symmetry breaking around the dipole axis vanishes, can be made evident by rescaling the effective absolute vorticity field in the $\mathbf{y}$ coordinate system, which is the system having axes in the direction of the dipole axis and the effective linear absolute vorticity gradient; see (5.11). Rescaling y_1 and y_2 by L we have for the effective absolute vorticity field

$$\zeta^* = \beta_e y_2 + \gamma_\times y_1 y_2 + \tfrac{1}{2}\gamma_+ (y_1^2 - y_2^2) \,, \tag{7.10}$$

where now

$$\beta_e = \frac{1}{E(1+\epsilon^2)^{1/2}} \,, \tag{7.11}$$

$$\gamma_\times = \frac{\epsilon^2}{2E^2(1+\epsilon^2)^2}, \quad \gamma_+ = \frac{-\epsilon}{2E^2(1+\epsilon^2)^2} \,. \tag{7.12}$$

Asymptotically, for $k,T \to 0$, $T/k = \text{const}$, such that $\epsilon \to \infty$ and $E \to 0$, we get

$$\begin{aligned} &\beta_e \to (\epsilon E)^{-1} = 2\delta^{-1} = \text{const} \,, \\ &\gamma_\times \to \tfrac{1}{2}(\epsilon E)^{-2} = 2\delta^{-2} = \text{const} \,, \\ &\gamma_+ \to \tfrac{1}{2}\epsilon^{-3}E^{-2} = 2\delta^{-2}\epsilon^{-1} \to 0 \,. \end{aligned} \tag{7.13}$$

As $\beta_e = 2\delta^{-1}$ is the e-folding distance that gives the primary asymmetry of the circulation pattern, and $\gamma_\times, \gamma_+$, as coefficients in the effective absolute vorticity field (7.10), stand, respectively, for the induction of the symmetric spreading, giving the butterfly pattern and the induction of symmetry breaking around the dipole axis, we see that in the almost free inertial limit the pattern keeps a finite primary symmetry breaking, a vanishing secondary symmetry breaking, and a finite strength of the butterfly shape.

The results (7.8) and (7.9) suggest that in the inertial limit the pertinent internal length scale of the problem is $T/\beta_* k$, rather than $2k/\beta_*$. Indeed, actually we deal with a renormalized internal length scale $2k(1+\epsilon^2)^{1/2}/\beta_*$, of which $T/\beta_* k$ is the limit for $\epsilon \to \infty$. The length scale $T/\beta_* k$ is exactly the one inferred formerly[8] for an almost free inertial boundary layer in the F-mode limit of oceanic circulation in a closed basin. This length scale could be obtained by scaling arguments on the basis of an integral constraint around a closed streamline:

$$\oint \boldsymbol{\sigma} \cdot d\mathbf{l} = k \oint \mathbf{u} \cdot d\mathbf{l} \,, \tag{7.14}$$

which can easily be obtained by integrating (1.1) over an area enclosed by a streamline and an application of Stoke's theorem [here $\boldsymbol{\sigma}$ is the wind-stress vector divided by an effective depth of the fluid and $T\tau \equiv \mathbf{k} \cdot (\nabla \times \boldsymbol{\sigma})$, whereas $\mathbf{u}$ is the quasigeostrophic velocity, $\mathbf{u} \equiv \mathbf{k} \times \nabla \psi$]. Obviously the solution for the Green's function satisfies (7.14) as well. If we now rescale the basic equation (1.1) with $T/\beta_* k$ as the length scale for the coordinates and with $T^3/k^3\beta_*^2$ as the scale for the stream function ψ, we get

$$\mathfrak{J}(\psi, \Delta\psi + x_2) = \epsilon^{-1}(\tau - \Delta\psi) \,. \tag{7.15}$$

This is precisely the canonical form of an "almost free" inertial mode on the β plane, which also serves for other nearly inertial geophysical flow systems like modons in the atmosphere.[9] To zeroth order in ϵ^{-1}, (7.15) describes the conservation of absolute vorticity. The zeroth-order equation is degenerate in the sense that any single valued function of the absolute vorticity in terms of the stream function is a solution. The degeneration has either to be removed in higher orders of ϵ^{-1} by using integral constraint (7.14), which has not led to any closed solutions up to now, or by using an *a priori* assumption about the relationship between ψ and $\Delta\psi + x_2$, usually a linear one, as in the first discussion of the fully inertial F mode.[3] It is interesting to note that we have circumvented both problems here by coming the other way round: starting from the fully nonlinear equation, effectively linearizing it by β-plane renormalization, and finally taking the limit $\epsilon \to \infty$, we recover the almost free inertial mode. The symmetry properties of this mode are exactly those of the inertial solution to zeroth order in ϵ^{-1}, obtained by using the *a priori* linear relationship between the stream function and the absolute vorticity.[3]

In conclusion, then, the three basic symmetry properties of nonlinear forced circulation on the β plane can be explained by global and local renormalization of the β plane, using the dipole and quadrupole character of the induced relative vorticity distribution. This once more supports the notion that the dynamics of mid-ocean gyres is primarily vorticity dynamics, even in the simple context of barotropic circulation with "bottom frictional" vorticity damping. In a subsequent paper we hope to apply the idea of β-plane renormalization to a closed basin of simple geometry in order to reproduce the circulation's boundary-layer character.

ACKNOWLEDGMENTS

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APPENDIX A

Here we give a brief summary of the derivation[7] of the solution to

$$\Delta G + 2\gamma \left[z_1 \frac{\partial G}{\partial z_1} - z_2 \frac{\partial G}{\partial z_2} \right] = -\delta(\mathbf{z} - \mathbf{a}) \tag{A1}$$

on an unbounded plane. Introducing an integrating factor, we write

$$G=\exp[\tfrac{1}{2}\gamma(z_2^2-z_1^2)]\Gamma(\mathbf{z}) , \quad \text{(A2)}$$

which gives an equation for Γ:

$$\Delta\Gamma-\gamma^2\rho^2\Gamma=-\frac{1}{\rho}\exp[\tfrac{1}{2}\gamma(a_1^2-a_2^2)]\delta(\rho-a)\delta(\theta-\alpha) , \quad \text{(A3)}$$

where $\rho=|\mathbf{z}|$. Transforming to $\sigma=\frac{1}{4}\rho^2$, using the transformation rules for the δ function, (A3) becomes

$$\left[\frac{\partial^2}{\partial\sigma^2}+\frac{1}{\sigma}\frac{\partial}{\partial\sigma}+\frac{1\partial^2}{4\sigma^2\partial\theta^2}-4\gamma^2\right]\Gamma = -\frac{\hat{a}}{4\sigma^{3/2}}\exp[\tfrac{1}{2}\gamma(a_1^2-a_2^2)]\delta(\sigma-\sigma_a)\delta(\vartheta) , \quad \text{(A4)}$$

where $\sigma_a=\hat{a}^2/4$ and $\vartheta=\theta-\alpha$. Expanding now both $\delta(\vartheta)$ and Γ in a complex Fourier series in ϑ, we get the Bessel equation

$$\left[\frac{\partial^2}{\partial\sigma^2}+\frac{1}{\sigma}\frac{\partial}{\partial\sigma}-\frac{m^2}{4\sigma^2}-4\gamma^2\right]\Gamma_m = -\frac{\hat{a}}{4\sigma^{3/2}}\exp[\tfrac{1}{2}\gamma(a_1^2-a_2^2)]\delta(\sigma-\sigma_a) , \quad \text{(A5)}$$

where $\Gamma_m(\sigma)$ is the mth-order Fourier amplitude. The solution to the homogeneous part of Eq. (A5) is given in terms of modified Bessel functions of fractional order by

$$A_m I_{m/2}(2\gamma\sigma)+B_m K_{m/2}(2\gamma\sigma) , \quad \text{(A6)}$$

where the integration constants A_m and B_m follow from regularity conditions for $\sigma\to 0$ and $\sigma\to\infty$ and from requiring continuity at $\sigma=\sigma_a$. This gives

$$A_m=\tfrac{1}{2}\exp[\tfrac{1}{2}\gamma(a_1^2-a_2^2)]K_{m/2}(2\gamma\sigma_a) , \quad \text{(A7)}$$

$$B_m=\tfrac{1}{2}\exp[\tfrac{1}{2}\gamma(a_1^2-a_2^2)]I_{|m|/2}(2\gamma\sigma_a) . \quad \text{(A8)}$$

The solution of Γ in terms of a Fourier-Bessel series then reads

$$\Gamma=(4\pi)^{-1}\exp[\tfrac{1}{2}\gamma(a_1^2-a_2^2)][S_1(\sigma,\vartheta)+S_2(\sigma,\vartheta)] , \quad \text{(A9)}$$

where

$$S_1(\sigma,\vartheta)=I_0(2\gamma\sigma_<)K_0(2\gamma\sigma_>) +2\sum_{m=1}^{\infty} I_m(2\gamma\sigma_<)K_m(2\gamma\sigma_>)\cos(2m\vartheta) , \quad \text{(A10)}$$

$$S_2(\sigma,\vartheta)=2\sum_{m=0}^{\infty} I_{m+1/2}(2\gamma\sigma_<)K_{m+1/2}(2\gamma\sigma_>) \times\cos[(2m+1)\vartheta] , \quad \text{(A11)}$$

in which $\sigma_<=\sigma_a$ for $\sigma>\sigma_a$ and $\sigma_<=\sigma$ for $\sigma<\sigma_a$ and vice versa for $\sigma_>$. The first series, S_1, can be summed to the zeroth-order modified Bessel function, K_0:

$$S_1(\sigma,\vartheta)=K_0(2\gamma(\sigma^2+\sigma_a^2-2\sigma\sigma_a\cos2\vartheta)^{1/2}) =K_0(\tfrac{1}{2}\gamma R) \equiv\int_0^{\infty}\exp(-\tfrac{1}{2}\gamma R\cosh u)du , \quad \text{(A12)}$$

where

$$R=|\mathbf{z}-\mathbf{a}||\mathbf{z}+\mathbf{a}| . \quad \text{(A13)}$$

The second sum, S_2, can be reduced to an analogous expression[7]

$$S_2=\int_{\eta(\sigma,\vartheta)}^{0}\exp(-\tfrac{1}{2}\gamma R\cosh u)du , \quad \text{(A14)}$$

where the integration domain is given by

$$\eta(\sigma,\vartheta)=a\cosh\left[\frac{\rho^2+\hat{a}^2}{R}\right]=\ln\frac{|\mathbf{z}-\mathbf{a}|}{|\mathbf{z}+\mathbf{a}|} . \quad \text{(A15)}$$

Defining now the incomplete modified Bessel function of the second kind of order ν by

$$\mathcal{K}_\nu(p;q)=\int_q^{\infty}\cosh(\nu u)\exp(-p\cosh u)du , \quad \text{(A16)}$$

the solution of (A1) reads

$$G=(4\pi)^{-1}\exp[\tfrac{1}{2}\gamma(z_2^2-a_2^2-z_1^2+a_1^2)] \times\mathcal{K}_0[\tfrac{1}{2}\gamma R(\mathbf{z});\eta(\mathbf{z})] . \quad \text{(A17)}$$

APPENDIX B

Here we consider the calculation of the global derivatives B and C_{ij} for the solution of (A1), as given by (A17). Let $\mathcal{D}$ be a linear differential operator and consider $\int\int\mathcal{D}\Delta\psi\, dA$. For $\int\int\Delta\psi\, dA$ we have either, by Green's theorem, $\int\int\Delta\psi\, dA=\oint\mathbf{n}\cdot\nabla\psi\, ds$, where $\mathbf{n}$ is the outward normal unit vector $(\cos\theta,\sin\theta)$ and ds is the arc length of the contour, or, introducing the vector $\mathbf{v}=\mathbf{k}\times\nabla\psi$, which is the geostrophic velocity in the present context, we have by Stoke's theorem

$$\int\int\Delta\psi\, dA=\int\int\mathbf{k}\cdot(\nabla\times\mathbf{v})dA =\oint\mathbf{v}\cdot d\mathbf{l}=\int_0^{2\pi}r\,\partial\psi/\partial r\, d\theta ,$$

where $\mathbf{k}$ is the vertical unit vector and $d\mathbf{l}=r(-\sin\theta,\cos\theta)d\theta$. For a singular distribution that extends infinitely far, but drops to zero sufficiently rapidly, the only contribution to the contour integral comes from the part surrounding the pole. Then we may calculate $\int\int\mathcal{D}\Delta\psi\, dA$ by one of the two equalities

$$\int\int\mathcal{D}\Delta\psi\, dA=\lim_{r\to 0}\oint\mathbf{n}\cdot\mathcal{D}\nabla\psi\, ds =\lim_{r\to 0}\int_0^{2\pi}r\frac{\partial}{\partial r}\mathcal{D}\psi\, d\theta . \quad \text{(B1)}$$

If now ψ has a logarithmic singularity the only contributions to the integral come from terms in the expansion of ψ around the singularity that are independent of θ and proportional to $\ln r$, whereas one has to prove that, in order that the integral converges, all higher-order singularities, $\propto r^{-n}$, that may arise by applying the operator $\mathcal{D}$ on

ψ, do not have contributions independent of θ. We shall now apply these rules to (A17) in order to determine the coefficients in the renormalization of the β plane. For brevity we write

$$G=(4\pi)^{-1}e^{h}\mathcal{K}_0(\tfrac{1}{2}\hat{\gamma}R;\eta),\quad h=\tfrac{1}{2}\hat{\gamma}(z_2^2-z_1^2-a_2^2+a_1^2)\ , \tag{B2}$$

where R and η are defined by (A13) and (A15) and $\mathbf{a}$ is given by (6.8), which we shall use frequently.

Starting now with the determination of the global linear vorticity gradient $\mathbf{B}$, we need

$$\nabla G=(4\pi)^{-1}(\mathcal{K}_0\nabla h+\nabla\mathcal{K}_0)e^{h}\ . \tag{B3}$$

Only the first term in the power-series expansion of e^h is of use. Thus we have only to evaluate the terms between parentheses. These read

$$\hat{\gamma}\begin{bmatrix}-z_1\\ z_2\end{bmatrix}\mathcal{K}_0-\tfrac{1}{2}\hat{\gamma}\mathcal{K}_1\nabla R+\exp(-\tfrac{1}{2}\hat{\gamma}R\xi)\nabla\eta\ , \tag{B4}$$

where

$$\xi=\frac{\rho^2+\hat{a}^2}{R}\ . \tag{B5}$$

As the singularity of (B2) is located in $\mathbf{z}=\mathbf{a}$, it is more convenient to translate our coordinate system to $\mathbf{z}'=\mathbf{z}-\mathbf{a}$, in which the singularity is located at $\mathbf{z}'=0$. Then, for $r=|\mathbf{z}'|\to 0$, we have

$$\eta=\ln\frac{r}{|\mathbf{z}'+2\mathbf{a}|}\to-\infty \tag{B6}$$

and

$$R=r|\mathbf{z}'+2\mathbf{a}|=2\hat{a}r\left[1+\frac{\mathbf{a}\cdot\mathbf{z}'}{2\hat{a}^2}+\cdots\right]\ . \tag{B7}$$

Then the first term of (B4) gives

$$\mathcal{K}_0\to 2K_0(\tfrac{1}{2}\hat{\gamma}R)\to -2\ln r\ . \tag{B8}$$

Upon substitution of $\mathbf{z}=\mathbf{z}'+\mathbf{a}$ the final evaluation of this part of the contribution to the contour integral gives

$$\lim_{r\to 0}\int_0^{2\pi}r\frac{\partial}{\partial r}\nabla G\,d\theta=\hat{\gamma}\begin{bmatrix}a_1\\ -a_2\end{bmatrix}\ . \tag{B9}$$

As to the second term in (B4) for $\eta\to-\infty$ we have

$$\mathcal{K}_1\to 2K_1(\tfrac{1}{2}\hat{\gamma}R)\to 2K_1(\hat{\beta}r)=2(\hat{\beta}r)^{-1}+\cdots \tag{B10}$$

and

$$\nabla R=2\hat{a}\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}(1+\cdots)+\frac{\mathbf{a}}{\hat{a}}r+\cdots\ , \tag{B11}$$

from which it can be checked that $\mathcal{K}_1\nabla R$ does not contribute to the integral in (B9). As to the third term in (B4) we have

$$\nabla\eta=-\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}r^{-1}+\frac{\mathbf{a}}{2\hat{a}^2}+\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}\frac{r}{4\hat{a}^2}-\frac{(\mathbf{a}\cdot\mathbf{z}')\mathbf{a}}{2\hat{a}^4}+\cdots\ , \tag{B12}$$

which in product with $\exp(-\hat{\gamma}R\xi/2)$ does not contribute to the integral in (B4) either. Thus for the evaluation of $\mathbf{B}$ we are left with (B9), from which we get (in the z' coordinate system)

$$\mathbf{B}=\lim_{r\to 0}\int_0^{2\pi}r\frac{\partial}{\partial r}\nabla G\,d\theta=\hat{\beta}\begin{bmatrix}\sin(\beta+\phi)\\ -\cos(\beta+\phi)\end{bmatrix}\ . \tag{B13}$$

Next we have to evaluate the components of C_{ij} for which we need the second derivatives of G. Using $\partial_i\equiv\partial/\partial z_i$ as a shorthand notation for derivatives to z_i (or z_i'), we have

$$\begin{bmatrix}\partial_{11}G\\ \partial_{12}G\\ \partial_{22}G\end{bmatrix}=\frac{e^h}{4\pi}\left[\begin{bmatrix}\hat{\gamma}^2z_1^2-\hat{\gamma}\\ -z_1z_2\hat{\gamma}\\ \hat{\gamma}^2z_2^2+\hat{\gamma}\end{bmatrix}\mathcal{K}_0+\begin{bmatrix}-z_1\partial_1\mathcal{K}_0\\ \tfrac{1}{2}(z_2\partial_1\mathcal{K}_0-z_1\partial_2\mathcal{K}_0)\\ z_2\partial_2\mathcal{K}_0\end{bmatrix}\hat{\gamma}+\begin{bmatrix}\partial_{11}\mathcal{K}_0\\ \partial_{12}\mathcal{K}_0\\ \partial_{22}\mathcal{K}_0\end{bmatrix}\right]\ . \tag{B14}$$

As to the first term in (B14), after transition to $\mathbf{z}'=\mathbf{z}-\mathbf{a}$, only the terms independent of z_i' in the factor of $\mathcal{K}_0$ contribute to the contour integral, such that for this term

$$\lim_{r\to 0}\int_0^{2\pi}r\frac{\partial}{\partial r}\partial_{ij}G\,d\theta=\begin{bmatrix}-\hat{\beta}^2\sin^2(\beta+\phi)+\hat{\gamma} & \tfrac{1}{2}\hat{\beta}^2\sin 2(\beta+\phi)\\ \tfrac{1}{2}\hat{\beta}^2\sin 2(\beta+\phi) & -\hat{\beta}^2\cos^2(\beta+\phi)-\hat{\gamma}\end{bmatrix}\ . \tag{B15}$$

After transforming to the $\mathbf{z}'$ system using (B11) and (B12), it can be checked that there is no contribution from the second term of (B14) to the integral in the left-hand side of (B15). Finally the third term can be written as

$$\partial_{ij}\mathcal{K}_0=-\tfrac{1}{2}\hat{\gamma}\partial_{ij}R\mathcal{K}_1+\frac{\hat{\gamma}^2}{4}\partial_iR\,\partial_jR\left[\mathcal{K}_0+\frac{2}{\hat{\gamma}R}\mathcal{K}_1-\frac{2}{\hat{\gamma}R}(\sinh\eta)\exp(-\tfrac{1}{2}\hat{\gamma}R\xi)\right]+\exp(-\tfrac{1}{2}\hat{\gamma}R\xi)[-\tfrac{1}{2}\hat{\gamma}\partial_iR\,\partial_j\eta\cosh\eta+\partial_{ij}\eta-\tfrac{1}{2}\hat{\gamma}\partial_i\eta\partial_j(R\xi)]\ . \tag{B16}$$

Using again (B8), (B10), (B11), and (B12), one can check that only the first part of the second term contributes to the evaluation of C_{ij} by means of the integral in (B15). Its contribution reads

$$-\hat{\beta}^2\begin{bmatrix}\tfrac{1}{2} & 0\\ 0 & \tfrac{1}{2}\end{bmatrix}\ , \tag{B17}$$

whence we have in the end for C_{ij} (in the $\mathbf{z}'$ coordinate system)

$$C_{ij}=\begin{bmatrix} -\frac{1}{2}\hat{\beta}^2[2\sin^2(\beta+\phi)+1]+\hat{\gamma} & \frac{1}{2}\hat{\beta}^2\sin2(\beta+\phi) \\ \frac{1}{2}\hat{\beta}^2\sin2(\beta+\phi) & -\frac{1}{2}\hat{\beta}^2[2\cos^2(\beta+\phi)+1]-\hat{\gamma} \end{bmatrix}, \tag{B18}$$

which gives

$$C_+=\tfrac{1}{2}\hat{\beta}^2\cos2(\beta+\phi)+\hat{\gamma},\quad C_\times=\tfrac{1}{2}\hat{\beta}^2\sin2(\beta+\phi)\ . \tag{B19}$$

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