A Closed Form Green Function Describing Diffusion in a Strained Flow Field

Leo R. M. Maas

A CLOSED FORM GREEN FUNCTION DESCRIBING DIFFUSION IN A STRAINED FLOW FIELD*

LEO R. M. MAAS†

Abstract. Stationary concentration profiles resulting from the two-dimensional diffusion of material away from a continuous source in an advective flow field on the infinite plane are considered. The advection field contains a straining component, in addition to a spatially uniform part. The inhomogeneous advection-diffusion equation describing the spreading away from the source can be transformed to a noncentrally forced Schrödinger equation with a two-dimensional harmonic oscillator potential. The exact solution of this equation is given in terms of a definite integral, being an incomplete integral representation of the zeroth order modified Bessel function of the second kind (to which it reduces in appropriate circumstances). The field dependence is present not only in the kernel function, but also in one of the limits of integration. The near-field limit leads to concentration profiles in which curvature effects of the straining flow field may be neglected in comparison to the uniformly advecting part. Taking the far-field limit, it is found that the noncentral aspect of the forcing vanishes, leading to a symmetric spreading of material.

Key words. advection-diffusion equation, incomplete modified Bessel function, straining advection field, Green function, closed form solution, two-dimensional harmonic oscillator

AMS(MOS) subject classifications. 34A05, 35C05, 35C10, 35C15, 76R50

1. Introduction. The aim of this paper is to study the equation describing the concentration field due to diffusion of a substance from a continuous source in the presence of a straining flow field. Exact solutions of the time-dependent advection-diffusion equation in the presence of an instantaneous source have been documented for some particular advective flow fields (e.g., Fischer et al. [5]) like a uniform flow, or a flow varying linearly in *one* spatial dimension with, or without, oscillatory time-dependence (Okubo [10]). Each of these can be turned into the steady response to a continuous source by convolution. In a realistic two-dimensional advective flow field, the advective velocity field in the vicinity of the source can be expanded in a Taylor series for each of the two velocity components. For general circumstances this leads, up to first order in the series, to rotational, divergent, and straining flow field components, in addition to a uniform zeroth-order flow. In the present study we assume only the straining component to be present as well as a uniform flow. This leads to an equation in which shear occurs in both orthogonal velocity components in general, such that the velocity field is hyperbolic with a stagnation point that does not coincide with the source position in the case of a nonzero uniform flow. This equation has been investigated by various authors, who have obtained solutions to it for particular circumstances. Thus, Young, Rhines, and Garrett [12], following Townsend [11], have obtained a time-dependent solution to the initial value problem in a case in which the initial profile consists of a doubly periodic concentration pattern. Okubo [9] assumes the source to be moving with the zeroth-order constant flow component, in which case the solute spreads out, relative to this moving coordinate system in elliptical concentration contours. The present study, in contrast, aims to solve the steady problem for a point source not coinciding with the stagnation point of the flow field (thus not moving with the constant flow field), which by the asymmetry introduced in this way, turns

out to pose a much more intricate mathematical problem. Applications of this advection-diffusion equation (or its transformed version) are not restricted to genuine advection-diffusion problems. Indeed, Zimmerman and Maas [13] have derived the same equation to describe certain symmetry aspects of the steady, general circulation of the ocean. Their equation, derived from the barotropic vorticity equation, is formally equivalent to the advection-diffusion equation studied here, in which a streamfunction is the "stuff" that is "advected" by the absolute vorticity gradient, expanded in a Taylor series around the source position.

In § 2 we discuss the advection-diffusion equation for a straining field and show how it can easily be transformed to a noncentrally forced Schrödinger equation, with a two-dimensional harmonic oscillator potential. In § 3 this equation is solved, first as an infinite series, which is subsequently contracted into a closed form expression in terms of a definite integral, which may be termed "incomplete modified Bessel function of the second kind." Several limits of the resulting expression for the concentration field can be discussed (§ 4), the interest of which partly derives from the intriguing fact that the field dependency occurs not only in the kernel function, but is also present in one of the limits of integration. Indeed, the subtlety of the closed form expression is revealed, when we verify the correctness of the obtained solution, by directly substituting it into the governing equation (see the Appendix). Some speculations on possible generalizations are discussed in § 5 and the paper is summarized in the last section.

2. Advection-diffusion equation. The diffusion in an infinite two-dimensional orthogonal $\mathbf{x} = (x, y)$ space of a substance, whose concentration is denoted by C, in a velocity field $\mathbf{u}$, with components parallel to the orthogonal grid, away from a continuous Dirac-delta point source at the origin, $\delta(\mathbf{x})$, is considered. It is determined by a balance of the advective and diffusive flux:

$$(1) \qquad\qquad \mathbf{u} \cdot \nabla C = \Delta C + \delta(\mathbf{x}),$$

here in nondimensional form, where Δ denotes the Laplacian operator ($\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$). For nondivergent flows ($\nabla \cdot \mathbf{u} = 0$), $\mathbf{u}$ can be derived from a streamfunction ψ, $\mathbf{u} = (-\partial\psi/\partial y, \partial\psi/\partial x)$, such that (1) is given by

$$(2) \qquad\qquad \Delta C + \mathfrak{J}(C, \psi) = -\delta(\mathbf{x})$$

where $\mathfrak{J}(a, b)$ is the Jacobian

$$\mathfrak{J}(a, b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}.$$

We propose use of the following irrotational streamfunction

$$(3) \qquad\qquad \psi = \alpha + \boldsymbol{\beta} \cdot \mathbf{x} + \tfrac{1}{2}\gamma_x(x^2 - y^2) + \gamma_y xy$$

where α is a constant having no dynamical significance; $\boldsymbol{\beta} \equiv \hat{\beta}(\cos \beta, \sin \beta) = (\beta_x, \beta_y)$ is the uniform flow component; and $\boldsymbol{\gamma} \equiv \hat{\gamma}(\cos \gamma, \sin \gamma) = (\gamma_x, \gamma_y)$ is the straining field. This yields a velocity field of the following form:

$$\mathbf{u} = (-\beta_y + \gamma_x y - \gamma_y x, \beta_x + \gamma_x x + \gamma_y y).$$

In the absence of straining ($\hat{\gamma} = 0$), (2) reduces to

$$\Delta C + \beta_y \frac{\partial C}{\partial x} - \beta_x \frac{\partial C}{\partial y} = -\delta(\mathbf{x}).$$

Since the only "special" point in the plane in this case is the source point, it is chosen as the origin. The solution of this equation is (e.g., Carslaw and Jaeger [3, p. 267])

(4)
$$C = \frac{1}{2\pi} \exp\left(\frac{1}{2}(\beta_x y - \beta_y x)\right) K_0\left(\frac{1}{2}\hat{\beta}r\right)$$
$$= \frac{1}{2\pi} \exp\left[\frac{1}{2}\hat{\beta}r \sin(\theta - \beta)\right] K_0\left(\frac{1}{2}\hat{\beta}r\right)$$

where r and θ denote polar coordinates of $\mathbf{x}$ and K_0 is the zeroth-order modified Bessel function of the second kind (sometimes referred to as modified Bessel function of the third kind (Erdélyi et al. [4]), or MacDonald function (Agrest and Maksimov [2])). From the asymptotics of K_0, we infer the typical exponential decay except in the downstream direction ($\theta = \beta + \pi/2$), where the decay is much slower ($\approx r^{-1/2}$) giving a wakelike appearance to the concentration distribution. The profile is symmetric around a line through the source, pointing in the advection direction.

   If the straining field is not negligible, the current field $\mathbf{u}$ reveals that there exists a second "special" point in the plane, namely, where $\mathbf{u}$ vanishes. It is convenient to translate our origin to this point. Also, since the streamfunction field ψ in (3) contains (in general) two hyperbolic fields, it is appealing to rotate our coordinate frame such that one of these is absent. Since the Jacobian and Laplacian are invariant to rotation and translation, these operations do not affect the structure of (2). Thus we rotate our coordinate frame to $\mathbf{x}'$

(5)
$$\mathbf{x}' = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \mathbf{x},$$

with $c \equiv \cos\varphi$, $s \equiv \sin\varphi$ and the rotation angle

$$\varphi = \frac{\pi}{4} - \frac{\gamma}{2}$$

is such that one of the two hyperbolic expressions in ψ is eliminated. The transformed streamfunction ψ' is thus expressed as

$$\psi' = \alpha + \boldsymbol{\beta}' \cdot \mathbf{x}' + \hat{\gamma}x'y'$$

where

(6)
$$\boldsymbol{\beta}' = \hat{\beta}(\cos\beta', \sin\beta'), \qquad \beta' = \beta + \varphi.$$

A subsequent translation

(7)
$$\mathbf{x}'' = \mathbf{x}' + \mathbf{a}$$

where

(8)
$$\mathbf{a} \equiv \hat{a}(\cos\theta_a, \sin\theta_a) = (a_x, a_y) = \frac{\hat{\beta}}{\hat{\gamma}}(\sin\beta', \cos\beta')$$

transforms ψ' to

(9)
$$\psi'' = \alpha' + \hat{\gamma}x''y''$$

and (2) turns into its canonical form (dropping the double primes):

(10)
$$\Delta C + \hat{\gamma}\left(x\frac{\partial C}{\partial x} - y\frac{\partial C}{\partial y}\right) = -\delta(\mathbf{x} - \mathbf{a}).$$

This equation describes the diffusion of matter released continuously at the source point $\mathbf{x} = \mathbf{a}$ in a pure straining field

$$(11) \qquad \mathbf{u} = \hat{\gamma}(-x, y)$$

considered on the infinite plane. Both operations are summarized in Fig. 1.

Setting $C = \exp(h(\mathbf{x}))C'$ we obtain

$$(12) \qquad e^h(\Delta C' + (2\nabla h - \mathbf{u}) \cdot \nabla C' + (\Delta h + (\nabla h)^2 - \mathbf{u} \cdot \nabla h)C') = -\delta(\mathbf{x} - \mathbf{a}).$$

Thus if we now choose $\nabla h = \frac{1}{2}\mathbf{u}$ the first derivative terms drop out. Since $\nabla \cdot \mathbf{u} = 0$, this choice implies

$$\Delta h = 0,$$
$$(13) \qquad (\nabla h)^2 = \tfrac{1}{4}\mathbf{u} \cdot \mathbf{u} = \hat{\gamma}^2(x^2 + y^2)/4 = \hat{\gamma}^2 r^2/4$$
$$h = \hat{\gamma}(y^2 - x^2)/4$$

and $2h$ is recognized as the harmonic function conjugate to the streamfunction in (9), i.e., the velocity potential. Equation (12) reduces to

$$(14) \qquad \Delta C' - \frac{\hat{\gamma}^2}{4} r^2 C' = -\frac{e^{-h(\mathbf{a})}}{r} \delta(r - \hat{a})\delta(\theta - \theta_a)$$

where the argument of h, in the right-hand side, has been set equal to $\mathbf{a}$ due to the sampling action of the delta function. This can be interpreted as a noncentrally forced Schrödinger equation with a two-dimensional harmonic oscillator potential yielding a suitable Green function solution to obtain a formal expression for the standard Schrödinger equation:

$$\Delta C' - V(r)C' = -EC'$$

with $V(r)$ the potential and E the energy level (Merzbacher, [8, p. 224]).
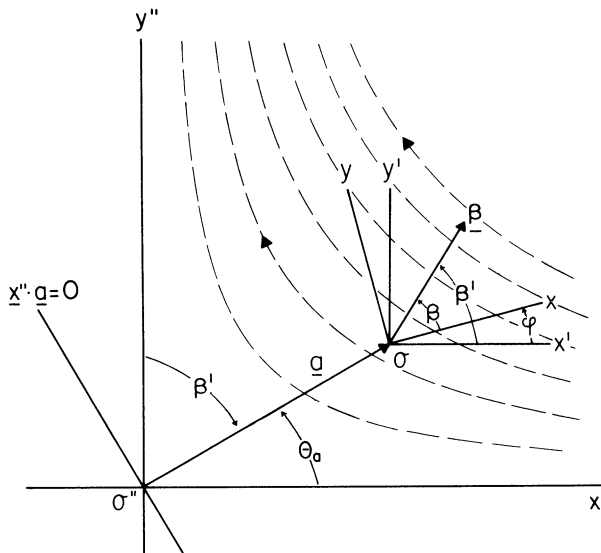
FIG. 1. *Coordinate frames* (x, y) *and* (x', y'), *centered at the source point* O, *are related by a rotation* φ, *frames* (x', y') *and* (x'', y'') *by a translation* $\mathbf{a}$. *The hyperbolic streamlines,* $\psi = $ *constant, of the straining advection field, are dashed.* $\boldsymbol{\beta}$ *is pointing in a direction normal to the local uniform flow direction. Also shown is a line through* O'', *normal to* $\mathbf{a}$ (*designated* $\mathbf{x}'' \cdot \mathbf{a} = 0$), *which is the line of symmetry, referred to later in the text.*

3. The incomplete Bessel function.

3.1. The formal infinite series solution.

In solving (14) it seems attractive to apply a conformal coordinate transformation z^2 ($z = x + iy$), whose real and imaginary parts are equivalent to the velocity-potential ($2h$) and the streamfunction (ψ''), respectively, since the equation then reduces to a standard modified Helmholtz equation. This transformation however is mapping each of two halfplanes onto the same plane making it difficult to distinguish the transformed source point (at $\mathbf{x} = \mathbf{a}$) from its transformed antipode (at $\mathbf{x} = -\mathbf{a}$). Thus we transform only the radial coordinate r:

$$(15) \qquad\qquad s = \tfrac{1}{4}r^2.$$

Equation (14) now turns into

$$(16) \quad \frac{\partial^2 C'}{\partial s^2} + \frac{1}{s}\frac{\partial C'}{\partial s} + \frac{1}{4s^2}\frac{\partial^2 C'}{\partial \theta^2} - \hat{\gamma}^2 C' = -\frac{\hat{a}}{4s^{3/2}}\exp\left(\frac{\hat{\gamma}}{4}(a_x^2 - a_y^2)\right)\delta(s - s_a)\delta(\theta - \theta_a)$$

where we used (Jackson [6, p. 30])

$$\delta(f(s)) = \frac{1}{|df/ds(s_a)|}\,\delta(s - s_a)$$

since $f(s)$ ($= 2s^{1/2} - \hat{a}$) has one simple zero at

$$(17) \qquad\qquad s = s_a = \hat{a}^2/4.$$

Now

$$(18) \qquad\qquad \delta(\theta - \theta_a) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-im(\theta - \theta_a)}$$

(Jackson [6, p. 117]). Thus if we expand

$$(19) \qquad\qquad C' = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} C_m(s)e^{im\theta'}$$

where

$$(20) \qquad\qquad \theta' \equiv \theta - \theta_a$$

we find that the C_m satisfy

$$(21) \quad \frac{\partial^2 C_m}{\partial s^2} + \frac{1}{s}\frac{\partial C_m}{\partial s} - \left(\frac{(m/2)^2}{s^2} + \hat{\gamma}^2\right)C_m = -\frac{\hat{a}}{4s^{3/2}}\exp\left(\frac{\hat{\gamma}}{4}(a_x^2 - a_y^2)\right)\delta(s - s_a).$$

The solution of the homogeneous equation, valid if $s \neq s_a$, is given by half-order modified Bessel functions of the first and second kind, $I_{m/2}(\hat{\gamma}s)$ and $K_{m/2}(\hat{\gamma}s)$, respectively. The solution is defined in the two regions where $s \neq s_a$:

$$(22) \qquad\qquad C_m = \begin{cases} A_m^{(1)}I_{m/2}(\hat{\gamma}s) + A_m^{(2)}K_{m/2}(\hat{\gamma}s), & s < s_a, \\ B_m^{(1)}I_{m/2}(\hat{\gamma}s) + B_m^{(2)}K_{m/2}(\hat{\gamma}s), & s > s_a. \end{cases}$$

Requiring regularity of the solution as $s \to 0$ and $s \to \infty$ puts some constraints on the coefficients. The modified Bessel functions of the second kind satisfy the symmetry property $K_{-\nu} = K_\nu$ for all real numbers ν. The modified Bessel functions of the first kind, however, share this property only for integer values, n, ($n \geq 0$): $I_{-n} = I_n$. For the half-order values this is replaced by

$$I_{-(n+1/2)}(\hat{\gamma}s) = I_{n+1/2}(\hat{\gamma}s) + (-1)^n\frac{2}{\pi}K_{n+1/2}(\hat{\gamma}s)$$

(Abramowitz and Stegun [1]). For the solution to vanish as $s \to \infty$, we require $B_m^{(1)} = 0$ for all values of m. A similar regularity restriction at the origin requires the separate treatment of two cases. In Case 1, in which m is either positive or negative but even $(m = -2n)$, we simply require that $A_m^{(2)} = 0$ $(m \geq 0)$ and $A_{-2n}^{(2)} = 0$. In Case 2, in which m is odd and negative $m = -(2n + 1)$, we must require that

$$\frac{2}{\pi} A_{-(2n+1)}^{(1)}(-1)^n + A_{-(2n+1)}^{(2)} = 0,$$

which assures the vanishing of $K_{n+1/2}$ terms for $s < s_a$. This then may be expressed succinctly as

$$(23) \qquad C_m = \begin{cases} A_m I_{|m|/2}(\hat{\gamma}s), & s < s_a, \\ B_m K_{m/2}(\hat{\gamma}s), & s > s_a. \end{cases}$$

The solution in the two regions can be connected by requiring continuity of C_m at $s = s_a$ and discontinuity of the derivative dC_m/ds at $s = s_a$ by an amount H given by the value of the right-hand side of (21) integrated over s from $s_a - \varepsilon$ to $s_a + \varepsilon$, for $\varepsilon \to 0$; i.e.,

$$(24) \qquad H = -\frac{2}{\hat{a}^2} \exp\left(\frac{\hat{\gamma}}{4}(a_x^2 - a_y^2)\right).$$

From these conditions we obtain

$$A_m I_{|m|/2}(\hat{\gamma}s_a) = B_m K_{m/2}(\hat{\gamma}s_a),$$

$$\hat{\gamma} B_m K'_{m/2}(\hat{\gamma}s_a) - \hat{\gamma} A_m I'_{|m|/2}(\hat{\gamma}s_a) = H$$

where the prime denotes differentiation to the argument. These imply

$$(25) \qquad (A_m, B_m) = \frac{H}{\hat{\gamma}} \frac{(K_{m/2}(\hat{\gamma}s_a), I_{|m|/2}(\hat{\gamma}s_a))}{W}$$

where the Wronskian $W(I_{|m|/2}(\hat{\gamma}s_a), K_{m/2}(\hat{\gamma}s_a)) = -1/\hat{\gamma}s_a$. Together with (24) this leads to the following expression for concentration:

$$(26) \qquad C = \frac{1}{4\pi} e^{h(\mathbf{x}) - h(\mathbf{a})} \sum_{m=-\infty}^{\infty} e^{im\theta'} I_{|m|/2}(\hat{\gamma}s_<) K_{m/2}(\hat{\gamma}s_>)$$

where $s_<$ is equal to s for $s < s_a$ and equal to s_a for $s > s_a$ and vice versa for $s_>$.

3.2. The closed form expression. Although (26) is a formal solution of our advection-diffusion problem (whose numerical convergence takes some effort to establish), its infinite series representation obscures its asymptotic behavior in the near- and far-field limits. For this reason we search for a closed form expression in which the physically interesting limits can be evaluated readily.

Equation (26) can be split in two parts, containing even and odd numbers m, respectively:

$$(27) \qquad C = \frac{1}{4\pi} e^{h(\mathbf{x}) - h(\mathbf{a})} \left\{ I_0(\hat{\gamma}s_<) K_0(\hat{\gamma}s_>) + 2 \sum_{n=1}^{\infty} I_n(\hat{\gamma}s_<) K_n(\hat{\gamma}s_>) \cos(2n\theta') \right.$$

$$\left. + 2 \sum_{n=0}^{\infty} I_{n+1/2}(\hat{\gamma}s_<) K_{n+1/2}(\hat{\gamma}s_>) \cos((2n+1)\theta') \right\}.$$

The first sum in enclosed brackets, including the zeroth-order term, can be evaluated by Gegenbauers' Addition Theorem (Magnus, Oberhettinger, and Soni [7]). It is the expansion of

$$(28) \qquad K_0(\hat{\gamma}(s^2 + s_a^2 - 2ss_a \cos 2\theta')^{1/2}).$$

This is the two-dimensional Green function solution of the modified Helmholtz equation in our transformed s-coordinate frame. Returning to our original coordinates, this can also be written as

$$(29) \qquad K_0(\hat{\gamma}R/4)$$

where

$$(30) \qquad R = |\mathbf{x} - \mathbf{a}||\mathbf{x} + \mathbf{a}|,$$

which reveals for this part of the solution a symmetric response for $\mathbf{x} \cdot \mathbf{a} \leq 0$ and $\mathbf{x} \cdot \mathbf{a} \geq 0$. This solution has a logarithmic singularity at $\mathbf{x} = \mathbf{a}$, corresponding to our desired source, *as well as* at $\mathbf{x} = -\mathbf{a}$ an undesired *virtual* source. We expect the second sum to annihilate this spurious virtual source and, possibly, to enhance the real source.

Inspection of the second sum S,

$$(31) \qquad S = 2 \sum_{n=0}^{\infty} I_{n+1/2}(\hat{\gamma}s_<) K_{n+1/2}(\hat{\gamma}s_>) \cos (2n+1)\theta',$$

reveals that it is indeed antisymmetric around the line defined by $\mathbf{x} \cdot \mathbf{a} = 0$ (i.e., $\theta' = \pm \pi/2$). The presence of a logarithmic singularity at $s = s_a$, however, is far from obvious, since each term individually is regular there. Remarkably, this sum can be evaluated from the *three-dimensional* Green function solution $\exp(-\hat{\gamma}R')/R'$ of the modified Helmholtz equation. Here

$$(32) \qquad R' = (s^2 + s_a^2 - 2ss_a \cos \nu)^{1/2}$$

where ν is an auxiliary variable. This is expanded (Magnus, Oberhettinger, and Soni [7]) as

$$(33) \qquad \frac{e^{-\hat{\gamma}R'}}{R'} = \frac{1}{\sqrt{ss_a}} \sum_{n=0}^{\infty} (2n+1) I_{n+1/2}(\hat{\gamma}s_<) K_{n+1/2}(\hat{\gamma}s_>) P_n(\cos \nu)$$

where $P_n(z)$ are Legendre polynomials of order n. By multiplying this expression by $-\sin \nu/(\cos 2\theta' - \cos \nu)^{1/2}$ and integrating ν from $-\pi$ to $2\theta'$, we find for the right-hand side of (33) that

$$(34) \qquad \begin{aligned} &\frac{1}{\sqrt{ss_a}} \sum_{n=0}^{\infty} (2n+1) \int_{-\pi}^{2\theta'} \frac{-\sin \nu P_n(\cos \nu)}{\sqrt{\cos 2\theta' - \cos \nu}} \, d\nu \, I_{n+1/2}(\hat{\gamma}s_<) K_{n+1/2}(\hat{\gamma}s_>) \\ &= \frac{1}{\sqrt{ss_a}} \sum_{n=0}^{\infty} (2n+1) \int_{-1}^{x} \frac{P_n(t)}{\sqrt{\chi - t}} \, dt I_{n+1/2}(\hat{\gamma}s_<) K_{n+1/2}(\hat{\gamma}s_>) \end{aligned}$$

where $\chi = \cos 2\theta'$ and $t = \cos \nu$. Due to a formula of Tricomi (Erdélyi et al. [4, p. 187]) the integral can be expressed in terms of Chebyshev polynomials of order n, $T_n(\chi)$:

$$(35) \qquad (2n+1) \int_{-1}^{x} \frac{P_n(t)}{\sqrt{\chi - t}} \, dt = 2 \left(\frac{T_n(\chi) + T_{n+1}(\chi)}{\sqrt{\chi + 1}} \right).$$

This in turn can be evaluated as

$$(36) \qquad (2n+1) \int_{-1}^{x} \frac{P_n(t)}{\sqrt{\chi - t}} \, dt = 2^{3/2} \cos (2n+1)\theta',$$

since the Chebyshev polynomials have the property $T_n(\cos \mu) = \cos n\mu$ (Erdélyi et al. [4, p. 184]). Inserting this into (34) we have rewritten the integrated right-hand side of (33) as

$$(37) \qquad \frac{2^{3/2}}{\sqrt{ss_a}} \sum_{n=0}^{\infty} I_{n+1/2}(\hat{\gamma}s_<) K_{n+1/2}(\hat{\gamma}s_>) \cos (2n+1)\theta' = \frac{2^{1/2}}{\sqrt{ss_a}} S.$$

Vice versa, we obtain S as

$$(38) \qquad S = \frac{\sqrt{ss_a}}{2^{1/2}} \int_{-1}^{\chi} \frac{e^{-\hat{\gamma}R'}}{R'} \frac{1}{\sqrt{\chi - t}} \, dt,$$

where

$$(39) \qquad R' = (s^2 + s_a^2 - 2ss_a t)^{1/2} = \sqrt{2ss_a}\sqrt{\zeta - t}$$

with $\zeta = \frac{1}{2}(s/s_a + s_a/s)$. Thus our second sum S, with $\gamma' = \hat{\gamma}\sqrt{2ss_a}$, becomes

$$S = \frac{1}{2} \int_{-1}^{\chi} \frac{\exp(-\gamma'\sqrt{\zeta - t})}{\sqrt{\zeta - t}\sqrt{\chi - t}} \, dt$$

and with $\xi^2 \equiv \zeta - t \geq 0$, we find (since $\zeta \geq 1 \geq \chi \geq t$)

$$S = \frac{1}{2} \int_{\zeta - \chi}^{\zeta + 1} \frac{\exp(-\gamma'\sqrt{\zeta - t})}{\xi\sqrt{\xi^2 - D^2}} \, d\xi^2 = \int_{D}^{(\zeta+1)^{1/2}} \frac{e^{-\gamma'\xi}}{\sqrt{\xi^2 - D^2}} \, d\xi$$

where $D^2 \equiv \zeta - \chi \geq 0$. Rescaling ξ with D ($\xi' = \xi/D$) gives

$$S = \int_{1}^{((\zeta+1)/(\zeta-\chi))^{1/2}} \frac{e^{-\gamma'D\xi'}}{\sqrt{\xi'^2 - 1}} \, d\xi'$$

and finally when we put $\xi' = \cosh u$, this reduces to

$$(40) \qquad S = \int_{0}^{\eta} \exp\left(-\hat{\gamma}\frac{R}{4}\cosh u\right) du$$

where

$$(41) \qquad \eta = \cosh^{-1}\left(\sqrt{\frac{\zeta + 1}{\zeta - \chi}}\right).$$

In the last step we reintroduced $\hat{\gamma}$ and R in favor of γ' and D using the identity

$$\gamma'D = \hat{\gamma}\sqrt{2ss_a}\sqrt{\frac{1}{2}\left(\frac{s}{s_a} + \frac{s_a}{s}\right) - \cos 2\theta'}$$

$$= \frac{\hat{\gamma}}{4}\sqrt{r^4 + \hat{a}^4 - 2r^2\hat{a}^2 \cos 2\theta'}$$

$$= \frac{\hat{\gamma}}{4}|\mathbf{x} - \mathbf{a}||\mathbf{x} + \mathbf{a}| = \frac{\hat{\gamma}}{4} R.$$

Also, η can be written as

$$(42) \qquad \eta = \cosh^{-1}\left(\frac{r^2 + \hat{a}^2}{R}\right).$$

Because the inverse hyperbolic cosine is a double-valued function, we have to specify which branch we take for which part of the plane. This choice is guided by our demand

that the virtual source generated by the first part of the solution is annihilated, which, from the antisymmetry of the solution, implies that the real source will be doubled. In the Appendix an unambiguous expression for η is derived that will be used in the discussion of the solution in the next section:

$$(43) \qquad \eta = \ln \frac{|\mathbf{x} + \mathbf{a}|}{|\mathbf{x} - \mathbf{a}|}.$$

To combine both sums (29) and (40) into one expression, we adopt the integral representation of the zeroth-order modified Bessel function of the second kind (Abramowitz and Stegun [1]) to replace (29):

$$K_0\left(\frac{\hat{\gamma}R}{4}\right) = \int_0^\infty \exp\left(-\frac{\hat{\gamma}}{4} R \cosh u\right) du.$$

We then combine the two sums and express the complete solution for C as

$$(44) \qquad \boxed{C = \frac{1}{4\pi} \exp\left(\frac{\hat{\gamma}}{4}(y^2 - a_y^2 - x^2 + a_x^2)\right) \int_{-\eta}^\infty \exp\left(-\frac{\hat{\gamma}}{4} R \cosh u\right) du,}$$

η given by (42) (with the convention that a minus sign is introduced when $\mathbf{x} \cdot \mathbf{a} \leq 0$), or (43); R by (30) and $\mathbf{a}$ by (8).

We recognize the definite integral in (44) as an *incomplete* integral representation of the zeroth-order modified Bessel function of the second kind. Incomplete cylindrical functions have been extensively discussed in the monograph by Agrest and Maksimov [2]. The incomplete Bessel function, referred to above, can be found there as incomplete MacDonald function, albeit with different functional forms of kernel and limit of integration. The authors remark that incomplete cylindrical functions often occur in transient problems, including the time-dependent response of the concentration field describing a diffusing substance due to an instantaneous source in a *uniformly* advecting flow field. Inhomogeneities, such as the *straining* flow component, considered presently, are excluded in their treatise.

4. Aspects of the closed form solution. The Green function solution (44) has a number of interesting properties. First we observe the symmetric appearance of source (**a**) and field point (**x**), which is expected from the symmetry in the governing equation. Considering the lower limit of integration

$$-\eta = \ln \frac{|\mathbf{x} - \mathbf{a}|}{|\mathbf{x} + \mathbf{a}|},$$

we can make the following observations:

(1) Close to the source (as $\mathbf{x} \to \mathbf{a}$) $-\eta \to -\infty$ and the integral reduces to twice the value of the complete integral representation of K_0 and its logarithmic infinite value matches with the source in **a**. More detail of the solution is obtained by translating our coordinate system back to the source region, while reintroducing the double-primes on our coordinates. This implies $R = |\mathbf{x}'' - \mathbf{a}| |\mathbf{x}'' + \mathbf{a}| = |\mathbf{x}'| |\mathbf{x}' + 2\mathbf{a}|$. Now, in the limit $\hat{\gamma} \to 0$, $\hat{a} \to \infty$. Thus, for $\mathbf{x}'$ having a fixed distance to the source point,

$$\lim_{\hat{\gamma} \to 0} R \to 2r'\hat{a}.$$

When we use (8), the argument of the Bessel function therefore correctly reduces to (see (4))

$$\lim_{\hat{\gamma}\to 0}\left(\frac{\hat{\gamma}R}{4}\right)\to\frac{\hat{\gamma}2r'\hat{a}}{4}=\frac{1}{2}\hat{\beta}r.$$

The exponent of the exponential leading the Bessel function reduces similarly to the corresponding one in (4):

$$\lim_{\hat{\gamma}\to 0}\frac{\hat{\gamma}}{4}(y''^2-x''^2+a_x^2-a_y^2)=\lim_{\hat{\gamma}\to 0}\frac{\hat{\gamma}}{4}[y'(y'+2a_y)-x'(x'+2a_x)]$$

$$\to\exp\left(\frac{\hat{\beta}}{2}(y'\cos\beta'-x'\sin\beta')\right)=\exp\left(\frac{1}{2}(\beta_x'y'-\beta_y'x')\right),$$

since $\beta_x'y'-\beta_y'x'=\beta_x y-\beta_y x$. Hence, close to the source, we may neglect the straining effect of the flow field and exactly retrieve the response due to a uniform advective flow, equation (4), as given by Carslaw and Jaeger [3].

(2) As our field point approaches the virtual source ($x\to -a$) the lower bound $-\eta\to\infty$, in which case the integral in (44) formally vanishes, reflecting the annihilation of the spurious source. In fact, the correct *finite* value of the integral at $x=-a$ is given by the exponential integral $E_1(\hat{\beta}^2/2\hat{\gamma})$ (Abramowitz and Stegun [1]) obtained by using $\lim_{u\to\infty}\cosh u\to\exp(u)/2$ in the exponential.

(3) As $x\cdot a=0$, i.e., on a line normal to the line connecting origin and source point, the lower limit $-\eta\to 0$, expressing the antisymmetry of the second sum S. The concentration value C is given entirely by the value of the symmetric part, but of a magnitude *half* of that expected close to the real source, where uniform advection dominates.

(4) As either $r/\hat{a}\to\infty$, or $r/\hat{a}\to 0$, the lower limit will also approach zero ($-\eta\to 0$). Therefore, far away from the source, or if the source is far away from the origin, the concentration is again given entirely by the symmetric part. Indeed as $\hat{a}=0$, corresponding to a source at the origin of the straining flow field (as the uniform flow part vanishes), we retrieve the solution obtained by Townsend [11] and Okubo [9], in which concentration profiles are given by ellipses.

As remarked in § 2, close to the source the concentration directly downstream is proportional to $r^{-1/2}$. However, due to the straining in the flow field, this path of "weakest" descent bifurcates into two branches, following the main advection directions along the positive and negative y-axis, and the concentration along it eventually ($r\gg\hat{a}$) drops at a much faster rate $\approx r^{-1}$.

An obvious thing to do, once an exact solution has been found, is to verify it by direct substitution into the differential equation it has to satisfy. This is usually done without much interest, but, in this case, it gives some valuable clues as to the peculiar nature of the solution. In the Appendix it is shown that the existence of the solution relies on the following properties of R and η:

$$R\Delta R=(\nabla R)^2=4r^2,$$

independent of a, while both η as well as $R\sinh\eta$ satisfy a Laplace equation

$$\Delta\eta=0,$$

$$\Delta(R\sinh\eta)=0.$$

Solution (44), finally, is in a form that can be readily evaluated by numerical integration. The analytical form of the kernel function guarantees rapid convergence.
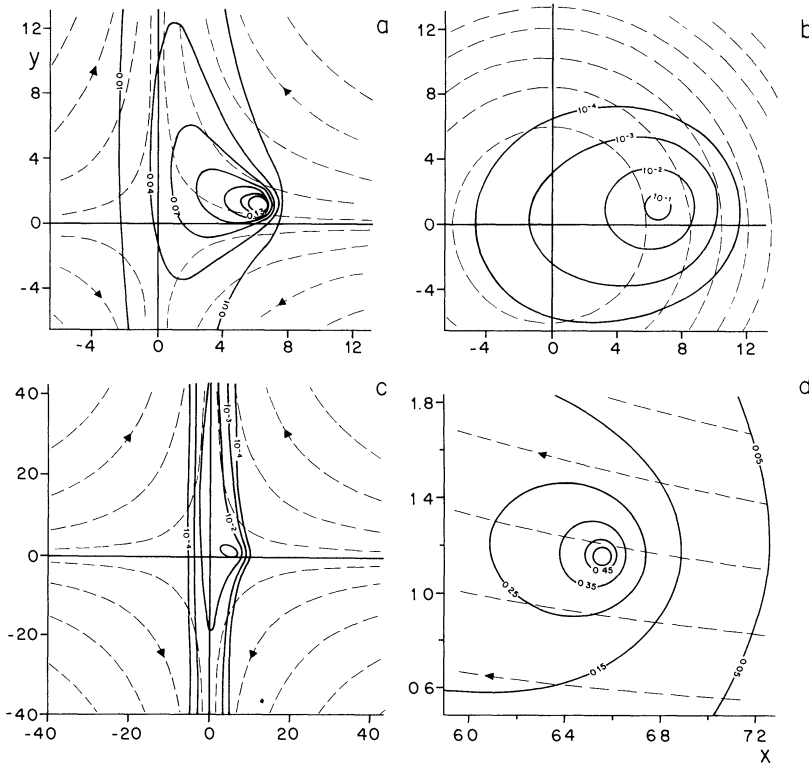
FIG. 2. *Stationary concentration profiles* (————) *in the horizontal* (x, y) *plane* (*dropping double primes*), *calculated by numerically integrating* (44), *are shown for a source position,* **a**, *arbitrarily located at* $\hat{\beta} = 2$, $\hat{\gamma} = 0.3$, $\beta' = 80°$, *with* (a) *and without* (b) *leading exponential of* (44). *Far- and near-field limits of the solution represented in* (a) *degenerate to elliptical* (c) *and wakelike* (d) *profiles, respectively. Dashed lines indicate streamlines of advection field* (a), (c), (d), *or equidistant* r^2 *values* (b).

In this way we have calculated concentration profiles for a few special positions of the source point, displayed in Fig. 2. Indeed, the resulting profiles confirm exactly our intuitive, physical notions about the distributions that we expect will occur: advection is transporting "stuff" downstream away from the source, while simultaneously spreading, due to diffusion that ultimately mixes it across flow field separation lines.

5. Discussion. Another, closely related problem exists that is now solved by inspection. This is given by the differential equation

$$\Delta C + r^2/4 C = -\delta(\mathbf{x} - \mathbf{a})$$

in which, in comparison with (14), the sign in front of $r^2/4$ has been changed; a constant factor has been dropped and $\hat{\gamma}$ has, without loss of generality, been set equal to one. Its solution is

$$C = \int_{-\eta}^{\infty} \exp\left(-iR4 \cosh u\right) du$$

where R and η have their prior meaning and i is the imaginary unit. If $\eta \to \infty$ this reduces to $-2iH_0^{(2)}(R)$, where $H_0^{(2)}(R)$ is the zeroth-order Hankel function of the second kind (Abramowitz and Stegun [1]). The solution is verified by direct substitution (as in the Appendix, appropriately changing signs and incorporating the imaginary unit). This is the Green function solution for a two-dimensional potential hill.

Conceivably a large number of generalizations can be given to the presented solution. Yet, they present new problems to be touched on now.

A first desirable generalization would deal with the time-dependent solution for an instantaneous release, of which the present solution is giving an integral aspect, since it can be viewed as the convolution of the time-dependent Green function with a source function that is stationary in time (in analogy to (4) being the result of this convolution for the case of a uniformly advecting flow $\hat{\gamma} = 0$).

A second generalization would be to obtain solutions of

$$\Delta C + r^l/4C = -\delta(\mathbf{x} - \mathbf{a}),$$

where l is an integer. This equation could arise, e.g., due to a more complicated advecting flow field $\mathbf{u} = \hat{k} \times \nabla\psi$, with $\psi = r^{1+l/2} \sin(1 + l/2)\theta$ for which $l = 0$, gives the uniform advecting flow and $l = 2$ the straining flow field. Parameter values $l = 1$ and 4 correspond with straining flows through three and six segments of the plane, respectively, of which the former has a discontinuity across one branch that must be viewed as a fixed slipping wall. This equation can again be transformed to a Bessel equation and has formal infinite series solutions

$$C = \sum_{m=0}^{\infty} \nu_m I_{|m/(1+l/2)|}(\hat{\gamma}s_<) K_{m/(1+l/2)}(\hat{\gamma}s_>) \cos m\theta'$$

with ν_m the Neumann symbol ($\nu_0 = 1$; $\nu_m = 2$, $m \neq 0$). The absolute value of the order of the modified Bessel function of the first kind again expresses the regularity condition. A closed form solution that would directly reveal the asymptotics is hampered by the fact that there exists no general inverse of Mehler's integral (Erdélyi et al. [4, p. 177]), which gives an integral representation of the Gegenbauer polynomials $C_n^\lambda(\cos\theta)$. For the special case $\lambda = \frac{1}{2}$, the Gegenbauer polynomials are equivalent to Legendre polynomials $P_n(\cos\theta)$, whose integral representations do have an inverse (Erdélyi et al. [4, p. 182]), Tricomi's formula, equation (35) (Erdélyi et al. [4, p. 187]), which relates it to Chebyshev polynomials. With an inverse of Mehler's integral, sums of products of real order Bessel functions multiplied by Gegenbauer polynomials (Magnus, Oberhettinger, and Soni [7, p. 107]), related to eigenfunctions of the hyperspherical modified Helmholtz equation, could be integrated and might lead to expressions for the desired summation of series.

As a last generalization we could consider divergent, or rotating advection fields, explicitly ignored in the present study. For the latter case of uniform rotation, we cannot expect to find any steady state solution, for similar reasons as the absence of such a steady solution in the case when there is no advection at all (see (4), in which we take the amplitude of the advecting flow $\hat{\beta} \to 0$). The former case, however, is interesting and can be readily solved for a source at the origin of the con-, or divergent flow. The general solution of $\Delta C \pm r \partial C/\partial r = \delta(\mathbf{x} - \mathbf{a})$, however, awaits a definite answer.

6. Conclusions. The diffusion of material away from a continuous source in a straining advective flow field is calculated by transforming this problem to a plane where the equation is given as a Green function problem for a two-dimensional harmonic oscillator potential. Its formal solution is readily obtained in terms of two infinite series of products of Bessel functions of integer and half integer order, respectively. The first series is recognized as the expansion of the Green function of the two-dimensional modified Helmholtz equation. It is symmetric in a line through the origin normal to the connection of the center of the flow field and the source point.

The second sum (as the first sum) consists of terms that are everywhere regular. The logarithmic singularity, which it is required to produce in order to annihilate the spurious source in the first sum, is therefore of an algebraic nature (i.e., it is due to the summation of the series). Curiously, the second sum can be related to the expansion of the Green function of the three-dimensional modified Helmholtz equation. This yields an expression that we identify as an incomplete integral representation of the zeroth-order modified Bessel function of the second kind. Indeed the combination of the two sums can be expressed concisely as an incomplete Bessel function, in which the field dependence now enters not only in the kernel function, but also in the lower limit of integration. The solution degenerates to the complete integral representation in the near- and far-field limits (where the ratio of the distances of field point and source point approaches zero or infinity) and at the symmetry line, referred to above. In the case for field points close to the source point, the straining flow component is of vanishing influence compared with the uniform advection part.

Finally, it may be remarked that the most striking property of the solution obtained is that there exists such a wide disparity in effort needed to come from a qualitative, physical understanding of the solution to its mathematical quantification.

Appendix. In this Appendix we verify, by substitution, that the solution to the transformed advection-diffusion equation (14):

(A1) $$\Delta C' - r^2/4\, C' = -\delta(\mathbf{x} - \mathbf{a})$$

(setting $\hat{\gamma} = 1$, without loss of generality, and removing the amplitude factor) is given by

(A2) $$C' = \frac{1}{4\pi} \int_{-\eta}^{\infty} \exp(-W \cosh u)\, du$$

where

(A3) $$W = \tfrac{1}{4}R, \qquad \eta = \cosh^{-1} \xi,$$
$$\xi = \frac{\hat{a}^2 + r^2}{R} = \frac{\hat{a}^2 + r^2}{|\mathbf{x} - \mathbf{a}|\,|\mathbf{x} + \mathbf{a}|}.$$

For convenience we introduce

(A4) $$\mu = R^2 = (|\mathbf{x} - \mathbf{a}|\,|\mathbf{x} + \mathbf{a}|)^2.$$

We need derivatives of μ in the course of the substitution process. Let us, therefore, rewrite μ as

(A5) $$\mu = (\hat{a}^2 + r^2)^2 - 4(\mathbf{a} \cdot \mathbf{x})^2$$

where, recall, $\hat{a} = |\mathbf{a}|$. Then

(A6) $$\nabla \mu = 4\mathbf{x}(\hat{a}^2 + r^2) - 8\mathbf{a}(\mathbf{a} \cdot \mathbf{x})$$

and

(A7) $$\Delta \mu = 16r^2.$$

Also,

(A8) $$(\nabla \mu)^2 = \nabla \mu \cdot \nabla \mu = 16\mu r^2.$$

In terms of $W = R/4 = \mu^{1/2}/4$, this means

(A9) $$\nabla W = \nabla \mu / (8\sqrt{\mu}), \qquad (\nabla W)^2 = (\nabla \mu)^2/(64\mu) = r^2/4,$$

while

$$\Delta W = \Delta \mu / (8\sqrt{\mu}) - (\nabla \mu)^2 / (16\mu^{3/2}).$$

This implies

(A10) $$W \Delta W = \Delta \mu / 32 - (\nabla \mu)^2 / (64\mu) = r^2 / 4.$$

Now, operating with $\mathscr{L} \equiv \Delta - r^2/4$ on C' we obtain from (A2), by Leibnitz' rule:

$$\mathscr{L} C' = \int_{-\eta}^{\infty} [-\Delta W \cosh u + (\nabla W)^2 \cosh^2 u - r^2/4] e^{-W \cosh u} \, du$$
$$- [2\nabla \eta \cdot \nabla W \cosh \eta + W \sinh \eta (\nabla \eta)^2 - \Delta \eta] e^{-W \cosh \eta},$$

which, on integrating the first term in the integrand by parts, yields:

(A11)
$$\mathscr{L} C' = \int_{-\eta}^{\infty} [-W \Delta W \sinh^2 u + (\nabla W)^2 \cosh^2 u - r^2/4] e^{-W \cosh u} \, du$$
$$- [\Delta W \sinh \eta + 2\nabla \eta \cdot \nabla W \cosh \eta + W \sinh \eta (\nabla \eta)^2 - \Delta \eta] e^{-W \cosh \eta}.$$

Using (A9) and (A10), we observe that the integrand vanishes identically! Thus, it is left to show that the part in brackets is also identically zero.

Now η can be rewritten as

(A12) $$\eta = \ln \frac{|\mathbf{x} + \mathbf{a}|}{|\mathbf{x} - \mathbf{a}|},$$

which we recognize as the real part of a complex function

$$F(z) = \frac{1}{2} \log \frac{z + a}{z - a}$$

where $z = x + iy$ and (momentarily) $a = a_x + ia_y$. Therefore η satisfies a Laplace equation

(A13) $$\Delta \eta = 0.$$

The remaining terms inside brackets in (A11) can be combined into $\Delta(W \sinh \eta)$, by adding a term $W \Delta \eta \cosh \eta$ (that is zero because of (A13)). However, since $\sinh \eta = (\xi^2 - 1)^{1/2} = 2(\mathbf{a} \cdot \mathbf{x})/\mu^{1/2}$ and $W = \mu^{1/2}/4$, the remaining term also vanishes:

(A14) $$\Delta W \sinh \eta = 0,$$

which confirms that (A2) solves (A1).

REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN, *Handbook of Mathematical Functions*, Dover, New York, 1965.
[2] M. M. AGREST AND M. S. MAKSIMOV, *Theory of Incomplete Cylindrical Functions and their Applications*, Springer-Verlag, Berlin, Heidelberg, New York, 1971.
[3] H. S. CARSLAW AND J. C. JAEGER, *Conduction of Heat in Solids*, Clarendon Press, Oxford, 1959.
[4] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER, AND F. G. TRICOMI, *Higher Transcendental Functions*, Vol. II, McGraw-Hill, New York, 1953.

[5] H. B. FISCHER, E. J. LIST, R. C. Y. KOH, J. IMBERGER, AND N. H. BROOKS, *Mixing in Inland and Coastal Waters*, Academic Press, New York, 1979.

[6] J. D. JACKSON, *Classical Electrodynamics*, John Wiley, New York, 1975.

[7] W. MAGNUS, F. OBERHETTINGER, AND R. P. SONI, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Springer-Verlag, New York, 1966.

[8] E. MERZBACHER, *Quantum Mechanics*, John Wiley, New York, 1970.

[9] A. OKUBO, *A note on horizontal diffusion from an instantaneous source in a nonuniform flow*, J. Ocean. Soc. Japan, 22 (1966), pp. 35-40.

[10] ———, *The effect of shear in an oscillatory current on horizontal diffusion from an instantaneous source*, Internat. J. Oceanol. Limnol. 1, 3 (1967), pp. 194-204.

[11] A. A. TOWNSEND, *The diffusion of heat spots in isotropic turbulence*, Proc. Roy. Soc. London Ser. A, 209 (1951), pp. 418-430.

[12] W. R. YOUNG, P. B. RHINES, AND C. J. R. GARRETT, *Shear-flow dispersion, internal waves and horizontal mixing in the ocean*, J. Phys. Oceanography, 12 (1982), pp. 515-527.

[13] J. T. F. ZIMMERMAN AND L. R. M. MAAS, *A renormalized Green's function for nonlinear circulation on the beta plane*, Phys. Rev. A, 39 (1989).