Computing Brauer groups via coarse moduli – draft version

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Throughout let $\mathcal{X}$ be a separated Deligne–Mumford stack and $q\colon \mathcal{X} \to X$ its coarse moduli space. The goal of this note is to compute the Brauer group of $\mathcal{X}$ in terms of invariants of X and I want to thank Ben Antieau and Minseon Shin for helpful discussions leading to our solution. The key question will be under which conditions $R^2q_*\mathbb{G}_m$ vanishes.

Convention 1. *All quotients will be stack quotients. If not marked otherwise, cohomology of schemes or stacks is étale cohomology.*

Example 2. Let l be a prime and G be a finite group such that $\mathrm{H}^3(G;\mathbb{Z}[\frac{1}{p}]) \cong \mathrm{H}^2(G;\mathbb{Q}/\mathbb{Z}[\frac{1}{p}])$ is nontrivial. For example, we can take $G = S_4$ and $p \geq 3$. We claim that $R^2q_*\mathbb{G}_m$ does not vanish for $\mathcal{X} = \operatorname{Spec}\overline{\mathbb{F}}_p/G$.

The claim is indeed equivalent to $\mathrm{Br}(\mathcal{X})$ nonvanishing. We can compute it via the descent spectral sequence

$$\mathrm{E}_2^{i,j} = \mathrm{H}^i(G, \mathrm{H}^j(\operatorname{Spec}\overline{\mathbb{F}}_p, \mathbb{G}_m)) \Rightarrow \mathrm{H}^{i+j}(\mathcal{X}, \mathbb{G}_m).$$

Clearly, $\mathrm{H}^j(\operatorname{Spec}\overline{\mathbb{F}}_p, \mathbb{G}_m)$ is zero for $j > 0$ and $\mathbb{G}_m(\overline{\mathbb{F}}_p) \cong \mathbb{Q}/\mathbb{Z}[\frac{1}{p}]$ if $j = 0$. Thus, $\mathrm{Br}(\mathcal{X}) = H^2(G;\mathbb{Q}/\mathbb{Z}[\frac{1}{p}])$.

We can generalize this example.

Definition 3. Let l be a prime and G be a finite group. We call this group *l-rich* if $H^3(G;\mathbb{Z})_{(l)} \neq 0$ and *l-poor* if $H^3(G;\mathbb{Z})_{(l)} = 0$, where the action on $\mathbb{Z}$ is trivial. We call a group *poor* if it is l-poor for every l.

Example 4. Clearly, every cyclic group is poor. According to `GAP`, this is also true for some other groups like $SL_2(\mathbb{F}_3)$ and Dic_{12}, which are the automorphism groups of the supersingular points of characteristic 2 and 3 in $\mathcal{M}_{1,1}$. Thus, all automorphism groups of geometric points of $\mathcal{M}_{1,1}$ are poor.

The following lemma motivates the definition.

Lemma 5. *A finite group G is l-poor if and only if $\mathrm{H}^2(G;\mathbb{G}_m(k))_{(l)} = 0$ for an (or, equivalently, every) algebraically closed field of characteristic not l. If k has characteristic l, then $\mathrm{H}^2(G;\mathbb{G}_m(k))_{(l)} = 0$ for all finite groups G.*

Proof. By Theorem 127.3 in [Fuc73], an abelian group is of the form $\mathbb{G}_m(k)$ for an algebraically closed field of characteristic $p > 0$ if and only if it is of the form $\mathbb{Q}/\mathbb{Z}[1/p] \oplus \bigoplus_I \mathbb{Q}$, where I is either infinite or empty. Thus,

$$\mathrm{H}^2(G;\mathbb{G}_m(k)) \cong \mathrm{H}^2(G;\mathbb{Q}/\mathbb{Z}[\tfrac{1}{p}]) \cong H^3(G;\mathbb{Z})[\frac{1}{p}].$$

An abelian group is of the form $\mathbb{G}_m(k)$ for an algebraically closed field of characteristic 0 if and only if it is of the form $\mathbb{Q}/\mathbb{Z} \oplus \bigoplus_I \mathbb{Q}$ where I is infinite. The argument is similar. □

So we could have taken $\mathcal{X} = \operatorname{Spec} k/G$ for an arbitrary algebraically closed field k of characteristic $p \geq 0$ and G a l-rich group for $l \neq p$ in the Example 2. Our main aim is to show that this kind of example is essentially the only obstruction for the vanishing of $R^2 q_* \mathbb{G}_m$.

Theorem 6. *Let $\mathcal{X}$ be a separated Deligne–Mumford stack of finite type over a locally noetherian $\mathbb{Z}[\frac{1}{l}]$-scheme S and assume that the automorphism group of every geometric point is l-poor. Then $(R^2 q_* \mathbb{G}_m)_{(l)}$ vanishes.*

Question 7. Can we replace the hypothesis with the assumption that $S_{\mathbb{Z}[\frac{1}{l}]}$ is dense in S?

We need the following proposition essentially proven in [Ols06, Theorem 2.12] and [AV02, Lemma 2.2.3].

Proposition 8. *Let $\mathcal{X}$ be a separated Deligne–Mumford stack of finite type over a locally noetherian scheme S with coarse moduli $\mathcal{X} \to X$. Let X^{sh} be the spectrum of the (strictly Henselian) local ring of a geometric point $x\colon \operatorname{Spec} k \to X$ in the étale topology. Then $\mathcal{X}^{sh} = \mathcal{X} \times_X X^{sh}$ is of the form $\operatorname{Spec} R/\Gamma$ for a strictly Henselian local ring R with residue field k and Γ is the automorphism group of x (or rather its pendant in $\mathcal{X}$). The group Γ acts trivially on the residue field k.*

Proof. The cited sources prove that after base change to an étale neighborhood V of x, the stack $\mathcal{X}$ is of the form U/Γ for Γ as above. This U is finite over V and $U \times_X X^{sh}$ is the spectrum of a strictly Henselian ring R. Its residue field must be finite over k and thus equals k. By definition of Γ, the field k is elementwise fixed by it.

As $\operatorname{Spec} R \to \mathcal{X}^{sh}$ is the pullback of the Γ-torsor $U \to \mathcal{X}$ along $\mathcal{X}^{sh} \to \mathcal{X}$, we see that it is a Γ-torsor as well, i.e. that $\mathcal{X}^{sh} \simeq \operatorname{Spec} R/\Gamma$. $\square$

Lemma 9. *Let R be a strictly Henselian domain with residue field k of characteristic $p \geq 0$ and with an action by a finite group G. Set $\mathcal{X} = \operatorname{Spec} R/G$. Then $\mathrm{H}^2(\mathcal{X}; \mathbb{G}_m)_{(l)} = \mathrm{H}^2(G; \mathbb{G}_m(k))$ if $l \neq p$.*

Proof. We will use the descent spectral sequence

$$\mathrm{E}_2^{i,j} = \mathrm{H}^i(G, \mathrm{H}^j(\operatorname{Spec} R, \mathbb{G}_m)) \Rightarrow \mathrm{H}^{i+j}(\mathcal{X}, \mathbb{G}_m).$$

Because R is strictly Henselian, $\mathrm{H}^j(\operatorname{Spec} R, \mathbb{G}_m)$ vanishes for $j > 0$. Thus, $\mathrm{H}^2(\mathcal{X}; \mathbb{G}_m) \cong \mathrm{H}^2(G, \mathbb{G}_m(R))$. Let K be the kernel of the natural map $\mathbb{G}_m(R)[\frac{1}{p}] \to \mathbb{G}_m(k)[\frac{1}{p}]$. We obtain a G-equivariant short exact sequence

$$0 \to K \to \mathbb{G}_m(R)[\frac{1}{p}] \to \mathbb{G}_m(k)[\frac{1}{p}] \to 0.$$

For $u \in \mathbb{G}_m(R)$ and n a natural number not divisible by p, the equation $x^n = u$ has a solution in R because R is strictly Henselian. Thus, $\mathbb{G}_m(R)[\frac{1}{p}]$ (or just $\mathbb{G}_m(R)$ if $p = 0$) is divisible and thus by [Fuc70, Theorem 23.1] a direct sum of groups of the form $\mathbb{Q}_r/\mathbb{Z}_r$ (for primes r) or $\mathbb{Q}$. The same is true for $\mathbb{G}_m(k)[\frac{1}{p}]$. By [Aut, Tag 06RR], the torsion of $\mathbb{G}_m(R)[\frac{1}{p}]$ maps isomorphically onto the torsion of $\mathbb{G}_m(k)[\frac{1}{p}]$. Thus K is also the kernel of $\mathbb{G}_m(R)[\frac{1}{p}]/\mathrm{tors} \to \mathbb{G}_m(k)[\frac{1}{p}]/\mathrm{tors}$, which is map of $\mathbb{Q}$-vector spaces. Thus, K is a $\mathbb{Q}$-vector space as well. We deduce that

$$\mathrm{H}^2(G, \mathbb{G}_m(R))_{(l)} \cong \mathrm{H}^2(G, \mathbb{G}_m(R)[\frac{1}{p}])_{(l)} \cong \mathrm{H}^2(G, \mathbb{G}_m(k)[\frac{1}{p}])_{(l)} \cong \mathrm{H}^2(G, \mathbb{G}_m(k))_{(l)}. \quad \square$$

Proof of theorem: To show that $(R^2 q_* \mathbb{G}_m)_{(l)}$ vanishes, it is enough to show that $\mathrm{H}^2(\mathcal{X}^{sh}; \mathbb{G}_m)_{(l)}$ vanishes for every geometric point x of X (with $\mathcal{X}^{sh}$ as in Proposition 8). By the same proposition, $\mathcal{X}^{sh}$ is of the form $\operatorname{Spec} R/G$ with R strictly Henselian and G the stabilizer group of x. Thus, we are exactly in the situation of the last lemma, where we use that G is l-poor. $\square$

Corollary 10. *Let S be a separated, regular and noetherian scheme over $\mathbb{Z}[\frac{1}{l}]$. Then we have a short exact sequence*

$$0 \to \mathrm{Br}'(S)_{(l)} \to \mathrm{Br}'(\mathcal{M}_S)_{(l)} \xrightarrow{r} \mathrm{H}^1(S; \mathbb{Z}/12)_{(l)} \to 0,$$

which is split (up to isomorphism) by the map

$$s \colon \mathrm{H}^1(S; \mathbb{Z}/12)_{(l)} \to \mathrm{Br}'(\mathcal{M}_S)_{(l)}, \qquad [\chi] \mapsto [(\chi, \Delta)_{12}].$$

Here, $[(\chi, \Delta)_{12}]$ is the cup product with the class of the μ_{12}-torsor defined by taking a 12-th root of Δ.

Proof. We have $q_*\mathbb{G}_m = \mathbb{G}_m$ and $R^1q_*\mathbb{G}_m \cong \mathbb{Z}/12$ by [FO10]. Indeed, $\mathbb{Z}/12 \to R^1q_*\mathbb{G}_m$ is a morphism of sheaves, which is an isomorphism after base change to an arbitrary local ring of S. By our main theorem, we have $(R^2q_*\mathbb{G}_m)_{(l)} = 0$. By $\mathbb{A}^1$-invariance of étale cohomology [AM16, Proposition 2.5], [Mil80, Corollary VI.4.20], we have $\mathrm{H}^i(\mathbb{A}^1_S; \mathbb{G}_m)_{(l)} \cong \mathrm{H}^i(S, \mathbb{G}_m)_{(l)}$ and $\mathrm{H}^i(\mathbb{A}^1_S; \mathbb{Z}/12)_{(l)} \cong \mathrm{H}^i(S; \mathbb{Z}/12)_{(l)}$. This shows the existence of an exact sequence

$$0 \to \mathrm{Br}'(S)_{(l)} \to \mathrm{Br}'(\mathcal{M}_S)_{(l)} \to \mathrm{H}^1(S; \mathbb{Z}/12)_{(l)}.$$

The composition rs defines a natural transformation of $\mathrm{H}^1(S; \mathbb{Z}/n)$ to itself, where $n = 4$ if $l = 2$, $n = 3$ if $l = 3$ and zero else. The map s certainly makes sense for $S = \mathrm{B}C_{n,\mathbb{Z}[\frac{1}{n}]}$ as well and r does so as well: Consider the map $q \colon \mathcal{M}_{\mathrm{B}C_{n,\mathbb{Z}[\frac{1}{n}]}} \to \mathbb{A}^1_{\mathrm{B}C_{n,\mathbb{Z}[\frac{1}{n}]}}$ given by base changing the map $\mathcal{M} \to \mathbb{A}^1$. As étale locally $\mathrm{B}C_{n,\mathbb{Z}[\frac{1}{n}]}$ is a separated, regular and noetherian scheme, our computation from above applies to show that $R^2q_*\mathbb{G}_m = 0$ and $R^1q_*\mathbb{G}_m = \mathbb{Z}/12$; thus, we obtain the required map r from the Leray spectral sequence.

Let $[\chi]$ be the tautological class in $\mathrm{H}^1(\mathrm{B}C_{n,\mathbb{Z}[\frac{1}{n}]}, \mathbb{Z}/n)$. Clearly, $rs([\chi])$ becomes zero after base change to $\operatorname{Spec}\mathbb{Z}[\frac{1}{n}]$. By the descent spectral sequence, the kernel $\mathrm{H}^1(\mathrm{B}C_{n,\mathbb{Z}[\frac{1}{n}]}, \mathbb{Z}/n) \to \mathrm{H}^1(\operatorname{Spec}\mathbb{Z}[\frac{1}{n}], \mathbb{Z}/n)$ is isomorphic to $\mathbb{Z}/n$ and generated by $[\chi]$. Thus, we see that there is an element $u \in \mathbb{Z}/n$ such that rs is multiplication by u.

We claim that u is a unit. It is enough to provide an $\mathbb{Z}[\frac{1}{2}]$-scheme S, where the image of s has an element of order 4, and an $\mathbb{Z}[\frac{1}{3}]$-scheme, where the image of s has an element of order 3. Examples abund in [AM16]. For example, we can take $S = \operatorname{Spec}\mathbb{F}_p$ for $p > 3$.

In particular, this shows that r is surjective.[1] □

Remark 11. The $\mathbb{A}^1$-invariance of the Brauer group is indeed more generally true than used in the last corollary. Let R be a regular noetherian ring such that $\operatorname{Spec} R[\frac{1}{p}]$ is dense in $\operatorname{Spec} R$. We claim that $\mathrm{Br}(R)_{(p)} \cong \mathrm{Br}(\mathbb{A}^1_R)_{(p)}$. Indeed, consider the diagram

$$\begin{tikzcd}
\mathrm{Br}(\mathbb{A}^1_R)_{(p)} \arrow[r] \arrow[d] & \mathrm{Br}(\mathbb{A}^1_{R[\frac{1}{p}]})_{(p)} \arrow[d, "\cong"] \\
\mathrm{Br}(R)_{(p)} \arrow[r] & \mathrm{Br}(R[\frac{1}{p}])_{(p)}
\end{tikzcd}$$

induced by choice of an R-point of $\mathbb{A}^1_R$. The right vertical morphism is an isomorphism by classical $\mathbb{A}^1$-invariance. The horizontal arrows are injections by density (using that $\mathbb{A}^1_R \to \operatorname{Spec} R$ is open). Thus, $\mathrm{Br}(\mathbb{A}^1_R)_{(p)} \to \mathrm{Br}(R)_{(p)}$ must be an injection as well. On the other hand, it is a split surjection. This implies that it is an isomorphism.

[1] If we did not want to make the splitting s so explicit, there would have been an easier proof, without recourse to [AM16]. Indeed, the split surjectivity of r is only a question if $l = 2$ or 3. Then there is a section of $\mathcal{M}_S \to S$ and we can use the induced map $\mathrm{Br}'(\mathcal{M}_S) \to \mathrm{Br}'(S)$ for the collapse of the Leray spectral sequence and the splitting.

Corollary 12. *Let S be a separated, regular and noetherian scheme such that $S_{\mathbb{Z}[\frac{1}{l}]} \subset S$ is dense (e.g. if S is an integral domain and $l \neq 0$). Then the map*

$$s\colon \mathrm{H}^1(S;\mathbb{Z}/12)_{(l)} \to \mathrm{Br}'(\mathcal{M}_S)_{(l)}, \qquad [\chi] \mapsto [(\chi, \Delta)_{12}]$$

is injective.

Proof. Consider the commutative square

$$\begin{array}{ccc} \mathrm{H}^1(S;\mathbb{Z}/12)_{(l)} & \longrightarrow & \mathrm{H}^1(S_{\mathbb{Z}[\frac{1}{l}]};\mathbb{Z}/12)_{(l)} \\ \downarrow & & \downarrow \\ \mathrm{Br}'(\mathcal{M}_S)_{(l)} & \longrightarrow & \mathrm{Br}'(\mathcal{M}_{S_{\mathbb{Z}[\frac{1}{l}]}})_{(l)} \end{array}$$

The right vertical map is an isomorphism by Corollary 10. We claim that the upper horizontal arrow is injective. We can assume that S is connected and hence integral. Let $\eta\colon \operatorname{Spec} K \to S$ be the generic point of S (and of $S_{\mathbb{Z}[\frac{1}{l}]}$) and $\overline{\eta}\colon \operatorname{Spec} K^{sep} \to S$ the corresponding map from the separable closure. By [Aut, Tag 0BQM], the map $\mathrm{Gal}(K^{sep}/K) \to \pi_1^{et}(S, \overline{\eta})$ is surjective and hence also the map $\pi_1^{et}(S_{\mathbb{Z}[\frac{1}{l}]}, \overline{\eta}) \to \pi_1^{et}(S, \overline{\eta})$. This implies that the induced map

$$\mathrm{H}^1(S_{\mathbb{Z}[\frac{1}{l}]};\mathbb{Z}/12) \cong \mathrm{Hom}(\pi_1^{et}(S_{\mathbb{Z}[\frac{1}{l}]}, \overline{\eta}), \mathbb{Z}/12) \to \mathrm{Hom}(\pi_1^{et}(S, \overline{\eta}), \mathbb{Z}/12) \cong \mathrm{H}^1(S;\mathbb{Z}/12)$$

is injective.

It follows that $\mathrm{H}^1(S;\mathbb{Z}/12)_{(l)} \to \mathrm{Br}'(\mathcal{M}_S)_{(l)}$ is injective as well. $\square$

One might conjecture that the map $\mathrm{H}^1(S;\mathbb{Z}/12)_{(l)} \to \overline{\mathrm{Br}'}(\mathcal{M}_S)_{(l)}$ is an isomorphism under the conditions of the last corollary, where $\overline{\mathrm{Br}'}(\mathcal{M}_S)$ denotes the cokernel of the map $\mathrm{Br}'(S) \to \mathrm{Br}'(\mathcal{M}_S)_{(l)}$. Note however that it is not an isomorphism for $S = \overline{\mathbb{F}}_2$ and $l = 2$ as Minseon Shin has recently computed that $\mathrm{Br}(\mathcal{M}_{\overline{\mathbb{F}}_2}) \cong \mathbb{Z}/2$ [Shi17].

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