

Topographic Filtering and Reflectionless Transmission of Long Waves

LEO R. M. MAAS

Netherlands Institute for Sea Research, Texel, the Netherlands

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ABSTRACT

The equation governing the passage of linear monochromatic, long waves over variable topography can be transformed into a Schrödinger equation. There are several transformations accomplishing this. First, a “naive” transformation (in which only the horizontal coordinate is stretched) yields a potential energy function (“potential”) that is nonvanishing, even if the slope in topography vanishes. Second, a transformation in which also the surface elevation field is stretched leads to a potential that does vanish outside the sloping region. The latter has the property that it displays scattering against a background of adiabatic variations.

For smooth bottom profiles, typical for the continental slope, it is shown that the potential has a positive lobe, the top of which acts as a “topographic cutoff frequency.” This lobe is missed by piecewise-linear topographies.

Despite that the topography, in general, acts as a high-pass filter it is shown that some particular, smooth bottom profiles exist for which long waves, obeying certain conditions, can pass reflectionless.

1. Introduction

The study of waves on water of variable depth is a classical topic (see, e.g., Lamb 1932; Stoker 1953; Provis and Radok 1977; Mei 1989) that has been studied at several scales, levels of complexity, and incorporating different physical mechanisms.

The topic is relevant for the description of small-scale wind waves climbing a shore, as well as for intermediate-scale tsunamis (Kajiura 1963) and large-scale tidal waves (Hendershott 1980) propagating across subsurface ridges and trenches onto the continental shelves.

The level of complexity at which the topic has been addressed ranges from the linearized long wave equations (Meyer 1979; Zhang and Zhu 1994) and linearized potential theory (Roseau 1952), through nonlinear, approximate descriptions of solitary (Shen and Keller 1973; Cai and Shen 1985; Goring and Raichlen 1992) and periodic (Svendsen and Buhr Hansen 1978) waves over variable depth, to a consideration of the exact, nonlinear, shallow-water equations for wave propagation on a linearly sloping beach (Carrier and Greenspan 1957). Depending on the scale of the phenomenon and the physics incorporated, each of these descriptions has its merits.

Waves over variable depth have generally been studied in homogeneous water and in nonrotating frames of reference, for which gravity is the sole restoring force. The depth dependence of phase and group velocity lead

to refraction and diffraction, which, over variable topography, imply the existence of trapped wave motions (e.g., edge waves) and caustics (Shen et al. 1968). Short, irrotational surface waves are described in terms of a velocity potential, from which the long-wave equations follow when the wavelength greatly exceeds depth (Roseau 1952). Depth changes in the latter description are required to extend over still greater scales, where the topography, in a traditional long-wave context, is slowly varying. An important trait in the literature on potential theory of the past decades, however, has been to employ a conformal mapping of the fluid domain over variable depth to a strip with a more complicated free-surface condition (Kreisel 1949; Roseau 1952; Fitz-Gerald 1976). Hamilton (1977) noticed that, on employing this method, he was able to give a much wider applicability to the long-wave equations, since they could be rederived—but now with a *smoothed* version of the topography replacing true depth—even in cases in which the water depth would vary abruptly in comparison to the wavelength.

Inclusion of inhomogeneities in the density of water leads to a consideration of interfacial and internal waves, as when the stratification is either layered or continuous. The interfacial waves are (apart from a rescaling) akin to surface waves and show similar phenomena when considered over variable depth. Internal waves in uniformly stratified media, however, behave quite differently. They propagate along characteristics that have a fixed angle with respect to the vertical, set by the ratio of wave frequency to stability frequency (Phillips 1977). In particular, this angle is retained upon reflection from boundaries, leading to focusing of in-

Corresponding author address: Dr. Leo R. Maas, Netherlands Institute for Sea Research, P.O. Box 59, 1790AB Texel, the Netherlands.
E-mail: maas@nioz.nl

ternal waves. The waves can either get focused into the corner of a fluid domain, such as in the case of a stratified wedge (Wunsch 1969) and stratified canyon (Hotchkiss and Wunsch 1982), or onto a limit set of characteristics, located in the interior of an enclosed stratified fluid domain (Maas and Lam 1995).

The restoring force of gravity, relevant for the wave motions discussed above, can, for long waves, be supplemented by an additional restoring force due to the conservation of potential vorticity. This occurs when there is a gradient in the background vorticity field. Such background vorticity can, for instance, be supplied by current shear, or, as is more commonly addressed, by the earth rotation (planetary vorticity). Besides a natural latitudinal variability in background planetary vorticity (leading to planetary Rossby waves), changes in vorticity are also produced by variations in water depth. The topographic Rossby waves (shelf waves) that result due to this variation are necessarily confined to the sloping regions where this restoring force is operating (Mysak 1978; Hendershott 1980). By assuming the transport of the shelf waves to be nondivergent, the conformal mapping technique has been successfully exploited in this context too (Johnson 1985).

The topic of waves on water of variable depth in itself belongs to the far broader class of problems that studies waves in inhomogeneous media (Tolstoy 1973; Brekhovskikh 1980). Consequently, approaches that have been developed in a particular area of wave propagation usually find their way into different wave propagation contexts. One example is offered by the wave ray, or geometric optics approach, applicable for waves in slowly varying media. This approach, introduced in optics, has successfully been formulated, applied, and extended in a variety of water wave problems (Keller 1958; Shen et al. 1968; Shen and Keller 1975; Christiansen 1977). One should recognize its limitations, however, as it neglects reflection, which can be better addressed with the long-wave equation. Another example is the conceptual framework of the Schrödinger equation, developed in quantum mechanics (e.g., Merzbacher 1970; Bender and Orszag 1978), which enables the qualitative and quantitative study of wave scattering (see section 2). In an oceanographic context this equation has, for instance, been discussed in the description of the scattering of Rossby waves (LeBlond and Mysak 1978, p. 205), internal waves (Eckart 1951), and gravity waves (Mei and Lo 1984), where changes in the “medium” are presented by depth changes, (vertical) density gradient changes, and current changes respectively.

We will follow this second example in section 3, where we will apply it to the scattering of linear long waves by smooth topography, such as is relevant to the propagation of tides and tsunamis over sedimentary slopes. Although this will be done at a very elementary level, three inferences can be drawn from this approach. First, it will be shown that there exists a transformation of variables that leads to a frame of reference in which

adiabatic variations, such as determined by the Liouville-Green (WKB) approach, are contained in the transformation. Variations that nevertheless appear in this frame of reference are thus truly reflecting scattering properties of the topography. Second, it is argued that a variation in depth in general acts as a filter such that for any given shape of the topography a single frequency can be identified that acts as spectral cutoff. This cutoff frequency is directly related to the existence of a positive maximum of the “potential energy function” (potential) in the time-independent Schrödinger equation: a maximum that itself depends on the existence of a convex (upward) curvature of the topography. This frequency should be an observable when comparing energy spectra obtained at neighboring locations in regions of variable depth. Third, it is obvious that such an estimate of the cutoff frequency is absent in a piecewise-linear model of the topography, which lacks such a positive maximum of the potential because of vanishing curvature.

Even though the topography acts as a filter, some particular smooth topographies will be presented that may be transparent for waves of particular frequencies and coming from particular directions (section 4). It is relevant to recognize the existence of this phenomenon, albeit for a particular shape of the bottom, as it may well be a generic property dominating the filtering properties of arbitrarily shaped topographies. Reflectionless transmission of water waves is generally held to be an unphysical feature (Mei 1989, p. 140), a consequence of modeling an inhomogeneity in water depth to extend over a finite width. Two specific smooth, infinitely wide topographies will, however, be discussed here. The first is a trench, reflectionless for normally incident waves. The second is a shelf-edge type of topography, which is reflectionless for obliquely incident waves coming from a direction determined by the depth contrast between shelf and deep sea. Reflection properties of obliquely incident waves have usually been discussed for beaches (Ryrie and Peregrine 1982; Carrier and Noizeaux 1983). An exception is the passage of waves obliquely incident on a partially immersed vertical barrier (Evans and Morris 1972). No reflectionless transmission was obtained in this case however.

2. Schrödinger equation

It is often instructive to consider scattering problems in terms of a Schrödinger equation, because it allows one to qualitatively assess the local nature of the wave field under consideration. The relevance of this equation for the long-wave equations will be discussed more extensively in the next section. The Schrödinger equation,

$$\frac{d^2\Psi}{dx^2} + [E - V(x)]\Psi = 0, \quad (1)$$

describes the shape of the state variable $\Psi(x)$, related to the wave field, $\psi(x, t) = \Psi(x)e^{-i\sigma t}$, (with t denoting time and σ the frequency), due to inhomogeneities of

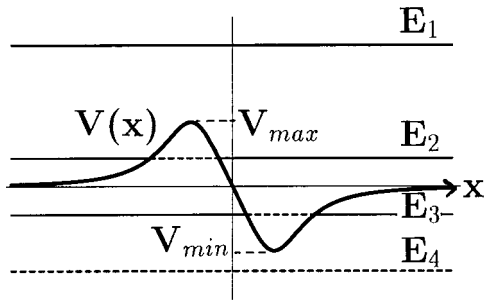


FIG. 1. Sketch of a typical potential $V(x)$ and four levels of the energy, E_1 to E_4 . Solid (dashed) parts of the energy levels refer to sinusoidal (exponential) behavior of the wave field.

the “medium” through which the wave propagates as a function of the coordinate x . The variations of the medium are here represented by the “potential” $V(x)$, related to variations in depth for the long waves to be considered. For localized variations of the medium it is natural to expect the potential to vanish outside the x region for which these variations occur. For values of the “energy” E greater than the maximum value of the potential V_{max} (like E_1 in Fig. 1), the quantity within square brackets in (1) is everywhere positive, and hence the solution is locally sinusoidal. It is therefore expected that the wave will not be greatly attenuated by the scattering potential. If E drops below this maximum, but is otherwise positive (E_2 in Fig. 1), this quantity is negative over some x interval and hence the wave field will be exponentially decaying over this range, leading to attenuation of an incoming wave field: waves can pass only through “tunneling” (Smith 1975; Bender and Orszag 1978, p. 528). For negative values of the energy, $V_{min} < E < 0$ (like E_3 in Fig. 1), trapped waves can exist. Finally, for still lower values, like E_4 , wave solutions no longer exist. Here we will consider only positive values of the energy parameter E . This quantity E actually is usually not related to the true energy of the wave field, but rather is to be regarded as a metaphor for the frequency of the wave involved ($E \propto \sigma^2$ in the long-wave context, see section 3). Therefore, for an incoming spectrum of waves, the existence of a maximum in the potential V_{max} can directly be interpreted as a (soft) cutoff frequency. For waves with energy (frequency) well above V_{max} , waves can pass unimpeded, while for waves with energy (frequency) below V_{max} waves are attenuated. The cutoff frequency is *soft*, however, since no rigorous cutoff (zero transmission) of the incoming wave field below this frequency is implied. A potential like the one shown in Fig. 1 thus acts as a high-pass filter.

3. Topographic filtering

Consider a linear, long plane wave propagating on an f plane at the surface of a homogeneous fluid incident on a smoothly varying topography $H_*(x_*)$, where an as-

terisk denotes a dimensional variable. Let the surface elevation take the form $\zeta_*(x_*) \exp(il_*y_* - i\sigma_*t_*)$. Here l_* indicates the wavenumber in the alongslope direction y_* . We nondimensionalize with length (L) and depth (H_0) scales appropriate to the shelf edge and with the inertial frequency f such that the relevant nondimensional quantities (without asterisks) are obtained from

$$\begin{aligned} \mathbf{x}_* &= L\mathbf{x}, & l_* &= l/L, \\ H_*(x) &= H_0h(x), & \sigma_* &= f\sigma. \end{aligned}$$

Here $h(x)$ is a nondimensional shape function modeling the shelf edge. The cross-isobath structure of the elevation field $\zeta(x)$ is then determined by (LeBlond and Mysak 1978)

$$\frac{d}{dx} \left(h \frac{d\zeta}{dx} \right) + \left[\epsilon^2(\sigma^2 - 1) - l^2h - \frac{l}{\sigma} \frac{dh}{dx} \right] \zeta = 0. \quad (2)$$

Here $\epsilon = L/R$ is the ratio of the external scale L and the Rossby deformation scale, $R = \sqrt{gH_0}/f$. The square of ϵ is known as the divergence parameter. In general, this is a small quantity. For instance, taking scales relevant for the shelf edge, $H_0 = 1$ km, $L = 100$ km, $g = 10 \text{ m}^2 \text{ s}^{-2}$, $f = 10^{-4} \text{ s}^{-1}$, one obtains $\epsilon = 10^{-1}$. However, since the theory may equally be applied to interfacial waves (with rigid-lid surface), this quantity may be of order one. For instance, in the two-layer case [with upper- and lower-layer depths h_1 and $h_2(x)$, respectively] depth is replaced by equivalent depth, $h_e(x) = h_1h_2(x)/(h_1 + h_2(x))$ and gravity g , by reduced gravity g' , being gravity multiplied by the relative density difference of the two layers (LeBlond and Mysak 1978). Typical values of these lead to a phase speed of about one-hundredth of its value in the barotropic case, or $\epsilon = O(1)$. The latter quantity being relatively large may raise some questions about the validity of the long-wave model. As argued in the introduction, Hamilton (1977) has shown (for normally incident waves, $l = 0$), however, that (2) still applies in case the horizontal length scale of the topography variations is much shorter than a typical wavelength, provided $h(x)$ represents a properly smoothed version of the actual topography.

Equation (2) can be “naively” transformed to a Schrödinger equation by multiplying it with h and identifying hd/dx with $d/d\xi$, which amounts to a stretching of the horizontal coordinate

$$\xi = \int_0^x \frac{1}{h(x')} dx', \quad (3a)$$

in inverse proportion to water depth, or, inversely to

$$x = \int_0^\xi h(\xi') d\xi'. \quad (3b)$$

The equation then takes the form

$$\frac{d^2\zeta}{d\xi^2} + \left[\epsilon^2(\sigma^2 - 1)h(\xi) - l^2h^2(\xi) - \frac{l}{\sigma} \frac{dh}{d\xi} \right] \zeta = 0, \quad (4)$$

where $h(\xi) = h(x(\xi))$. This equation was employed by Saint-Guilly (1976) for the depth profile

$$h(\xi) = 1 + \lambda \tanh \xi, \quad (5)$$

with $\lambda \in (-1, 1)$ the ‘‘depth-contrast’’ parameter. Together with (3b) this gives a parametrically defined depth profile $h(x)$. This shape of the topography enabled him to calculate the spectrum of *trapped* modes exactly. These topographic Rossby waves are trapped, however, only in cross-slope direction but may freely propagate along the slope. Equation (4), with (5) substituted, can be viewed as a Schrödinger equation if we identify the constant and spatially varying parts with the energy E and potential $V(\xi)$ respectively. For the case of freely *propagating* waves, however, this potential is not ‘‘physically realistic’’ in the sense that it does not vanish at infinity and therefore does not satisfy the requirement that the scattering be localized. For instance, for normally incident waves ($l = 0$), (4) simplifies to

$$\frac{d^2\zeta}{d\xi^2} + \epsilon^2(\sigma^2 - 1)h(\xi)\zeta = 0,$$

which, for the tanh-shaped topography (5) considered, leads to a potential that is nonvanishing at infinity. For this reason a transformation is employed that stretches not only the horizontal but also the vertical coordinate (the elevation)

$$\xi = \int_0^x \frac{1}{\sqrt{h(x')}} dx', \quad Z = \zeta h^{1/4} \quad (6)$$

(see Morse and Feshbach 1953, p.730 and, in a planetary wave context, Krauss 1973). Discussion is for the sake of simplicity here again limited to waves of normal incidence ($l = 0$), for which (2) takes the form

$$\frac{d}{dx} \left(h \frac{d\zeta}{dx} \right) + \epsilon^2(\sigma^2 - 1)\zeta = 0. \quad (7)$$

The general case of obliquely incident waves can be treated likewise. With (6) this equation takes the form

$$\frac{d^2Z}{d\xi^2} + \left[\epsilon^2(\sigma^2 - 1) - \frac{1}{h^{1/4}} \frac{d^2h^{1/4}}{d\xi^2} \right] Z = 0,$$

which is a Schrödinger equation once we identify energy and potential as $E = \epsilon^2(\sigma^2 - 1)$ and $V(\xi) = h^{-1/4} d^2h^{1/4}/d\xi^2$ respectively. This shows that energy E is indeed related to frequency σ . This expression for E identifies positive values with freely propagating, superinertial ($\sigma > 1$) waves and negative values with subinertial ($\sigma < 1$) waves (trapped in the cross-slope direction). Since the second ξ derivative of $h^{1/4}$ is related to first and second x derivatives of the topography $h(x)$ (see below), which vanish away from the sloping region, it is evident that this form of the

potential *does* vanish outside that region and hence is of the expected localized form.

Without actually solving the resulting equation, a number of inferences can be drawn from the form it has.

First, assume that we are dealing with energy values (frequencies) much greater than the maximum value of the potential $E \gg V_{\max}$, such as occurs when the depth varies only weakly in terms of the stretched horizontal distance ξ . Then, we can approximate the potential by assuming that it vanishes identically, $V(\xi) \equiv 0$. Depth variations exist but have a negligible impact on the scattering process. In this case, the solutions in the transformed plane consist, of course, of plane waves of the form $Z = Z_0 \exp(\pm i\sqrt{E} \xi)$. In the original frame this solution reduces to the adiabatic variations presented by the Liouville-Green (WKB) approximation

$$\zeta = \frac{Z_0}{h^{1/4}} \exp\left(\pm i\sqrt{E} \int_0^x \frac{1}{\sqrt{h(x')}} dx' - i\sigma t \right).$$

This contains wavenumber variations (the x derivative of the phase factor) inversely proportional to \sqrt{h} (consistent with group velocity, c_g , and phase velocity proportional to \sqrt{h}) and amplitude variations inversely proportional to $h^{1/4}$ (consistent with the conservation of energy flux, proportional to $|\zeta|^2 c_g$). Since amplitude (and wavenumber) variations associated with adiabatic changes do not form part of the scattering process, the physically appropriate frame of reference in which to consider scattering is that based on (7). Any amplitude and wavenumber variations obtained in that frame are truly associated with scattering.

Second, consider a typical monotonic shelf edge like $h^{1/4} = 1 + \lambda \tanh \xi$, which is similar to the topography employed by Saint-Guilly (1976) except that it now applies to the quarter power of the topography. Then the potential takes the form

$$V(\xi) = -2\lambda \frac{T(1 - T^2)}{1 + \lambda T},$$

where $T \equiv \tanh \xi$ (see Figs. 2 and 3a). The potential is observed to have the typical two-lobed shape adopted in the discussion of Fig. 1, the positive lobe extending over the top of the shelf edge. The position of the maximum value of the potential T_{\max} , as well as its value at this position, V_{\max} , can be determined analytically as a function of the depth-contrast parameter λ . Their expressions are rather cumbersome and are therefore just shown graphically (Fig. 3b). Now, since the topology of the problem would not change when we vary the shape of the monotonically sloping topography, we may expect the occurrence of a positive lobe [$V(\xi) > 0$ for some range of ξ] to be a generic feature of this scattering problem. Therefore, following the discussion in the introduction, one may infer that there exists a topographic cutoff frequency $\sigma_T \equiv (1 + \epsilon^2 V_{\max}(\lambda))^{1/2}$, for *any* monotonically sloping topography, which is a function of the

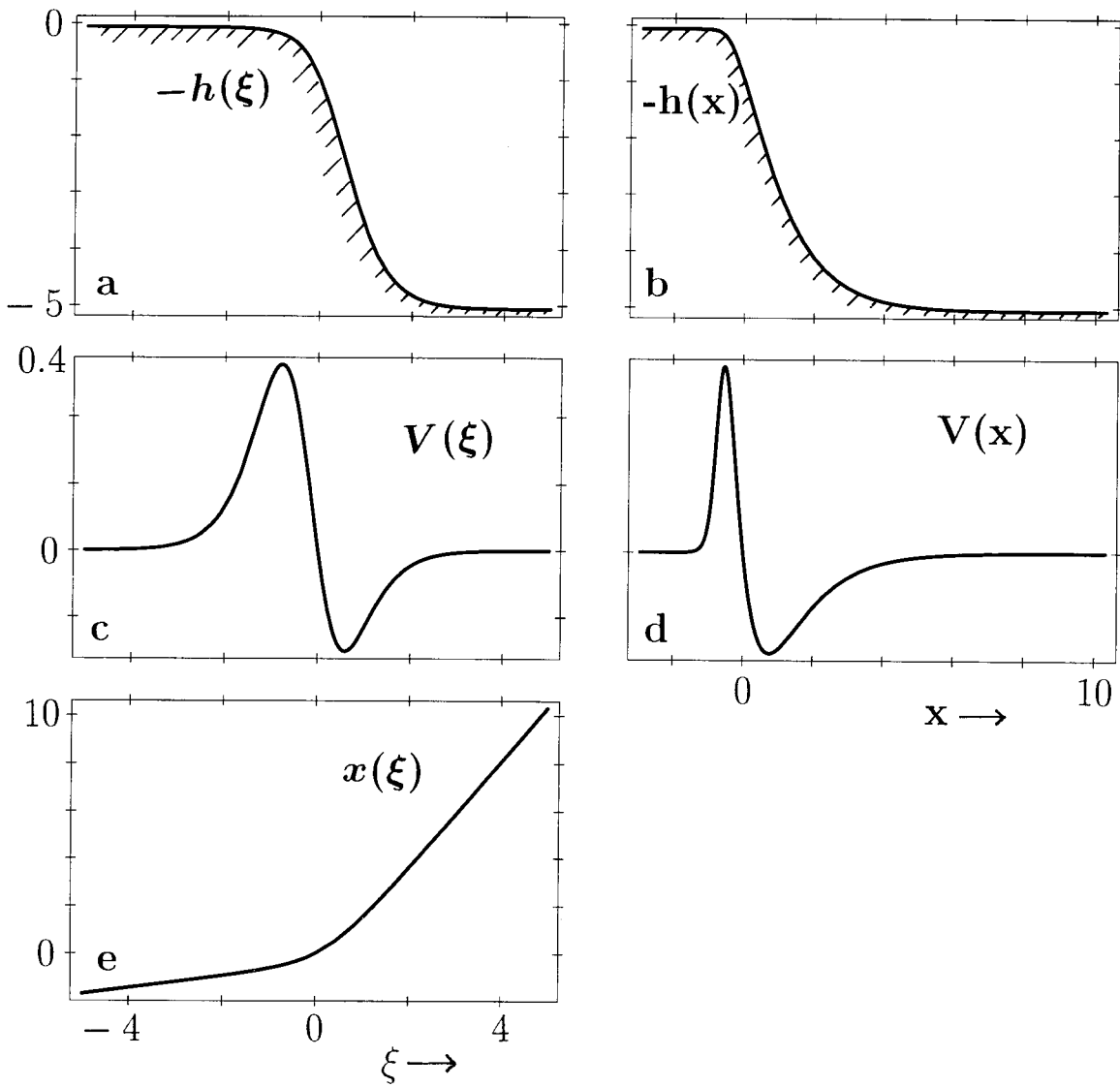


FIG. 2. Sketches of the topography $h(\xi) = (1 + \lambda \tanh \xi)^4$ for $\lambda = 1/2$ as a function of ξ (a) and, parametrically, of x (b). The potential $V(\xi) = h^{-1/4} d^2 h^{1/4} / d\xi^2$ can be obtained from these, both in the transformed (c) as well as, again parametrically, in the original frame (d). Here the x dependence on ξ is given by $x = \int_0^\xi \sqrt{h(\xi)} d\xi = \xi(1 + \lambda^2) + 2\lambda \ln[\cosh \xi] - \lambda^2 \tanh \xi$; see (e).

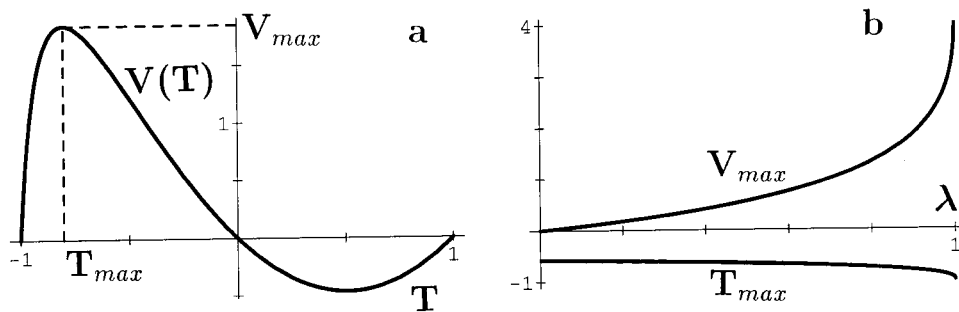


FIG. 3. (a) Potential V as a function of T for $\lambda = 0.9$. Since $T \equiv \tanh \xi$ is a monotonic function of ξ , this is a compact way of representing the ξ -dependence of the potential. In this figure the position, T_{max} , and value of the peak of the potential, V_{max} , have been indicated, which are shown as a function of λ in (b).

geometrical parameters (the divergence parameter ϵ^2 and the depth-contrast parameter λ). For the parameters considered previously $\epsilon = 10^{-1}$, and for $\lambda = 1/2$, which has $V_{\max} \approx 1/2$, we obtain $\sigma_T \approx 7$. For interfacial waves, with $\epsilon \approx 1$, σ_T approaches the inertial frequency even closer ($\sigma_T \downarrow 1$). It is likely that this quantity σ_T , which is easily computed for any given shape of a depth profile, must be observable as a dividing frequency when comparing adjacent (directional) deep-sea and shelf spectra, such that for $\sigma > \sigma_T$, waves can pass the sloping region fairly easy (and vice versa).

Third, expanding the expression of the potential we find

$$V = \frac{1}{4h} \frac{d^2h}{d\xi^2} - \frac{3}{16} \left(\frac{1}{h} \frac{dh}{d\xi} \right)^2 = \frac{1}{4} \frac{d^2h}{dx^2} - \frac{1}{16h} \left(\frac{dh}{dx} \right)^2.$$

Since $h > 0$ for all x , the second term in the last expression at the right is always negative. Hence the potential is positive only because of the existence of the first term in that expression, which is related to the (convex upward) curvature of the topography at the top of the shelf slope (Meyer 1979). It is clear that this term would be absent in piecewise-linear topographies, which would therefore exclude the phenomenon of tunneling and, to some extent, topographic filtering.

4. Reflectionless transmission

Another artifact would actually be introduced when approximating a smooth, monotonic topography by a piecewise-linear topography. This is the phenomenon of reflectionless transmission at certain discrete wave frequencies that have an integral number of half-waves fitting over a linearly sloping region (LeBlond and Mysak 1978, p. 209). This, in fact, also applies for more generally shaped slopes (Kajiura 1963) as well as for (long) symmetrical hills (Newman 1965; Fitz-Gerald 1976) as long as they are of finite width. It is generally regarded that the occurrence of these reflectionless frequencies is a spurious feature, which is a consequence of the piecewise approximation of the topography employed (Kajiura 1963; Meyer 1979; Mei 1989). It is expected that such complete transmission would be absent for *smooth* depth profiles. Indeed, for some particular, smooth monotonic profiles Kajiura (1963) showed that the reflection coefficient is a steadily decreasing function of frequency for normally incident waves. Reflectionless transmission of normally incident long waves over smooth topographies of infinite extent is found only for symmetrical ridges (Fitz-Gerald 1976), a result obtained semianalytically, without restriction on wavelength, by an iterative scheme (see also Roseau 1952).

Surprisingly, reflectionless transmission may also occur over a smooth trench for normally incident long waves, as well as over a monotonically increasing depth profile for obliquely incident long waves that come from

a particular direction related to the depth contrast between deep sea and shelf. Ironically, this can be demonstrated most easily by employing the naive transformation (3).

Kay and Moses (1956) showed that the Schrödinger equation

$$\frac{d^2\zeta}{d\xi^2} + [E' + n(n+1)\text{sech}^2\xi]\zeta = 0, \quad (8)$$

is reflectionless, for $n = 0, 1, 2, \dots$, for any value of $E' > 0$. Now consider a ridge ($\mu < 0$), or trench ($\mu > 0$) topography

$$h(\xi) = 1 + \mu \text{sech}^2\xi,$$

that, with (3b), is parametrically related to the spatial coordinate

$$x = \xi + \mu \tanh \xi.$$

Then, for normally incident waves ($l = 0$), a substitution of this topography in (4) yields equation (8) for the discrete set of frequencies determined by

$$E' \equiv \epsilon^2(\sigma^2 - 1) = n(n+1)/\mu.$$

This leads to propagating gravity waves ($\sigma^2 > 1$) provided $n > 0$ and $\mu > 0$, that is, provided the topography is a *trench*. Expressions for transmission and reflection coefficients for incoming waves of arbitrary frequency and approximate expressions for the surface elevation field can be found in Lamb (1980).

Using, alternatively, the Saint-Guilly (1976) topography (5), Eq. (4) reads

$$\frac{d^2\zeta}{d\xi^2} + \left[E - l^2(1 + \lambda^2) + \lambda(E - 2l^2)\tanh \xi - l\lambda \left(\frac{1}{\sigma} - l\lambda \right) \text{sech}^2\xi \right] \zeta = 0.$$

This tells us that long waves are able to pass the tanh-shaped shelf edge without reflection provided this again reduces to (8). Identifying coefficients between these equations we find that this requires

$$E = 2l^2, \quad (9a)$$

and

$$l\lambda \left(\frac{1}{\sigma} - l\lambda \right) = -n(n+1). \quad (9b)$$

Because $E' = E - l^2(1 + \lambda^2)$, we verify from (9a) that E' satisfies the positivity constraint: $E' = l^2(1 - \lambda^2)$. Since, by definition,

$$E = \epsilon^2(\sigma^2 - 1), \quad (10)$$

which from the dispersion relation applied in the far field equals $h(l^2 + k^2)$, we find, assuming the waves to enter from the deep region where $h = 1 + \lambda$,

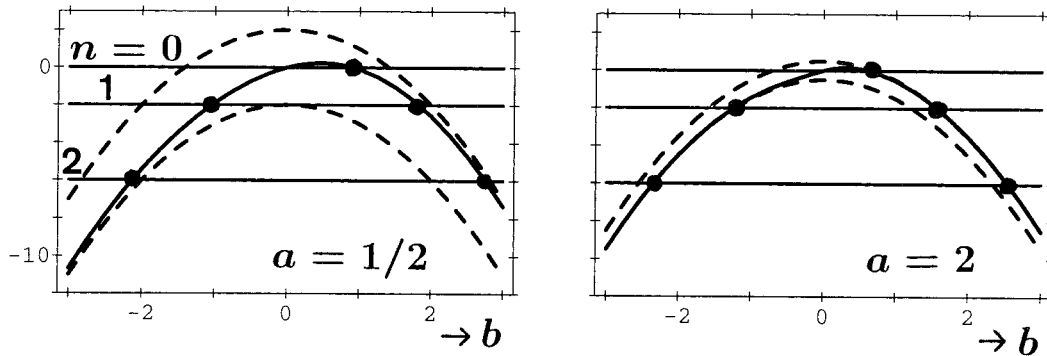


FIG. 4. Plot of $b/\sqrt{1+a^2b^2} - b^2$ and $-n(n+1)$ (solid lines) as a function of $b \equiv \lambda l$ for two values of $a \equiv 2/\lambda^2\epsilon^2$ and integer $n \in \{0, 1, 2\}$. Heavy dots indicate values of b (i.e., scaled along-isobath wavenumber l) for which waves, coming from a direction α_* given by (11), are able to pass the tanh-shaped shelf edge reflectionless. Dashed lines indicate the two asymptotes $\pm 1/a - b^2$.

$$E = (1 + \lambda) \frac{l^2}{\sin^2 \alpha}.$$

Here the absolute value κ of the wavenumber vector $\mathbf{k} = (k, l) = \kappa(\cos \alpha, \sin \alpha)$ —that makes an angle α with the x direction—has been replaced in terms of l and α . Hence, inserting this expression for E in (9a) and eliminating l^2 , we find that waves coming from directions α_* determined by

$$\sin^2 \alpha_* = \frac{1 + \lambda}{2} \tag{11}$$

are able to pass this shelf edge reflectionless, provided they satisfy also the second constraint (9b). From (10) and (9a) we obtain σ as a function of $b \equiv l\lambda$:

$$\sigma = \sqrt{1 + a^2b^2}, \tag{12}$$

where $a^2 = 2/\lambda^2\epsilon^2$. Inserting this into (9b) and plotting both the left- and right-hand sides of this equation as a function of b (Fig. 4), we find at their intersections the wavenumbers for which waves, coming from directions determined by (11), are able to pass the shelf edge without any reflection. From (12) their frequencies can be obtained. For $a > 1$, these are approximately determined by the two asymptotic (dashed) curves in Fig. 4, which leads to

$$\sigma_n^2 \approx 1 + \frac{l}{|l|} a + a^2 n(n + 1), \quad n \in \mathbf{N}.$$

For $n = 0$, only one physically realistic solution ($\sigma_n^2 > 0$) is obtained, having the coast at its left (in the Northern Hemisphere), seen from the alongshelf propagation direction ($l > 0$). The expression of the frequency for general values of a is slightly more involved. Note that the same result applies to the reflectionless transmission of interfacial waves, provided gravity and depth (both depth scale and shape function) are replaced by their internal equivalents (see section 3). In a continuously stratified fluid, reflectionless transmission of internal

waves propagating over a shelf edge has been addressed in Sandstrom (1976).

5. Conclusions

It is shown qualitatively that a monotonically sloping shelf edge generally acts as a topographic filter for incoming, long superinertial waves. This filter can be characterized by a (soft) cutoff frequency above (below) which waves can pass the topography without (with) much attenuation. This frequency is solely dependent on parameters characterizing the geometry of the problem (topographic scales, latitude, and earth rotation rate). The filtering properties are crucially dependent on the existence of a convex (upward) part of the bottom shape at the top of the shelf edge. It provides the positive lobe of the localized potential in the Schrödinger equation to which the scattering problem can be transformed. It is attractive to view this positive lobe of the potential of a shelf edge as providing a natural shield by which the shelf region is “protected” against incoming waves. Each shelf edge may, however, also have an Achilles’ heel. It is suggested by the study of a particular, continuous topography that a shelf edge may be completely transparent for waves of particular discrete frequencies, coming from two directions determined by the “depth-contrast parameter.” The shelf edge under consideration is particularly “vulnerable” to waves propagating from these two directions. (It was noted that a particular form of a trenchlike topography is transparent for normally incident waves.) Although no parameter sensitivity analysis of these results has been made yet, it is conjectured that a true shelf edge (susceptible to depth-dependent frictional and nonlinear effects) should show its vulnerability over some range of directions and frequencies around those calculated. Within the limits of an inviscid and linear analysis it might be useful, therefore, to make a catalog for shelf edges around the world identifying these reflectionless angles and frequencies at each location.

REFERENCES

- Bender, C. M., and S. A. Orszag, 1978: *Advanced Mathematical Methods for Scientists and Engineers*. McGraw-Hill, 593 pp.
- Brekhovskikh, L. M., 1980: *Waves in Layered Media*. Academic Press, 240 pp.
- Cai, Y. H., and M. C. Shen, 1985: The development of a solitary wave in a channel of variable elliptical cross section. *Phys. Fluids*, **28**, 2352–2356.
- Carrier, G. F., and H. P. Greenspan, 1957: Water waves of finite amplitude on a sloping beach. *J. Fluid Mech.*, **4**, 97–109.
- , and C. F. Noizeaux, 1983: The reflection of obliquely incident tsunamis. *J. Fluid Mech.*, **133**, 147–160.
- Christiansen, P. L., 1977: Diffraction of gravity waves by ray methods. *Waves on Water of Variable Depth*, D. G. Provis and R. Radok, Eds., Springer, 28–38.
- Eckart, C., 1951: Internal waves in the ocean. *Phys. Fluids*, **4**, 791–799.
- Evans, D. V., and C. A. N. Morris, 1972: The effect of a fixed vertical barrier on obliquely incident surface waves in deep water. *J. Inst. Math. Its Appl.*, **9**, 198–204.
- Fitz-Gerald, G. F., 1976: The reflexion of plane gravity waves traveling in water of variable depth. *Philos. Trans. Roy. Soc. London A*, **284**, 49–89.
- Goring, D. G., and F. Raichlen, 1992: Propagation of long waves onto shelf. *J. Waterw., Port, Coastal Ocean Div., ASCE*, **118**, 43–61.
- Hamilton, J., 1977: Differential equations for long-period gravity waves on fluid of rapidly varying depth. *J. Fluid Mech.*, **83**, 289–310.
- Hendershott, M. C., 1980: Long waves and ocean tides. *Evolution of Physical Oceanography*, B. Warren and C. Wunsch, Eds., The MIT Press, 292–341.
- Hotchkiss, F. S., and C. Wunsch, 1982: Internal waves in Hudson Canyon with possible geological implications. *Deep-Sea Res.*, **29**, 415–442.
- Johnson, E. R., 1985: Topographic waves and the evolution of coastal currents. *J. Fluid Mech.*, **160**, 499–509.
- Kajiura, K., 1963: On the partial reflection of water waves passing over a bottom of variable depth. *Proc. Tsunami Meetings 10th Pacific Science Congress, IUGG Monogr.*, No. 24, IUGG, 206–234.
- Kay, I., and H. E. Moses, 1956: Reflectionless transmission through dielectrics and scattering potentials. *J. Appl. Phys.*, **27**, 1503–1508.
- Keller, J. B., 1958: Surface waves on water of non-uniform depth. *J. Fluid Mech.*, **4**, 607–614.
- Krauss, W., 1973: *Methoden und Ergebnisse der Theoretischen Oceanographie, II Interne Wellen*. Gebrüder Borntraeger, 248 pp.
- Kreisel, G., 1949: Surface waves. *Quart. Appl. Math.*, **7**, 21–44.
- Lamb, G. L., Jr., 1980: *Elements of Soliton Theory*. Wiley-Interscience, 289 pp.
- Lamb, Sir H., 1932: *Hydrodynamics*. Dover, 738 pp.
- LeBlond, P. H., and L. A. Mysak, 1978: *Waves in the Ocean*. Elsevier, 602 pp.
- Maas, L. R. M., and F.-P. A. Lam, 1995: Geometric focusing of internal waves. *J. Fluid Mech.*, **300**, 1–41.
- Mei, C. C., 1989: *The Applied Dynamics of Ocean Surface Waves*. World Scientific, 740 pp.
- , and E. Lo, 1984: The effects of a jet-like current on gravity waves in shallow water. *J. Phys. Oceanogr.*, **14**, 471–477.
- Merzbacher, E., 1970: *Quantum Mechanics*. Wiley, 621 pp.
- Meyer, R. E., 1979: Surface-wave reflection by underwater ridges. *J. Phys. Oceanogr.*, **9**, 150–157.
- Morse, P. M., and H. Feshbach, 1953: *Methods of Theoretical Physics I*. McGraw-Hill, 997 pp.
- Mysak, L. A., 1978: Recent advances in shelf wave dynamics. *Rev. Geophys. Space Phys.*, **18**, 211–241.
- Newman, J. N., 1965: Propagation of water waves past long two-dimensional obstacles. *J. Fluid Mech.*, **23**, 23–29.
- Phillips, O. M., 1977: *Dynamics of the Upper Ocean*. Cambridge University Press, 336 pp.
- Provis, D. G., and R. Radok, Eds., 1977: *Waves on Water of Variable Depth*. Springer, 231 pp.
- Roseau, M., 1952: Contribution à la théorie des ondes liquides de gravité en profondeur variable. Publication Scientifique et Technique du Ministre de l'air 275, 91 pp.
- Ryrie, S., and D. H. Peregrine, 1982: Refraction of finite-amplitude water waves obliquely incident on a uniform beach. *J. Fluid Mech.*, **115**, 91–104.
- Saint-Guily, B., 1976: Sur la propagation des ondes de seconde classe le long d'un talus continental. *C.R. Acad. Sci. Paris*, **282**, 141–144.
- Sandstrom, H., 1976: On topographic generation and coupling of internal waves. *Geophys. Fluid Dyn.*, **7**, 231–270.
- Shen, M. C., and J. B. Keller, 1973: Ray method for nonlinear wave propagation in a rotating fluid of variable depth. *Phys. Fluids*, **16**, 1565–1572.
- , and —, 1975: Uniform ray theory of surface, internal and acoustic wave propagation in a rotating ocean or atmosphere. *SIAM J. Appl. Math.*, **28**, 857–875.
- , R. E. Meyer, and J. B. Keller, 1968: Spectra of water waves in channels and around islands. *Phys. Fluids*, **11**, 2289–2304.
- Smith, R., 1975: Second-order turning point problems in oceanography. *Deep-Sea Res.*, **22**, 837–852.
- Stoker, J. J., 1953: *Water Waves*, Interscience, 567 pp.
- Svendsen, I.A., and J. Buhr Hansen, 1978: On the deformation of periodic long waves over a gently sloping bottom. *J. Fluid Mech.*, **87**, 433–448.
- Tolstoy, I., 1973: *Wave Propagation*. McGraw-Hill, 466 pp.
- Wunsch, C., 1969: Progressive internal waves on slopes. *J. Fluid Mech.*, **35**, 131–141.
- Zhang, Y., and S. Zhu, 1994: New solutions for the propagation of long water waves over variable depth. *J. Fluid Mech.*, **278**, 391–406.