# Forms of $BP\mathbb{R}\langle n\rangle$ – Preliminary Version

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## 1 Introduction

Let $R$ be a ring spectrum that is a form of $BP\langle n\rangle$ and has with a $C_2$-action that strictly preserves the ring structure and acts on $\pi_{2i}$ as $(-1)^i$. For example, $R$ could be $ku$ or $tmf_1(3)$. We will show that this information alone suffices to compute the homotopy fixed point spectral for the $C_2$-action on $R$. The basic method is to transfer differentials from the computation of Real bordism $M\mathbb{R}$.

Given $R$ as above, we can define a genuine $C_2$-spectrum that is the connective cover of the cofree version $R^{(EC_2)_+}$. We will deduce that this is already a form of BPℝ⟨n⟩, i.e. can be obtained from $BP\mathbb{R}$ by quotiening out a sequence of (generalized) $\overline{v}_{n+1}, \overline{v}_{n+2}, \ldots$.

## 2 Forms of $BP\mathbb{R}\langle n\rangle$

For general background on Real homotopy theory, we refer to [HK01], [HM15] and [GM16]. Recall in particular the following definition.

**Definition 1.** Let $R$ be an even 2-local commutative and associative $C_2$-ring spectrum up to homotopy.[1] By [HM15, Lemma 3.3], $R$ has a Real orientation and after choosing one, we obtain a formal group law on $\pi^{C_2}_{*\rho}R$. The 2-typification of this formal group law defines a map $\pi^e_{2*}BP \cong \pi^{C_2}_{*\rho}BP\mathbb{R} \to \pi^{C_2}_{*\rho}R$. We call $R$ a *form of $BP\mathbb{R}\langle n\rangle$* if the map

$$\underline{\mathbb{Z}_{(2)}[\overline{v}_1, \ldots, \overline{v}_n]} \subset \underline{\pi}_{*\rho}BP\mathbb{R} \to \underline{\pi}_{*\rho}R$$

is an isomorphism of constant Mackey functors.

For the purposes of this paper, we also introduce the notion of a BPℝ⟨n⟩-like spectrum.

**Definition 2.** A BPℝ⟨n⟩*-like $C_2$-spectrum* is a $C_2$-equivariant homotopy commutative ring spectrum with the following properties:

1. Its underlying spectrum is a form of $BP\langle n\rangle$,
2. The non-trivial element of $C_2$ acts on $\pi_{2i}R$ as $(-1)^i$,

[1] Here and in the following we could easily relax the unitality, associativity and commutativity conditions to be up to phantoms. For example, the Real Johnson–Wilson spectra are known to satisfy this latter condition by [KLW17], but are in general not known to be homotopy commutative and associative $C_2$-ring spectra.

3. $R$ is the connective cover of its cofree version $R^h = R^{(EC_2)_+}$.

A useful fact about connective covers is the following:

**Lemma 3.** *Let $E$ be a $C_2$-spectrum and $\tau_{\geq 0}E$ be its connective cover. Then $\pi^{C_2}_{a+b\sigma}\tau_{\geq 0}E = 0$ if $a < 0$ and $a + b \leq 0$. Furthermore, the map*

$$\pi^{C_2}_{a+b\sigma}\tau_{\geq 0}E \to \pi^{C_2}_{a+b\sigma}E$$

*is an isomorphism if $a \geq 0$ and $a + b \geq 0$.*

*Proof.* The cofiber sequence $(C_2)_+ \to S^0 \to S^\sigma$ induces a long exact sequence

$$\pi^{C_2}_{(a+1)+b\sigma}E \to \pi^e_{a+1+b}E \to \pi^{C_2}_{a+(b+1)\sigma}E \to \pi^{C_2}_{a+b\sigma}E \to \pi^e_{a+b}E.$$

and likewise for $\tau_{\geq 0}E$. This implies that if $a+b < 0$ and $\pi^{C_2}_{a+(b+1)\sigma}\tau_{\geq 0}E = 0$, then $\pi^{C_2}_{a+b\sigma}\tau_{\geq 0}E = 0$. Likewise, if $a + b + 1 < 0$ and $\pi^{C_2}_{a+b\sigma}E = 0$, then also $\pi^{C_2}_{a+(b+1)\sigma}E = 0$. This implies the first part since we already know that $\pi^{C_2}_{a}\tau_{\geq 0}E = 0$ for $a < 0$.

A similar five lemma argument implies the second part. □

We will later show that every $\mathrm{BP}\mathbb{R}\langle n\rangle$-like $C_2$-spectrum is a form of $\mathrm{BP}\mathbb{R}\langle n\rangle$ using the following lemma.

**Lemma 4.** *If the $RO(C_2)$-graded homotopy fixed point spectral sequence[2] of a $\mathrm{BP}\mathbb{R}\langle n\rangle$-like $C_2$-spectrum $R$ agrees with that of $\mathrm{BP}\mathbb{R}\langle n\rangle$, then $R$ is a form of $\mathrm{BP}\mathbb{R}\langle n\rangle$.*

*Proof.* By [GM16, Section 4.C], we just have to show that $R$ is strongly even, i.e. that $\underline{\pi}^{C_2}_{(k-1)\rho}R$ is constant and $\pi^{C_2}_{k\rho-1}R = 0$ for all $k \in \mathbb{Z}$, where $\rho$ denotes the real regular representation of $C_2$. For $k \leq 0$, these groups are zero by the last lemma since $R$ is connective. For $k > 0$, the last lemma implies that we have a chain

$$\underline{\pi}^{C_2}_{(k-1)\rho}R \cong \underline{\pi}^{C_2}_{(k-1)\rho}R^h \cong \underline{\pi}^{C_2}_{(k-1)\rho}\mathrm{BP}\mathbb{R}\langle n\rangle^h \cong \underline{\pi}^{C_2}_{(k-1)\rho}\mathrm{BP}\mathbb{R}\langle n\rangle$$

of isomorphisms as [GM16, Proposition 4-9] implies $\mathrm{BP}\mathbb{R}\langle n\rangle$ is the connective cover of $\mathrm{BP}\mathbb{R}\langle n\rangle^h$. Here, we use that in these degrees there can be no additive extension issues in the homotopy fixed point spectral sequence as the groups are free $\mathbb{Z}_{(2)}$-modules.

We have a similar chain of isomorphisms for $k\rho - 1$. As $\mathrm{BP}\mathbb{R}\langle n\rangle$ is strongly even [GM16, Proposition 4-6], the result follows. □

# 3 The right unit for $BP_*$

There are two unit maps $\eta_L, \eta_R \colon BP_* \to BP_*BP$. We can identify $BP_*BP$ with $BP_*[t_1, t_2, \dots]$ so that $\eta_L$ is the obvious inclusion and thus we will leave it out from the notation. The right unit $\eta_R$ is more difficult to describe. There are some refined results in [Rav86, Section 4.3], but we will need only two rather simple-minded facts about it.

Let $F$ be the universal $p$-typical formal group law on $BP_*$. We can identify $BP_*$ with $\mathbb{Z}_{(p)}[v_1, v_2, \dots]$, where we use the Araki generators for definiteness. We have $|v_i| = |t_i| = 2(p^i - 1)$.

[2] We will explain more about this in Section 4.

As usual, we set formally $v_0 = p$ and $t_0 = 1$. In these terms, we have the following formula for the right unit [Rav86, Thm A.2.2.5]:

$$\sum_{i,j\geq 0}^{F} t_i \eta_R(v_j)^{p^i} = \sum_{i,j\geq 0}^{F} v_i t_j^{p^i} \tag{1}$$

This has the following well-known consequence.

**Lemma 5.** *We have $\eta_R(v_k) \equiv v_k \mod I_k$ for $I_k = (p, v_1, \ldots, v_{k-1})$.*

*Proof.* We can assume by induction that $\eta_R(v_i) \in I_k$ for $i < k$. If $x, y \in I$ for an ideal $I \subset BP_*$, then $x +_F y \in I$ again. Thus, we see that the left-hand side of Equation (1) is up to degree $|v_k|$ congruent to $\eta_R(v_k)$ modulo $I_k$. The right-hand side is obviously congruent to $v_k$ modulo $I_k$ in degrees $\leq |v_k|$. □

We need indeed also the following generalization, which is a slight strengthening of a statement in the proof of [Rav86, Theorem 4.3.2].

**Lemma 6.** *Let $M$ be the nonunital subring of $BP_*BP$ generated by $t_1, \ldots, t_{r-1}, v_k, \ldots, v_{k+r-1}$ and $I_k \cdot BP_*BP$. Then we have*

$$\eta_R(v_{k+r}) \equiv v_{k+r} + v_k t_r^{p^k} - v_k^{p^r} t_r \mod M,$$

*i.e. the difference between the left and right-hand side is in $M$. Here, $r$ does not have to be positive when we set $t_j = 0$ for $j < 0$.*

*Proof.* We assume by induction that the statement has been shown already for all smaller $r$ (the case $r \leq 0$ being the last lemma). Contemplating again Equation (1), we claim that in degrees up to $|v_{k+r}|$ the only formal summands in the left-hand side that are not in $M$ are $\eta_R(v_{k+r})$ and $t_r \eta_R(v_k)^{p^r}$. Indeed, observe first that $|t_i \eta_R(v_j)^{p^i}| = |v_{i+j}|$, i.e. we care only about the terms with $i + j \leq r + k$. If $j < k$, we know that $\eta_R(v_j) \in I_k$ and thus $t_i \eta_R(v_j)^{p^i} \in M$. If $j < k + r$, we still know that $\eta_R(v_j) \in M$ by induction and thus $t_i \eta_R(v_j)^{p^i} \in M$ if $0 \leq i < r$. The only cases with $i + j \leq r + k$ that remain are $(i = 0, j = k + r)$ and $(i = r, j = k)$.

As $x +_F y \equiv x + y$ modulo higher terms and all elements of $BP_*$ in degrees smaller than $|v_{k+r}|$ (and hence all coefficients of $F$ in these degrees) are in $M$, we conclude that the left-hand side is congruent to

$$\eta_R(v_{k+r}) + t_r \eta_R(v_k)^{p^r} \equiv \eta_R(v_{k+r}) + t_r v_k^{p^r} \mod M,$$

where we use again Lemma 5.

By a similar argument, the only formal summands of the right-hand side that are not in $M$ are $v_{k+r}$ and $v_k t_r^{p^k}$. The lemma follows. □

# 4 The main theorem

Let $R$ be a homotopy commutative $C_2$-ring spectrum with underlying homotopy concentrated in even degrees such that $C_2$ acts as $(-1)^n$ on $\pi_{2n}R$. By Section 2.3 and Corollary 4.7 of [HM15], we obtain a multiplicative spectral sequence

$$\overline{\pi_{2*}R} \otimes \mathbb{Z}[a, u^{\pm 1}]/2a \Rightarrow \pi_{\bigstar}^{C_2} R^h,$$

which is called the $RO(C_2)$*-graded homotopy fixed point spectral sequence*. Here $R^h = R^{(EC_2)_+}$, $|a| = -\sigma = 1 - \rho$ and $|u| = 2 - 2\sigma = 4 - 2\rho$. Furthermore, $\overline{\pi_{2n}R} \cong \pi_{2n}R$, but viewed as sitting in degree $n\rho$. We denote the pendant of a class $x \in \pi_{2n}R$ by $\overline{x}$. Po Hu [Hu02] has computed how the $RO(C_2)$-graded homotopy fixed point spectral sequence for $\mathrm{BP}\mathbb{R}\langle \mathrm{n}\rangle$ looks like (see also Proposition 4-1 of [GM16] for a source using the same notation as we do here). With this preparation, we are ready to attack our main theorem.

**Theorem 7.** *Every* $\mathrm{BP}\mathbb{R}\langle \mathrm{n}\rangle$*-like* $C_2$*-spectrum is a form of* $\mathrm{BP}\mathbb{R}\langle \mathrm{n}\rangle$.

*Proof.* Let $R$ be a $\mathrm{BP}\mathbb{R}\langle \mathrm{n}\rangle$-like $C_2$-spectrum. By Lemma 4, we only need to determine the differentials in the $RO(C_2)$-graded homotopy fixed point spectral sequence for $R$ and see that they agree with those for $\mathrm{BP}\mathbb{R}\langle \mathrm{n}\rangle$. For this purpose, consider the diagram

$$BP\mathbb{R} \xrightarrow{\eta_R} BP\mathbb{R} \wedge R \xleftarrow{\eta_L} R$$

of $C_2$-spectra induced by the unit maps $\mathbb{S} \to BP\mathbb{R}$ and $\mathbb{S} \to R$. We will denote the corresponding $RO(C_2)$-graded homotopy fixed point spectral sequences by $E^B$, $E^{BR}$ and $E^R$ and the corresponding differentials by $d^B$, $d^{BR}$ and $d^R$. The differentials in $E^B$ are completely known (see [HK01] or [GM16, Appendix]).

As the underlying spectrum of $R$ is complex orientable, there is an isomorphism

$$\pi_*^e(BP\mathbb{R} \wedge R) \cong (\pi_*^e R)[t_1, t_2, \dots],$$

where $C_2$ acts as $(-1)$ on the $t_i$. This implies that we have $E_2^{BR} \cong E_2^R[\bar{t}_1, \bar{t}_2, \dots]$ with $|\bar{t}_i| = |\overline{v}_i|$. We need to show the following three claims (where the last one is only auxiliary):

1. $d_i^R(u^{2^{j-1}}) = \begin{cases} a^{2^{j+1}-1}\overline{v}_j & \text{for } i = 2^{j+1}-1, \\ 0 & \text{for } i < 2^{j+1}-1, \end{cases}$
2. $d_i^R(\overline{v}_j) = 0$ for all $i \geq 2$ and all $j$,
3. $d_i^{BR}(\bar{t}_j) = 0$.

We will prove this via induction on $i$ (where the claims are meant to be void for $i < 2$). So assume that the three claims are already proven for all $d_i$ for $i < k$. Then we know by the third claim that $E_k^{BR} \cong E_k^R[\bar{t}_1, \bar{t}_2, \dots]$ and the map $E_k^R \xrightarrow{(\eta_L)_*} E_k^{BR}$ is given by the obvious inclusion. We will leave out $\eta_L$ from the notation in the following.

Let us show the first claim. If $u^{2^{j-1}}$ is a $d_{k-1}$-cycle in $E^R$, then we have

$$d_k^R(u^{2^{j-1}}) = d_k^{BR}(u^{2^{j-1}}) = \eta_R(d_k^B(u^{2^{j-1}})).$$

If $k < 2^{j+1} - 1$, this is 0. If $k = 2^{j+1} - 1$, this is $a^{2^{j+1}-1}\eta_R\overline{v}_j$. As by Lemma 5, we know that $\eta_R\overline{v}_j \equiv \overline{v}_j \mod I_j E_k^{BR}$ and $I_j$ is killed by $a^{2^{j+1}-1}$ in $E_k^{BR}$, the first claim follows.

For the second claim, assume that $d_k^R(\overline{v}_j) = a^k u^s \overline{v} \neq 0$, where $\overline{v}$ is a polynomial in the $\overline{v}_r$ and $s$ is determined by $4s + k = -1$; note here that we write down the formula in $E_2^R$ for a representative of the $E_k^R$-element $d_k^R(\overline{v}_j)$. Let us write $s = 2^b c$ with $c$ odd. We claim that $k = 2^{b+2} - 1$. Indeed, note first that $4s + k = -1$ implies that $k \geq 2^{b+2} - 1$. Furthermore, $a^k u^s \overline{v}$ is only a $(k-1)$-cycle if either $k \leq 2^{b+2} - 1$ or every monomial in $\overline{v}$ is divisible by some $\overline{v}_m$ with

$m \leq b$. But if $m \leq b$, then $a^{2^{m+1}-1}\overline{v}_m = 0$ and thus also $a^k u^s \overline{v} = 0$.[3] Thus, $k \leq 2^{b+2} - 1$ and hence $k = 2^{b+2} - 1$. This implies $c = -1$ and $|\overline{v}| = |\overline{v}_j| + |\overline{v}_{b+1}|$.

Now choose $m$ such that $2^m \leq k \leq 2^{m+1} - 1$. Assume that $d_k(\overline{v}_i) = 0$ for $i < m + r$ and $d_k(\overline{t}_i) = 0$ for $i < r$ (where $r$ can be also nonpositive, in which case the latter condition is empty). We know by the first claim that $I_m = (2, \overline{v}_1, \ldots, \overline{v}_{m-1})$ kills all $d_k$-differentials and thus $I_m E_k^{BR}$ consists of $d_k$-cycles. By Lemma 6, we know that

$$0 = d_k^{BR}(\eta_R(\overline{v}_{m+r})) = d_k^{BR}(\overline{v}_{m+r} + \overline{v}_m \overline{t}_r^{2^m} - \overline{v}_m^{2^r} \overline{t}_r) = d_k^{BR}(\overline{v}_{m+r}) + \overline{v}_m^{2^r} d_k^{BR}(\overline{t}_r), \qquad (2)$$

where we use that we know that $\overline{v}_m$ is a $d_k$-cycle if $r > 0$; the cases $r < 0$ and $r = 0$ are easily dealt with and in these cases we directly see that $\overline{v}_{m+r}$ is a $d_k$-cycle and $\overline{t}_r$ is a cycle anyhow. Thus, we assume in the following that $r > 0$.

We claim that all $\overline{v}_m$-torsion in $E_k^{BR}$ is zero. This is true as $E_2^{BR}$ is $\overline{v}_m$-torsion free and there is no $d_i$-differential with $i < k$ whose target involves $\overline{v}_m$. It follows that if $d_k^{BR}(\overline{v}_{m+r}) = 0$ (e.g. if $k \neq 2^{m+1} - 1$), then $d_k^{BR}(\overline{t}_r) = 0$. Now assume $k = 2^{m+1} - 1$ so that $d_k^{BR}(\overline{v}_{m+r}) = a^k u^{-2^{m-1}} \overline{v}$ with $|\overline{v}| = |\overline{v}_{m+r}| + |\overline{v}_m|$ and no monomial of $\overline{v}$ contains a factor $\overline{v}_i$ with $i \leq m-1$ if $\overline{v}$ is nonzero.

For degree reasons, $d_k^{BR}(\overline{t}_r)$ must be of the form $a^k u^{-2^{m-1}} \overline{v}'$ and thus $a^k u^{-2^{m-1}} \overline{v}_m^{2^r} = a^k u^{-2^{m-1}} \overline{v}$ by Equation (2). As all $(a^k u^{-2^{m-1}})$-torsion must have factors of the form $\overline{v}_j$ for $j < m$, it follows that $\overline{v} = \overline{v}_m^{2^r} \overline{v}'$. Clearly, $|\overline{v}'| < |\overline{v}_{m+r}|$ and thus $\overline{v}'$ is a polynomial in the $\overline{v}_i$ for $i < m + r$ and hence a $d_k$-cycle. Thus,

$$0 = d_k(a^k u^{-2^{m-1}} \overline{v}) = a^k \overline{v} d_k(u^{-2^{m-1}}) = a^{2k} u^{-2^m} \overline{v}\overline{v}_m.$$

As noted before, no monomial of $\overline{v}$ contains a factor $\overline{v}_i$ with $i \leq m - 1$ if $\overline{v}$ is nonzero. Thus, $\overline{v}$ has to be zero. This means that $d_k^{BR}(\overline{v}_{m+r}) = 0$ and hence also $d_k^{BR}(\overline{t}_r) = 0$.

□

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[3] Note that it is not necessarily true that $a^{2^{m+1}-1}\overline{v}_m$ divides the corresponding summand of $a^k u^s \overline{v}$ in $E_k^R$ as $u^s$ is perhaps not a $(k-1)$-cycle. It might be that only some $u^{2^m t}\overline{v}_m$ divides the summand, but $a^{2^{m+1}-1} u^{2^m t} \overline{v}_m$ is zero as well in $E_k^R$.