Computing Brauer groups via coarse moduli – draft version

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Throughout let $X$ be a separated Deligne–Mumford stack and $q: X \rightarrow X$ its coarse moduli space. The goal of this note is to compute the Brauer group of $X$ in terms of invariants of $X$ and I want to thank Ben Antieau and Minseon Shin for helpful discussions leading to our solution. The key question will be under which conditions $R^2q_*G_m$ vanishes.

**Convention 1.** All quotients will be stack quotients. If not marked otherwise, cohomology of schemes or stacks is étale cohomology.

**Example 2.** Let $l$ be a prime and $G$ be a finite group such that $H^3(G; \mathbb{Z}[\frac{1}{p}]) \cong H^2(G; \mathbb{Q}/\mathbb{Z}[\frac{1}{p}])$ is nontrivial. For example, we can take $G = S_4$ and $p \geq 3$. We claim that $R^2q_*G_m$ does not vanish for $X = \text{Spec} \mathbb{F}_p/G$.

The claim is indeed equivalent to $\text{Br}(X)$ nonvanishing. We can compute it via the descent spectral sequence

$$E_2^{i,j} = H^i(G, H^j(\text{Spec} \mathbb{F}_p, G_m)) \Rightarrow H^{i+j}(X, G_m).$$

Clearly, $H^j(\text{Spec} \mathbb{F}_p, G_m)$ is zero for $j > 0$ and $G_m(\mathbb{F}_p) \cong \mathbb{Q}/\mathbb{Z}[\frac{1}{p}]$ if $j = 0$. Thus, $\text{Br}(X) = H^2(G; \mathbb{Q}/\mathbb{Z}[\frac{1}{p}]).$

We can generalize this example.

**Definition 3.** Let $l$ be a prime and $G$ be a finite group. We call this group $l$-rich if $H^3(G; \mathbb{Z})[l] \neq 0$ and $l$-poor if $H^3(G; \mathbb{Z})[l] = 0$, where the action on $\mathbb{Z}$ is trivial. We call a group poor if it is $l$-poor for every $l$.

**Example 4.** Clearly, every cyclic group is poor. According to GAP, this is also true for some other groups like $\text{SL}_2(\mathbb{F}_3)$ and $\text{Dic}_{12}$, which are the automorphism groups of the supersingular points of characteristic 2 and 3 in $\mathcal{M}_{1,1}$. Thus, all automorphism groups of geometric points of $\mathcal{M}_{1,1}$ are poor.

The following lemma motivates the definition.

**Lemma 5.** A finite group $G$ is $l$-poor if and only if $H^2(G; G_m(k))_{(l)} = 0$ for an (or, equivalently, every) algebraically closed field of characteristic not $l$. If $k$ has characteristic $l$, then $H^2(G; G_m(k))_{(l)} = 0$ for all finite groups $G$.

**Proof.** By Theorem 127.3 in [Fuc73], an abelian group is of the form $G_m(k)$ for an algebraically closed field of characteristic $p > 0$ if and only if it is of the form $\mathbb{Q}/\mathbb{Z}[\frac{1}{p}] \oplus \bigoplus I \mathbb{Q}$, where $I$ is either infinite or empty. Thus,

$$H^2(G; G_m(k)) \cong H^2(G; \mathbb{Q}/\mathbb{Z}[\frac{1}{p}]) \cong H^3(G; \mathbb{Z})[\frac{1}{p}].$$

An abelian group is of the form $G_m(k)$ for an algebraically closed field of characteristic 0 if and only if it is of the form $\mathbb{Q}/\mathbb{Z} \oplus \bigoplus I \mathbb{Q}$ where $I$ is infinite. The argument is similar. 

So we could have taken $\mathcal{X} = \text{Spec } k/G$ for an arbitrary algebraically closed field $k$ of characteristic $p \geq 0$ and $G$ a $l$-rich group for $l \neq p$ in the Example [2]. Our main aim is to show that this kind of example is essentially the only obstruction for the vanishing of $R^2 q_* G_m$.

**Theorem 6.** Let $\mathcal{X}$ be a separated Deligne–Mumford stack of finite type over a locally noetherian $\mathbb{Z}[\frac{1}{l}]$-scheme $S$ and assume that the automorphism group of every geometric point is $l$-poor. Then $(R^2 q_* G_m)(l)$ vanishes.

**Question 7.** Can we replace the hypothesis with the assumption that $S_{\mathbb{Z}[\frac{1}{l}]}$ is dense in $S$?

We need the following proposition essentially proven in [Ols06, Theorem 2.12] and [AV02, Lemma 2.2.3].

**Proposition 8.** Let $\mathcal{X}$ be a separated Deligne–Mumford stack of finite type over a locally noetherian scheme $S$ with coarse moduli $\mathcal{X} \to X$. Let $X^{sh}$ be the spectrum of the (strictly Henselian) local ring of a geometric point $x$: Spec $k \to X$ in the étale topology. Then $\mathcal{X}^{sh} = X \times_X X^{sh}$ is of the form Spec $R/\Gamma$ for a strictly Henselian local ring $R$ with residue field $k$ and $\Gamma$ is the automorphism group of $x$ (or rather its pendant in $\mathcal{X}$). The group $\Gamma$ acts trivially on the residue field $k$.

**Proof.** The cited sources prove that after base change to an étale neighborhood $V$ of $x$, the stack $\mathcal{X}$ is of the form $U/\Gamma$ for $\Gamma$ as above. This $U$ is finite over $V$ and $U \times_X X^{sh}$ is the spectrum of a strictly Henselian ring $R$. Its residue field must be finite over $k$ and thus equals $k$. By definition of $\Gamma$, the field $k$ is elementwise fixed by it.

As Spec $R \to X^{sh}$ is the pullback of the $\Gamma$-torsor $U \to X$ along $\mathcal{X}^{sh} \to \mathcal{X}$, we see that it is a $\Gamma$-torsor as well, i.e. that $X^{sh} \simeq \text{Spec } R/\Gamma$. □

**Lemma 9.** Let $R$ be a strictly Henselian domain with residue field $k$ of characteristic $p \geq 0$ and with an action by a finite group $G$. Set $\mathcal{X} = \text{Spec } R/G$. Then $H^2(\mathcal{X}; G_m)(l) = H^2(G; G_m(k))$ if $l \neq p$.

**Proof.** We will use the descent spectral sequence

$$E_2^{i,j} = H^i(G, H^j(\text{Spec } R, G_m)) \Rightarrow H^{i+j}(\mathcal{X}, G_m).$$

Because $R$ is strictly Henselian, $H^j(\text{Spec } R, G_m)$ vanishes for $j > 0$. Thus, $H^2(\mathcal{X}; G_m) \cong H^2(G; G_m(R))$. Let $K$ be the kernel of the natural map $G_m(R)[\frac{1}{p}] \to G_m(k)[\frac{1}{p}]$. We obtain a $G$-equivariant short exact sequence

$$0 \to K \to G_m(R)[\frac{1}{p}] \to G_m(k)[\frac{1}{p}] \to 0.$$

For $u \in G_m(R)$ and $n$ a natural number not divisible by $p$, the equation $x^n = u$ has a solution in $R$ because $R$ is strictly Henselian. Thus, $G_m(R)[\frac{1}{p}]$ (or just $G_m(R)$ if $p = 0$) is divisible and thus by [Fun70, Theorem 23.1] a direct sum of groups of the form $Q_r/Z_r$ (for primes $r$) or $Q$. The same is true for $G_m(k)[\frac{1}{p}]$. By [AMU, Tag 06RR], the torsion of $G_m(R)[\frac{1}{p}]$ maps isomorphically onto the torsion of $G_m(k)[\frac{1}{p}]$. Thus $K$ is also the kernel of $G_m(R)[\frac{1}{p}]/\text{tors} \to G_m(k)[\frac{1}{p}]/\text{tors}$, which is map of $Q$-vector spaces. Thus, $K$ is a $Q$-vector space as well. We deduce that

$$H^2(G, G_m(R))(l) \cong H^2(G, G_m(R)[\frac{1}{p}]) \cong H^2(G, G_m(k)[\frac{1}{p}]) \cong H^2(G, G_m(k))(l).$$

**Proof of theorem:** To show that $(R^2 q_* G_m)(l)$ vanishes, it is enough to show that $H^2(\mathcal{X}^{sh}; G_m)(l)$ vanishes for every geometric point $x$ of $X$ (with $\mathcal{X}^{sh}$ as in Proposition [8]). By the same proposition, $\mathcal{X}^{sh}$ is of the form Spec $R/G$ with $R$ strictly Henselian and $G$ the stabilizer group of $x$. Thus, we are exactly in the situation of the last lemma, where we use that $G$ is $l$-poor. □
Corollary 10. Let S be a separated, regular and noetherian scheme over \( \mathbb{Z}[\frac{1}{n}] \). Then we have a short exact sequence

\[
0 \to Br'(S)_{(l)} \to Br'(M_S)_{(l)} \xrightarrow{s} H^1(S; \mathbb{Z}/12)_{(l)} \to 0,
\]

which is split (up to isomorphism) by the map

\[
s : H^1(S; \mathbb{Z}/12)_{(l)} \to Br'(M_S)_{(l)}, \quad [\chi] \mapsto [(\chi, \Delta)_{12}].
\]

Here, \( [(\chi, \Delta)_{12}] \) is the cup product with the class of the \( \mu_{12} \)-torsor defined by taking a 12-th root of \( \Delta \).

Proof. We have \( q_*G_m = G_m \) and \( R^1q_*G_m \cong \mathbb{Z}/12 \) by [10]. Indeed, \( \mathbb{Z}/12 \to R^1q_*G_m \) is a morphism of sheaves, which is an isomorphism after base change to an arbitrary local ring of \( S \). By our main theorem, we have \( (R^2q_*G_m)_{(l)} = 0 \). By \( \mathbb{A}^1 \)-invariance of étale cohomology [AM16, Proposition 2.5], [Mil80, Corollary VI.4.20], we have \( H^1(\mathbb{A}^1_S; G_m)_{(l)} \cong H^1(S; G_m)_{(l)} \) and \( H^2(\mathbb{A}^1_S; \mathbb{Z}/12)_{(l)} \cong H^2(S; \mathbb{Z}/12)_{(l)} \). This shows the existence of an exact sequence

\[
0 \to Br'(S)_{(l)} \to Br'(M_S)_{(l)} \to H^1(S; \mathbb{Z}/12)_{(l)}.
\]

The composition \( sr \) defines a natural transformation of \( H^1(S; \mathbb{Z}/n) \) to itself, where \( n = 4 \) if \( l = 2 \), \( n = 3 \) if \( l = 3 \) and zero else. The map \( s \) certainly makes sense for \( S = BC_{n, \mathbb{Z}[\frac{1}{n}]}(\mathbb{A}^1) \) as well and \( r \) does so as well: Consider the map \( q : M_{BC_{n, \mathbb{Z}[\frac{1}{n}]}(\mathbb{A}^1)} \to \mathbb{A}^1_{n, \mathbb{Z}[\frac{1}{n}]} \) given by base changing the map \( M \to \mathbb{A}^1 \). As étale locally \( BC_{n, \mathbb{Z}[\frac{1}{n}]}(\mathbb{A}^1) \) is a separated, regular and noetherian scheme, our computation from above applies to show that \( R^2q_*G_m = 0 \) and \( R^1q_*G_m = \mathbb{Z}/12 \); thus, we obtain the required map \( r \) from the Leray spectral sequence.

Let \( [\chi] \) be the tautological class in \( H^1(BC_{n, \mathbb{Z}[\frac{1}{n}]}(\mathbb{A}^1), \mathbb{Z}/n) \). Clearly, \( rs([\chi]) \) becomes zero after base change to \( \text{Spec} \mathbb{Z}[\frac{1}{n}] \). By the descent spectral sequence, the kernel \( H^1(BC_{n, \mathbb{Z}[\frac{1}{n}]}(\mathbb{A}^1), \mathbb{Z}/n) \to H^1(\text{Spec} \mathbb{Z}[\frac{1}{n}], \mathbb{Z}/n) \) is isomorphic to \( \mathbb{Z}/n \) and generated by \( [\chi] \). Thus, we see that there is an element \( u \in \mathbb{Z}/n \) such that \( rs \) is multiplication by \( u \).

We claim that \( u \) is a unit. It is enough to provide an \( \mathbb{Z}[\frac{1}{n}] \)-scheme \( S \), where the image of \( s \) has an element of order 4, and an \( \mathbb{Z}[\frac{1}{n}] \)-scheme, where the image of \( s \) has an element of order 3. Examples abound in [AM16]. For example, we can take \( S = \text{Spec} \mathbb{F}_p \) for \( p > 3 \).

In particular, this shows that \( r \) is surjective. \( \square \)

Remark 11. The \( \mathbb{A}^1 \)-invariance of the Brauer group is indeed more generally true than used in the last corollary. Let \( R \) be a regular noetherian ring such that \( Spec R[\frac{1}{p}] \) is dense in \( Spec R \). We claim that \( Br(R)_{(p)} \cong Br(\mathbb{A}^1_R)_{(p)} \). Indeed, consider the diagram

\[
\begin{array}{ccc}
Br(\mathbb{A}^1_R)_{(p)} & \longrightarrow & Br(\mathbb{A}^1_{R[\frac{1}{p}]}(p)
\end{array}
\]

\[
\begin{array}{c}
\approx
\end{array}
\]

\[
\begin{array}{ccc}
Br(R)_{(p)} & \longrightarrow & Br(R[\frac{1}{p}])_{(p)}
\end{array}
\]

induced by choice of an \( R \)-point of \( \mathbb{A}^1_R \). The right vertical morphism is an isomorphism by classical \( \mathbb{A}^1 \)-invariance. The horizontal arrows are injections by density (using that \( \mathbb{A}^1_R \to \text{Spec} R \) is open). Thus, \( Br(\mathbb{A}^1_R)_{(p)} \to Br(R)_{(p)} \) must be an injection as well. On the other hand, it is a split surjection. This implies that it is an isomorphism.

\footnote{If we did not want to make the splitting \( s \) explicit, there would have been an easier proof, without recourse to [AM16]. Indeed, the split surjectivity of \( r \) is only a question if \( l = 2 \) or 3. Then there is a section of \( M_S \to S \) and we can use the induced map \( Br'(M_S) \to Br'(S) \) for the collapse of the Leray spectral sequence and the splitting.}
Corollary 12. Let $S$ be a separated, regular and noetherian scheme such that $S_{\mathbb{Z}[\frac{1}{l}]} \subset S$ is dense (e.g. if $S$ is an integral domain and $l \neq 0$). Then the map
\[ s: H^1(S; \mathbb{Z}/12)_l \rightarrow Br'(M_S)_l, \quad [\chi] \mapsto (\chi, \Delta)_{12} \]
is injective.

Proof. Consider the commutative square
\[
\begin{array}{ccc}
H^1(S; \mathbb{Z}/12)_l & \longrightarrow & H^1(S_{\mathbb{Z}[\frac{1}{l}]; \mathbb{Z}/12})_l \\
\downarrow & & \downarrow \\
Br'(M_S)_l & \longrightarrow & Br'(M_\mathbb{Z}[\frac{1}{l}])_l
\end{array}
\]

The right vertical map is an isomorphism by Corollary 10. We claim that the upper horizontal arrow is injective. We can assume that $S$ is connected and hence integral. Let $\eta: Spec K \rightarrow S$ be the generic point of $S$ (and of $S_{\mathbb{Z}[\frac{1}{l}]}$) and $\overline{\eta}: Spec K^{sep} \rightarrow S$ the corresponding map from the separable closure. By [Aut Tag 0BQM], the map $Gal(K^{sep}/K) \rightarrow \pi^et_1(S, \eta)$ is surjective and hence also the map $\pi^et_1(S_{\mathbb{Z}[\frac{1}{l}]; \eta}) \rightarrow \pi^et_1(S, \eta)$. This implies that the induced map
\[ H^1(S_{\mathbb{Z}[\frac{1}{l}]; \mathbb{Z}/12}) \cong \text{Hom}(\pi^et_1(S_{\mathbb{Z}[\frac{1}{l}]; \eta}), \mathbb{Z}/12) \rightarrow \text{Hom}(\pi^et_1(S, \eta), \mathbb{Z}/12) \cong H^1(S; \mathbb{Z}/12) \]
is injective.

It follows that $H^1(S; \mathbb{Z}/12)_l \rightarrow Br'(M_S)_l$ is injective as well. \hfill \qED

One might conjecture that the map $H^1(S; \mathbb{Z}/12)_l \rightarrow Br'(M_S)_l$ is an isomorphism under the conditions of the last corollary, where $Br'(M_S)$ denotes the cokernel of the map $Br'(S) \rightarrow Br'(M_S)_l$. Note however that it is not an isomorphism for $S = \mathbb{F}_2$ and $l = 2$ as Minseon Shin has recently computed that $Br(M_{\mathbb{F}_2}) \cong \mathbb{Z}/2$ [Shi17].

References


