On a new conservation law in Hydrodynamics

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Abstract

Distinguishing between globally and locally (materially) conserved quantities, existing conservation laws in hydrodynamics are reviewed. A brief comment is made on the possibility of recursively applying Ertel's theorem. A new flux conservation law, leading under certain conditions to a globally conserved quantity, is derived for both 2D and 3D flows. When position and velocity fields are uniquely related, this quantity can be considered as the conservation of mass in velocity space.

1 Introduction

The nonlinear equations of hydrodynamics, describing the flow of a fluid, contain a number of invariants, or conserved quantities, which constrain this flow in some sense. Knowledge of these invariants is of paramount importance in restricting the solution space. Indeed it has been shown for some simpler, but still nonlinear subsystems of the general equations of hydrodynamics (*e.g.* the $1D$ shallow water equations (Whitham, 1974; Miura, 1974) and the Korteweg-de Vries equation (Miura, Gardner and Kruskal, 1967), that they contain an *infinite* number of conserved quantities. This seems to restrict the flow to the extent that one is able to obtain exact solutions of these systems of equations. Whether the existence of an infinite set of conserved quantities always implies integrability is a question not fully resolved (Miura 1976).

Quantities can be conserved *globally* and/or *locally*. A globally conserved quantity is obtained from a (flux) conservation law:

$$\frac{\partial T}{\partial t} + \nabla \cdot \mathbf{X} = 0, \tag{1}$$

which relates the local time (t) evolution of a quantity T to the spatial ($\mathbf{x}$) divergence of a flux $\mathbf{X}$. By applying Gauss' theorem, the integral of (1) over a volume V of fixed size generates the globally conserved quantity $\int\int\int T dV$, whenever the integrated flux normal to the boundary of V (denoted by A), $\int\int \mathbf{X}\cdot d\mathbf{A}$, vanishes (or, in case the integration domain extends to infinity, drops off with distance rapidly enough). The conservation of energy in hydrodynamics often appears in this form. One may be tempted to apply this integration not over the entire fluid domain, but rather over some restricted part of it. There is no basis however to expect this to lead to any conserved quantities in a fixed, Eulerian space, but, once applied to a material fluid 'element', conservation of some properties of the fluid packet in Lagrangian space is feasible. As the material fluid packet itself is moving with the fluid flow, $\mathbf{u}, a$ local, or more properly speaking, *materially* conserved quantity Q is satisfying

$$\frac{dQ}{dt} = 0, \tag{2}$$

where the material derivative d/dt, now containing the advective operator

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla,$$

denotes the evolution in time following the motion of this packet. Materially conserved quantities can be obtained by integration of equations similar to (1), but which differ depending on the dimension n over which the integration is performed (*i.e.* whether one is dealing with a line, surface or volume element). The reason for this difference is that the fluid domain - and in particular its boundary - is now itself evolving, in contrast with the integration of (1) over a fixed domain. Fortak (1956), generalizing earlier results by Ertel(1942a,b, 1954) and Ertel and Rossby (1949), showed that in three-dimensional space for any tensor quantity $A_{i..}$,

$$\frac{d}{dt}\left[\int A_{i..} dV_n\right] = \int \frac{D_n}{Dt} A_{i..} dV_n, \tag{3}$$

where dV_n denotes the line (dx_i), surface (dA_i), or volume (dV) element of dimension $n = 1, 2$ or 3, over which the integration is performed. The operators D_n/Dt are defined by

$$\frac{D_n}{Dt} A_{i..} \equiv \begin{cases} \frac{d}{dt} A_{i..} + (\partial_i u_l) A_{l..} \quad (\mathrm{n} = 1) \\ \frac{d}{dt} A_{i..} + (\partial_l u_l) A_{i..} - (\partial_l u_i) A_{l..} (\mathrm{n} = 2) \\ \frac{d}{dt} A_{i..} + (\partial_l u_l) A_{i..} \quad (\mathrm{n} = 3) \end{cases} \tag{4}$$

using the abbreviation $\partial_i \equiv \partial/\partial x_i$. Here, as in the subsequent part of this paper, repeated occurence of indices implies summation over their entire range. Whenever a variable satisfies the equation $D_n A_{i..}/Dt = 0$, for some n, Eq. (3) shows that we can obtain a materially conserved quantity $\int A_{i..} dV_n$. One example is the conservation of density, ρ, given by (Batchelor, 1967)

$$(\frac{d}{dt} + \nabla \cdot \mathbf{u})\rho = 0 \tag{5}$$

which, in Fortak's terminology, is nothing but $D_3\rho/Dt = 0$, whence we obtain the material conservation of mass: $d/dt[\int\int\int \rho dV] = 0$. Often one encounters an equation expressing the material conservation of density itself, but this is just a consequence of making the approximation that the flow itself is non-divergent. Another example is afforded by the well-known vorticity equation derived by Helmholtz (Lamb, 1932, p.205), also known as the d'Alembert-Euler equation (Truesdell, 1954) for an inviscid, barotropic (pressure, p, being a function of the density) fluid:

$$\frac{d}{dt}\left(\frac{\omega}{\rho}\right) = \left(\frac{\omega}{\rho}\right) \cdot \mathbf{u}, \tag{6}$$

where the vorticity, $\omega = \nabla \times \mathbf{u}$, the curl of the velocity field. Together with Eq. (5) this can be rewritten as

$$\frac{d}{dt}\omega + (\nabla \cdot \mathbf{u})\, \omega - (\omega \cdot \nabla)\mathbf{u} = 0, \tag{7}$$

which is the vectorial equivalent of Eq. (4b), or

$$\frac{D_2}{Dt}\omega = 0, \tag{8}$$

from which we obtain, :

$$\frac{d}{dt}\int\int_A \omega \cdot d\mathbf{A} = 0, \tag{9}$$

which, by Stokes' theorem, expresses just Kelvin's material conservation of circulation $C = \oint_\Gamma \mathbf{u} \cdot d\mathbf{x}$, with Γ the boundary of the open surface A.

Intuition suggests that materially conserved quantities impose stronger constraints on a flow than their global counterparts do, as the former are obeyed by every single fluid element rather than by just the entire volume of fluid. Indeed, in geophysical fluid dynamics (Pedlosky, 1987), the material conservation of potential vorticity (Section 2) controls a great deal of the existing flow fields and is a central principle in many modelling studies in this field. In passing we remark on a recursive application of Ertel's theorem, which, ideally, generates an infinite number of materially conserved quantities. The main emphasis will be on the derivation of a new conservation law (Section 3), whose impact needs to be assessed in future studies.

In order to put these new conservation laws into proper perspective, as a prelude, we review existing conservation laws and materially conserved quantities appearing in fluid dynamics (Section 2). As there are many situations in which conservation laws arise, depending on the precise assumptions, this review is necessarily incomplete and sketchy. Because of the great practical interest in the shallow water equations on a rotating plane a ($2D$ subset of the general hydrodynamic equations, particularly arising for a hydrostatic, homogeneous fluid) conservation laws of this system of equations are discussed separately.

2 Review of conservation laws in hydrodynamics

With an application to GFD in mind, in this paper we will frequently refer to the equations of motion on an f-, or β-plane[8]. The f-plane can be considered as a plane tangent to the earth, of which the rotation rate, $f/2$, is constant and which is determined by the projection of the earth's rotation vector, $\mathbf{\Omega}_0$, on the local vertical: $f_0 = 2 \mid \mathbf{\Omega}_0 \mid \times \sin(\varphi_0)$. Here φ_0 denotes the central latitude around which the approximation is made. For relatively large-scale features the Coriolis parameter, f, varies with latitude, which, in β-plane approximation $-\beta \equiv \partial f/\partial\varphi(\varphi_0)$ - is mimicked by adding a linearly varying part, βy, to the constant part f_0 :

$$f = f_0 + \beta y, \tag{10}$$

in this case one speaks of $a\beta$-plane.
Coordinates $\mathbf{x}$, *i.e.* x, y and z are denoting the "East", "North" and vertical (positive upwards) direction respectively. Velocities along these Cartesian axes will be denoted by $\mathbf{u} = (u, v, w)$.

2.1 Conservation laws in three Dimensions

Starting point for any hydrodynamical problem are the conservation laws which constitute the equations of motion. They are
1) *Conservation of momentum*; an integral relation which leads to the momentum equations

$$\rho\frac{d}{dt}\mathbf{u} + \rho 2\mathbf{\Omega} \times \mathbf{u} + \nabla p - \rho\mathbf{g} = \mathbf{F}. \tag{11}$$

Here ∇p is the pressure gradient, $\mathbf{g} = -g\hat{k}$ is the acceleration of gravity, pointing downwards (where $\hat{k}$ is the vertical unit vector) and the rotation vector $\mathbf{\Omega} = (0, 0, f/2)$, according to the β-plane approximation referred to above. $\mathbf{F}$ contains any other surface or volume forces present and in particular the viscous forces, which, as is common when considering *conservation* laws, are neglected throughout the main part of this paper.

2) *Conservation of mass* as given by Eq. (5).

From these two we can derive

3) *Conservation of energy.* Special forms of this conservation law are derived under particular circumstances (see Batchelor, 1967). Thus, for example, a flux-conservation law for the mechanical energy is derived by taking the dot-product of (11) with $\mathbf{u}$. Then, in an *incompressible* ($\nabla \cdot \mathbf{u} = 0$) medium, it becomes

$$\frac{\partial}{\partial t}(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u} - \rho\mathbf{g}\cdot\mathbf{x}) + \nabla\cdot(\mathbf{u}(p + \frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u} - \rho\mathbf{g}\cdot\mathbf{x})) = 0. \tag{12}$$

When the fluid is considered compressible, usage of the internal energy, E, related to the previously introduced dynamical variables by

$$\frac{dE}{dt} = -\frac{p}{\rho}\nabla\cdot\mathbf{u},$$

establishes the material conservation of

$$1/2\mathbf{u}\cdot\mathbf{u} + p/\rho + E \tag{13}$$

(Bernoulli's theorem) *provided* the pressure is time-independent.

4) *Conservation of vorticity*, as given in Eqs. (7 - 9), can be considered the hydrodynamical analogue of the conservation of spin-angular momentum in classical particle mechanics (see Appendix A). This follows by taking the curl of the momentum equations (11), except that on a rotating frame it is the

absolute vorticity ω_a, being the vectorsum of the relative vorticity ω and the planetary vorticity $2\mathbf{\Omega}$, which is conserved. Remark that whenever pressure and density are not uniquely related a solenoidal forcing term appears in the right-hand side of (9), given by $\int \nabla p \times \nabla(1/\rho) dA$; an equality known as Bjerknes' theorem[8].

5) *Conservation of potential vorticity.* Potential vorticity is a term really encompassing a *class* of materially conserved quantities. Ertel (1942a) ascertained that for *any* conservative property λ (*i.e.* with $d\lambda/dt = 0$), the dot-product of Eq. (6) (where ω is replaced by absolute vorticity ω_a) with $\nabla\lambda$ yields, in the absence of any forcing and dissipation, a materially conserved quantity

$$\Pi_0 = \frac{\omega_{\mathbf{a}} \cdot \nabla\lambda}{\rho}, \tag{14}$$

provided $\nabla\lambda \cdot (\nabla p \times \nabla(1/\rho))$ vanishes, a condition particularly met with whenever $\lambda = \lambda(p, \rho)$ and trivially satisfied for a barotropic fluid. This assertion has an interesting recursive property: once we have decided on a conserved property λ to generate a Π_0 according to (14) we may then use Π_0 as conserved quantity λ to generate Π_1 and so on according to the scheme

$$\Pi_n = \frac{\omega_{\mathbf{a}} \cdot \nabla\mathbf{\Pi_{n-1}}}{\rho} \quad (n = 1, 2, 3 \ldots).(15)$$

6) *Conservation of angular momentum*, is, as discussed in Appendix A, related to the conservation of orbital angular momentum of a portion of fluid. It is obtained in flux conservation form from the equations of motion (11) and the kinematical relation $d\mathbf{x}/dt = \mathbf{u}$ for a non-rotating frame only:

$$\frac{\partial}{\partial t}\rho(\mathbf{x} \times \mathbf{u}) + \nabla \cdot [\mathbf{u}\rho(\mathbf{x} \times \mathbf{u}) + Xp] = 0, \tag{16}$$

where X is the 'biposition' tensor, the skew-symmetric tensor conjugate to the position vector $\mathbf{x}$:

$$X = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}. \tag{17}$$

A similar conservation law on a rotating plane does exist for planar flow in a plane normal to the axis of rotation (next subsection). This property is lost however, for general $3D$ flows on rotating f- or β-planes, despite the fact that it exists on a rotating globe when evaluated with respect to its rotation axis, which is well-known and applied *e.g.* in meteorology (Holton, 1979).

7) *Conservation of helicity.* In barotropic conditions, $p = p(\rho)$, the equations of motion (11) can be written, again for the nonrotating case, as

$$\frac{d}{dt}\mathbf{u} = \nabla(P + \Phi), \tag{18}$$

where $P = /\rho$ and where all forces are assumed to be conservative and hence derivable from a potential Φ. Under these conditions the vorticity equation takes the d'Alembert-Euler form, Eq. (6). Then the dot-product of $\mathbf{u}$ with Eq. (6), in combination with the dot-product of ω/ρ with (18), yields

$$\frac{d}{dt}\left(\frac{\mathbf{u}\cdot\omega}{\rho}\right) = \frac{\omega}{\rho}\cdot\nabla(\frac{1}{2}\mathbf{u}\cdot\mathbf{u} - P - \Phi). \tag{19}$$

By employing the definition of a Lagrange function, L, defined in terms of an action function (*Wirkungsfunktion*), W:

$$L \equiv \frac{dW}{dt} = \frac{1}{2}\mathbf{u}\cdot\mathbf{u} - P - \Phi, \tag{20}$$

this can, of course, be rewritten as

$$\frac{d}{dt}\left(\frac{\mathbf{u}\cdot\omega}{\rho}\right) = \frac{\omega}{\rho}\cdot\nabla\frac{dW}{dt}. \tag{21}$$

With the aid of the continuity equation (5) this is taken into flux conservation form

$$\frac{\partial}{\partial t}(\mathbf{u}\cdot\omega) + \nabla\cdot\left[\mathbf{u}(\mathbf{u}\cdot\omega) - \omega\frac{dW}{dt}\right] = 0, \tag{22}$$

from which we immediately retrieve the global conservation of the quantity $\int \mathbf{u}\cdot\omega dV$, termed helicity (Moffatt, 1969. Conservation of potential vorticity and helicity have later been generalized (see Mobbs, 1981, Gaffet, 1989) (in analogy to its use in particle physics), provided either $\omega\cdot\mathbf{n}$ vanishes on the solid surface S bounding the volume V, or ω decays sufficiently rapid ($|\,\omega\,| = \mathcal{O}(|\,\mathbf{x}\,|^{-4})$, when S is taken at infinity.

It apparently went by unnoticed that, prior to the establishment of the importance of the global conservation of helicity, Ertel and Rossby had, already in 1949, derived its *materially* conserved counterpart (Ertel and Rossby, 1949). This is retrieved from (21) by reversing the order of the derivatives in its right-hand side, so that it becomes

$$\frac{d}{dt}\left(\frac{\mathbf{u}\cdot\omega}{\rho}\right) = \frac{\omega}{\rho}\cdot\frac{d}{dt}\nabla W + \left[\left(\frac{\omega\cdot\mathbf{u}}{\rho}\right)\mathbf{u}\right]\cdot\nabla W. \tag{23}$$

By subtracting the dot-product of ∇W with the d'Alembert-Euler vorticity equation (6) from Eq. (23), we obtain the material conservation law

$$\frac{d}{dt}\left(\frac{\omega}{\rho}\cdot(\mathbf{u}-\nabla W)\right) = 0. \tag{24}$$

They readily extended their result to an application in a rotating frame of reference by substituting the 'absolute velocity' $\mathbf{u}_a = \mathbf{u} + \mathbf{\Omega}\times\mathbf{x}$ and absolute vorticity $\omega_a = \omega + 2\mathbf{\Omega}$ instead of the relative velocity $\mathbf{u}$ and vorticity ω proper, and by adding a term $\mathbf{\Omega}\cdot(\mathbf{x}\times\mathbf{u})$ to the definition of dW/dt, Eq. (20). The impact of the Ertel-Rossby material conservation law (exclusively present in $3D$ flows) has remained largely undiscussed (Truesdell, 1954). A flux conservation equation for helicity on a rotating plane then is obtained similarly by replacing the same quantities in (22), except for the advective velocity appearing in the flux (*i.e.* the first $\mathbf{u}$ in between square brackets), which remains unchanged.

2.2 Conservation laws in two dimensions

In the particular circumstance that the fluid under consideration is 1) homogeneous, 2) hydrostatic, 3) inviscid and 4) initially free of shear in the vertical, the equations of motion can be replaced by the shallow-water, or long-wave equations. Condition 2) especially has prompted the naming of this approximate set of equations as it requires the wavelengths of the phenomena involved to be larger than the water depth. By condition 3) one may circumvent the introduction of boundary layers near horizontal boundaries, while condition 4) is a necessary requirement for the continued vanishing of any vertical shear in the field variables. Under these conditions the horizontal velocity field is replaced by its vertically-averaged counterpart, such that we obtain

1) *Conservation of momentum,*

$$\frac{d}{dt}\mathbf{u} + f\hat{\mathbf{k}}\times\mathbf{u} + g\nabla\zeta = 0, \tag{25}$$

where, in this subsection, $d/dt = \partial/\partial t + u\partial_x + v\partial_y$, $\mathbf{u} = (u, v)$, $f = 2|\mathbf{\Omega}| \times \sin\varphi$ is the Coriolis parameter, $\hat{\mathbf{k}}$ is the vertical unit vector, $\mathbf{x} = (x, y)$, $\nabla = (\partial_x, \partial_y)$ and ζ is the vertical elevation of the free surface above the mean position.

We obtain similarly

2) *Conservation of mass,*

$$\frac{\partial\zeta}{\partial t} + \nabla \cdot [\mathbf{u}(H + \zeta)] = 0, \tag{26}$$

where $H(x, y)$ is denoting the bottom profile. This latter equation can be recast in terms of the total depth,

$$(\frac{d}{dt} + \nabla \cdot \mathbf{u})h = 0. \tag{28}$$

The operator in brackets is the $2D$ equivalent of the D_3/Dt - operator, introduced in (4). In analogy with (3), Eq. (28) generates the material conservation of mass as

$$\frac{d}{dt} \int\int h dA = 0. \tag{29}$$

Using (28), for a uniformly rotating (f = constant), uniform depth (H = constant) sea, the vertically integrated momentum equation (25) can be cast in a true flux conservation form

$$\frac{\partial}{\partial t} [\,(\mathbf{u} + f\hat{k} \times \mathbf{x})h\,] + \nabla \cdot [\,\mathbf{u}h(\mathbf{u} + f\hat{k} \times \mathbf{x}) + \tfrac{1}{2}gh^2\,] = 0. \tag{30}$$

It may be remarked that in a non-rotating ($f = 0$) fluid of uniform depth the one-dimensional analogs of (25) and (26) are satisfied by an *infinity* of conserved quantities, which are generated by an appropriate algorithm. Whitham[1] suggested this to be related to the fact that the equations can be solved exactly by means of a hodograph transformation, which switches the roles of dependent and independent variables. In a rotating frame ($f \neq 0$) this algortihm breaks down, the equations remain two-dimensional and the invariants come one by one.

By taking the curl of (25) we obtain

3) *Conservation of vorticity*

$$(\frac{d}{dt} + \nabla \cdot \mathbf{u})\omega_a = 0, \tag{31}$$

where $\omega_a \equiv f + \omega$, with $\omega = v_x - u_y$, now expressing only the vertical (z) component of its $3D$ counterpart. The actual conserved property, again is the horizontally integrated vorticity

$$\frac{d}{dt}\left[\int\int \omega_a dA\right] = 0, \tag{32}$$

a $2D$ version of Kelvin's theorem. Combining (31) with the conservation of mass (28) we obtain one member of an ensemble of conserved quantities known as

4) *Conservation of potential vorticity,*

$$\frac{d}{dt}\left(\frac{\omega_a}{h}\right) = 0. \tag{33}$$

Indeed, from (33) it is obvious that *any* differentiable function $G(\omega_a/h)$ is materially conserved, a feature applied by Stern (Stern, 1975) in obtaining a description of modons by taking a particular polynomial form $G = (\omega_a/h)^\gamma$, where in his special case γ was assumed to be integer. This generalized form of the potential vorticity is sometimes applied to its integrated counterpart, where it takes the form

$$\frac{d}{dt}\left[\int\int hG(\omega_a/h)dA\right] = 0. \tag{34}$$

The potential vorticity, appearing in Eq. (33), ω_a/h, corresponding to the polynomial form of G with $\gamma = 1$, can again be interpreted as the fluid dynamical analog of the conservation of spin angular momentum, $I\omega_a$, once $1/h$ is interpreted as the moment of inertia, I, of a fluid cylinder of "infinitesimal radius R". The latter phrase is placed in between quotes as it refers to a (for didactical purposes useful, but *qua* definition of R) nebulous concept of a cylinder of fluid. In terms of polar coordinates r and θ, $I = \int_0^R \int_0^{2\pi} h\rho r^2 r dr d\theta = \frac{1}{2}\pi\rho h R^4$.

$$M = \int_0^R \int_0^{2\pi} h\rho r dr d\theta = \pi\rho h R^2, \tag{36}$$

therefore $I = \left(\frac{M^2}{2\pi\rho}\right)\frac{1}{h} \propto \frac{1}{h}$.

Using $\gamma = 2$, we obtain what is most properly described as the conservation of potential enstrophy, $(\omega_a/h)^2$. The conservation of enstrophy (vorticity squared) proper does not, as often loosely stated, follow from the shallow-water equations, but is conserved only on the sphere, or, on the plane within the quasi-geostrophic approximation. The flux conservation form of the potential enstrophy conservation law is

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\frac{\omega_a^2}{h}\right) + \nabla\cdot\left(\mathbf{u}\frac{1}{2}\frac{\omega_a^2}{h}\right) = 0, \tag{38}$$

which is best interpreted as the fluid dynamical analog of the global conservation of 'spin' kinetic energy, associated with the solid body rotation of the fluid column, once we again interpret $1/h$ as moment of inertia, I.

This is not to be confused with

5) *Conservation of energy*, which refers to the energy associated with the linear momentum, and which is conserved (in a global sense) separately. We obtain this conservation equation by taking the dot-product of $h\mathbf{u}$ with the momentum equations (25) and adding $1/2\mathbf{u}\cdot\mathbf{u} + gh$ times the conservation of mass, Eq. (28), to it

$$\frac{\partial}{\partial t}\left[\,\tfrac{1}{2}h\mathbf{u}\cdot\mathbf{u} + \tfrac{1}{2}gh^2\,\right] + \nabla\cdot\left[\,\mathbf{u}(\tfrac{1}{2}h\mathbf{u}\cdot\mathbf{u} + gh^2)\,\right] = 0. \tag{39}$$

As the Coriolis force is normal to the momentum vector this force is doing no work and hence does not appear in (39).

6) *Conservation of angular momentum*, is again referring to *orbital* angular momentum, defined on a rotating plane as

$$Q \equiv xv - yu + \frac{1}{2}fr^2. \tag{40}$$

The flux conservation equation then takes the form

$$\frac{\partial}{\partial t}(Qh) + \nabla\cdot[\mathbf{u}Qh + \frac{1}{2}gh^2\hat{k}\times\mathbf{x}] = 0. \tag{41}$$

Therefore, contrary to the $3D$ case, it is possible to define the conservation of orbital angular momentum on a rotating plane as ω is directed perpendicular to $\mathbf{x}$, which strictly lies in the horizontal plane. The integral constraint, derived from (41), has for instance been used in a study on isolated elliptical vortices (Cushman-Roisin et al 1985, Young, 1986).

3 A new flux conservation law

Another conserved property can be added to the list given in the previous section, for which it is most instructive to follow the derivation in $2D$, that is, starting from the shallow-water equations. These results, discussed in section 2.1, suggest the way to generalize the conservation law in 3 dimensions, the topic of Section 3.2.

3.1 The conservation law in 2D

By taking the spatial derivatives of the momentum equations (25) we may derive evolution equations for, what are called (Molinari and Kirwan, 1975) the differential kinematic properties (DKP's) of the flow, *i.e.* for the vorticity, ω, divergence, δ, stretching deformation, s_+ and shearing deformation, $s_\times$, defined as

$$\begin{aligned} \omega &= v_x - u_y \\ \delta &= u_x + v_y \\ s_+ &= u_x - v_y \\ s_\times &= v_x + u_y \end{aligned} \tag{42}$$

The subscripts of s_+ and $s_\times$ can be thought of as pictorially referring to the principal axes, along which the deformation takes place. These equations, on a rotating β-plane ($f = f_0 + \beta y$) become (Petterssen, 1953; Kirwan, 1975)

$$\begin{aligned} \frac{d}{dt}\omega + (f_0 + \omega)\delta + \beta v &= 0 \\ \frac{d}{dt}\delta + \frac{1}{2}(s_+^2 + s_\times^2 + \delta^2 + \omega^2) - f_0\omega + \beta u &= -g\Delta\zeta \\ \frac{d}{dt}s_+ + s_+\delta - f_0 s_\times - \beta u &= -g\tilde{\Delta}\zeta \\ \frac{d}{dt}s_\times + s_\times\delta + f_0 s_+ - \beta v &= -2g\zeta_{xy}, \end{aligned} \tag{43}$$

where

$$\tilde{\Delta} \equiv \partial_{xx} - \partial_{yy}. \tag{44}$$

In the traditional treatment of these equations it is argued that the complications in solving equations (43) arise from the left-hand side of the divergence equation (43*b*). It is subsequently argued that approximate solutions can be obtained by neglecting the local evolution in time of the divergence

- thus rendering a *diagnostic* equation, instead of its full *prognostic* form (43b). This approximation, which filters out time-dependent gravity waves (Holton, 1979), is often defended (Petterssen, 1953) by noting that the numerical magnitude of the neglected term is much smaller than those of the remaining terms (at least in the application to large scale planetary waves).

However, this approximation is unnecessary and, moreover, destroys the existing symmetry within Eqs. (43), since a flux conservation law is obtained by multiplying equation (43.a, b, c and d) with ω, δ, s_+ and $s_\times$ respectively, and by subsequently adding the first two of theseand subtracting the last two from them. We then obtain

$$\frac{d}{dt}\frac{1}{2}\left((\delta^2+\omega^2)-(s_+^2+s_\times^2)\right)+\frac{\delta}{2}\left((\delta^2+\omega^2)-(s_+^2+s_\times^2)\right)+\beta\partial_x(\mathbf{u}\cdot\mathbf{u})=$$

$$g(-\delta\Delta\zeta+2s_\times\zeta_{xy}+s_+\tilde{\Delta}\zeta), \tag{45}$$

which, remark, is independent of the local rate of rotation, f_0. This can be expressed in a more compact form, as the variable in square brackets is twice the Jacobian of u and v:

$$J(u,v)=u_xv_y-u_yv_x=\frac{1}{4}((\delta^2+\omega^2)-(s_+^2+s_\times^2)), \tag{46}$$

whereas, the right-hand side of (45) is also expressible in terms of Jacobians. Thus (47) becomes

$$\frac{d}{dt}J(u,v)+\delta J(u,v)+\beta\partial_x\frac{1}{2}(\mathbf{u}\cdot\mathbf{u})=g(J(v,\zeta_x)-J(u,\zeta_y)). \tag{47}$$

On a uniformly rotating plane ($\beta=0$), Eq. (47), using (28), reveals the existence of a materially conserved quantity,

$$J(u,v), \tag{48}$$

once $\mathbf{u}$ is geostrophic, *i.e.* once $\mathbf{u}$ is in a *dynamical* steady state, in which case Eq. (25) is satisfied by

$$\frac{d}{dt}\mathbf{u}=0, \qquad f\hat{k}\times\mathbf{u}+g\nabla\zeta=0, \tag{49}$$

seperately. Using (49b) to eliminate the velocity components in (48), we find that under these conditions

$$\zeta_{xx}\zeta_{yy} - \zeta_{xy}^2 = \mathcal{J}(x,y), \tag{50}$$

where $\mathcal{J}(x,y)$ specifies the initial value of the Jacobian of the material element which initially resides at position (x,y). This equation is somewhat similar to a form which the conservation of potential vorticity takes in a study on frontogenesis by Hoskins and Bretherton (1972). Considered as an equation for ζ it is an elliptic equation of the Monge-Ampère type (Courant and Hilbert, 1953; Cheng and Yua, 1980), in which discontinuities in the second derivative of ζ(the DKP's) arise only at the boundaries, or at positions where h(or its first or second derivatives) are discontinuous. As geostrophy is a force balance frequently met with in oceanic and atmospheric applications it would be interesting to test the material conservation of (50). Preliminary investigations by the author of moving drogues, which allow a Lagrangian (or material) evaluation of the DKP's, lends some experimental support to the conservation of the quantity in (48). The sign of the 'constant', $\mathcal{J}$, appearing in Eq. (48), is of some importance as it separates what may be termed 'elliptic motions' (along r and θ contours) - as when the divergent and vorticity terms dominate - from 'hyperbolic motions' (along hyperbola's) - as when the deformation terms are most strongly present.

The way in which the pressure gradient force, $g\nabla\zeta$, enters Eq. (47) is representative of the way in which any force term, $\mathbf{F} = (F^x, F^y)$, appears in this equation. Introducing, for instance, a Rayleigh friction,

$$\mathbf{F} = -\kappa\mathbf{u}, \tag{51}$$

with κa friction coefficient, then the geostrophic equilibrium is replaced by a three-term force balance and some down-gradient flow is generated:

$$\mathbf{u} = a\nabla\zeta + b\hat{k} \times \nabla\zeta, \tag{52}$$

where constants a and b depend on the relative degree of friction. In this case, the right-hand side of (47) is proportional to $J(u,v)$ and there is no true conservation of (48) anymore. However this quantity still satisfies a simple decay law along its trajectory.

Often, some simple type of flow field is considered, which, in terms of the present conservation law, may be called *degenerate*. Define a degenerate flow field as one for which

$$J(u,v) = 0. \tag{53}$$

Consideration of such a flow field then implicitly puts constraints on the elevation field, as it requires the vanishing of

$$J(v, \zeta_x) - J(u, \zeta_y) = 0. \tag{54}$$

In complex notation, with $\mathbf{u} = u + iv$ and $\nabla^* = \partial_x - i\partial_y$, this reads

$$Re\,[\,J(\mathbf{u}, i\nabla^*\zeta)\,] = 0, \tag{55}$$

where $Re[..]$ denotes the real part of the quantity within brackets. Hence, this implies a functional relation

$$\mathbf{u} = \mathbf{U}(i\nabla^*\zeta), \tag{56}$$

with $\mathbf{U}(s)$ an arbitrary function, which can be considered as a generalized geostrophic relation (to which it reduces once $\mathbf{U}(s) = s$). As an example consider a spatially uniform shear flow ($u_y \neq 0, u_x = v_x = v_y = 0$), which satisfies (53), then from (54), ζ has to satisfy a hyperbolic equation $\zeta_{xy} = 0$.

Eq. (47) can be brought in flux conservation form

$$\frac{\partial}{\partial t}J(u, v) + \nabla \cdot [\mathbf{u}J(u, v) + \frac{1}{2}\beta(\mathbf{u} \cdot \mathbf{u})\hat{i} + \mathbf{Z}] = 0, \tag{57}$$

where $\hat{\mathbf{i}}$ is the unit vector in the x-direction and $\mathbf{Z}$ can take either one of the following forms

$$\mathbf{Z} = \begin{cases} g(v_y\zeta_x - u_y\zeta_y, u_x\zeta_y - v_x\zeta_x) \\ -g(v\zeta_{xy} - u\zeta_{yy}, u\zeta_{xy} - v\zeta_{xx}), \end{cases} \tag{58}$$

as we can arbitrarily absorb a divergenceless vector in it. The latter expression in particular is useful when considering the integral of (57) over a fixed area A. Conservation of

$$\int\int J(u, v)dxdy = constant \tag{59}$$

is guaranteed whenever the flux normal to the boundary vanishes. The term originating from the nonlinear advection, $\mathbf{u}J$, automatically satisfies this requirement on solid boundaries. The other two terms, however, vanish only when the velocity itself is zero on the solid boundary, such as occurs in a viscous flow. This additional, more stringent condition is akin to the one which had to be imposed in the conservation of helicity (Section 3.1, #7), where it was required that the vorticity vanishes. In the absence of the β-term, this no-slip condition can be somewhat relaxed to the requirement that the boundary coincides with a geostrophic contour. Its most general application, however, is the case where the disturbance is localized, so that velocities vanish far away from it, and the integral value of the Jacobian is conserved.

As $J(u,v)$ stands for the Jacobian $\partial(u,v)/\partial(x,y)$, it is tempting to interpret the globally conserved quantity in (59) as the total area which the flow occupies in *velocity space*:

$$\int\int dudv = constant, \tag{60}$$

where the integration area is the physical area *mapped* onto the velocity space by the time-dependent transformation $\mathbf{u}(\mathbf{x},t)$. This, of course, applies only when the mapping is one-to-one, a situation generally *not* met with in reality, where similar velocities occur at different positions (*e.g.* the velocity in the core of a vortex and that far away from the vortex both tend to zero).
We may verify (see Appendix B) that the general Eq. (47) is related to a well-known materially conserved quantity. This is (perhaps) more readily recognized from an examination of a similar flux conservation law for the Jacobian in $3D$, which, therefore, is the topic considered below. *3.2 The conservation law in* $3D$

Consider the momentum equations (11), written in the form

$$\frac{d}{dt}\mathbf{u} = \mathbf{F}, \tag{60}$$

where, obviously, the density is for the moment absorbed in the description of the forcing terms, $\mathbf{F}$. Remark that $\mathbf{F}$ is now supposed to contain Coriolis force, pressure gradient force, gravity force and, when present, any other force terms. Let the velocity, $\mathbf{u}$ have components u, v and w and let $\mathbf{F} = (F^x, F^y, F^z)$, and let us further denote differentiations, $\partial/\partial x_i$, by a single subscript i, (*i.e.* i = 1,2 and 3 stand for $\partial/\partial x, \partial/\partial y$, and $\partial/\partial z$ respectively) and similarly for other dummy subscripts, j and k, then, taking the derivative of the u, v and w equation to x_i, x_j and x_k respectively, we obtain

$$\frac{\partial u_i}{\partial t} + [(\mathbf{u}\cdot\nabla)u]_i = F_i^x$$
$$\frac{\partial v_j}{\partial t} + [(\mathbf{u}\cdot\nabla)v]_j = F_j^y \tag{61}$$

$$\frac{\partial w_k}{\partial t} + [(\mathbf{u}\cdot\nabla)w]_k = F_k^z.$$

Multiplying (61.a) with $v_j w_k \epsilon_{\rm ijk}$ -where a summation, running from one to three, over repeated indices is implied-, Eq. (61.b) with $u_i w_k \epsilon_{\rm ijk}$ and (61.c) with $u_i v_j \epsilon_{\rm ijk}$ and adding them yields,

$$\frac{\partial J}{\partial t} + (\mathbf{u}\cdot\nabla)J + J\nabla\cdot\mathbf{u} = (F_i^x v_j w_k + F_j^y u_i w_k + F_k^z u_i v_j)\epsilon_{\rm ijk}, \tag{62}$$

in terms of the Jacobian

$$J \equiv J(u,v,w) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = u_i v_j w_k \epsilon_{\rm ijk}.$$

Here $\epsilon_{\rm ijk}$ is the alternating, or Levi-Civita tensor, which is zero when two of the indices are equal, while $\epsilon_{\rm ijk} = +1, -1$ for an even / odd permutation of 1,2 and 3 respectively. The nonlinear advection term in (62) is obtained by noting that terms in which a velocity component occurs twice, such as u in $u_i u_k v_j \epsilon_{\rm ijk}$, are zero.

Eq. (62) can be written in flux conservation form as

$$\frac{\partial J}{\partial t} + \nabla\cdot[\mathbf{u}J - (F^x\mathbf{s}^x + F^y\mathbf{s}^y + F^z\mathbf{s}^z)] = 0, \tag{64}$$

where,

$$\mathbf{s}^x = (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1) = \left(\tfrac{\partial(v,w)}{\partial(y,z)}, \tfrac{\partial(v,w)}{\partial(z,x)}, \tfrac{\partial(v,w)}{\partial(x,y)}\right). \tag{65}$$

Similar expressions are obtained for $\mathbf{s}^y$ and $\mathbf{s}^z$ by cyclic permutations of u, v and w. Note that $\nabla\cdot\mathbf{s}^x = \nabla\cdot\mathbf{s}^y = \nabla\cdot\mathbf{s}^z = 0$. This can be written as

$$\frac{D_3 J}{Dt} + \nabla\cdot(-F^i\mathbf{s}^i) = 0,$$

which implies, upon using (3),

$$\frac{d}{dt}\int J dV = \oint F^i(\mathbf{s}^i\cdot\mathbf{n})dA$$

for a material volume V, with boundary A, moving with the flow. With $\beta = 0$, this would yield an analogous 2D result from (57),

$$\frac{D_3 J}{Dt} + \nabla\cdot\mathbf{z} = 0, \rightarrow \frac{d}{dt}\int J dA = -\oint_\Gamma \mathbf{z}\cdot\mathbf{n} dl.$$

These two results will be particularly useful when volumes of fluid are chosen, such that on their boundaries the righthand sides continue to vanish, as time progresses.

Eq. (64) also assures the existence of a globally conserved quantity

$$\int\int\int J dxdydz = constant, \tag{66}$$

whenever $F^x \mathbf{s}^x + F^y \mathbf{s}^y + F^z \mathbf{s}^z$ vanishes on the boundary S, bounding the volume V over which the integration is performed, or, more likely, when the disturbamce is localized. In analogy to the $2D$ case, Sect. 3.1, this may for uniquely related position and velocity fields be interpreted as the total volume which the flow occupies in velocity space, $\int\int\int dudvdw$. Note that, together with Eq. (5), a materially conserved quantity,

$$\frac{J}{\rho} = constant, \tag{67}$$

may again be obtained once the right-hand side of (62) itself vanishes, such as occurs in geostrophic flow. Also, as in $2D$, remark that the Coriolis term does not contribute on an f-plane (*i.e.* there is no f_0-dependence), but only on $a\beta$-plane, where the Coriolis force $2\mathbf{\Omega} \times \mathbf{u}$ produces a term

$$-\beta\nabla \cdot [(u^2 + v^2)(w_z\hat{i} - w_x\hat{k})]$$

once we assume, as is customary, that only the projection of the Ω^z- term (whose y-derivative is β) is of importance.
By noting that the Jacobian can itself be written as the divergence of a vector (for instance in $2D$: $J(u,v) = \nabla \cdot (uv_y, -uv_x)$), it could be suggested that (57) and (64) are void statements, it is the vanishing of terms like those incorporating Coriolis effects which render (59) and (66) useful conserved quantities: they belong to the kernel of the divergence operator, which cannot be recovered by integration. In fact the argument would also have applied to the vorticity equation, since, in *e.g.* $2D$, the vorticity can be written as $\omega = \nabla \cdot (v, -u)$, which together with (31) equally suggests that the divergence operator can be removed. This however does not lead us back to the momentum equations as, in this case, the Bernoulli potential is not recovered. Related results on the evolution of velocity gradient components can be found in Kirwan (1975) and Cantwell (1992). As in this study, their results get amplified once the external forcing and pressure forcing terms vanish.

4 Discussion

The material conservation of the Jacobian, that can be derived from the flux conservation law in the absence of external forcing and vanishing pressure forces, has previously been experimentally observed in a laboratory study on the extension of large polymer-chains dissolved in a fluid, whose flow was stretched in between two co-rotating parllel rollers (Frank and Mackley, 1975). The two-dimensional, non-divergent flow is kinematically characterized by the difference between teh squared vorticity and squared principal strain rate (the squared combination of the two deformation rates s_+ and $s_\times$). It is observed that when the flow is strain-dominated, it is characterized by the existence of a singular point, and the fluid particles that are close to the outgoing plance of symmetry of the flow field experience persistent straining. The continued extension of the dissolved polymer-chains which is the result of this, is manifested by (an observable) change in optical properties of the fluid along this plane (double refraction). The authors aptly refer to (the square root of) the previously introduced kinematical quantity as the *persistence of strain.*

Although this feature has been utilized in more complicated 2D flows (in a 6 roll mill system by Berry and Mackley, 1976), and its application in dynamical systems theory has been proposed (Dresselhaus and Tabor, 1989), its status as a conserved quantity does not seem to have been rigorously established. Also, the generalization in 3 dimensions in the latter work — quite different from the one derived in Sectin 4.2— is, though well motivated, merely proposed. Again, the latter authors do not seem to take its name as implying a *conserved* property too literally, as they propose to monitor its *evolution* for different 'flows' and use this as an indication of the steady, periodic, or chaotic nature of it.In this respect, their approach is very similar to that taken earlier by Okubo (1970), who characterized particle trajectories in a 2D flow field on the basis of the nature of the 'singularity' of the flow. This is done by plotting the observed (squared) persistence of strain (as defined above) versus the observed divergence of the flow field. By subsequently noting in which region of this phase-plane the observation lies, one is able to predict the behaviour of its particle trajectories.

In a previous field study, employed in the North Sea, observations of horizontally moving drifters have been used to calculate the evolution of the differential kinematic properties of the flow field (Maas, 1989). When these are plotted as proposed by Okubo (1970), a slight tendency to fall along the central parabola, is evident (se Figure 1). This lends some (weak) support to the (approximate) material conservation of the Jacobian (points which are exactly on this central parabola have $J(u,v)=0$). The implication of its conservation is first, that the Jacobian is (apparently) initially zero, and, second, that, since it remains zero, even though the drifters disperse, the flow must be nearly geostrophic. The impact of the (more general) global constraint, obtained in this study, however, remains to be assessed.

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6 Appendices

6.1 Analogy of vorticity to spin

In this appendix it is argued that the individually conserved quantities 'vorticity' and 'angular momentum' are the hydrodynamical analogues of the 'spin-' and 'orbital angular momentum' the sum of which is known to be conserved for an ensemble of particles in classical mechanics .

For simplicity we consider a uniform density, $2D$ fluid. Let a circular, infinitesimal (radius R) fluid portion be centred at a position $\bar{\mathbf{x}}$ (Fig. 1). Let the positions of fluid particles within this fluid element have positions $\mathbf{x}=\bar{\mathbf{x}}+\mathbf{x}'$, while their velocities are given by $\mathbf{u}=\bar{\mathbf{u}}+\mathbf{u}'$, being an average value over the fluid element. The total angular momentum for continuous fields $\mathbf{x}$ and $\mathbf{u}$(supposed to be bounded by the circle of radius R) is a (normalized) integral over each of the individual angular momenta within the disk:

$$\frac{1}{A}\int \mathbf{x}\times\rho\mathbf{u}dA=\frac{\rho}{A}\int(\bar{\mathbf{x}}+\mathbf{x}')\times(\bar{\mathbf{u}}+\mathbf{u}')\ \mathrm{dA}.$$

Now, the primed fields are defined to have a spatial average zero. Therefore this expression has only two contributions $1)\rho\bar{\mathbf{x}}\times\bar{\mathbf{u}}$, corresponding to the *orbital* angular momentum and $2)\rho/A\int\mathbf{x}'\times\mathbf{u}'dA$, corresponding to the *spin* angular momentum. The former involves only the gross-scale features (especially the *linear* momentum) of the fluid element, while the latter is equivalent to the area-averaged vorticity as we can observe by making a Taylor-expansion of $\mathbf{u}'$ around the central position $\bar{\mathbf{x}}$:

$$\mathbf{u}'(\bar{\mathbf{x}} + \mathbf{x}') = \mathbf{u}'(\bar{\mathbf{x}}) + (\mathbf{x}' \cdot \nabla)\mathbf{u}'(\bar{\mathbf{x}}) + O(x_i' x_j').$$

As angle-dependent terms all drop out in the averaging procedure only the terms linear in $\mathbf{x}'$ are sampled and we find

$$\frac{\rho}{A} \int \mathbf{x}' \times \mathbf{u}' dA = \frac{1}{4\pi} \rho A \omega,$$

with $\omega = v_x - u_y$. This is (proportional to) the limit of the area averaged vorticity, $\int \omega dA$(see Eq. (9)), applied to an infinitesimal fluid element, such that ω can be considered to become constant.

In fluid dynamics these two contributions to the total angular momentum are found to be conserved *separately* (section 2.1, no.'s 4 and 6).

6.2 Recursive Application of Ertel's Theorem

Thus, for a barotropic fluid an infinity of conserved quantities is generated provided each of the Π_n has a nonzero gradient, which is not perpendicular to ω_a. As an example we may apply this idea to a barotropic, weakly fluctuating ($\mid \omega \mid << f$), uniformly stratified ($N^2 \equiv -g/\rho d\rho/dz =$ constant) and incompressible fluid, where we take $\lambda = \rho$. Then

$$\Pi_0 \approx -\frac{f N^2}{g},$$

and with f varying linearly with y, Eq. (10), we find (using the previously introduced notation for the derivative) the materially conserved quantity

$$\Pi_1 = \frac{\beta N^2}{g}(u_z - w_x),$$

associated with the large-scale vertical circulation in a zonal plane. Thus, any decrease of the vertical density stratification along the trajectory of the fluid parcel may be associated with an intensification of any preexisting vorticity in the horizontal plane. Indeed, such an intensification will also result for uniform stratified flow, when the horizontal component of the earth rotation is taken into account.

6.3 Relation to Conservation of Mass

It is observed that Eqs. (62) and (64) have been derived for very general circumstances, *i.e.* irrespective of the detailed form of the forcing. In fact, a similar remark is valid for their $2D$ analogs, Eqs. (47) and (57), once we replace Coriolis and pressure gradient force terms by a general forcing term, $\mathbf{F} = (F^x, F^y)$. Then Eq. (47) reads

$$\frac{d}{dt}J(u,v) + J(u,v)\nabla \cdot \mathbf{u} = J(F^x, v) + J(u, F^y), \tag{68}$$

where the 2D Jacobian $J(u,v) = u_x v_y - v_y u_x$. In Sect. 3 we observed that the globally conserved quantity could be interpreted as an area (in $2D$), or volume (in $3D$) in *velocity space.* This suggests that we should map our equations from Cartesian $\mathbf{x}$ space to velocity $\mathbf{u}$ space. Indeed, following this suggestion, we will obtain an even more compact form of the conservation Eqs. (64) and (68). To that end, rewrite derivatives to a scalar G in terms of derivatives in $\mathbf{u} = (u,v)$ space

$$\begin{pmatrix} G_x \\ G_y \end{pmatrix} = \begin{pmatrix} G_u u_x + G_v v_x \\ G_u u_y + G_v v_y \end{pmatrix} = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} \begin{pmatrix} G_u \\ G_v \end{pmatrix}$$

from which we obtain, by inversion

$$\nabla_u G \equiv \begin{pmatrix} G_u \\ G_v \end{pmatrix} = \frac{1}{J(u,v)} \begin{pmatrix} v_y & -v_x \\ -u_y & u_x \end{pmatrix} \begin{pmatrix} G_x \\ G_y \end{pmatrix} = \frac{1}{J(u,v)} \begin{pmatrix} J(G,v) \\ J(u,G) \end{pmatrix}.$$

Applying this to F^x and F^y respectively we can rewrite Eq. (68) as

$$\frac{d}{dt}J + J\nabla \cdot \mathbf{u} = J\nabla_u \cdot \mathbf{F}, \tag{69}$$

where we use the abbreviation J to denote the Jacobian. Indeed, performing a similar analysis in $3D$ leads to the (symbolically) same equation, except that the Jacobian, J, total derivative, d/dt, and gradient, ∇_u, attain their equivalent $3D$ expressions. With the use of the respective continuity equations, Eqs. (28) and (5), this can still further be simplified to

$$\frac{d}{dt}\frac{h}{J} + \frac{h}{J}\nabla_u \cdot \mathbf{F} = 0, \tag{70}$$

and

$$\frac{d}{dt}\frac{\rho}{J} + \frac{\rho}{J}\nabla_u \cdot \mathbf{F} = 0, \qquad (71)$$

for the $2D$ and $3D$ cases respectively. Since the material derivative is referring to the coordinate system in which the evolution of the quantity concerned is evaluated this operator in the present case should be read as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{du_i}{dt}\frac{\partial}{\partial u_i} = \frac{\partial}{\partial t} + \mathbf{F} \cdot \nabla_u,$$

so that flux conservation laws become

$$\frac{\partial}{\partial t}\frac{h}{J} + \nabla_u \cdot \left[\left(\frac{h}{J}\right)\mathbf{F}\right] = 0, \qquad (72)$$

and

$$\frac{\partial}{\partial t}\frac{\rho}{J} + \nabla_u \cdot \left[\left(\frac{\rho}{J}\right)\mathbf{F}\right] = 0. \qquad (73)$$

Eq. (71) is the analog of the operator D_3/Dt in velocity space. Therefore, after an integration in velocity space over the mapped volume of a material element, we obtain, from (3),

$$\frac{d}{dt}\left(\int\int\int \frac{\rho}{J}\mathrm{dudvdw}\right) = 0. \qquad (74)$$

The interpretation of this conserved quantity is facilitated once we recognize that ρ/J is in fact the ratio of two Jacobians. This is so because, except for a normalizing factor, ρ_0, the density ρ is giving the ratio of the initial volume of the material element concerned (whose location is fixed in Eulerian space by its initial coordinates (a, b, c)) to the volume which it occupies at a later instant

$$\rho = \frac{\partial(a, b, c)}{\partial(x, y, z)}\rho_0. \qquad (75)$$

As the ratio of two Jacobians is just another Jacobian,

$$\frac{\rho}{J} = \frac{\partial(a, b, c)}{\partial(u, v, w)}\rho_0, \qquad (76)$$

Eq. (74) just expresses the conservation of mass in velocity space, or, as we can arbitrarily multiply with a constant for which it is appropriate to choose the ratio of the initial volume occupied by the fluid element in velocity space (denoted by subscripts 0) to the initial mass , *i.e.* by

$$\frac{1}{\rho_0}\frac{\partial(u_0, v_0, w_0)}{\partial(a,b,c)},$$

it may also be interpreted as *the conservation of volume occupied by a material fluid element in velocity space.*

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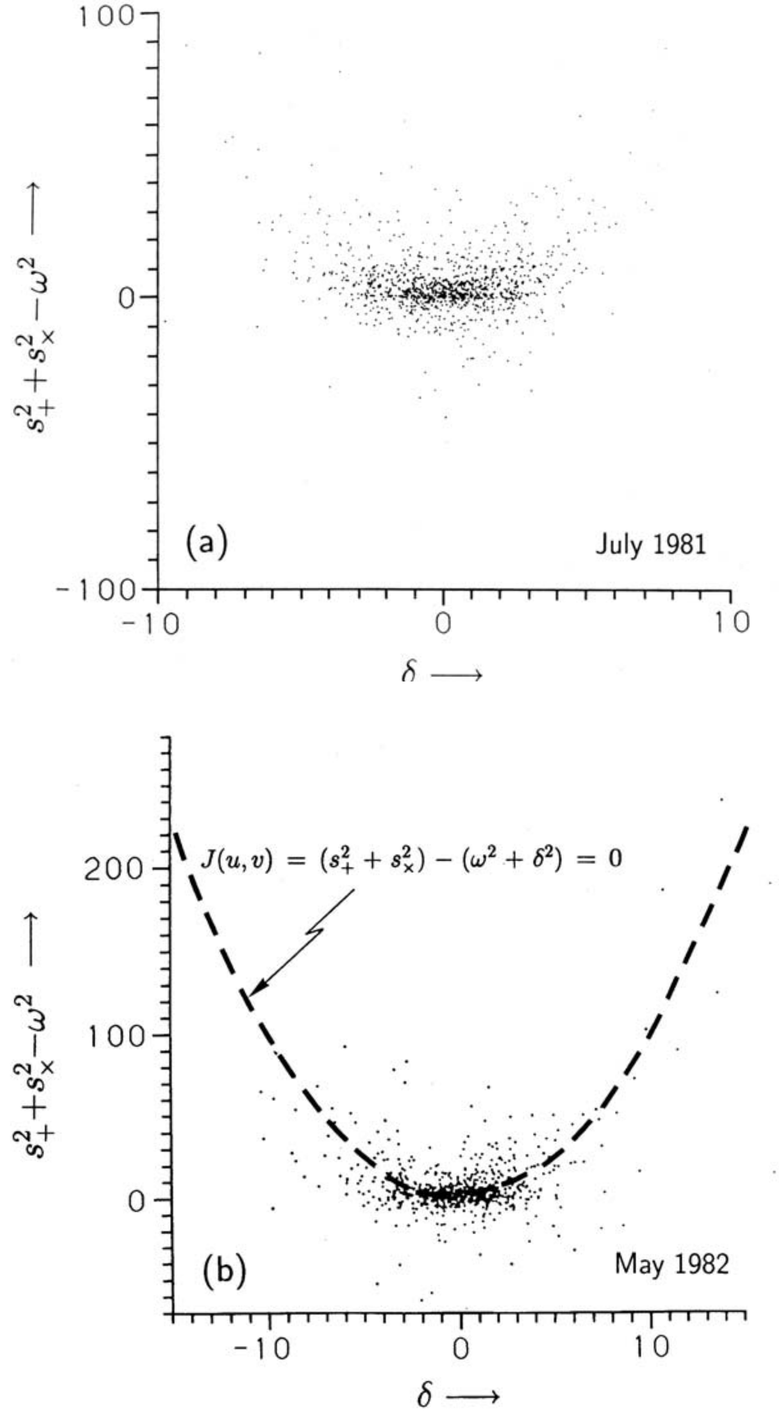

Figure 1: The quantity $s_+^2 + s_\times^2 - \omega^2$ — the persistence of strain, in the terminology of Frank and Mackley (1976)— versus divergence δ for two drifter experiments carried out in the North Sea in (a) july 1981 and (b) May 1982 (Maas, 1989). Each of the shear terms has been scaled and nondimensionalized by a factor $10^{-5}s^{-1}$. In (b) the dashed line represents the case that $4J(u,v) \equiv (s_+^2 + s_\times^2) - (\omega^2 + \delta^2) = 0$.