DIPLOMARBEIT

Eine geometrische Betrachtung der String-Topologie

(A Geometric View on String Topology)

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Contents

1 Introduction

String Topology is the study of algebraic structures on the homology of mapping spaces between manifolds, especially on the free loop space.

Historically the first algebraic structure on the homology of a space was the intersection product on the homology of a manifold M, which can be seen as the Poincare dual of the cup product. In the absence of Poincare duality, there is in general no product on homology. So it came as a surprise when in 1999 Chas und Sullivan discovered a product on the homology of the free loop space LM of a manifold (now called the *Chas-Sullivan product*), although because of the infinite-dimensionality of the free loop space there cannot be any Poincare duality.

Recall that one can describe the intersection of two homology classes on a manifold which are represented by manifolds as their transversal intersection. Chas and Sullivan mimicked that definition at the level of free loop spaces. Unfortunately, their definitions and proofs were not always clear and rigorous. Cohen and Jones ([C-J]) later found a way to describe the algebraic structures Chas and Sullivan found via homotopy theory and the Thom isomorphism. Furthermore, they were able to do everything in general homology theories.

If one could represent any homology class by manifolds, it would be easy to give a more geometric treatment of the Chas-Sullivan product. But sadly enough, this is not true as Thom showed. Chataur ([Cha]) found a way to circumvent this problem by Jakob's geometric homology (2.2) where one equips the manifolds with cohomology classes and suddenly every class is representable. We will introduce an alternative way via Kreck's theory of stratifolds, a possibly singular variant of manifolds (see 2.1). This has the advantage to be even more geometric concrete than Chataur's description. For example, we are able to present a completely finite-dimensional way to define the Chas-Sullivan product without using any infinite-dimensional spaces (3.2.1). To show the equivalence to the Cohen-Jones approach we have to use intersection theory on the free loop space, whereto we have to use a homotopy model which is a Hilbert manifold (i.e. a manifold modeled on a Hilbert space). Since Chataur does not address the question whether his approch is equivalent to the one Cohen and Jones we will also discuss this problem.

We will present the Chas-Sullivan product and further algebraic structures (3.4) via a systematic use of so called *Gysin maps*. While homology is usually covariant, for a Hilbert submanifold $L \subset X$ of finite codimension d they give a morphism $h_*(X) \to h_{*-d}(L)$. Using stratifolds or geometric homology this can be described as an intersection.

It is nice to define algebraic structures on homologies, but a question immediately arises: Are they computable? In [CJY] Cohen, Jones and Yan construct a multiplicative structure on the Serre spectral sequence of $\Omega M \to LM \to M$ in ordinary homology where on the E^2-term $H_*(M; H_*(\Omega M))$ the product is given by the intersection product where the ring structure on $H_*(\Omega M)$ is induced by its structure of an H-group. This way they were able to compute the Chas-Sullivan product for the free loop spaces of spheres and complex projective spaces.

Two of the main themes of this diploma thesis take this paper as a starting point. The first is to make their calculations more concrete. This is realized in 3.5 and 3.6 by giving explicit manifold generators for the homologies of LS^n and $L\mathbb{KP}^n$.

The second is to increase the level of generality of their spectral sequence (4). First, we do everything in generalized homology (by using Jakob's geometric homology). Secondly, we give also spectral sequences for the coproduct and module structure and compare not only to ΩM but also allow comparision between the algebraic structures on the free loop spaces of the fibre, the total space and the base of a bundle $M \to N \to O$. We achieve this by constructing Gysin maps of spectral sequences (4.3). For the case of an infinite-dimensional base, we approximate the base by finite-dimensional manifolds (see 2.42). An important special case of our multiplicative spectral sequences is a multiplicative structure on the Atiyah-Hirzebruch spectral sequence.

At the end we present some calculations in K-theory and oriented and complex bordism. Furthermore, we start an investigation of the homology of the free loop space of a sphere bundle by comparing it to rational homotopy theory (4.5.1).

All manifolds are assumed to be smooth in this diploma thesis.

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2 Preliminaries

2.1 Stratifolds

There is a famous problem by Steenrod which asks if every homology class is representable by a manifold. It is well known that Thom answered this question to the negative. Furthermore, not all manifolds which represent the same homology class are bordant. This and the next section will show two constructions which deal with these two problems. The first one substitutes manifolds by more general spaces, which are possibly singular, while the second one equips the manifolds with the additional structure of a cohomology class. Except for some details about simplicial complexes, everything in this section is due to Matthias Kreck although not everything is published yet. Only all errors are mine.

2.1.1 Definitions

Before we come to stratifolds, we define a preliminary notion, which captures the minimal notion for a smooth structure on a space.

Definition 2.1 (Differential Space)**.** A *differential space* is a pair $(X, \mathbf{C})$, where X is a topological space and $\mathbf{C}$ is a subalgebra of $C^0(X;\mathbb{R})$ such that

1. the restrictions of $\mathbf{C}$ form a sheaf and
2. for all $f_1, \dots f_n \in \mathbf{C}$ and $g\colon \mathbb{R}^n \to \mathbb{R}$ smooth, the function

$$x \mapsto g(f_1(x), \dots f_n(x))$$

   is in $\mathbf{C}$.

A *morphism* of differential spaces $(X, \mathbf{C})$ and $(X', \mathbf{C'})$ is a continuous map $f\colon X \to X'$, such that $\rho \circ f \in \mathbf{C}$ for every $\rho \in \mathbf{C'}$.

A simple example takes for X a manifold and for $\mathbf{C}$ just the smooth functions. For a general differential space $(X, \mathbf{C})$ and a point $x \in X$ we can define as in the case of manifolds a tangent space T_xX as the vector space of all derivations of function germs at x. Define strata $X_i = \{x \in X : T_xX \text{ has dimension } i\}$. We call the union $\bigcup_{i \le r} X_i$ the r-skeleton X^r or $X^{(r)}$ of X. For an n-dimensional manifold M we have that $M_n = M$ and all other strata are empty. In the following, we will allow our spaces to have more than one non-empty stratum.

Definition 2.2 (Stratifolds)**.** A k-dimensional *stratifold* is a differential space $(S, \mathbf{C})$, where S is a locally compact Hausdorff space with countable base of topology. All skeleta should be closed. In addition we assume:

1. The $(S_i, \mathbf{C}|_{S_i})$ are i-dimensional manifolds.
2. For all $x \in S$, restriction defines an isomorphism $\mathbf{C}_x \to C^\infty(S_i)_x$.
3. All tangent spaces have dimension $\le k$.
4. For each $x \in S$ and every neighbourhood U of x, there exists a function $\rho\colon U \to \mathbb{R}$ with $\rho(x) \neq 0$ and $\operatorname{supp}(\rho) \subset U$ (a bump function).

To define a bordism theory, we have to define the notion of a stratifold with boundary.

Definition 2.3. An n-dimensional *c-stratifold* T (a collared stratifold) is a pair of topological spaces $(T, \partial T)$ with ∂T closed, where $\mathring{T} = T - \partial T$ is equipped with the structure of an n-dimensional stratifold and ∂T with that of an $(n-1)$-dimensional stratifold, together with a germ of collars $[c]$. By a collar we mean an isomorphism of stratifolds $c\colon \partial T \times [0, \epsilon) \to V$ (for a neighbourhood V of ∂T), which is the identity on ∂T. We call ∂T the *boundary* of T.

In particular, we can consider every stratifold as a c-stratifold with empty boundary. To get the right homology theory at the end we have to impose two further (technical) conditions.

Definition 2.4. A stratifold S is called a *regular* stratifold if for each $x \in S_i$, there is an open neighborhood U of x in S, a stratifold F with F_0 a single point pt and an isomorphism $\varphi\colon (U \cap S_i) \times F \to U$ whose restriction to $(U \cap S_i) \times pt$ is the identity. A c-stratifold T is called *regular*, if $\mathring{T}$ and ∂T are regular.

Definition 2.5. An *oriented* m-dimensional c-stratifold is an m-dimensional c-stratifold T with $\mathring{T}_{m-1} = \varnothing$ and an orientation on $\mathring{T}_m$. We denote by $-T$ the same c-stratifold with the opposite orientation on $\mathring{T}_m$. An oriented c-stratifold induces an orientation of its boundary by requiring the collar isomorphism to be orientation preserving.

Now we are ready to define the notion of the oriented bordism relation. Two k-dimensional compact oriented (regular) stratifolds S, S' are called bordant iff there exists a compact oriented (regular) c-stratifold T with boundary $S \coprod (-S')$. Two maps $g\colon S \to X$ and $g'\colon S' \to X$ are called bordant, if there is a bordism T between S and S' and an extension $G\colon T \to X$ whose restriction to the boundary is equal to $g + (-g')$.

Definition 2.6. Let X be a topological space. We define:

$$SH_k(X) = \{g\colon S \to X\colon S \text{ compact regular oriented } k\text{-dim stratifold}\}/\text{bordism}$$

Definition 2.7. For X, Y topological spaces define the homology cross product $SH_k(X) \otimes SH_l(Y) \to SH_{k+l}(X \times Y)$ by sending $[S, g] \otimes [S', g']$ to $[S \times S', g \times g']$. One can show that this is well defined ([Kre], 10.1). For the definition of the cross product of stratifolds, see [Kre], 2.3, Example 6.

Theorem 2.8. *For every X which is homotopy equivalent to a CW-complex we have $SH_k(X) \cong H_k(X)$. This isomorphism commutes with the homology cross product.*

Proof. A detailed proof can be found in [Kre], chapters 4, 5.1, 8.1 and 20 and in appendix B. The idea is the following: first we show that the SH_k define a homology theory. This proof is similar to this for the usual bordism theories, if one developes differential topology in the context of stratifolds. Now we have only to compute the coefficients. Since the codimension 1 stratum is empty, the 0- and 1-dimensional oriented c-stratifolds are the same as 0- and 1-dimensional oriented manifolds with boundary. Therefore we have $SH_0(pt) \cong \mathbb{Z}$. We will see in the next section that every oriented stratifold of positive dimension is the boundary of an oriented c-stratifold. Therefore, we have $SH_k(pt) = 0$ for $k > 0$. □

2.1.2 Examples

There are plenty of examples of stratifolds. For example, all (real) algebraic varieties with only isolated singularities admit the structure of a stratifold (see [Grin], p.28). We focus on some constructions which can be done with stratifolds and examples obtained by these methods.

Example 2.9 (Cones)**.** Let S be a k-dimensional stratifold. Let

$$CS = (S \times [0,1])/S \times \{1\}$$

be the cone of S. We define a stratifold structure on $\mathring{C}S = CS - S \times \{0\}$ by

$$\mathbf{C} = \{f \colon \mathring{C}S \to \mathbb{R} \colon g|_{S\times(0,1)} \text{ smooth and } g \text{ constant in a nbhd of } S \times \{1\}\}.$$

Clearly we get $\mathring{C}S_0 = pt$, where $pt = S \times \{1\} \subset CS$, and $\mathring{C}S_{k+1} = S \times (0,1)$. We see that it was necessary to impose local constance at pt on g because of condition (2) in the definition of stratifolds. It is easy to check that $\mathring{C}S$ is a $(k+1)$-dimensional stratifold and therefore CS a $(k+1)$-dimensional c-stratifold (see [Kre], 2.3). Furthermore, if S is regular, the cone CS is regular, too (see [Kre], 4.3). The cone is oriented if S is oriented and $\dim(S) > 0$. Therefore, every (compact, oriented, regular) stratifold of positive dimension is the boundary of a (compact, oriented, regular) c-stratifold of one dimension higher.

Example 2.10 (p-stratifolds)**.** Let $(S, \mathbf{C})$ be an n-dimensional stratifold and W a k-dimensional smooth manifold together with a collar $c \colon \partial W \times [0, \epsilon) \to W$. We assume that $k > n$. Let $f \colon \partial W \to S$ be a morphism, which we call *attaching map*. We further assume that the attaching map f is proper, which in our context is equivalent to requiring that the preimages of compact sets are compact. Then we define a new space $S' = W \cup_f S$ by gluing W to S via f.

On this space, we consider the algebra **C'** consisting of those functions $g \colon S' \to \mathbb{R}$ whose restriction to S are in $\mathbf{C}$, whose restriction to $\mathring{W}$ are smooth, and such that for some $\delta < \epsilon$ we have $gc(x,t) = g(x)$ for all $x \in \partial W$ and $t < \delta$. One can check that this defines the structure of a stratifold on S'.

We can use the procedure above to get an inductive method for constructing stratifolds: A 0-dimensional *stratifold of p-type* is defined to be a 0-dimensional manifold. An n-dimensional *stratifold of p-type* is defined as an $(n-1)$-dimensional stratifold of p-type glued together with an n-dimensional manifold with boundary in the way described above. The name "of p-type" is chosen because the manifolds W provide some kind of parametrization. If one fixes such a parametrization, one speaks of *p-stratifolds*. Since we have no need for this distinction, we will speak for short always of p-stratifolds.

One advantage of p-stratifolds is that we have something like a cellular approximation theorem:

Theorem 2.11 (Approximation Theorem)**.** *Let $f \colon S \to X$ be a map between an n-dimensional p-stratifold and a CW-complex. Then one can homotope f to a map $f' \colon S \to X$ whose image lies in the n-skeleton of X.*

Proof. We use the fact that every smooth manifold with boundary is a relative CW-complex. We may e.g. triangulate the boundary and by [Mun], 10.6, we can extend this triangulation to the whole manifold. We will use induction. Clearly the theorem holds for a collection of points. Now assume that $f|_{S'}\colon S' \to X$ is a map from a p-stratifold where the theorem holds and that $S = S' \cup_g W$ for an n-dimensional bounded manifold W and a morphism $g\colon \partial W \to S$. Then we apply the (relative) cellular approximation theorem ([Bre], IV.11.5), by which we can homotope $f|_W$ to a map with image in the n-skeleton of X by a homotopy which leaves ∂W fixed. □

Example 2.12 (Simplicial complexes)**.** Let S be an n-dimensional finite simplicial complex. Then we can give S the structure of a p-stratifold by an inductive procedure. For a 0-dimensional simplicial complex, we have simply a collection of points, which we give the usual smooth structure. If S is a simplicial complex of dimension k, we have a map $\coprod \partial D^k_\alpha \to S^{(k-1)}$ to the $(k-1)$-skeleton. Note that this map is smooth, because the smooth functions on the boundary of a simplex are a subset of these of the homeomorphic sphere. If for all simplicial complexes of dimension $\leq (k-1)$ a stratifold structure is defined, we simply define a stratifold structure on S by the procedure of the last example. More concretely, the stratifold structure looks as follows: Let $c_i\colon S^i \times [0,\varepsilon) \to U_i \subset S^{(i+1)}$ be collars for all $i < n$ of the i-skeleta into the $(i+1)$-skeleta. Then a smooth function on S is a function f which restricts to smooth functions on all S_i and commutes with all collars, i.e. we have $fc_i(x,t) = f(x)$ for all $t < \delta$ for a $\delta < \varepsilon$.

Let $\Delta_0 \subset \Delta$ be a subsimplex of a simplex. The opposite of Δ_0 is a subsimplex Δ_1 with $\Delta_0 \cap \Delta_1 = \varnothing$ such that the convex hull of Δ_0 and Δ_1 is the whole of Δ. The link $\mathrm{lk}(\Delta_0)$ of a simplex Δ_0 in a simplicial complex is defined to be the union of the opposites of Δ_0 in all simplices containing Δ_0. Let $x \in S_i$ be a point in a simplex Δ_0. There are open neighbourhoods U and V of x in S and Δ_0 respectively such that $V \times \mathring{C}\,\mathrm{lk}(\Delta_0) \cong U$ where $\mathring{C}\,\mathrm{lk}(\Delta_0)$ denotes the open cone over the link. Therefore, S is regular.

Let T be the subset of all points in S_{n-1} which lie in two boundary faces. We now want to modify the stratifold structure to a stratifold $S' = (S, \mathbf{C}')$ as follows: f is smooth iff it restricts to smooth functions on all S_i and satisfies $fc_i(x,t) = f(x)$ for all $t < \delta$ for a $\delta < \varepsilon$ and all $x \in U \cap S_i$ for an open neighbourhood U of $S^{n-1} - T$. Clearly we have now $\dim T_x S' = n$ for all $x \in T$ since x has a neighbourhood diffeomorphic to D^n on which all smooth functions can be extended to smooth functions of all of S'. All local rings $\mathbf{C}'_x$ stay the same for $x \notin T$. Therefore, S' is a stratifold again. If the simplicial complex S is oriented and $T = S_{n-1}$, S' is oriented as a stratifold, too, since its codimension 1 stratum is empty. Furthermore, for every $x \in S_i$ in some simplex Δ_0, there are neighbourhoods U and V in S and Δ_0 respectively such that $U \cong V \times (\mathring{C}\,\mathrm{lk}(\Delta_0))'$ where $(\mathring{C}\,\mathrm{lk}(\Delta_0))'$ denotes the modification of the stratifold structure of the open cone in the sense above. Therefore, S' is regular.

By this means, we may make the natural isomorphism from singular homology to stratifold homology more explicit. So let $z \in Z_*(X)$ be a cycle for singular homology. This z is a formal sum of maps $g_i\colon \Delta_i \to X$. Since z is a cycle, one can group the faces of the Δ_i to pairs which cancel. We glue the Δ_i along these faces and get a simplicial complex K. By the procedure of the last paragraph, this defines an oriented regular stratifold S. If $z = \partial a$, we can glue in the simplices of a into K and get a c-stratifold with boundary S. More precisely we can group the top dimension simplices of z and the boundary faces of a to pairs which cancel. We glue along them and modify the stratifold structure of this simplicial complex according to the last paragraph. This defines an oriented and regular c-stratifold with boundary K. Therefore, S is nullbordant if z is zero in homology. Hence, we get a functor $\theta\colon H_*(X) \to SH_*(X)$.

The suspension of z in $H_{*+1}(\Sigma X)$ is represented by the suspension of K with canonical triangulation (this can be seen by investigating the Mayer-Vietoris sequence). Therefore, θ commutes with suspensions. Clearly it is an isomorphism on $H_*(pt)$, so it is an isomorphism for every space which is homotopy equivalent to a CW-complex.

2.1.3 Equivariance

For every homology theory h and compact Lie group G, we can define equivariant homology groups $h_*^G(X) := h_*(EG \times_G X)$ for a G-space X, where EG is a contractible space with a free G-action or equivalently the total space of the universal G-principal bundle over the classifying space BG. You can choose the space EG to be a CW-complex with cellular G-action. If the G-action on X is free, we have that $h_*^G(X) = h_*(EG \times X/G) \cong h_*(X/G)$. For non-free G-actions, this may be far from true. For example $h_*^{\mathbb{Z}/2}(pt) \cong h_*(B\mathbb{Z}/2) \cong h_*(\mathbb{RP}^\infty)$, which is non-zero in infinitely many degrees. We want to define equivariant homology in a more geometric way.

Denote by $SH_k^G(X)$ bordism classes of equivariant maps from compact oriented regular stratifolds with an orientation preserving free G-operation into X. Here a bordism has to be equipped with an orientation preserving free G-action, too. Before we compare it to usual equivariant homology, we have to prove a little lemma.

Lemma 2.13. *The space EG has a filtration by finite dimensional manifold EG_n on which G acts smooth and free. This induces also a filtration BG_n of BG by finite dimensional manifolds.*

Proof. Embed G into a $GL(N)$. Recall $EGL(N) = V_{N,\infty}$, the infinite-dimensional Stiefel manifold. Since the induced G-action on $V_{N,\infty}$ is free, $V_{N,\infty}$ is a model for EG and $V_{N,\infty}/G$ is a model for BG. The G-action restricts to the finite-dimensional Stiefel manifolds $V_{N,n}$. Now set $EG_n = V_{N,n}$. □

Theorem 2.14. *For a G-space X homotopy equivalent to a CW-complex holds: $SH_k^G(X) \cong H_{k-dim(G)}^G(X)$.*

Proof. Let $f\colon S \to X$ be a G-equivariant map from a compact oriented regular stratifold. Then S is a principal G-bundle over S/G, so we get a classifying map (well-defined up to homotopy) $g\colon S/G \to BG$, which lifts to a bundle map $\tilde{g}\colon S \to EG$. Therefore, we get a map $h\colon S \to EG \times X$ and since both f and $\tilde{g}$ are G-equivariant, this descends to the quotient and we get a map $\bar{h}\colon S/G \to EG \times_G X$. One can show that S/G is oriented and regular again. It is clear that a bordant choice of f or a homotopic choice of g define a bordant h. So the bordism class of h is well-defined and we get a map $SH_k^G(X) \to H_{k-dim(G)}(EG \times_G X)$ via the isomorphism of stratifold homology and singular homology.

To define a map backwards, let $f\colon S \to EG \times_G X$ be a map from a compact oriented regular stratifold S. Consider the pullback diagramm

$$\begin{array}{ccccc} S' & \xrightarrow{\tilde{f}} & EG \times X & \xrightarrow{\mathrm{pr}_2} & EG \\ \downarrow & & \downarrow & & \downarrow \\ S & \xrightarrow{f} & EG \times_G X & \xrightarrow{\mathrm{pr}_2} & EG/G = BG \end{array}$$

Being a principal G-bundle, S' is equipped with a free G-action. Since S is compact, the map $\mathrm{pr}_2 \circ f\colon S \to BG$ factors over one of the BG_n. Because the bundle $EG_n \to BG_n$ is smooth, the bundle $S' \to S$ is smooth, too, and therefore S' is again a (regular, oriented) stratifold. The composition $\mathrm{pr}_2 \circ \tilde{f}\colon S' \to X$ defines an equivariant map which is well defined up to bordism. This gives a map $H_{k-dim(G)}(EG \times_G X) \to SH_k^G(X)$.

Both are inverse to each other, which is clear since $S = S'/G$. □

By this construction of equivariant homology, we get for every G-space X homomorphisms

$$\begin{aligned} E\colon &\quad H^G_{k-dim(G)}(X) \cong SH^G_k(X) \to H_k(X) \text{ and} \\ M\colon &\quad H_k(X) \to SH^G_{k+dim(G)}(X) \cong H^G_k(X). \end{aligned}$$

The first one is defined by forgetting the G-action and the second one is defined by sending $f\colon S \to X$ to $\tilde{f}\colon S \times G \to X, (s,g) \mapsto g \cdot f(s)$. Without using stratifolds, these morphisms can be understood via spectral sequences. The letters E and M stand for *erase* and *mark*.

2.2 Geometric Homology

Now we want to carry out a different bordism description for homology due to Martin Jakob ([Jak]) which works for every (generalized) homology theory. It can be thought as a geometric way to build out of a cohomology theory the corresponding homology theory. Recall that the usual definition associates to a cohomology theory h^* first the representing spectrum E and then defines the homology theory as $h_*(X) = \pi_*(E \wedge X)$ (at least for all X of the homotopy type of a CW-complex). We call this the spectral homology associated to h^*. Since our applications in section 3 demand to consider also relative homology, we want also to recall that the relative homology of a pair $\iota\colon A \hookrightarrow X$ is defined to be the homology of the cone $C\iota$.

Definition 2.15 (Geometric cycles)**.** Let h^* be a cohomology theory and (X, A) a pair of topological spaces. A *geometric cycle* is a triple (P, a, f) where: $f\colon P \to X$ is a continuous map from a compact connected h^*-oriented manifold P with boundary to X, such that $f(\partial P) \subset A$ and $a \in h^*(P)$.

If P is of dimension p and $a \in h^m(P)$ then (P, a, f) is a geometric cycle of *degree* $p - m$.

Take the free abelian group generated by all the geometric cycles and impose the following relation:

$$(P, \lambda a + \mu b, f) = \lambda(P, a, f) + \mu(P, b, f).$$

Thus, we get a graded abelian group. In order to recover the spectral homology we must impose the additional relations on geometric cycles:

1. (Bordism relation) We call two triples (P, a, f) and (P', a', f') bordant, if there is a geometric cycle (W, b, g), such that $P \coprod (-P') \subset \partial W$ is a regularly embedded submanifold of codimension 0 which inherits the h^*-orientation of W. We require further that $b|_P = a, b|_{P'} = a',\ g|_P = f,\ g|_{P'} = f'$ and $g(\partial W - P \coprod P') \subset A$. Two bordant cycles are defined to be equivalent.

2. (Vector bundle modification) Let (P, a, f) be a geometric cycle and consider a smooth h^*-oriented d-dimensional vector bundle $\pi\colon E \to P$, take the unit sphere bundle $S(E \oplus$

1) of the Whitney sum of E with a copy of the trivial line bundle over P. The bundle $S(E \oplus 1)$ admits a section s. By $s_!\colon h^*(P) \to h^{*+d}(S(E \oplus 1))$ we denote the Gysin morphism in cohomology associated to this section (this will be defined in the next section). Then we impose: $(P, a, f) \sim (S(E \oplus 1), s_!(a), fp)$.

We lay upon the group of cycles the equivalence relations generated by the relations 1 and 2. An equivalence class of geometric cycle is denoted by $[P, a, f]$, called a geometric class. We define $gh_q(X, A)$ to be the abelian group of geometric classes of degree q.

Theorem 2.16 ([Jak], Corollary 4.3)**.** *The is a natural isomorphism*

$$gh_q(X, A) \to h_q(X, A)$$

defined by

$$[P, a, f] \mapsto f_*(a \cap [P]),$$

where $[P]$ *is the fundamental class of* $(P, \partial P)$*.*

For later applications, we give an explicit description of the excision isomorphism: So let $[P, a, f] \in h_*(X, A)$ be a geometric class and $B \subset A$, such that $\overline{B} \subset \mathring{A}$. The preimages $f^{-1}(\overline{B})$ and $f^{-1}(X - \mathring{A})$ are closed and we can choose a smooth Urysohn function $g\colon P \to \mathbb{R}$ separating them which is a submersion. Choose a regular value x between 0 and 1. Then $Q := g^{-1}([0, x])$ is a manifold with boundary in $A - B$. The restriction $[Q, a|_Q, f|_Q]$ is the image of the excision isomorphism in $h_*(X - B, A - B)$. Indeed, $(P \times [0, 1], \mathrm{pr}_1^*(a), f \circ \mathrm{pr}_1)$ is a bordism between $[P, a, f]$ and $i_*[Q, a|_Q, f|_Q]$ since $Q \coprod P$ is a regular submanifold of codimension 0 in $P \coprod P$.

We now want to mimic our definition of equivariant homology via stratifolds (which is, in turn, mimicked after equivariant bordism). So let G be a fixed compact Lie group, X be a G-space and A a G-invariant subspace.

Definition 2.17. A *geometric G-equivariant cycle* is a triple (P, a, f) where $f\colon P \to X$ is an equivariant map from a compact connected h^*-oriented manifold P with boundary and a free h^*-orientation preserving G-action to X, such that $f(\partial P) \subset A$ and $a \in h^*_G(P) \cong h^*(P/G)$. If P is of dimension p and $a \in h^m_G(P)$ then (P, a, f) is a geometric cycle of *degree* $p - m$.

The relations must be G-equivariant in the sense that the bordism has to be equipped with a h^*-orientation preserving free G-action and the vector bundles have also to be equipped with an orthogonal free h^*-orientation preserving G-action, which descends to the G-action on the base. The abelian group of equivalence classes of G-equivariant geometric cycles of degree k is denoted by $gh^G_k(X, A)$.

Theorem 2.18. *We have an isomorphism* $gh^G_k(X, A) \cong h^G_{k-dim(G)}(X, A)$*.*

Proof. The proof is the same as that of 2.14. □

By this construction of equivariant homology we get for every G-space X homomorphisms

$$E\colon \quad h^G_{k-dim(G)}(X) \cong gh^G_k(X) \to h_k(X) \text{ and}$$
$$M\colon \quad h_k(X) \to gh^G_{k+dim(G)}(X) \cong h^G_k(X).$$

The first one is on a representing cycle (P, a, f) defined by forgetting the G-action on P and sending a to $p^*(a)$, where $p\colon P \to P/G$ is the projection. The second one is defined by sending $[P, a, f]$ to $[P, a, \tilde{f}]$ with $\tilde{f}\colon P \times G \to X$ defined by $(p, g) \mapsto (g \cdot f(p))$. These generalize the homomorphisms of the preceding subsection to arbitrary homology theories.

2.3 The Thom Isomorphism and Gysin Morphisms

Let h^* be a multiplicative cohomology theory and h_* its associated homology theory (see the subsection above). First, we want to recall the theory of the Thom isomorphism.

Definition 2.19. Let $\xi = (E \to B)$ be a (real) d-dimensional vector bundle over a space B with projection map p and zero section $s\colon B \to E$. Then the Thom space $Th(\xi)$ of ξ is defined to be DE/SE where an euclidean metric is chosen on E and DE and SE are the associated disc and sphere bundles. A class $\Theta \in h^d(Th(\xi)) \cong h^d(E, E - s(B))$ is called *Thom class* for ξ if for every $b \in B$,

$$j^*(\Theta) \in h^d(p^{-1}(b), p^{-1}(b) - \{s(b)\}) \cong h^n(S^n)$$

is a generator as an $h^*(pt)$-module. Here $j\colon (p^{-1}(b), p^{-1}(b) - \{b\}) \to (E, E - s(B))$ is the inclusion. A bundle is called *h^*-orientable* if there exists a Thom class and *h^*-oriented* if a Thom class is chosen. A manifold is called *oriented* if a Thom class for its tangent bundle is chosen.

Let now $\xi = E \to B$ be a d-dimensional oriented vector bundle. Then we can define maps (called the homology and cohomology *Thom isomorphism*)

$$\begin{aligned} h_*(Th(\xi)) \cong h_*(E, E - s(B)) \to h_{*-d}(E) \xrightarrow{\cong} h_{*-d}(B), \\ h^*(B) \cong h^*(E) \to h^{*+d}(E, E - s(B)) \cong h^{*+d}(Th(\xi)). \end{aligned}$$

Here the middle map is defined by capping and cupping with the Thom class Θ respectively. For a homology theory with $h_i(pt) = 0$ for $i < 0$ or if B is a finite-dimensional CW-complex, this map is an isomorphism ([Swi], 14.6).

We now want to see what the Thom class looks like in the stratifold and the Jakob description for B a manifold. We will give only the stratifold case in detail.

First, one has to define *stratifold cohomology* for an n-dimensional oriented manifold X (see [Kre], chapters 12 and 13, for details). A k-cycle is given by a pair (S, f) where S is an $(n-k)$-dimensional (not necessarily compact) regular, oriented stratifold and $f\colon S \to X$ a proper map. To get stratifold cohomology, one has to divide out the equivalence relation of bordism as in 2.1. Let $g\colon X \to Y$ be a continuous map, $(S, f) \in SH^k(Y)$ and $g'\colon X \to Y$ a map which is homotopic to g, smooth and transverse to f (i.e. transverse to every stratum). Then we can define the induced map $SH^k(Y) \to SH^k(X)$ by $(S, f) \mapsto (S \times_Y X, \mathrm{pr}_2)$. This is isomorphic to ordinary cohomology of X by [Kre], 20.3. We get the relative cohomology for a pair (X, A) by allowing S to have a boundary which is mapped into A as in 2.2. In the same way, one can define relative homology. We will usually identify stratifold cohomology with singular cohomology since they are naturally isomorphic.

Let $\xi = (E \to B)$ be an oriented d-dimensional vector bundle over an m-dimensional oriented manifold B as base space. We clearly have that $(B, s) \in H^d(E, E - B)$ is a Thom class since $j^*(B, s) = 1 \in H^0(p^{-1}(b), p^{-1}(b) - \{b\})$. Because the stratifold cap product is given by transversal intersection, the homology Thom isomorphism is given by intersection with the base (known to the author by [Kre2]). In analogy, the Thom isomorphism in the Jakob description is given by $[P, a, f] \mapsto [P \cap B, a|_{P \cap B}, f|_{P \cap B}]$ (see [Jak2]).

Now let N be an h^*-oriented manifold and $g\colon M \hookrightarrow N$ be the inclusion of an h_*-oriented submanifold of codimension d. We want to define Gysin maps $g_!\colon H^k(M) \to H^{k+d}(N)$ and $g^!\colon H_k(N) \to H_{k-d}(M)$. If U is a tubular neighbourhood of M and ν its normal bundle, we get a map $c\colon N \to N/(N-U) \cong Th(\nu)$, called the *Thom collapse*. Now we simply compose c^* or c_* with the Thom isomorphism and get the above Gysin maps. It can be shown that these Gysin morphisms coincide with those given by Poincare duality. In the homology case, we can furthermore interprete the Gysin map as transversal intersection with M.

Suppose N is triangulated and g is transverse to the triangulation. Triangulate M as in 2.5. Let $C_*(M)$ and $C_*(N)$ denote the cellular chain complexes. By sending a simplex Δ of the triangulation of N to all the simplices of $\Delta \cap M$, we get a chain map $s\colon C_*(N) \to C_{*-d}(M)$. A cycle z in $C_*(N)$ is a formal sum of simplices; the same homology class is represented by the corresponding formal sum of simplices in singular homology. The corresponding element of stratifold homology is represented by the simplicial complex glued together from these simplices (see 2.12). The transversal intersection of this stratifold with M represents exactly the homology class $\theta(s(z))$. Therefore, s is a cellular description of the Gysin morphism as can be seen by the descriptions of the Thom class above.

This cellular description of the Gysin morphism is also suitable to describe Gysin morphisms for homology with local coefficients. So let $\mathcal{G}$ be a local system. Let $x = x_\Delta$ be the midpoint of a simplex Δ. In the cellular complex with respect to $\mathcal{G}$ the coefficient of Δ lies in $\mathcal{G}_x$. Choose arbitrary paths from x to the midpoints of the simplices of $\Delta \cap M$ in Δ and map via them the coefficient of Δ to coefficients for these simplices (note that all possible choices of these paths are homotopic). This describes a Gysin map $g^!\colon H_*(N; \mathcal{G}) \to H_{*-d}(M; \mathcal{G}|_M)$.

Suppose one has a local system $\mathcal{R}$ of rings on a d-manifold M (we require all induced maps to be ring homomorphisms). Then one can define an intersection product as the following composition:

$$H_p(M;\mathcal{R}) \otimes H_q(M;\mathcal{R}) \stackrel{\times}{\to} H_{p+q}(M \times M; \mathcal{R} \otimes \mathcal{R}) \stackrel{\Delta^!}{\to} H_{p+q-d}(M; (\mathcal{R} \otimes \mathcal{R})|M) \to H_{p+q-d}(M;\mathcal{R})$$

Here $R \otimes R$ is a $\pi_1(M \times M)$-module via $\gamma_*(x \otimes y) = (\mathrm{pr}_1\, \gamma)_*(x) \otimes (\mathrm{pr}_2\, \gamma)_*(y)$ for a loop γ in $M \times M$.

2.4 Hilbert Manifolds

2.4.1 Definitions

Definition 2.20. A *Hilbert manifold* is a metrizable space which is locally homeomorphic to a separable Hilbert space $\mathbb{E}$. Smooth Hilbert manifolds one defines in analogy to the usual finite dimensional smooth manifolds.

We assume every Hilbert manifold to be smooth. Since (at least in principle) all results in this section are well known in the finite-dimensional case, we assume all Hilbert manifolds to be infinite-dimensional.

As in the finite-dimensional case, there are different ways to define tangent spaces which are all equivalent (e.g. one can show this by the use of Hilbert space bases). We choose one which is in analogy to [Bre], II.5.1.

Definition 2.21. Let M be a Hilbert manifold. Tangent vectors at $p \in M$ are defined to be equivalence classes of germs of smooth curves $\gamma\colon \mathbb{R} \to M$ with $\gamma(0) = p$, where $\gamma \sim \gamma'$ if the operations $D_\gamma\colon C^\infty(p) \to \mathbb{R}, f \mapsto \frac{d}{dt} f(\gamma(t))|_{t=0}$ and $D_{\gamma'}$ are the same. The set of all tangent vectors at p is called the *tangent space at* p and is denoted by T_pM

As usual one can now define for a given morphism (i.e. smooth map) differentials between the tangent spaces. An especially nice class of morphisms between Hilbert manifolds is the following:

Definition 2.22. A morphism $f\colon M \to N$ between Hilbert manifolds is called *Fredholm* if the differential $df_p\colon T_pM \to T_{f(p)}N$ is Fredholm everywhere, i.e. has finite-dimensional kernel and cokernel for every $p \in M$.

2.4.2 Differential Topology

Proposition 2.23. *Let N be a Hilbert manifold and X be a topological space. Then one can choose a positive function $\varepsilon\colon X \to \mathbb{R}$ such that $d(f(x), g(x)) < \varepsilon(x)$ for two continuous functions $f, g\colon X \to N$ implies $f \simeq g$.*

Proof. The proof is virtually the same as in the finite-dimensional case (see e.g. [B-J], 12.9). For the Hilbert manifold cases of the results used in the proof, look at [La], IV §§3-5. □

Proposition 2.24. *Let S be an n-dimensional stratifold and $f\colon S \to N$ a smooth map to a Hilbert manifold which is transverse (i.e. transverse on every stratum) to a closed submanifold $L \subset N$ with d-dimensional normal bundle. Then $f^{-1}(L)$ is an $(n-d)$-dimensional stratifold.*

Proof. The proof is virtually the same as in [Kre], 2.7 and 4.2. For the results on manifolds used there in the case of Hilbert manifolds, see [La], II §2. □

Proposition 2.25. *Assume we have a positive function $\varepsilon\colon S \to \mathbb{R}$. Then one can approximate every continuous map $f\colon S \to N$ from a compact stratifold to a Hilbert manifold by a smooth map $g\colon S \to N$ with distance smaller than ε. If f is already smooth on an open neighbourhood U of a closed subset A, one can assume that $g|_A = f|_A$.*

Proof. By [E-E1] every Hilbert manifold is diffeomorphic to an open subset U of the standard separable Hilbert space $\mathbb{E}$. Since one can find a $\delta\colon S \to \mathbb{R}$ with $\delta < \varepsilon$ such that a δ-neighbourhood of the image of S is contained in U, one can assume $N = \mathbb{E}$. Then the proof is virtually the same as in [B-J], 14.8. □

Remark 2.26. Let S be c-stratifold with boundary S' and $g\colon S \to N$ a continuous map such that $g|_{S'}$ is smooth. Let $c\colon U \to S' \times [0, 1)$ be a collar for S'. There is an isomorphism $h\colon (S - U) \to S$. Define $g'\colon S \to N$ by gh on $(S - U)$ and by gc on U. By the preceding proposition, we can homotope g' to a map f which is smooth and $f|_{S'} = g'|_{S'} = g|_{S'}$. By the same method, one can argue for a closed submanifold A of a manifold S.

An important substitute for the usual theorem of Sard is the theorem of Smale-Sard:

Theorem 2.27. *Let $f\colon M \to N$ be a Fredholm morphism between Hilbert manifolds*[1] *Then the set of regular values contains the intersection of a countable family of open dense subsets.*

[1] It is enough to assume them to be separable Banach manifolds, indeed.

2.4.3 The Transversality Theorems

In this section, we want to prove our transversality results. We prove some of the theorems in greater generality than we actually need. We hope that they are maybe interesting for other applications, though, especially our piece of relative differential topology.

Let S be a c-stratifold. We say that a subset $A \subset S$ has the property (P) if there is a differentiable function $\alpha\colon S \to \mathbb{R}$ such that $\alpha^{-1}(0) = A$, $\alpha < 1$ and all derivatives of α vanish on A. For example, this property holds if A is a submanifold and S a manifold or if A is the boundary of S in the general case.

Lemma 2.28. *Let $E \to S$ be a (smooth) Hilbert space bundle over a compact c-stratifold S with a Riemannian metric. Furthermore, let $L \subset E$ be a subspace such that $L \cap E|_{S^i} \subset E|_{S^i}$ is a closed sub Hilbert manifold of finite codimension for every $i \geq 0$ and $\varepsilon\colon S \to \mathbb{R}$ a positive function on S. Then there is a smooth section $s\colon S \to E$ with $|s(p)| < \varepsilon(p)$ for all $p \in S$ such that s is transverse to L. If $A \subset S$ is a subset fulfilling property (P) and the zero section is already transverse to L on A, we can choose $s|_A = 0$.*

Proof. The theorem of Kuiper ([Kui]) says that every Hilbert space bundle over a compact space is trivial. So we have $E \cong S \times \mathbb{E}$. Define $U = S - A$ and $\delta = \varepsilon \cdot \alpha\colon S \to \mathbb{R}$. Now we have a bundle map

$$g\colon E|_U = U \times \mathbb{E} \to U \times \mathbb{E},\ (p, v) \mapsto (p, (\delta(p))^{-1} \cdot v).$$

Consider the map

$$h\colon L \cap (E|_U) \xrightarrow{g} U \times \mathbb{E} \xrightarrow{\mathrm{pr}_2} \mathbb{E}.$$

For every stratum S^i of S we have that $L^i = E|_{S^i} \cap L$ is a Hilbert manifold and $h|_{L^i}$ is Fredholm. Furthermore, by the theorem of Smale-Sard the regular values of $h|_{L^i}$ form a countable intersection of open dense subsets. Therefore, there is a common regular value $w \in \mathbb{E}$ of all $h|_{L^i}$ of norm smaller than 1, thus also one for h. Now define a section $s\colon S \to E$, $p \mapsto (p, \delta(p) \cdot w)$. This section is transverse to L and has norm smaller than ϵ. □

We will prove two versions of of the transversality theorem. The first one is simply a generalization of the usual finite-dimensional transversality theorem to the Hilbert manifold case. The second one is a relative transversality theorem (in the sense of algebraic geometry), i.e. we map into a fibre bundle and make our map transverse to a sub fibre bundle without changing the projection to the base. Although the relative version includes the absolute case, we prove for the reader's convenience the easier absolute case first.

Theorem 2.29 (absolute case)**.** *Let $f\colon S \to N$ be a smooth map between a compact c-stratifold S and a Hilbert manifold N. Furthermore let $L \subset N$ be a closed sub Hilbert manifold of finite codimension. Then there is a homotopy $H\colon S \times I \to N$ between f and a map $g\colon S \to N$ which is transverse to L. If $A \subset S$ is subset fulfilling property (P) with $f|_A$ transverse to L, we can choose $H|_{A \times I} = f \circ \mathrm{pr}_1$.*

Proof. The map $\mathrm{id} \times f\colon S \to S \times N$ is a closed embedding. We want to construct a tubular neighbourhood of S in $S \times N$. By [Kli1], 1.9.11, there is a continuous function $\kappa\colon N \to \mathbb{R}$, such that the $\kappa(x)$-ball around every $x \in N$ is strongly convex, i.e. every two points are connected by a unique geodesic. As in 2.25, one can find a positive smooth function on N

which is smaller than κ on $f(S)$ (note that $f(S)$ is compact). So we can assume $\kappa \circ f$ to be smooth. Since the normal bundle of S in $S \times N$ is isomorphic to f^*TN, we have that $T\colon = \{(s,n) \in S \times N \colon n \in B_{\kappa(f(s))}\}$ is isomorphic to the κ-neighbourhood of S in f^*TN. To apply the lemma we identify T with a neighbourhood in f^*TN.

So we can find for every positive function $\epsilon\colon S \to \mathbb{R}$ a section $s\colon S \to T$ (of norm smaller than κ) which is transverse to $L \cap T$, therefore transverse to $S \times L$. Since $\mathrm{pr}_2\colon S \times N \to N$ is a submersion, $g\colon = \mathrm{pr}_2 \circ s$ is transverse to L and homotopic to f. □

In the relative case, we want to work with smooth fibre bundle over a differentiable space (see 2.1). We define the notion of smoothness here in complete analogy to the usual case:

Definition 2.30. Let $(X, \mathbf{C})$ and $(F, \mathbf{D})$ be differentiable spaces. A fibre bundle $F \to E \xrightarrow{\pi} X$ is called *smooth* iff there is a cover by open subsets $U_i \subset X$ and there are homeomorphisms $\Phi_i\colon U_i \times F \to \pi^{-1}(U_i)$ such that $\pi \circ \Phi_i = \mathrm{pr}_1$ and

$$\Phi_i^{-1}\Phi_j|_{(U_i \cap U_j)\times F)}\colon (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$$

is smooth for every i and j.

For X and F manifolds, this definition clearly coincides with the usual definition of a smooth fibre bundle. If we fix the choice of the U_i and Φ_i, there is an induced structure of a differentiable space on E where we call a function $g\colon E \to \mathbb{R}$ smooth iff $g \circ \Phi_i$ is smooth for all i.

Let $(X', \mathbf{C}')$ be a second differentiable space and $f\colon X' \to X$ smooth and Let $F \to X' \times_X E \to X'$ be the pullback bundle. Set $V_i = f^{-1}(U_i)$. Now define $\Psi_i\colon V_i \times F \to \mathrm{pr}_1^{-1}(V_i)$ by $\Psi_i(x,a) = (x, \Phi_i(f(x),a))$. This is a homeomorphism. We have that $\Psi_i^{-1}\Psi_j(x,a) = (x, \mathrm{pr}_2\,\Phi_i^{-1}\Phi_j(f(x),a))$ which is smooth. Therefore, the pullback bundle is smooth again.

Theorem 2.31 (relative case)**.** *Let $\pi\colon E \to B$ be a fibre bundle where the fibre is a Hilbert manifold and the base is a differentiable space. Furthermore, let $E_0 \subset E$ be a subbundle which is in every fibre a closed sub Hilbert manifold of finite codimension. Let $f\colon S \to E$ be a smooth map from a compact c-stratifold S. Then there is a homotopy $H\colon S \times I \to E$ between f and a map $g\colon S \to E$ which is transverse to E_0 such that $\pi \circ H = \pi \circ f \circ \mathrm{pr}_1$. If $A \subset S$ fulfilling property P is a subset with $f|_A$ transverse to L, we can choose $H|_{A\times I} = f \circ \mathrm{pr}_1$.*

Proof. Consider the closed embedding $\mathrm{id} \times f\colon S \to S \times_B E$, where the pullback is over $\pi \circ f\colon S \to B$. Construct as in the previous theorem a tubular neighbourhood T of S in $S \times_B E$ and choose a section $s\colon S \to T$ transverse to $S \times_B E_0$. Since $\pi|_{E_0}\colon E_0 \to B$ is a submersion, we have that $s\colon S \to S \times E$ is transverse to E_0, too. Therefore, we get as in the last theorem that $g := \mathrm{pr}_2 \circ s$ transverse to E_0. □

Since the crucial ingredient in the proof of the lemma was the Sard-Smale theorem and a countable intersection of countable intersections is a countable intersection again, in the lemma and the following theorems we can substitute L by a countable collection of L_i. It was only for simplicity that we concentrated on the case of a single submanifold.

We want to have a version of 2.29 for arbitrary finite-dimensional manifolds instead of the compact c-stratifold. To this aim, first we have to prove a simple lemma:

Lemma 2.32. *Let $A \subset D^n$ be a finite union of closures of convex open subsets with differentiable boundary. Then A has the property (P).*

Proof. Let A be the closure of an open convex set itself. Choose a point $q \in \mathring{A}$. By considering the rays coming from q, we see that $\mathbb{R}^n - \mathring{A}$ is diffeomorphic to $S^{n-1} \times [0, \infty)$. Choose a function $g\colon [0,\infty) \to \mathbb{R}$ with $g(0) = 0$, $g < 1$ and all derivatives of g vanish at 0. So we get a function $\alpha\colon D^n \to \mathbb{R}$ defined by

$$p \mapsto \begin{cases} 0 \text{ for } p \in A \\ g(\mathrm{pr}_2(p)) \text{ for } p \in (D^n - \mathring{A}) \end{cases} .$$

Here we consider $(D^n - \mathring{A})$ as a subset of $S^{n-1} \times [0,\infty)$. This function α fulfills the conditions for property (P).

If $A = A_1 \cup \cdots \cup A_k$ where the A_i have property (P) with respect to functions α_i, then A has property (P) with respect to $\alpha = \alpha_1 \cdots \alpha_n$. □

Theorem 2.33. *Let $f\colon M \to N$ be a smooth map from a finite-dimensional manifold M to a Hilbert manifold N. Furthermore let $L \subset N$ be a closed sub Hilbert manifold of finite codimension. Then there is a homotopy $H\colon M \times I \to N$ between f and a map $g\colon M \to N$ which is transverse to L. If $A \subset M$ is subset with $f|_A$ transverse to L, then we can choose $H|_{A\times I} = f \circ \mathrm{pr}_1$.*

Proof. The manifold M possesses a countable cover by closed geodesic balls D_i. We will construct the wanted homotopy by induction. Set $f^0 = f$ and assume f^{k-1} restricted to D_i for $i < k$ is already transverse to L where D_i is seen as a stratifold. The intersection $\bigcap_{i=1}^k D_i$ is a union of closures of open convex sets in D_k. So we can homotope f^{k-1} to a map f^k which is transverse to L on D_k (and therefore on $\bigcup_{i=1}^k D_i$) and is equal to f^{k-1} on $\bigcup_{i=1}^{k-1} D_i$. Here we use that $\bigcup_{i=1}^k D_i \hookrightarrow M$ is a cofibration. Since for every point $p \in M$, there is an K with $f^K(p) = f^{K+1}(p) = \cdots$, we can compose all these homotopies and get a homotopy between f and a map $g\colon M \to N$ which is transverse to L. □

2.5 Triangulations

Let $A \subset B$ be a submanifold of a finite-dimensional manifold B. We want to construct a triangulation $\mathcal{T}$ transverse to A, i.e. every stratum $\mathcal{T}^k - \mathcal{T}^{k-1}$ is transverse to A. So first choose any triangulation $\mathcal{T}'$ of B. By 2.28 we can make the inclusion ι of A transverse to $\mathcal{T}'$ by a section s in a tubular neighbourhood U. By [Bre], II.15.5, there is a diffeomorphism of U leaving the boundary fixed which takes s to ι. This diffeomorphism defines a new triangulation $\mathcal{T}$, which is transverse to A. Now we want to find a suitable triangulation of A in the following sense:

Proposition 2.34. *There exists a (non-necesserily smooth) triangulation $\overline{\mathcal{T}}$ of A such that $\mathcal{T}^k \cap A \subset \overline{\mathcal{T}}^{k-d}$, where d denotes the codimension of A and $\mathcal{T}^k$ the k-skeleton. More specifically, it is glued from triangulations of the $A \cap \Delta_i$ for the simplices Δ_i of $\mathcal{T}$*

The proof is most naturally formulated in the language of $\langle n\rangle$-manifolds, a notion due to Jänich ([Jän]).

Definition 2.35. A *differentiable manifold with corners* is a topological manifold X with boundary, together with a C^∞-structure with corners, i.e. X is covered by charts of the form

$$\varphi\colon \Omega \to \mathbb{R}^n_+ = [0,\infty)^n$$

where the transition maps are diffeomorphisms. For $x \in X$, the number of zeros in $\varphi(x)$ is for every chart (Ω, φ) with $x \in \Omega$ the same and is called $c(x)$. A *connected face* of X is the closure of one component of $\{x \in X | c(x) = 1\}$. It is called *manifold with faces* if all $x \in X$ are in $c(x)$ distinct connected faces.

An $\langle n \rangle$*-manifold* is a manifold with faces X together with an n-tuple

$$(\partial_0 X, \partial_1 X, \ldots, \partial_{n-1} X)$$

of faces (i.e. disjoint unions of connected faces) fulfilling the following conditions:

1. $\partial_0 X \cup \partial_1 X \cup \ldots \cup \partial_{n-1} X = \partial X$

2. $\partial_i X \cap \partial_j X$ is a face of both $\partial_i X$ and $\partial_j X$ for all $i \neq j$.

Every simplex Δ_i has the structure of an $\langle n \rangle$-manifold with the usual faces. Since A is transverse to $\mathcal{T}$, $A_i = A \cap \Delta_i$ is an $\langle n \rangle$-manifold, too. Since every face of an $\langle n \rangle$-manifold is an $\langle n-1 \rangle$-manifold, we can define the triangulation inductively. We simply use the fact that a triangulation of the boundary of a manifold can be extended to a triangulation of the whole manifold (see [Mun], 10.6). Since A is glued together from the A_i along faces, this gives a triangulation of the whole of A which finishes the proof.

Proposition 2.36. *If $A \subset B$ is a closed submanifold and A is triangulated, there is a triangulation of B extending that of A.*

Proof. By [Mun], 10.7, we only need to show that we can find an extension of the triangulation to some neighbourhood of A. So consider a bounded tubular neighbourhood $p \colon T \to A$. Since a simplex Δ is contractible, the normal bundle over Δ is trivial and so $p^{-1}(\Delta) \cong \Delta \times D^d$ where d is the codimension of A. Choose a triangulation of S^{d-1} and cone it off to get a triangulation of D^d. For every simplex Δ of A we triangulate $\Delta \times D^d$ by first triangulating the product of Δ with the zero skeleton of D^d by $\coprod \Delta$ and then inductively extending the triangulation over the skeleta to $\Delta \times D^d$ (so it extends Δ). Triangulate T by first triangulating the zero skeleton of $A \times D^d$ and then extending this inductively over the skeleta of A. Here we use [Mun], 10.6, again. This triangulation extends that of A. □

2.6 Mapping Spaces

To every two spaces X and Y we can associate a mapping space $Map(X, Y)$ via the compact open topology. The object of the whole thesis is the homology of this mapping space in the case when both $X = M$ and $Y = N$ are manifolds, which will be assumed for the rest of this section. The aim of this section is threefold:

1. Show that $Map(M, N)$ is a fibre bundle over N,

2. Exhibit via the theory of Sobolev spaces a Hilbert manifold which is a homotopy model for $Map(M, N)$ and

3. Study how one can approximate mapping spaces by finite dimensional manifolds.

We will start with the first point.

Theorem 2.37. *Let M be a pointed manifold with base point x. Then the map* $\mathrm{ev} \colon Map(M, N) \to N, f \mapsto f(x)$ *is a locally trivial fibration.*

Proof. Choose a disc $D \subset N$ with centre $y = 0$. We want to define a map G from $\mathrm{ev}^{-1}(\mathring{D})$ to $\mathrm{ev}^{-1}(y)$, which is an isomorphism on each fibre. To that end, it is enough to define a map $g\colon \mathring{D} \to \mathrm{Homeom}(D)$ which sends each $p \in \mathring{D}$ to a homeomorphism F fixing the boundary with $F(0) = p$. Extend g to a map $\tilde{g} : \mathring{D} \to \mathrm{Homeom}(N)$ by the identity on $N - D$. Then $f \mapsto (\tilde{g}(\mathrm{ev}(f)))^{-1} \circ f$ is the map G we want to define.

Define $F : D \to D$ by $F(q) = q + (1 - |q|) \cdot p$. This is injective, for assume $F(q) = F(q')$. Then $q - q' = (|q| - |q'|) \cdot p$ and therefore $|q - q'| = ||q| - |q'|| \cdot |p| < ||q| - |q'||$ for $|p| < 1$ and $q \neq q'$ which is in contradiction to the triangle inequality. The map F is also surjective as it defines a self map of S^n and every injective map of closed manifolds of the same dimension is also surjective. ([Bre], IV.19.9). □

2.6.1 The Hilbert Manifold Structure

In this section, we want to exhibit a Hilbert manifold $H^n(M, N)$ which is homotopy equivalent to the space $Map(M, N)$ of continuous maps. We will use the same methods as those exhibited in [Kli1] and [Kli2] in the case of the free loop space.

Let M be an n-dimensional compact manifold and N be an arbitrary Riemannian manifold. We define $H^n(M, N) \subset Map(M, N)$ as the space of all continuous maps $f\colon M \to N$ such that there is for every $p \in M$ a chart (U, ϕ) around p and a chart (V, ψ) around $f(p)$ such that $\psi f \phi^{-1}\colon \phi(U) \to \psi(V)$ is of Sobolev class H^n. Note that every L^2-map $\phi(U) \to \psi(V)$ of class H^n has a unique continuous representative (see [Alt], 8.13).

Choose some $f \in \mathcal{C}^\infty(M, N)$. Since the image of f is compact, there exists an $\varepsilon > 0$ such that the exponential map cross the projection restricted to the ε-neighbourhood $O = O_\varepsilon \subset TN$ of the zero section is a diffeomorphism onto an open neighbourhood of the diagonal in $N \times N$. Let $f_*\colon f^*TN \to TN$ be the pullback and $O_f = (f_*)^{-1}(O)$. We define,

$$\begin{aligned} \exp_f\colon H^n(O_f) &\to H^n(M, N), \\ \xi &\mapsto (p \mapsto \exp(f_*\xi(p))). \end{aligned}$$

Here $H^n(O_f)$ denotes the following: all continuous sections ξ of $\pi\colon f^*TN \to M$ with image in O_f such that there is a chart (U, ϕ) around every point of M and a local trivialization $\pi^{-1}(U) \to U \times \mathbb{R}^k$ such that $\mathrm{pr}_2\, \xi\phi^{-1}\colon \phi(U) \to \mathbb{R}^k$ is of class H^n.

Lemma 2.38. *The map* $\exp_f$ *is injective and its image is the open set*

$$\mathcal{U}_f = \{g \in H^n(M, N)\colon g(p) \in \exp(O \cap T_{f(p)}N)\}$$

Proof. Both the injectivity and the fact that $\mathrm{im}\exp_f$ is contained in $\mathcal{U}_f$ are clear. Now we want to show that ξ is H^n if $\exp_f(\xi)$ is H^n. If $\exp_f(\xi)$ is H^n, then also $\mathrm{id} \times \exp_f(\xi)\colon M \to M \times N$ is H^n. The map $\mathrm{pr} \times (\exp \circ f_*)$ defines a diffeomorphism from O_f onto an open set in $M \times N$. Composing with the reversed diffeomorphism, we get ξ which is therefore also H^n.

To show that $\mathcal{U}_f$ is open, it is enough to show that $U_f = \{g \in \mathcal{C}^0(M, N)\colon g(p) \in \exp(O \cap T_{f(p)}N)\}$ is open in $\mathcal{C}^0(M, N)$. But U_f is just the ε-ball around f in the maximum metric. □

Theorem 2.39. *The space* $H^n(M, N)$ *is a (smooth) Hilbert manifold.*

Proof. Note that $H^n(O_f)$ is an open subset of the separable Hilbert space $H^n(f^*TN)$. Therefore, $\exp_f^{-1}\colon \mathcal{U}_f \to H^n(O_f)$ is a chart. Now we want to show that for every $g \in H^n(O_f)$, there is a $\mathcal{U}_f$ with $g \in \mathcal{U}_f$. Approximate g by smooth functions f_n in the maximum metric.

Choose $\delta > 0$. Let ε be the maximal injectivity radius on the closed δ-neighbourhood of $\mathrm{im}(g)$ and choose k with $d_\infty(f_k, g) < \varepsilon$. Then we have $g \in \mathcal{U}_f$.

For the more technical aspects, namely the smoothness of the atlas and that there is a countable base of topology, see [Kli1] or [Kli2] in the case of the free loop space or have a look at [Mei]. □

We cite the following special case of a theorem by Palais:

Proposition 2.40 ([Pal], Thm 16)**.** *Let X be a Banach space, Y a dense subspace and $U \subset X$ open. Then the inclusion $Y \cap U \hookrightarrow U$ is a homotopy equivalence.*

This allows us to prove the following:

Proposition 2.41. *The inclusion $H^n(M, N) \hookrightarrow \mathcal{C}^0(M, N)$ is a homotopy equivalence.*

Proof. Embed N as a closed submanifold in some euclidean space $\mathbb{R}^m$. Let T be a tubular neighbourhood of N in $\mathbb{R}^m$. Then $H^n(M, N)$ is homotopy equivalent to $H^n(M, T)$ and $\mathcal{C}^0(M, N)$ is homotopy equivalent to $\mathcal{C}^0(M, T)$. Since $\mathcal{C}^0(M, T)$ is an open subset of the Banach space $\mathcal{C}^0(M, \mathbb{R}^m)$ and $H^n(M, \mathbb{R}^m)$ is dense in $\mathcal{C}^0(M, \mathbb{R}^m)$ (already $\mathcal{C}^\infty(M, \mathbb{R}^m)$ is dense), we get our result. □

Since from the view of algebraic topology there is no difference between $H^n(M, N)$ and $\mathcal{C}^0(M, N)$ and the former is geometrically much more convenient, we will reserve the notation $Map(M, N)$ in the following for $H^n(M, N)$. It is easy to see that in this context, $Map(M, N) \to N$ is still a fibre bundle.

2.6.2 The Approximation Theorem

Theorem 2.42. *Let M, N be manifolds and assume M to be compact. Then there exists a sequence of submanifolds $P_1 \subset P_2 \subset \cdots \subset Map(M, N)$ such that one can deform every map $X \to Map(M, N)$ from a compact X to a map into one of the P_i.*

To that end, we will follow ideas of [Mil], §16, and start with some preliminary considerations. We fix Riemannian metrics on M and N.

Definition 2.43. Define for $f \in Map(M, N)$ the *energy* $E(f) := \int_M ||T_p f||^2$ and the *length* $L(f) := \int_M ||T_p f||$ of f. Here $||T_p f|| = \max_{v \in T_p M, |v|=1} |Tf(v)|$.

Let $g\colon O \to M$ be a map from a Riemannian manifold O. Then $||T_p(fg)|| \leq ||T_p f|| \cdot ||T_p g||$. Therefore, by Cauchy-Schwarz we have $L(fg) \leq \sqrt{E(f) \cdot E(g)}$.

Choose for every k a finite triangulation $\mathcal{T}^k$ of M such that every simplex has diameter smaller than $1/k$ and $\mathcal{T}^k$ is a refinement of $\mathcal{T}^{k-1}$.

Definition 2.44. Let B be a strongly convex ball on a Riemannian manifold N. A *geodesic simplex* is a map $h\colon \Delta^n \to B$ such that for every point $q \in B$ the map $\exp_q^{-1} \circ h\colon \Delta^n \to T_q N$ is the inclusion of an euclidean simplex. Since $\exp_q$ is an isometry, this is independent of the choice of q.

Now we define for each k the space $P_k \subset Map(M, N)$ to be the subspace of all $f \in Map(M, N)$ whose restriction to any simplex of $\mathcal{T}^k$ is a geodesic simplex. Since a geodesic simplex is determined by the images of the vertices, P_k is an (open) submanifold of $N^{|V(\mathcal{T}^k)|}$ (where $V(\mathcal{T}^k)$ denotes the set of vertices) and therefore finite dimensional.

Let now $F : X \to Map(M, N)$ be a continuous map from a compact space X. Note that the energy functional is continuous since $Map(M, N)$ is topologized via the Sobolev norm. Therefore, the energy functional has a maximum on $F(X)$, which we denote by $E(F)$. Consider the adjoint map $\tilde{F} : M \times X \to N$. Since M is compact, too, we have an $\varepsilon(F) > 0$ such that every point in the image of $\tilde{F}$ has injectivity radius greater than $\varepsilon(F)$.

Let $1/k < \frac{\varepsilon(F)}{\sqrt{E(F)}}$. Then we have for every $p, q \in M$ of distance smaller than $1/k$ and every $f \in F(X)$ the inequality

$$\begin{aligned} d(f(p), f(q)) &\leq L(f\gamma) \leq \sqrt{E(f)E(\gamma)} \\ &= \sqrt{E(F)}\sqrt{E(\gamma)} = \sqrt{E(F)} \cdot L(\gamma) = \sqrt{E(F)} \cdot d(p, q) \\ &< \varepsilon(F). \end{aligned}$$

Here $\gamma : [0, 1] \to M$ denotes a minimal geodescis connecting p and q and we use that geodesics are parametrized proportional to arc-length. Our claim is now that F can be homotoped to a map with image in P_k. We denote the restriction $\tilde{F}|_{M\times\{x\}}$ by f_x. Then the following lemma will do the job.

Lemma 2.45. *For every simplex Δ^n of $\mathcal{T}^k$, there is a homotopy $H \colon \Delta^n \times X \times I \to N$ such that $H(a, x, 0) = \tilde{F}(a, x)$ for all $(a, x) \in \Delta^n \times X$, $H|_{\Delta^n\times\{x\}\times\{1\}}$ a geodesic simplex and $H(a, x, t) = \tilde{F}(a, x)$ for $a \in \partial\Delta^n, x \in X$ and $t < 1$.*

Proof. Note first that every $f_x(p)$ lies in the injectivity radius of every $f_x(q)$ with $p, q \in \Delta^n$ by Cauchy-Schwarz as above. For simplicity, we will assume that the distance of the barycenter to the faces is 1. To prove our lemma, we use induction:

For $n = 0$, we take the constant homotopy.

For $n > 0$, denote by Δ_s the simplex parallel to Δ^n with faces of distance s from the barycenter. We have $\Delta^n = \Delta_1$. For $s_0 < s_1$ we have $\Delta_{s_1} - \mathring{\Delta}_{s_0} \cong \bigcup \Delta^{n-1} \times [0, 1]$, where the union is over $(n + 1)$ copies glued at $\partial\Delta^{n-1} \times [0, 1]$. Now assume, our homotopy $H^j \colon \Delta^j \times X \times [0, 1] \to N$ is already defined for $j = n - 1$. Then define H^n by

$$\begin{aligned} H^n_t|_{\Delta_{t/3}\times\{x\}} &= \Gamma_x \\ H^n_t|_{(\Delta_{2t/3}-\Delta_{t/3})\times X} &= \tilde{H}^{n-1} \\ H^n_t|_{(\Delta-\Delta_{2t/3})\times X} &= F|_{(\Delta-\Delta_{2t/3})\times X}. \end{aligned}$$

Here Γ_x is the geodesic simplex defined by the images of the vertices of $\Delta_{t/3}$ under f_x and $\tilde{H}^{n-1}$ is defined by glueing the H^{n-1} together (remember they leave the boundary fixed for $t < 1$ and for $t = 1$ everything is geodesic, so they agree on the intersections). □

Now we simply glue these homotopies to a homotopy $H : M \times X \times I \to N$ and then take the adjoint $H' : X \times I \to Map(M, N)$.

3 The Chas-Sullivan Product

In this section, we will define the main objects of this diploma thesis, namely algebraic structures on the homology of mapping spaces between manifolds. The most important one is the Chas-Sullivan product on the homology of the free loop space. This product and most of the other algebraic structures depend on the definition of Gysin morphisms (2.3) in an infinite-dimensional context whose importance is stressed in [Cha]. We will discuss three descriptions of Gysin maps: via the Thom isomorphism as used in [C-V], via Jakob's theory of geometric homology as used in [Cha] and via Kreck's theory of stratifolds due to the author. We will show that these are equivalent in the cases we need.

We want to fix the notation $L^nM = Map(S^n, M)$ for the unpointed and $\Omega^nM = Map^\bullet(S^n, M)$ for the pointed maps. Furthermore, we fix a homology theory h_*.

3.1 Gysin Morphisms

Let $\iota\colon Y \hookrightarrow X$ be the inclusion of a sub Hilbert manifold of finite codimension d with h_*-oriented normal bundle ν_ι.

1. (via Thom isomorphism) By [La], IV.5, Y has a tubular neighbourhood U in X, which is homeomorphic to an open neighbourhood of Y in ν_ι. We have $X/(X-U) \cong Th(\nu_\iota)$. Consider the composition

$$\iota^!\colon h_p(X) \to h_p(Th(\nu_\iota)) \to h_{p-d}(Y)$$

   where the first arrow is induced by the Thom collapse $X \to X/(X-U)$ and the second arrow is defined by capping with the Thom class of the normal bundle. We call $\iota^!_T\colon h_p(X) \to h_{p-d}(Y)$ the Thom-Gysin map of ι. This construction is independent of the choice of the tubular neighbourhood since all tubular neighbourhoods are isotopic ([La], Thm IV.6.2) and isotopic tubular neighbourhoods give homotopic Thom collapse maps.

2. (via stratifolds) Let $[S,f]$ represent a class in $H_p(X)$. By 2.29, we can assume that f is transversal to Y. Define $\iota^!_S([S,f]) := [\widetilde{S}, f|_{\widetilde{S}}] \in H_{p-d}(Y)$, where $\widetilde{S} = f^{-1}(Y)$. To check well-definedness, let $F\colon W \to X$ be a bordism between p-cycles (S,f) and (S',f'), such that f and f' are transversal to Y. Since $\partial W \subset W$ has the property (P) in the sense of 2.4.3, we can make W transversal to Y while leaving the boundary fixed. Thus, we get a bordism $F^{-1}(Y)$ between $\widetilde{S}$ and $\widetilde{S}'$.

3. (via Jakob's Geometric Homology) Let $[P,a,f]$ be a geometric cycle in $h_p(X)$. By 2.29, we can assume that f is transversal to Y. We now define $\iota^!_G([P,a,f]) := [\widetilde{P}, a|_{\widetilde{P}}, f|_{\widetilde{P}}] \in h_{p-d}(Y)$, where $\widetilde{P} := f^{-1}(Y)$. By the same argumentation as above, we see that this class is well-defined.

Theorem 3.1. *If there is a collection $X_0, X_1, \dots \subset X$ of finite-dimensional submanifolds such that every map $g\colon K \to X$ from a compact space K is homotopic to a map with image in one of the X_i, then all three definitions of the Gysin map coincide.*

Proof. We may assume that the X_i are transverse to Y (by a homotopy). Then the essential fact is that the following diagram commutes:

$$\begin{array}{ccc} h_p(X_i) & \longrightarrow & h_p(X) \\ \downarrow \iota_\bullet^! & & \downarrow \iota_\bullet^! \\ h_{p-d}(X_i \cap Y) & \longrightarrow & h_{p-d}(Y) \end{array}$$

Here $\bullet$ stands for either T, S or G. The commutativity follows for $\bullet = T$ by the naturality of the Thom isomorphism. For $\bullet = S$ and $\bullet = G$ we use the fact that it is not important if we first include and then intersect or if we first intersect and then include. Now let (P, a, f) be a cycle in $h_*(X)$. Then it can be assumed that f has image in one of the X_i. Then $\iota_\bullet^!([P, a, f])$ can be computed via the left corner path and this is independent of $\bullet$ by 2.3. $\square$

The conditions of the theorem are, of course, modeled on the situation of mapping spaces (see 2.42).

3.2 The Chas-Sullivan Product

Let M be h_*-oriented of dimension d. Consider the diagram

$$\begin{array}{ccc} L^nM \times_M L^nM & \xrightarrow{\iota} & L^nM \times L^nM \\ \downarrow \mathrm{ev} & & \downarrow \mathrm{ev} \times \mathrm{ev} \\ M & \xrightarrow{\Delta} & M \times M \end{array}$$

Here Δ stands for the diagonal, ι is the inclusion and ev the evaluation at the base point pt. Since ev is a submersion, $L^nM \times_M L^nM$ is a sub Hilbert manifold of $L^nM \times L^nM$ and the normal bundle of $L^nM \times_M L^nM$ in $L^nM \times L^nM$ is the pullback of the normal bundle of M in $M \times M$.

We have a map

$$\gamma\colon L^nM \times_M L^nM = Map(S^n \vee S^n, M) \to Map(S^n, M) = L^nM$$

induced by the collapse map $c\colon S^n \to S^n \vee S^n$. This is defined to be

$$(x_0, \ldots, x_n) \mapsto \begin{cases} (2x_0 - 1, \frac{\sqrt{1-(2x_0-1)^2}}{\sqrt{1-x_0^2}} x_1, \ldots, \frac{\sqrt{1-(2x_0-1)^2}}{\sqrt{1-x_0^2}} x_n)_1 \text{ for } x_0 \geq 0 \\ (-2x_0 - 1, -\frac{\sqrt{1-(2x_0+1)^2}}{\sqrt{1-x_0^2}} x_1, \ldots, -\frac{\sqrt{1-(2x_0+1)^2}}{\sqrt{1-x_0^2}} x_n)_2 \text{ for } x_0 \leq 0 \end{cases}$$

The strange looking sign convention comes from the 1-dimensional case where it corresponds to "traversing the figure 8" and ensures homotopy commutativity (see below). Look now at the following composition:

$$\begin{array}{ccc} h_p(L^nM) \otimes h_q(L^nM) & & h_{p+q-d}(L^nM) \\ \downarrow \times & & \uparrow \gamma_* \\ h_{p+q}(L^nM \times L^nM) & \xrightarrow{\iota^!} & h_{p+q-d}(L^nM \times_M L^nM) \end{array}$$

For notational convenience, define $\mathbf{h}_*(L^nM) = h_{*+d}(L^nM)$. If $h = H$ is ordinary homology, one usually chooses the notation $\mathbb{H}_*(L^nM) = \mathbf{h}_*(L^nM)$. In the same way, we set $\mathbb{H}_*(M) = H_{*+d}(M)$. The above composition defines now a product, called the *Chas-Sullivan product*:

$$\mu\colon \mathbf{h}_p(L^nM) \otimes \mathbf{h}_q(L^nM) \to \mathbf{h}_{p+q}(L^nM)$$

To study the associativity and commutativity of μ, we have first to study the behaviour of γ. We have

$$\gamma \circ (id \times \gamma) = \gamma \circ (\gamma \times id)\colon L^nM \times_M L^nM \times_M L^nM \to L^nM \times_M L^nM \to L^nM,$$

because only the height at which we collapse an "equator" differs, which plays no role up to homotopy. Let now $\tau\colon L^nM \times_M L^nM \to L^nM \times_M L^nM$ be the twist map which permutes both factors. We want to show $\gamma \circ \tau \simeq \gamma$. This holds since τ is induced by the twist map of $S^n \vee S^n$, which is in turn induced from the map

$$(x_0, \ldots, x_n) \mapsto (-x_0, -x_1, x_2, \ldots, x_n),$$

on S^n, which has degree 1 and is therefore homotopic to the identity.

Proposition 3.2. *The product μ is associative and commutative up to sign. For M compact, μ has a unit represented by the inclusion of the constant loops $\kappa\colon M \to LM$.*

Proof. Associativity is clear, since γ is homotopy associative and no orientations change. For commutativity, denote the twist map $L^nM \times L^nM \to L^nM \times L^nM$ also by τ. We have

$$\begin{aligned} \mu(b \otimes a) &= \gamma_* \iota^!(b \times a) \\ &= (-1)^{pq} \gamma_* \iota^! \tau_*(a \times b) \\ &= (-1)^{pq} \cdot (-1)^d \gamma_* \tau_* \iota^!(a \times b) \\ &= (-1)^{pq+d} \gamma_* \iota^!(a \times b) \\ &= (-1)^{pq+d} \mu(a \otimes b) \end{aligned}$$

The factor $(-1)^d$ comes in, because the orientation of the normal bundle of M in $M \times M$ and therefore of $L^nM \times_M L^nM$ in $L^nM \times L^nM$ changes by a factor of $(-1)^d$ if the two factors are interchanged. In addition we use that, since $L^nM \times_M L^nM$ stays invariant under τ, an invariant tubular neighbourhood might be chosen as well.

Now let $[P, a, f] \in h_p(L^nM)$ be a geometric cycle and consider the Chas-Sullivan product with $[M, 1, \kappa] \in h_d(L^nM)$. Their cross product is

$$[P \times M, pr_1^*(a), f \times \kappa] \in h_{p+d}(LM \times LM),$$

which is transverse to $L^nM \times_M L^nM$. So

$$\mu([P, a, f] \otimes [M, 1, \kappa]) = [P, a, f + \kappa]$$

where $f + \kappa$ denotes the map $p \mapsto f(p) * c_{(f(p))(pt)}$, i.e. the concatenation with constant loops. This is clearly homotopic to f. $\square$

Warning 3.3. One has to be careful with signs here since there are different conventions in the literature. This confusion exists already at the level of the intersection product. There are two basic methods to define the intersection product for a compact h_*-oriented d-dimensional manifold M via Poincare duality:

$$\begin{array}{ccc} h_p(M) \otimes h_q(M) & & h_{p+q-d}(M) \\ \downarrow{\scriptstyle D_M \otimes D_M} & & \uparrow{\scriptstyle \cap M} \\ h^{d-p}(M) \otimes h^{d-q}(M) & \xrightarrow{\cup} & h^{2d-p-q}(M) \end{array}$$

and

$$\begin{array}{ccccc} h_p(M) \otimes h_q(M) & & & & h_{p+q-d}(M) \\ \downarrow{\scriptstyle \times} & & & & \uparrow{\scriptstyle \cap M} \\ h_{p+q}(M \times M) & \xrightarrow{D_{M\times M}} & h^{2d-p-q}(M \times M) & \xrightarrow{\Delta^*} & h^{2d-p-q}(M) \end{array}$$

Here D_M denotes the inverse of capping with the fundamental class. If one works out the signs up to which the two products are commutative, one sees that first one defines a graded commutative product on $\mathbf{h}_*(M)$ while the latter corresponds to our sign convention as it is in the implementation of the Gysin morphism $\Delta^!$ (wherefore we use this convention for the intersection product, too). The first type of sign convention is used for example in [C-S] and also in [Cha] (where it is ensured by an "artificial" sign), while our sign convention agrees e. g. with that in [C-V] (Theorem 1.2.1 has the wrong sign as stated there).

The Chas-Sullivan product is in a way composed of the intersection product on the base manifold (corresponding to $\iota^!$) and the Pontrjagin product on the homology of the (based) loop space, i.e. the map on homology induced by $Map^\bullet(S^n, M) \times Map^\bullet(S^n, M) \to Map^\bullet(S^n, M)$ induced by $S^n \to S^n \vee S^n$. This will be made more precise in the sections 3.2.1 and 4.4. What we want to note for the moment is that $\mathrm{ev}_*\colon \mathbf{h}_*(L^nM) \to \mathbf{h}_*(M)$ and $j^!\colon \mathbf{h}_*(L^nM) \to h_*(\Omega^nM)$ (for $j\colon \Omega^nM \to L^nM$ the inclusion) are algebra homomorphisms. For the first, this is clear since $\mathrm{ev}_* \circ \iota^! = \Delta^! \circ \mathrm{ev} \times \mathrm{ev}$ and $\mathrm{ev}_* \circ \gamma_* = \mathrm{ev}_*$. The second is owed to the fact that $j \times j\colon \Omega^nM \times \Omega^nM \to L^nM \times L^nM$ factors over $\iota^!\colon L^nM \times_M L^nM \to L^nM \times L^nM$.

We now want to compare our definition of the Chas-Sullivan product with the way it is defined in [C-V], section 1.2. They work with piecewise smooth instead of H^n-maps and construct the tubular neighbourhood of $L^nM \times_M L^nM$ as the preimage of a tubular neighbourhood of M in $M \times M$ under $\mathrm{ev} \times \mathrm{ev}$. Then they apply the Thom isomorphism and γ_* as above. Since one can construct tubular neighbourhoods the same way in H^n and our definition of the Gysin map is independent of the chosen tubular neighbourhood, it is enough to show that the Thom collapse map commutes with the inclusion of piecewise smooth into H^n, which is clear.

3.2.1 A Finite-Dimensional Description

By giving a variation on the stratifold approach, we will exhibit in this section a description of the Chas-Sullivan product without using any infinite dimensional spaces in the case of ordinary homology.

By adjunction, $H_k(L^nM)$ is isomorphic to $\{f\colon S^n \times S \to M\}/\sim$ with S a stratifold, where $f \sim g$ if there exists a bordism of the form $S^n \times T$ between them. The image of $H_k(L^nM \times_M L^nM) \to H_k(L^nM \times L^nM)$ consists exactly of those singular stratifolds in $\{f\colon S^n \times S \to M \times M\}$ for which $f(\{0\} \times S) \subset \Delta_M$. We have now the following

Proposition 3.4. *Let $[S, \tilde{f}] \in H_k(L^nM)$ be a cycle and $0 \in S^n$ be some point. If the restriction of the adjoint map $f\colon \{0\} \times S \hookrightarrow S^n \times S \to M \times M$ is transverse to the diagonal, $f|_{S^n \times S'}\colon S^n \times S' = S^n \times (f^{-1}(\Delta_M) \cap \{0\} \times S) \to M \times M$ represents the image of $[S, \tilde{f}]$ under the Gysin morphism $H_k(L^nM \times L^nM) \to H_{k-d}(L^nM \times_M L^nM)$.*

Proof. We want to show that, if f is transverse to the diagonal on $\{0\} \times S$, the adjoint map $\tilde{f}\colon S \to L^n(M \times M)$ is transverse to $L^nM \times_M L^nM$. This is enough, because then $\tilde{f}|_{S'}$ represents the image of the Gysin map in $H_k(L^nM \times_M L^nM)$.

Recall the notion of a tangent vector: tangent vectors at $p \in M$ are equivalence classes of (germes of) smooth curves $\gamma\colon \mathbb{R} \to M$ with $\gamma(0) = p$ where $\gamma \sim \gamma'$ if the operations $D_\gamma\colon C^\infty(p) \to \mathbb{R}, f \mapsto \frac{d}{dt} f(\gamma(t))|_{t=0}$ and $D_{\gamma'}$ are the same.

We assume that $f|_{\{0\} \times S}$ is transverse to the diagonal of $M \times M$. We can find for every $p \in \mathrm{im}(\{0\} \times f) \cap \Delta_M$ exactly d curves $\gamma_i\colon \mathbb{R} \to S$ with $(\{0\} \times f) \circ \gamma_i(0) = p$, such that the family of tangent vectors represented by the curves $(\{0\} \times f) \circ \gamma_i$ is linear independent of every tangent vector of Δ_M.

Consider $\tilde{\gamma_i} := \tilde{f} \circ \gamma_i$. Assume, a linear combination of these $\tilde{\gamma_i}$ is equal to a tangent vector of $\tilde{\delta} : \mathbb{R} \to L^nM \times_M L^nM \to L^n(M \times M)$ in a point $p \in L^nM \times_M L^nM$. Choose an open neighbourhood U around $\mathrm{ev}(p) \in M \times M$. Then $\mathrm{ev}^{-1}(U)$ is an open neighbourhood of p in $L^n(M \times M)$. For every function $g \in C^\infty(U)$ we have $g \circ \mathrm{ev} \in C^\infty(\mathrm{ev}^{-1}(U))$. Therefore:

$$\begin{aligned}
&\frac{d}{dt} g((\{0\} \times f)(\mathrm{ev}(\delta(t))))|_{t=0} \\
&\quad = \lambda_1 \frac{d}{dt} g(\mathrm{ev}(\tilde{\gamma_1}(t)))|_{t=0} + \cdots + \lambda_d \frac{d}{dt} g(\mathrm{ev}(\tilde{\gamma_d}(t)))|_{t=0} \\
&\quad = \lambda_1 \frac{d}{dt} g((\{0\} \times f)(\gamma_1(t)))|_{t=0} + \cdots + \lambda_d \frac{d}{dt} g((\{0\} \times f)(\gamma_d(t)))|_{t=0}
\end{aligned}$$

for appropriate $\lambda_i \in \mathbb{R}$. Hence we have $\lambda_1 = \cdots = \lambda_d = 0$ since $\mathrm{ev}_0(\delta(t))$ lies in Δ_M.

□

Therefore, we have the following finite dimensional description of the Chas-Sullivan product of $[S_1, f_1]$ and $[S_2, f_2]$: intersect $\mathrm{ev} \circ f_1$ and $\mathrm{ev} \circ f_2$ in M and compose the loops at the intersection points.

3.3 The Equivariant Product

Using the S^1-structure on LM given by rotation of loops in [C-V] and [C-S], a product $[,]$ (called the *string bracket*) is defined on equivariant homology as the composition

$$\begin{array}{ccc}
h_q^{S^1}(LM) \otimes h_r^{S^1}(LM) & & h_{q+r+2-d}^{S^1}(LM) \\
\big\downarrow{\scriptstyle E \otimes E} & & \big\uparrow{\scriptstyle M} \\
h_{q+1}(LM) \otimes h_{r+1}(LM) & \xrightarrow{\ \mu\ } & h_{q+r+2-d}(LM)
\end{array}$$

Here μ denotes the Chas-Sullivan product. For the definition of E and M see 2.2 and 2.1.3. We want to show that this definition coincides with the definition in [C-V]. For simplicity we restrict in the rest of this section to ordinary homology and the stratifold description, but for generalized homology and Jakob's geometric homology the proofs are the same.

We have to show the commutativity of the following diagram:

$$\begin{array}{ccc}
H_p^{S^1}(LM) \otimes H_q^{S^1}(LM) & \xrightarrow{\Phi\otimes\Phi} & SH_{p+1}^{S^1}(LM) \otimes SH_{q+1}^{S^1}(LM) \\
\downarrow{\scriptstyle \tau\otimes\tau} & & \downarrow{\scriptstyle E\otimes E} \\
H_{p+1}(LM) \otimes H_{q+1}(LM) & \longrightarrow & SH_{p+1}(LM) \otimes SH_{q+1}(LM) \\
\downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} \\
H_{p+q+2-d}(LM) & \longrightarrow & SH_{p+q+2-d}(LM) \\
\downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle M} \\
H_{p+q+2-d}^{S^1}(LM) & \xrightarrow{\Phi} & SH_{p+q+3-d}^{S^1}(LM)
\end{array}$$

Here τ is the composition

$$\begin{aligned}
H_q^{S^1}(LM) \cong H_q(ES^1 \times_{S^1} LM) &\overset{\text{Thomiso}}{\longrightarrow} H_{q+2}(D, ES^1 \times LM) \\
&\overset{\partial}{\to} H_{q+1}(ES^1 \times LM) \cong H_{q+1}(LM)
\end{aligned}$$

where D denotes the D^2-bundle corresponding to $ES^1 \times LM \to ES^1 \times_{S^1} LM$. We denote by π_* the map induced by the bundle projection.

The commutativity of the middle square is already shown in 3.1. Therefore, we will identify (non-equivariant) stratifold homology with singular homology.

Recall that the isomorphism Φ is given as follows: We can represent an element in $H_p^{S^1}(LM)$ by a singular stratifold $[S, f] \in SH_p(LM \times_{S^1} ES^1)$. If one pulls back the bundle $LM \times ES^1 \to LM \times_{S^1} ES^1$ to S, one gets a S^1-bundle S' over S, in particular a free S^1-stratifold. The composition $S' \to LM \times ES^1 \overset{\mathrm{pr}_2}{\to} LM$ is then equal to $\Phi([S, f])$.

We want to show $\tau = E \circ \Phi$. Look at the stratifold $f^*(D)$ (see above). This defines a class in $h_{p+2}(D, D - LM \times_{S^1} ES^1) \cong h_{p+2}(Th(D))$. This class is mapped to $[S, f]$ under the Thom isomorphism, because its intersection with $LM \times_{S^1} ES^1$ equals S (see 3.1). If one applies the boundary operator, one gets $E \circ \Phi$.

Now we want to show that $\Phi \circ \pi_* = M$. Look at the diagram

$$\begin{array}{ccc}
(\pi \circ f)^*(ES^1 \times LM) & \xrightarrow{F} & ES^1 \times LM \\
\downarrow & \overset{f}{\nearrow} & \downarrow{\scriptstyle \pi} \\
S & \longrightarrow & ES^1 \times_{S^1} LM
\end{array}$$

By the universal property of the pullback we have a section from S to $(\pi \circ f)^*(ES^1 \times LM)$. Therefore, the bundle is trivial, since it's principal. Since F is equivariant (it is a morphism of S^1 principal bundles), it equals $M([S, f])$.

In this situation, there is a finite-dimensional description via stratifolds as well: By adjunction, we can represent a class in $SH_k^{S^1}(LM)$ by a map $f\colon S^1 \times S \to M$ where S is a free S^1-stratifold and f satisfies $f(t^{-1}\cdot s, t\cdot x) = f(s,x)$. We interpret E again as forgetting the S^1-action on S and the homomorphism M as substituting S by the free S^1-stratifold over S. Therefore, we get at the end out of two (transverse) classes represented by $f\colon S^1\times S\to M$ and $f'\colon S^1\times S'\to M$ the string bracket represented by $g\colon S^1\times \text{Intersection}(S,S')\times S^1\to M$ with $g(s,x,t) = f(st,x)$.

3.4 Further Algebraic Structure

The reader might have wondered why we have considered only maps out of spheres so far, while we characterized string topology as the study of algebraic structures on the homology of mapping spaces of manifolds in general. A closer look at the definition of the Chas-Sullivan product reveals that the only things we have used of S^n are: it is a manifold and an H-cogroup via the map $S^n\to S^n\vee S^n$. Since every n-dimensional manifold N has the structure of an H-comodule over S^n via the map $c: N\to N\vee S^n$, collapsing the boundary of a little disc, we get (for M h_*-oriented) the following module structure:

$$\begin{array}{ccc}
h_p(Map(N,M))\otimes h_q(L^nM) & & h_{p+q-d}(Map(N,M)) \\
\big\downarrow \times & & \big\uparrow \gamma_* \\
h_{p+q}(Map(N,M)\times L^nM) & \xrightarrow{\iota^!} & h_{p+q-d}(Map(N,M)\times_M L^nM)
\end{array}$$

Here $\iota\colon Map(N,M)\times_M L^nM\to Map(N,M)\times L^nM$ is the inclusion and $\gamma\colon Map(N,M)\times_M L^nM\to Map(N,M)$ the map induced by c. This defines a module structure over $\mathbf{h}_*(L^nS)$ on $h_*(Map(N,M))$ (this was first considered in [K-S]). This structure is independent of the chosen disc since all embeddings of a disc are isotopic. Note that we could substitute $h_*(Map(N,M))$ by $h'_*(Map(N,M))$ where h'_* is a module homology theory over h_*. For example, K-theory is a module over complex cobordism MU and every homology theory is a module over stable homotopy.

Besides the module structure, there is the structure of a coalgebra on $\mathbf{h}_*(LM)$ if $h_*(LM\times LM)\cong h_*(LM)\otimes h_*(LM)$ (e.g. for ordinary homology with field coefficients). To that aim, let $i\colon LM\times_M LM\to LM$ be the inclusion of all loops $\alpha\colon[0,1]\to M$ with $\alpha(0)=\alpha(\frac{1}{2})=\alpha(1)$ and $\iota\colon LM\times_M LM\to LM\times LM$ the usual inclusion. So we get a map

$$h_n(LM)\xrightarrow{i^!} h_{n-d}(LM\times_M LM)\xrightarrow{\iota_*} h_{n-d}(LM\times LM)\cong (h_*(LM)\otimes h_*(LM))_{n-d}$$

Note that one cannot mimic this definition for the higher L^nM, because the inclusion $i\colon L^nM\times_M L^nM\to L^nM$ has infinite codimension for $n>1$. Therefore, there are also no interesting comodule structures, since there are not many interesting 1-dimensional manifolds.

Remark 3.5. As spelled out in [C-V], the coproduct together with the Chas-Sullivan product is part of something like a 1-dimensional topological field theory. The homology $\mathbf{h}_*(LM)$ is the module associated to S^1. The product corresponds to the pair of pants, the coproduct to the opposite pair of pants, the unit is given by the inclusion of M as constant loops (for M compact), but the counit is missing.

The last algebraic structure we want to mention is the Batalin-Vilkovisky structure on $h_*(LM)$. Let $[S^1]\in h_1(S^1)$ be the fundamental class. Note that S^1 is orientable with respect to every homology theory since its tangent bundle is trivial. We get an operator $\Delta\colon h_p(LM)\to h_{p+1}(LM\times S^1)\xrightarrow{r_*} h_{p+1}(LM)$. Here r sends an (α,t) to $s\mapsto\alpha(s+t)$, i.e.

it is the rotation by t. (The same is possible replacing S^1 by S^3, but the other spheres are sadly enough no Lie groups) Together with the product structure this satisfies some axioms, making $\mathbf{h}_*(LM)$ into a so called Batalin-Vilkovisky algebra (see [C-V] for a definition). While the definition of Δ is simple, it is sometimes surprisingly hard to compute in practice.

3.5 Example 1: The Spheres

In this section, we will present concrete generators for the homology of LS^n. To achieve this, we consider first the simpler case of ΩS^n. The homology is equipped with the Pontrjagin product induced by composing loops. It is well known that $H_*(\Omega S^n) \cong \mathbb{Z}[x]$ with $x \in H_{n-1}(\Omega S^n)$ for $n > 0$ (see e.g. [Hat], 3C.8 and 4J.1). By adjunction from the identity, we get a map $f\colon S^{n-1} \to \Omega\Sigma S^{n-1} \cong \Omega S^n$. This represents a class in $H_{n-1}(\Omega S^n)$.

Proposition 3.6. *The class $f_*[S^{n-1}]$ is an additive generator of $H_{n-1}(\Omega S^n)$*

Proof. Consider the composition

$$\Sigma S^{n-1} \xrightarrow{\Sigma f} \Sigma\Omega\Sigma S^{n-1} \xrightarrow{g} \Sigma S^{n-1},$$

where g is given by adjunction. It is easy to check that $g \circ \Sigma f = id$. Taking homology H_n we get a diagram $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}$, where the composition is the identity. Therefore, the first map $(\Sigma f)_*$ has to be surjective and maps 1 to a generator. Hence, f_* maps $[S^{n-1}]$ to a generator of $H_{n-1}(\Omega S^n)$. □

To visualize x, think of the base point as the north pole. The points $p \in S^{n-1}$ of the equator parametrize the minimal geodesics γ_p between north and south pole. Now choose a distinguished minimal geodesic δ from the south to the north pole (the "way backwards"). Then $p \mapsto \delta * \gamma_p$ defines $f\colon S^{n-1} \to \Omega S^n$ (note that the suspensions above are reduced).

Now consider the free loop space. The case $n = 1$ is easy. The loop space ΩS^1 consists of contractible components which are indexed by $\mathbb{Z}$ (since $\pi_i(\Omega S^1) = 0$ for all $i > 0$ and it has the homotopy type of a CW-complex). As S^1 is a Lie group, we have $LS^1 \cong S^1 \times \Omega S^1$. The homology of S^1 is free. Therefore, we have by Künneth $\mathbb{H}_*(LS^1) \cong \Lambda(a) \otimes \mathbb{Z}[t, t^{-1}]$ additively, where $|a| = -1$ and $|t| = 0$. So let $f_n\colon S^1 \to S^1$ be some pointed map of degree n. Then the map $S^1 \cdot f_n\colon S^1 \times S^1 \to S^1$ given by rotation represents t^n. For a we choose as a representative an arbitrary constant loop. The multiplicative structure follows now immediately.

Cohen, Jones and Yan show in [CJY] that

$$\begin{aligned} \mathbb{H}_*(LS^n) &= \Lambda(a) \otimes \mathbb{Z}[u] \text{ for } n \text{ odd}, \\ \mathbb{H}_*(LS^n) &= \Lambda(b) \otimes \mathbb{Z}[a, v]/(a^2, ab, 2av) \text{ for } n \text{ even} \end{aligned}$$

with generators $a \in \mathbb{H}_{-n}(LS^n)$, $b \in \mathbb{H}_{-1}(LS^n)$, $u \in \mathbb{H}_{n-1}(LS^n)$ and $v \in \mathbb{H}_{2n-2}(LS^n)$. For this computation they use the multiplicative spectral sequence exhibited in 4.4 in the special case of singular homology. We now want to make the structure of the homology more transparent by finding explicit manifold generators.

Since a is in $H_0(LM)$ it can be represented by an arbitrary loop, e.g. a constant loop. Multiplication with a corresponds to $j_* j^!\colon \mathbb{H}_*(LM) \to H_*(\Omega M) \to \mathbb{H}_{*-n}(LM)$. By studying the Serre spectral sequence for $\Omega M \to LM \to M$ it can be shown that $j_*\colon H_{n-1}(\Omega M) \to$

$\mathbb{H}_{-1}(LM)$ is an isomorphism (see [CJY]). Therefore, $j_*(x)$ is a generator of $\mathbb{H}_{-1}(LM)$ and hence up to sign equal to b for n even and to au for n odd.

Consider now first the case n *odd*. We want to find a preimage of x under $j^!$. This is a fortiori a generator of $\mathbb{H}_{n-1}(LS^n)$ and therefore up to sign equal to u. Let S^n be equipped with the standard metric of the sphere of circumference 1 and STS^n be the unit sphere bundle in the tangent bundle TS^n. Let V be a vector field of unit length (which you can choose since n is odd). We define a map $F\colon STS^n \to LS^n$ by

$$(p,v) \mapsto \left(t \mapsto \begin{cases} \exp_p(tv) \text{ for } t \leq \frac{1}{2} \\ \exp_{-p}((t-\frac{1}{2})V(-p)) \text{ for } t \geq \frac{1}{2} \end{cases} \right)$$

Here p denotes a point in S^n and v is a unit tangent vector to p. By the description of x above it is clear that $j^!(F_*[STS^n]) = x$.

This construction cannot work for n even, since in this case we have $\mathbb{H}_{n-1}(LS^n) = 0$ for $n > 2$ and the generator $bv \in \mathbb{H}_{n-1}(LS^2)$ maps to zero under $j^!$, because a representative (S,f) can be chosen with $\operatorname{im}(\mathrm{ev} \circ f) = pt$. This gives an eccentric proof for the theorem of the hairy ball which says that there is no nowhere vanishing vector field on S^n for n even, because this was the only thing we used for n odd.

To construct an explicit representative of v we need an alternative description of a representative of x^2. By our description above we get as a representative:

$$(v_1,v_2) \mapsto \left(s \mapsto \begin{cases} \exp_p(2sv_1) \text{ for } s \leq \frac{1}{4} \\ \exp_{-p}(2(s-\frac{1}{4})w) \text{ for } \frac{1}{4} \leq s \leq \frac{1}{2} \\ \exp_p(2(s-\frac{1}{2})v_2) \text{ for } \frac{1}{2} \leq s \leq \frac{3}{4} \\ \exp_{-p}(2(s-\frac{3}{4})(-w)) \text{ for } \frac{3}{4} \leq s \leq 1 \end{cases} \right)$$

Here v_1 and v_2 are unit tangent vectors at the base point p and w is a unit tangent vector at $-p$. This map is now homotopic to

$$(v_1,v_2) \mapsto \left(s \mapsto \begin{cases} \exp_p(sv_1) \text{ for } s \leq \frac{1}{2} \\ \exp_{-p}((s-\frac{1}{2})(-v_2)) \text{ for } \frac{1}{2} \leq s \leq 1 \end{cases} \right)$$

via

$$(v_1,v_2,t) \mapsto \left(s \mapsto \begin{cases} \exp_p(\frac{2}{1+t}sv_1) \text{ for } s \leq \frac{1+t}{4} \\ \exp_{-p}(2(s-\frac{1+t}{4})tw) \text{ for } \frac{1+t}{4} \leq s \leq \frac{1}{2} \\ \exp_{q(t)}(\frac{2}{1+t}(s-\frac{1}{2})(\overline{\delta})_{1-t}(v_2)) \text{ for } \frac{1}{2} \leq s \leq \frac{3+t}{4} \\ \exp_{-p}(2(s-\frac{3}{4})(-w)) \text{ for } \frac{3+t}{4} \leq s \leq 1 \end{cases} \right)$$

Here $\overline{\delta}_{1-t}$ denotes the parallel transport along the opposite path of δ from $\delta(1)$ to $\delta(t)$, where $\delta(s) = \exp_{-p}(sw)$. The 0-end of this homotopy is not exactly the second map, but has some constant parts at p and $-p$, which can be easily homotoped out. By this description it is now easy to construct a preimage of x^2 under $j^!$: consider the bundle E over S^n with fibre

$ST_pS^n \times ST_pS^n$ over every $p \in S^n$ (to be more precise: the pullback of the product bundle of STS^n with itself via the diagonal). Then define $F\colon E \to LS^n$ by:

$$(p, v_1, v_2) \mapsto \left(s \mapsto \begin{cases} \exp_p(sv_1) \text{ for } s \leq \frac{1}{2} \\ \exp_{-p}((s-\frac{1}{2})(-v_2)) \text{ for } \frac{1}{2} \leq s \leq 1 \end{cases} \right)$$

Since x^2 is an additive generator of $H_{2n-2}(\Omega S^n)$ and $\mathbb{Z}\{v\}$ is the non-torsion part of $\mathbb{H}_{2n-2}$, the class $[E, F]$ is equal to v up to sign.

By the Chas-Sullivan product we get now explicit generators for every class in $H_*(LM)$. In particular every class in $H_*(LM)$ is represented by a manifold.

3.5.1 The Batalin-Vilkovisky structure

We recall that for every space X we can define an operator $\Delta\colon H_k(LX) \to H_{k+1}(LX)$ given by the composition

$$H_k(LX) \cong H_k(LX) \otimes H_1(S^1) \xrightarrow{\times} H_{k+1}(LX \times S^1) \longrightarrow H_{k+1}(LX)$$

where the last map is given by the rotation S^1-action on LX.

For $X = S^d$ with d odd holds $\Delta(u^k) = 0$ and $\Delta(au^k) = \lambda_k u^{k-1}$ for a $\lambda_k \in \mathbb{Z}$ for dimension reasons. If one intersects $\Delta(au^k)$ with the (pointed) loop space based at a "generic" point, i. e. at a point not lying on the "way backward" of u one counts exactly k disjoint families of loops all inducing the generator (one needs to take care about orientation), i.e. $\Delta(au^k) \cap \Omega S^d = k \cdot x^{k-1}$. Therefore $\lambda_k = k$.

For the same reasons $\Delta(bv^k) \cap \Omega S^d = (2k+1) \cdot x^{2k}$ for d even and all other Batalin-Vilkoviskies are zero intersected with ΩS^d for dimension or torsion reasons. This determines $\Delta(bv^k)$ for $d > 2$ as $(2k+1)v^k$. For $d = 2$ one can show (surprisingly) that $\Delta(bv^k) = (2k+1)v^k + av^{k-1}$ by comparing to the Batalin-Vilkovisky operator on $H_*(\Omega^2 S^3; \mathbb{F}_2)$ given by the rotation S^1-action on $\Omega^2 S^3$) induced by that on S^2 ([M-G]). Their study of Δ on $H_1(\Omega^2 S^3; \mathbb{F}_2)$ uses detailed study of further algebraic structure on $H_*(\Omega^2 S^3; \mathbb{F}_2)$; it would be nice to have a more geometric proof.

3.6 Example 2: Projective Spaces

In [CJY] the authors show that

$$\mathbb{H}_*(L\mathbb{CP}^n) \cong \Lambda(w) \otimes \mathbb{Z}[c, u]/(c^{n+1}, wc^n, (n+1)c^n u),$$

where $|w| = -1$, $|c| = -2$ and $|u| = 2n$. By the same methods one can show

$$\mathbb{H}_*(L\mathbb{HP}^n) \cong \Lambda(w) \otimes \mathbb{Z}[c, u]/(c^{n+1}, wc^n, (n+1)c^n u)$$

with $|w| = -1$, $|c| = -4$ and $|u| = 4n+2$. In [Wes] it is shown that for n even

$$\mathbb{H}_*(L\mathbb{RP}^n; \mathbb{Z}/2) \cong \Lambda(w) \otimes \mathbb{Z}[c, u]/(c^{n+1}, wc^n, c^n u)$$

with $|w| = -1$, $|c| = -1$ and $|u| = n-1$. We will find explicit generators for these homology classes. Furthermore, the Hopf maps from the spheres to the projective spaces induce maps on the free loop space and we will compute their effect in homology (this might be called "loop Hurewicz"). In the following, $\mathbb{K}$ will stand for one of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and d will be the $\mathbb{R}$-dimension of $\mathbb{K}$. If $\mathbb{K} = \mathbb{R}$, we will assume $n > 1$ and all homology with $\mathbb{Z}/2$-coefficients.

3.6.1 The (Pointed) Loop Space

There is a homotopy fibration $S^{d(n+1)-1} \to \mathbb{KP}^n \to \mathbb{KP}^\infty$ (see [CJY]). If we loop this, we get a homotopy section $\Omega(\mathbb{KP}^\infty) \to \Omega(\mathbb{KP}^n)$, since the map $\Omega\mathbb{KP}^\infty \simeq S^{d-1} \to S^{d(n+1)-1}$ is nullhomotopic. Therefore we get (additively):

$$H_*(\Omega\mathbb{KP}^n) \cong H_*(\Omega S^{d(n+1)-1}) \otimes H_*(S^{d-1}) \cong \mathbb{Z}[y] \otimes \Lambda(z),$$

where $|y| = d(n+1)-2$ and $|z| = d-1$. Here we use $\Omega\mathbb{KP}^\infty \simeq S^{d-1}$. The above isomorphism holds also multiplicatively, because the Serre spectral sequence is here multiplicative, since the Pontrjagin product is induced by a map and the Serre spectral sequence is natural (note that there are no filtration issues for dimension reasons). Thus, in particular $H_*(\Omega\mathbb{RP}^n) \cong \mathbb{Z}[f] \oplus \mathbb{Z}[x * f]$, where f is the generator of $H_*(\Omega S^n)$ described in 3.5, x is a homotopy non-trivial loop in $\mathbb{RP}^n$ and $*$ is the Pontrjagin product.

3.6.2 The Free Loop Space

In this whole section let the projective spaces be equipped with the metric coming from the standard metric on the unit sphere. The generator c is represented by $c\colon \mathbb{KP}^{n-1} \hookrightarrow \mathbb{KP}^n \hookrightarrow L\mathbb{KP}^n$. This holds because $\mathrm{ev} \circ c\colon \mathbb{KP}^{n-1} \to \mathbb{KP}^n$ represents a generator of $\mathbb{H}_{-d}(\mathbb{KP}^n)$.

Let $d > 1$ and $STS^{d(n+1)-1}$ be the sphere bundle of the tangent bundle of $S^{d(n+1)-1}$. Let E be the quotient of the sphere bundle via the (isometric) action of S^{d-1} on $S^{d(n+1)-1}$. Clearly the composite $STS^{d(n+1)-1} \to LS^{d(n+1)-1} \to L\mathbb{KP}^n$ of the generator u of $H_*(LS^{d(n+1)-1})$ and the looped Hopf map factors over E. This map $E \to L\mathbb{KP}^n$ is our generator u. To see this, we simply intersect u with $\Omega\mathbb{KP}^n$ and observe that this is the image of the generator of $\Omega S^{d(n+1)-1}$. Here we use the isomorphism described in the last subsection.

For $\mathbb{K} = \mathbb{R}$ we consider the map $u\colon ST\mathbb{RP}^n \to L\mathbb{RP}^n$ from the sphere tangent bundle given by $v \mapsto (t \mapsto exp(\pi t \cdot v))$. If we intersect with $\Omega\mathbb{RP}^n$ we get $x * f$. Mapping onto a generator u must be a generator, too. By the description of c and w one sees that intersecting a product of an element of $\mathbb{H}_*(L\mathbb{RP}^n)$ with c or w one certainly gets zero; so our u must be the generator u in the description at the beginning of this section.

For the last generator look at $T\mathbb{KP}^n|_{\mathbb{KP}^{n-1}}$. This has certainly a nonvanishing section s by obstruction theory, because the (dn)-th cohomology of $\mathbb{KP}^{n-1}$ vanishes. One can also construct such a section directly as follows: take the map $A\colon \mathbb{K}^{n-1} \to \mathbb{K}^n$ that sends each e_i to e_{i+1}. Take the geodesic on the sphere joining $x \in S^{dn-1-d}$ in the "equator" and Ax in S^{dn-1} and map it to $\mathbb{KP}^n$. Clearly $x \neq Ax$ even here and so this geodesic defines certainly a non-zero tangent vector. In either construction we can assume that $s(x)$ has unit length for every $x \in \mathbb{KP}^{n-1}$. Let now L be the (trivial) S^{d-1}-bundle in $T\mathbb{KP}^n|_{\mathbb{KP}^{n-1}}$ generated by s. Then we define our map $w'\colon L \to L\mathbb{KP}^n$ via

$$(p, l) \mapsto \begin{cases} t \mapsto \exp_p(\pi t \cdot l) \text{ for } t \leq \frac{1}{2} \\ t \mapsto \exp_p(\pi(1-t) \cdot s(p)) \text{ for } t \geq \frac{1}{2} \end{cases}$$

Now let $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$. If we multiply with c^{n-1}, i.e. intersect with a $\mathbb{KP}^1$ connecting a $p \in \mathbb{KP}^{n-1}$ with $\exp_p(\frac{\pi}{2} \cdot s(p))$, we get obviously the image of the generator $[f] \in H_{d-1}(\Omega S^d) \cong H_{d-1}(\Omega\mathbb{KP}^1)$ under the map $\Omega\mathbb{KP}^1 \hookrightarrow \Omega\mathbb{KP}^n \hookrightarrow L\mathbb{KP}^n$. The first is an isomorphism on H_{d-1} because of the description of the homology of $\Omega\mathbb{KP}^n$ in (4.1). The latter is also an isomorphism on H_1 because of the Serre Spectral Sequence (see [CJY]). So w' is an additive (non-torsion) generator since wc^{n-1} is. Since $\mathbb{H}_{-1}(L\mathbb{KP}^n) \cong \mathbb{Z}$, this settles w' (modulo sign) as the generator w described in [CJY]. In the case of $\mathbb{K} = \mathbb{R}$, L has two components, so w' is a sum $w_1 + w_2$. If $l = s(p)$, it is easy to see that the path $w'(p,l)$ is nullhomotopic and therefore we have $w_2 = c$. If $l = -s(p)$, we see that the path $w'(p,l)$ is not nullhomotopic. Therefore, we have that $c^{n-1}w_1$ is a point in the free loop space in the other component than c^n. Hence $w_1 = w$.

3.6.3 The Hopf Maps

We want to determine the morphisms $(L\eta)_*\colon H_*(LS^{d(n+1)-1}) \to H_*(L\mathbb{KP}^n)$ induced by the Hopf maps $\eta\colon S^{d(n+1)-1} \to \mathbb{KP}^n$ for $\mathbb{K}$ the complex numbers or the quaternions. First consider the pointed case: by 3.6.1 the map $(\Omega\eta)_*\colon H_*(\Omega S^{d(n+1)-1}) \to H_*(\Omega\mathbb{KP}^n)$ sends the generator x to the generator y. Because of the commutative diagram

$$\begin{array}{ccc} H_*(\Omega S^{d(n+1)-1}) & \xrightarrow{(\Omega\eta)_*} & H_*(\Omega\mathbb{KP}^n) \\ \downarrow & & \downarrow \\ H_*(LS^{d(n+1)-1}) & \xrightarrow{(L\eta)_*} & H_*(L\mathbb{KP}^n) \end{array}$$

we have $(L\eta)_*(au^k) = c^n u^k$. Since $\Delta(au^k) = ku^{k-1}$ and Δ is natural we have $k(L\eta)_*(u^{k-1}) = \Delta(c^n u^k) = 0$ since $c^n u^k$ is torsion and $H_{k(d(n+1)-2)+1}(L\mathbb{KP}^n)$ is torsionfree (note the only torsion is in degrees $k(d(n+1)-2)$). Therefore, $(L\eta)_*(u^{k-1}) = 0$ and the "loop Hurewicz" of the Hopf maps is determined.

4 Spectral Sequences

4.1 Exact Couples and Spectral Sequences

An *exact couple* consists of two objects D and E of an abelian category with morphisms

$$\begin{array}{ccc} D & \xrightarrow{\;i\;} & D \\ {\scriptstyle k}\nwarrow & & \swarrow{\scriptstyle j} \\ & E & \end{array}$$

such that $D \xrightarrow{i} D \xrightarrow{j} E \xrightarrow{k} D \xrightarrow{i} D$ is exact. Set $d = jk$, which fulfills $d^2 = 0$. We can define a *derived exact couple* with $E' = \ker d/\operatorname{im} d$ and $D' = i(D)$ on objects and $i' = i|_{D'}$, $j'(ia) = [j(a)]$ and $k'([e]) = k(e)$ on morphisms. These can all be checked to be well-defined and to define an exact couple again ([Hat2], lemma 1.1). By this procedure we can construct sequences $D^1, D^2, \ldots$ of objects and $E^1, E^2, \ldots$ of differential objects in our abelian category such that $H(E^i, d^i) = E^{i+1}$. Such a sequence is called a *spectral sequence.* We denote the morphisms in the n-th derived exact couple by i^n, j^n and k^n.

Usually one considers the abelian category of double graded abelian groups. In this thesis, we use the grading conventions that i is of bidegree $(1, -1)$, j of bidegree $(0, 0)$ and d of bidegree $(-1, 1)$ in the first exact couple and set $D'_{pq} = i(D_{pq}) \subset D_{p+1,q-1}$. The bidegrees of i' and k' are the same as the bidegrees of i and k, but the bidegree of j^n is $(-n-1, n)$. In general, the bidegree of d^n is $(-n-1, n)$. If there is for every bidegree (p, q) an n such that $D^n_{pq} = D^{n+1}_{pq} = \cdots$, we call both the exact couple and the induced spectral sequence *convergent* to

$$D^\infty_{p+q} = \operatorname{colim}(D^1_{pq} \xrightarrow{i} D^2_{pq} \xrightarrow{i} \cdots).$$

We get a filtration $F^p_{p+q} = F^p D^\infty_{p+q} = \operatorname{im}(D_{pq} \to D^\infty_{p+q})$. Furthermore, for every (p, q), there is an n with $E^n_{pq} = E^{n+1}_{pq} = \cdots$ and we set $E^\infty_{pq} = E^n_{pq}$. If $D^n_{p-1,q+1} = D^{n+1}_{p-1,q+1} = \cdots$ and $D^n_{pq} = D^{n+1}_{pq} = \cdots$, we have that $D^n_{p-1,q+1} = F^{p-1}D^\infty_{p+q}$ and $D^n_{pq} = F^p D^\infty_{p+q}$ and i^n is an injection between them. So we get short exact sequences

$$0 \to F^{p-1}_{p+q} \to F^p_{p+q} \to E^\infty_{pq} \to 0.$$

This is the way, one tries to read the groups D^∞ from the spectral sequence.

A morphism between an exact couple $C = (D, E, i, j, k)$ and a second exact couple $\tilde{C} = (\tilde{D}, \tilde{E}, \tilde{i}, \tilde{j}, \tilde{k})$ consists of morphisms $D \to \tilde{D}$ and $E \to \tilde{E}$ which commute with the i, j and k. It induces a morphism of spectral sequences of (level 1 and) suitable bidegree (a, b) in the sense that we have homomorphisms $f^n \colon E^n_{pq} \to \tilde{E}^n_{p+a,q+b}$ for all $n \geq 1$ which commute with the differentials and satisfy $H(f^n) = f^{n+1}$. If the exact couples are convergent, we get a morphism of convergent spectral sequences in the sense that we have in addition homomorphisms $D^\infty_r \to \tilde{D}^\infty_{r+a+b}$ which map F^p to F^{p+a} and induce f^∞ on E^∞.

4.2 The Serre Spectral Sequence

Now let $E \xrightarrow{\pi} B$ be a fibre bundle with B a path-connected CW-complex (the non-CW-case can be handled by CW-approximation). We define $E^{(p)}$ to be the preimage of the p-skeleton of B under π. If we choose a homology theory h_*, we get an exact couple C:

$$\begin{array}{ccc} \bigoplus_{p,q} h_{p+q}(E^{(p-1)}) & \xrightarrow{\quad i \quad} & \bigoplus_{p,q} h_{p+q}(E^{(p)}) \\ & {\scriptstyle k}\nwarrow \quad \swarrow{\scriptstyle j} & \\ & \bigoplus_{p,q} h_{p+q}(E^{(p)}, E^{(p-1)}). & \end{array}$$

Here i and j are defined by the inclusion of filtrations $E^{(p-1)} \to E^{(p)}$ and $E^{(p)} \to (E^{(p)}, E^{(p-1)})$ respectively and k is the boundary map. The spectral sequence induced by this exact couple is called the *Serre spectral sequence.*

Theorem 4.1. *The Serre spectral sequence converges to $h_*(E)$ on which the filtration is given by $F^p_* = \operatorname{im}(h_*(E^{(p)} \to E))$. Furthermore, we have $E^2_{pq} \cong H_p(B; \mathfrak{h}_q(F))$. Here $\mathfrak{h}_q(F)$ denotes the local system defined by the homology groups of the fibres.*

To see this, choose closed balls D^p_α in the interiors of the p-cells of B with midpoints x_α and boundaries S^{p-1}_α. Then we have

$$\begin{aligned} h_{p+q}(E^{(p)}, E^{(p-1)}) &\cong h_{p+q}\left(\coprod_{p-\text{cells } \alpha \text{ of } B} D^p_\alpha \times \pi^{-1}(x_\alpha), \coprod_{p-\text{cells } \alpha \text{ of } B} S^{p-1}_\alpha \times \pi^{-1}(x_\alpha) \right) \\ &\cong \bigoplus_{p-\text{cells } \alpha \text{ of } B} [(h_*(D^p_\alpha, S^{p-1}_\alpha) \otimes_{h_*(pt)} h_*(\pi^{-1}(x_\alpha))]_{p+q} \\ &\cong \bigoplus_{p-\text{cells } \alpha \text{ of } B} h_q(\pi^{-1}(x_\alpha)), \end{aligned}$$

where the first isomorphism is by excision after thickening up $E^{(p-1)}$. Thus, we see that we have as E^1-term the cellular chain complex for local coefficients $\mathfrak{h}_q(F)$. For later applications we want to make these isomorphisms more explicit. So let $[P, a, f] \in h_{p+q}(E^{(p)}, E^{(p-1)})$ be a geometric cycle. Then the image in $h_{p+q}(D^p_\alpha \times \pi^{-1}(x_\alpha), S^{p-1}_\alpha \times \pi^{-1}(x_\alpha))$ under the excision map is the intersection $[P_\alpha, a|_{P_\alpha}, f|_{P_\alpha}]$ of $[P, a, f]$ with $(D^p_\alpha \times \pi^{-1}(x_\alpha), S^{p-1}_\alpha \times \pi^{-1}(x_\alpha))$. We can choose an equivalent cycle $[(D^p, S^{p-1}) \times X, b, g]$. Since the class of the intersection with $\pi^{-1}(x_\alpha)$ depends only on the class of the cycle, we have

$$[\{x_\alpha\} \times X, b|_{\{x_\alpha\} \times X}, g|_{\{x_\alpha\} \times X}] = [P_\alpha \cap \pi^{-1}(x_\alpha), a|_{P_\alpha \cap \pi^{-1}(x_\alpha)}, f_{P_\alpha \cap \pi_{-1}(x_\alpha)}].$$

Therefore

$$[P \cap \pi^{-1}(x_\alpha), a|_{P \cap \pi_{-1}(x_\alpha)}, f|_{P \cap \pi_{-1}(x_\alpha)}]$$

is the image of $[P, a, f]$ in $h_q(\pi^{-1}(x_\alpha))$.

Theorem 4.2 ([Hat2], section 1.1, Supplements)**.** *Beginning with the E^2-term, the Serre spectral sequence is independent of the chosen CW-structure of B.*

We denote the Serre spectral sequence of a fibre bundle ξ by $E(\xi, h) = E(\xi)$ (usually we surppress the h). In the special case of $\xi = (pt \to B \to B)$, the Serre spectral sequence is also called *Atiyah-Hirzebruch spectral sequence* (or short: AHSS). This spectral sequence is very useful to compute h_* of a space when its ordinary homology is already known.

Theorem 4.3 ([Hat2], section 1.1, Supplements). *Let $\phi\colon E' \to E$ be a bundle map of fibre bundles $\xi' = (F' \to E' \to B')$ and $\xi = (F \to E \to B)$. Then there is a morphism of convergent spectral sequences $E(\xi') \to E(\xi)$ of level 2 which induces ϕ_* on $h_*(E)$ and $H_*(B; \mathfrak{h}_*(F))$. This morphism is invariant under fibre homotopies.*

Furthermore, we have obviously the following:

Proposition 4.4. *Let $\tau\colon h \to h'$ be a natural transformation of homology theories. Then for a fibre bundle ξ a morphism $\tau_*\colon E(\xi, h) \to E(\xi, h')$ of convergent spectral sequences is induced.*

4.3 Intersecting on Fibre and Base

The goal of this subsection is to define Gysin morphisms of Serre spectral sequences which "compute" the corresponding Gysin morphisms of the homology of the total space.

4.3.1 Intersecting on the Base

Let $\xi = (F \to E \xrightarrow{\pi} B)$ be a fibre bundle with B a (finite-dimensional) manifold and $A \subset B$ a closed submanifold of codimension d with h_*-oriented normal bundle. Choose a triangulation of B transverse to A and triangulate A as in 2.5.

Theorem 4.5 (Intersection on the Base). *There is a morphism $s_B(A)$ of convergent spectral sequences of level 1 and bidegree $(-d, 0)$ between $E(\xi)$ and $E(\xi|_A)$ where the spectral sequences are defined by the triangulations above. The morphism is canonical starting with level 2 and induces the usual Gysin morphism $H_p(B; \mathfrak{h}_q(F)) \to H_{p-d}(A; \mathfrak{h}_q(F))$ on this level.*

Proof of the Theorem. We want to construct a morphism of the corresponding exact couples $C(\xi)$ and $C(\xi|_A)$. The first thing we show is that, for a manifold P, every map $f\colon P \to E^p$ or relative map $f\colon (P, \partial P) \to (E^p, E^{p-1})$ can be homotoped in such a way that πf is transverse to A. We concentrate on the relative case since this is more difficult.

Let $[P, a, f] \in h_{p+q}(E^{(p)}, E^{(p-1)})$. We want to find a homotopy

$$(P, \partial P) \times I \to (E^{(p)}, E^{(p-1)})$$

from f to a map g such that πg is transverse to A. Consider open neighbourhoods U_p of B^p such that there are smooth retracts $r_p\colon U_p \to B^p$. There is a homotopy $H_1\colon \partial P \times I \to U_{p-1}$ from $\pi f|_{\partial P}$ to a smooth map (see 2.25). Extend this homotopy to a homotopy $H_1\colon P \times I \to U_p$ from πf to a map $\tilde{f}$. This map $\tilde{f}$ is smooth on ∂P, so we can find a homotopy $H_2\colon P \times I \to U_p$ to a smooth map such that $H_2|_{\partial P} = \tilde{f} \circ \mathrm{pr}_1$. We define $g'(x) = r_p \circ H_2(x, 1)$ which is smooth and homotopic to πf.

We can homotope $g'|_{\partial P}$ in $B^{(p-1)}$ to be transverse to $A \cap B^{(p-1)}$ since $A \cap B^{(p-1)}$ has constant codimension d in $B^{(p-1)}$ and therefore we can proceed as in 2.4.3.

Because $\partial P \hookrightarrow P$ is a cofibration, we can extend it to a map $\tilde{g}'$ on the whole of P. Since $\tilde{g}'$ is tranverse to A on ∂P, we can homotope it to a $\tilde{g}\colon P \to B^{(p)}$ in $B^{(p)}$ which is transverse to $A \cap B^{(p)}$ while leaving ∂P fixed. Since $E \to B$ is a fibration we can lift this homotopy to E and get a map $g\colon P \to E^{(p)}$, for which $\pi g = \tilde{g}$ is transverse to A.

Since $B^p \cap A \subset A^{p-d}$ we get homomorphisms

$$h_{p+q}(E^{(p)}) \to h_{p+q-d}(\pi^{-1}(A^{p-d}))$$

and

$$h_{p+q-d}(E^p, E^{p-1}) \to h_{p+q}(\pi^{-1}(A^{p-d}), \pi^{-1}(A^{p-1-d}))$$

by transverse intersection of our representative P with A. More precisely, define $Q := (\pi f)^{-1}(A)$ and send $[P, a, f]$ to $[Q, a|_Q, f|_Q] \in h_{p+q-d}(E|_A^{(p-d)}, E|_A^{(p-1-d)})$ and the same in the absolute case. Since $\partial Q = \partial P \cap A$, everything commutes and we get a morphism of exact couples and therefore of convergent spectral sequences $E^1_{**}(\xi) \to E^1_{(*-d)*}(\xi|_A)$.

We now have to check that it induces the usual Gysin morphism on E^2. Choose the x_α of the last subsection on A for every cell α intersecting A in its interior. Now apply the description of the "cellular" Gysin morphism of 2.3. □

Proposition 4.6 (Homotopy Invariance)**.** *Let $i_0, i_1\colon A \hookrightarrow B$ two isotopic embeddings. Then $s_B(i_0) = s_B(i_1)$.*

Proof. The isotopy $J\colon A \times I \to B$ defines an embedding $\tilde{J} = (J, \mathrm{pr}_2)\colon A \times I \to B \times I$. We simply have to check the commutativity of the following diagram:

$$\begin{array}{ccc} E(\xi) & \longrightarrow & E(\xi \times I) \\ \downarrow{\scriptstyle s(A)} & & \downarrow{\scriptstyle s(\tilde{J})} \\ E(\xi|_A) & \longrightarrow & E(\tilde{J}^*(\xi \times I)) \end{array}$$

for the horizontal arrows the inclusion at the bottom or the top since the horizontal morphisms are isomorphisms. Put on the bottom (or the top) a triangulation transverse to i_0 and on $B \times I$ the product CW-structure. Since every representative $f\colon P \to B \times I$ can be homotoped to B and the k-skeleton of B is contained in that of $B \times I$ we can do the intersection at the bottom and therefore the diagram commutes. □

Lemma 4.7. *Let $A \subset B$ be a submanifold and $B' \subset B$ be a second submanifold transverse to A. Then the normal bundle of B' in B can embedded as a tubular neighbourhood in a way that restriction of the normal bundle to $B' \cap A$ embeds into A.*

Proof. We want to choose a Riemannian metric on B such that a neighbourhood of $A \cap B'$ in B is geodesically embedded and A intersects B orthogonally. To achieve this, choose a Riemannian metric on A and a cover $\mathcal{C}$ of $A \cap B'$ by small balls diffeomorphic to $\mathbb{R}^n$. We can surely find such a diffeomorphism from one of these balls D which sends $A \cap D$ to $\mathbb{R}^k \times 0$ and $B' \cap D$ to $0 \times \mathbb{R}^m$ with $k + m \geq n$. So we can choose a Riemannian metric on the ball where A embeds geodesically and A and B intersect orthogonally.

Choose now a countable locally finite cover $\mathcal{D}$ of B which contains the cover above and such there is a neighbourhood of $A \cap B'$ which no open sets of $\mathcal{D} - \mathcal{C}$ intersect. Choose furthermore Riemannian metrics on all these open sets and define a Riemannian metric on the whole of B by a partition of unity. Since A and B' intersect orthogonally with respect to each of the ball metrics, they also intersect orthogonally with respect a linear combination of these metrics. For every intrinsic geodesic c of A, the velocity field can be locally extended to a vector field on B. One can see by the formula for the Levi-Cevita connection the following: if one has Riemannian metrics g_i and c is a geodesic for every g_i, it is a geodesic for $\Sigma\lambda_i g_i$, too.

Now one has simply to use the usual Riemannian geometry description of a tubular neighbourhood. □

Theorem 4.8 (Naturality)**.** *Let $\phi\colon E' \to E$ be a map of fibre bundles $\xi' = (F' \to E' \to B')$ and $\xi = (F \to E \to B)$. Let $A \subset B$ be a submanifold and the map on the bases $f\colon B' \to B$ be transverse to A. Then the following diagram is commutative:*

$$\begin{array}{ccc} E(\xi') & \xrightarrow{\phi_*} & E(\xi) \\ \downarrow{\scriptstyle s(\phi^{-1}(A))} & & \downarrow{\scriptstyle s(A)} \\ E(\xi'|_{\phi^{-1}(A)}) & \xrightarrow{\phi_*} & E(\xi|_A) \end{array}$$

Proof. First assume $f\colon B' \to B$ to be the inclusion of a closed submanifold. Choose a triangulation $\mathcal{T}'$ of B' and extend it to a triangulation of B (see 2.36). By 2.29 we can find an isotopy of $i\colon A \to B$ to a map $i'\colon A \to B$ transverse to B', $\mathcal{T}$ and $\mathcal{T}'$. By the lemma, we can assume that $B' \cap A$ is not homotoped out of B' during the whole homotopy and nothing is homotoped into B'. Due to the homotopy invariance, the vertical arrows in the diagram are not changed by this homotopy. Since the k-skeleton of B' lies in the k-skeleton of B and both are triangulated transverse to A, the commutativity of the diagram is clear now.

To see the general case, factorize f into $\mathrm{pr}_2 \circ g\colon B' \to B' \times B \to B$ where g is the graph morphism. The image is closed since f is continuous. Furthermore, g is transverse to $B' \times A$. The only thing still to show is therefore the naturality for projection maps. We cover B and B' by cubes instead of simplices (for example we could refine a triangulation) and choose the product cube covering on $B' \times B$ (the proof of 4.3.1 works of course also with these "cubical complexes"). The p-skeleton of $B' \times B$ projects into the p-skeleton of B. So if we choose a representative (P, a, f) of a homology class in $h_p(\mathrm{pr}_2^* E^{(p)}, \mathrm{pr}_2^* E^{(p-1)})$ which is transverse to $B' \times A$, then its projection $(P, a, \mathrm{pr}_2 f)$ is in $h_p(E^{(p)}, E^{(p-1)})$ and is transverse to A. Therefore, the diagram commutes. □

We now want to generalize the intersection morphism to an infinite-dimensional context. So let $\xi = (F \to E \to B)$ be a fibre bundle with B a Hilbert manifold and $A \subset B$ a closed sub Hilbert manifold of codimension d with h_*-oriented normal bundle. Assume furthermore that there is a collection of finite-dimensional manifolds $P_0 \subset P_1 \subset \cdots \subset B$ with inclusions $\iota_i\colon P_i \hookrightarrow B$ such that every map $f\colon X \to B$ from a compact space can be homotoped into one of the P_i and the homotopy can be assumed to be constant on $f^{-1}(P_i)$.

Proposition 4.9. *In the situation above, there is a morphism $s_B(A)$ of convergent spectral sequences of level 2 and bidegree $(-d, 0)$ between $E(\xi)$ and $E(\xi|_A)$. The morphism induces the usual Gysin morphism $H_p(B; \mathfrak{h}_q(F)) \to H_{p-d}(A; \mathfrak{h}_q(F))$ on E^2.*

Proof. For simplicity, we assume the local system $\mathfrak{h}_q(F)$ to be trivial (otherwise we would have to use geometric homology with local coefficients). So let x be in $E^n_{pq}(\xi)$. This element is represented by an element z in $E^2_{pq} \cong H_p(B; h_q(F))$ with $d_2(z) = d_3(z) = \cdots = d_{n-1}(z) = 0$. The cycle z in its turn is represented by some geometric cycle (X, a, f) and we have zero bordisms for $d_i(X, a, f)$ for $i < n$. All occuring manifolds are compact, so there is an N such that all occuring maps factor over $P_N \subset B$.

Consider the map of spectral sequences $(\iota_N)_*\colon E(\xi|_{P_N}) \to E(\xi)$. The above considerations yield that there is an $y \in E^n_{pq}$ mapping to x. By the transversality theorem, we can assume that P_N is transverse to A, so $P_N \cap A \subset P_N$ is a closed submanifold of P_N. We now define $s_B(A)(x) = (\iota_N)_* s_B(P_N \cap A)(y)$. We have to check that this is a well-defined map and that it defines a morphism of spectral sequences.

The map is independent of the choice of N because of the naturality of intersecting on the base. Now suppose $(\iota_N)_*(y_1) = (\iota_N)_*(y_2) = x$ where y_1 and y_2 are represented by geometric cycles (X_1, a_1, f_1) and (X_2, a_2, f_2). Then there is a bordism between these two in B which factors (up to homotopy) over some P_{N+M}. Now we use that ι_N factors over ι_{N+M}, that intersecting on the base is natural and that the two cycles become equal in P_{N+M} to deduce that our map is well-defined.

The map is a morphism of spectral sequences since intersecting on the base is a morphism of spectral sequences for each P_i.

□

4.3.2 Intersecting on the Fibre

Let $\xi = (F \to E \xrightarrow{\pi} B)$ be a smooth fibre bundle where fibre a Hilbert manifold and base a finite-dimensional manifold. Let $\xi_0 = (F_0 \to E_0 \to B)$ be a subbundle of constant codimension d and h_*-oriented normal bundle.

Before we come to our theorem, we want to discuss an alternative description of the Serre spectral sequence. Choose a triangulation on B and consider open neighbourhoods U_p of B^p such that there are deformation retracts $r_p\colon U_p \to B^p$ with $U_{p-1} \subset U_p$. Then we have an isomorphism from the usual exact couple $C(\xi)$ for the Serre spectral sequence to the exact couple

$$\begin{array}{ccc} \bigoplus_{p,q} h_{p+q}(\pi^{-1}(U_{p-1})) & \xrightarrow{\quad i \quad} & \bigoplus_{p,q} h_{p+q}(\pi^{-1}(U_p)) \\ & {\scriptstyle k}\nwarrow \qquad \swarrow{\scriptstyle j} & \\ & \bigoplus_{p,q} h_{p+q}(\pi^{-1}(U_p), \pi^{-1}(U_{p-1})) & \end{array}$$

Theorem 4.10 (Intersection on the Fibre)**.** *There is a morphism $s_F(E_0)$ of convergent spectral sequences of level 1 and bidegree $(0, -d)$ between $E(\xi)$ and $E(\xi_0)$. This induces the usual Gysin morphism $H_p(B; \mathfrak{h}_q(F)) \to H_p(B; \mathfrak{h}_{q-d}(F_0)))$ on E^2.*

Proof. We want to define a morphism of the corresponding exact couples. Let $[P, a, f_1] \in h_{p+q}(\pi^{-1}(U_p), \pi^{-1}(U_{p-1}))$ be a homology class. There is a homotopy $H_1\colon \partial P \to \pi^{-1}(U_{p-1})$ from $f_1|_{\partial P}$ to a smooth map (see 2.25). Extend this homotopy to a homotopy $H_1\colon P \times I \to \pi^{-1}(U_p)$ from f_1 to a map f_2. This map f_2 is smooth on ∂P, so we can find a homotopy

$H_2\colon P \times I \to \pi_{-1}(U_p)$ to a smooth map such that $H_2|_{\partial P \times I} = f_2 \circ \mathrm{pr}_1$. We define $f(x) = H_2(x,1)$ which is smooth and homotopic to f_1.

By 2.31 we can find a homotopy $H\colon (P, \partial P) \times I \to (\pi^{-1}(U_p), \pi^{-1}(U_{p-1}))$ from f to a g which is transverse to E_0. By the same argument we also get in the absolute case a representative which is transverse to E_0. By intersecting with E_0 we get now a morphism of the couples which commutes with the boundary operator as above. More precisely we map $[P, a, f]$ to $[Q, a|_Q, f|_Q] \in h_{p+q-d}(\pi^{-1}(U_p), \pi^{-1}(U_{p-1}))$ with $Q = f^{-1}(E_0)$. This induces a convergent morphism $E(\xi) \to E(\xi_0)$ of level 1 and bidegree $(0, -d)$. That we get the Gysin morphism on E^2 can be seen by the explicit isomorphism of the E^1-term to the cellular complex: there is no difference if we intersect first with F and then with F_0 or if we first intersect with E_0 and then with F_0 (if everything is transverse). □

As in the case of the intersection on the base, we can extend the cases we are interested in to an infinite-dimensional context. So let now B be a Hilbert manifold and the other notation as above and assume that there is a collection of finite-dimensional manifolds $P_1 \subset P_2 \subset \cdots \subset B$ such that every map $f\colon X \to B$ from a compact space can be homotoped into one of the P_i

Theorem 4.11. *In the situation above, there is a morphism $s_F(E_0)$ of convergent spectral sequences of level 2 and bidegree $(0, -d)$ between $E(\xi)$ and $E(\xi_0)$. This induces the usual Gysin morphism $H_p(B; \mathfrak{h}_q(F)) \to H_p(B; \mathfrak{h}_{q-d}(F_0)))$ on E^2. Furthermore, it converges to the Gysin morphism in the homology of the total spaces.*

Proof. As in the case of the intersection on the base. □

4.4 Multiplicative, Comultiplicative and Module Structures

To give a conceptual treatment of multiplicative, comultiplicative and module structures on spectral sequences, we want first to define a monoidal product on the category of bigraded (convergent) spectral sequences. Let (E, d) and (E', d') be spectral sequences. Denote by $E \otimes E'$ the spectral sequence defined by $(E \otimes E')^k_{pq} = \oplus_{i,j} E^k_{ij} \otimes E'^k_{(p-i)(q-j)}$ with differential $d''(a \otimes b) = d(a) \otimes b + (-1)^{i+j} a \otimes d'(b)$ where $a \in E^k_{ij}$ and $b \in E'^k_{(p-i)(q-j)}$. If E and E' converge to filtrations F^*_* and F'^*_*, then we define $(F \otimes F')^k_q = \oplus_{i,j} F^i_j \otimes F'^{k-i}_{q-j}$. It is easy to see that $E \otimes E'$ converges to $F \otimes F'$ (note that tensoring is right-exact).

A *multiplicative spectral sequence* is now simply a monoid in spectral sequences, i.e. a spectral sequence E with a morphism $E \otimes E \to E$ which satisfies associativity (we will not consider the unit maps). Define $E[(a,b)]$ to be the shifted spectral sequence with $E^k[(a,b)]_{pq} = E^k_{(p+a)(q+b)}$. Recall the notation $\mathbb{H}_*(M) = H_{*+d}(M)$.

Theorem 4.12. *Let M be a d-dimensional h_*-oriented manifold. Then*

$$E(\Omega^n M \to L^n M \to M)[(-d, 0)]$$

can be equipped with the structure of a multiplicative spectral sequence which converges to the Chas-Sullivan product on $\boldsymbol{h}_*(L^n M)$. Furthermore, the induced product on the E^2-term $\mathbb{H}_*(M; \mathfrak{h}_q(\Omega^n M))$ is equal to the intersection product (see 2.3) with coefficients in the local system of rings $\mathfrak{h}_*(\Omega^n M)$ whose multiplication is given by the Pontryagin product.

Proof. Let $\xi = (\Omega^n M \to L^n M \to M)$ and denote by $\Delta_M : M \to M \times M$ the diagonal. Consider the diagram

$$E\begin{pmatrix} \Omega^n M \\ \downarrow \\ L^n M \\ \downarrow \\ M \end{pmatrix} \otimes E\begin{pmatrix} \Omega^n M \\ \downarrow \\ L^n M \\ \downarrow \\ M \end{pmatrix} \overset{\times}{\to} E\begin{pmatrix} \Omega^n M \times \Omega^n M \\ \downarrow \\ L^n M \times L^n M \\ \downarrow \\ M \times M \end{pmatrix} \overset{s_B(\Delta_M)}{\to} E\begin{pmatrix} \Omega^n M \times \Omega^n M \\ \downarrow \\ L^n M \times_M L^n M \\ \downarrow \\ M \end{pmatrix} \overset{\gamma_*}{\to} E\begin{pmatrix} \Omega^n M \\ \downarrow \\ L^n M \\ \downarrow \\ M \end{pmatrix}$$

Here γ is defined as in 3.2. Note that the cross product is a map of spectral sequences. All claims are obvious. □

Theorem 4.13. [2]*Let $M \to N \to O$ be a fibre bundle of h_*-oriented manifolds of dimensions m, n and o, respectively, with projection map π. Then*

$$E(L^n M \to L^n N \to L^n O)[(-o, -m)]$$

can be equipped with the structure of a multiplicative spectral sequence which converges to the Chas-Sullivan product on $\boldsymbol{h}_*(L^n N)$. Furthermore, the induced product on the E^2-term $H_{p+m}(L^n O; \mathfrak{h}_{q+m}(L^n M))$ is equal to the Chas-Sullivan product with coefficients in the local system of rings $\mathfrak{h}_{*+m}(L^n O)$.

Proof. Consider the diagram

$$E\begin{pmatrix} L^n M \\ \downarrow \\ L^n N \\ \downarrow \\ L^n O \end{pmatrix} \otimes E\begin{pmatrix} L^n M \\ \downarrow \\ L^n N \\ \downarrow \\ L^n O \end{pmatrix} \overset{\times}{\to} E\begin{pmatrix} L^n M \times L^n M \\ \downarrow \\ L^n N \times L^n N \\ \downarrow \\ L^n O \times L^n O \end{pmatrix} \xrightarrow{s_B(L^n O \times_O L^n O)} E\begin{pmatrix} L^n M \times L^n M \\ \downarrow \\ X \\ \downarrow \\ L^n O \times_O L^n O \end{pmatrix}$$

$$\xrightarrow{s_F(L^n N \times_N L^n N)} E\begin{pmatrix} L^n M \times_M L^n M \\ \downarrow \\ L^n N \times_N L^n N \\ \downarrow \\ L^n O \times_O L^n O \end{pmatrix} \overset{\gamma_*}{\to} E\begin{pmatrix} L^n M \\ \downarrow \\ L^n N \\ \downarrow \\ L^n O \end{pmatrix}$$

Here γ is defined as in 3.2 and $X = \{(\alpha, \beta) \in L^n N \times L^n N \colon \pi(\alpha(pt)) = \pi(\beta(pt))\}$ where pt denotes the base point of S^n. By 2.42, we are in the situation of 4.3.1 and the intersection morphism is defined. □

There is also the notion of a *comultiplicative spectral sequence*, which is simply a comonoid in spectral sequences, i.e. one has a map $E \to E \otimes E$ which is coassociative (we will not consider counits).

[2]A similar spectral in the case of singular homology was already considered in [LBo].

In 3.4 we have defined coproducts on $h_*(LM)$. By the same method, there is also a coproduct on $h_*(\Omega M)$ if we assume h_* to have field coefficients:

$$h_*(\Omega M) \xrightarrow{i^!} h_{*-d}(\Omega M \times \Omega M) \cong [h_*(\Omega M) \otimes_{h_*} h_*(\Omega M)]_{*-d}$$

Here $i\colon \Omega M \times \Omega M \to \Omega M$ is the inclusion of loops α with $\alpha(\frac{1}{2}) = \alpha(0)$. Let V be a coalgebra over a field k. Then we have a coproduct on $H_*(M;V)$:

$$H_*(M;V) \to H_*(M; V \otimes V) \xrightarrow{\Delta_*} H_*(M \times M; V \otimes V) \cong H_*(M;V) \otimes_k H_*(M;V)$$

Theorem 4.14. *Let M be a d-dimensional h_*-oriented manifold. Then*

$$E(\Omega M \to LM \to M)[(0,-d)]$$

can be equipped with the structure of a comultiplicative spectral sequence which converges to the coproduct on $\boldsymbol{h}_*(LM)$. Furthermore, the induced coproduct on the E^2-term $H_{p+d}(M;\mathfrak{h}_q(\Omega M))$ is equal to the coproduct on M with coefficients in the coalgebra $\mathfrak{h}_*(\Omega M)$.

Proof. Consider the diagramm

$$E\begin{pmatrix} \Omega M \\ \downarrow \\ LM \\ \downarrow \\ M \end{pmatrix} \xrightarrow{s_F(LM \times_M LM)} E\begin{pmatrix} \Omega M \times \Omega M \\ \downarrow \\ LM \times_M LM \\ \downarrow \\ M \end{pmatrix} \xrightarrow{\Delta_*} E\begin{pmatrix} \Omega M \times \Omega M \\ \downarrow \\ LM \times LM \\ \downarrow \\ M \times M \end{pmatrix}$$

$$\xrightarrow{\cong} E\begin{pmatrix} \Omega^n M \\ \downarrow \\ L^n M \\ \downarrow \\ M \end{pmatrix} \otimes E\begin{pmatrix} \Omega^n M \\ \downarrow \\ L^n M \\ \downarrow \\ M \end{pmatrix}$$

All claims are obvious. □

Theorem 4.15. *Let $M \to N \to O$ be a fibre bundle of h_*-oriented manifolds of dimensions m, n and o, respectively. Then*

$$E(LM \to LN \to LO)[(-o,-m)]$$

can be equipped with the structure of a comultiplicative spectral sequence which converges to the coproduct on $\boldsymbol{h}_*(LN)$. Furthermore, the induced coproduct on the E^2-term $H_{p+o}(LO;\mathfrak{h}_{q+m}(LM))$ is equal to the coproduct with coefficients in the coalgebra $\mathfrak{h}_{*+m}(LO)$.

Proof. Analogous to the previous theorem and the one for the corresponding multiplicative spectral sequence. □

For E a multiplicative spectral sequence, there is also the notion of a module spectral sequence, i.e. a spectral sequence E' together with a morphism $E \otimes E' \to E'$ and the usual coherence diagrams. Recall that, if M is a d-manifold and N a module over a ring R, we have an $H_{*+d}(M;R)$-module structure on $H_{*+d}(M;N)$ defined analogous to the intersection product (note we have a cross product $H_*(M;R) \otimes H_*(M;N) \to H_*(M \times M;N)$.

Theorem 4.16. [3]*Let Z be a closed n-manifold and M be a d-dimensional h_*-oriented manifold. Then*

$$E(Map^\bullet(Z,M) \to Map(Z,M) \to M)[(-d,0)]$$

can be equipped with the structure of a module spectral sequence over $E(\Omega^n M \to L^n M \to M)[(-d,0)]$ which converges to the module structure on $\boldsymbol{h}_*(Map(Z,M))$. Furthermore, the induced module structure on the E^2-term $H_{p+d}(M;\mathfrak{h}_q(Map^\bullet(Z,M))$ coincides with the module structure described above.

Proof. As in the multiplicative case. □

Theorem 4.17. *Let Z be a closed n-manifold. Furthermore, let $M \to N \xrightarrow{\pi} O$ be a fibre bundle of h_*-oriented manifolds of dimensions m, n and o respectively. Then*

$$E(Map(Z,M) \to Map(Z,N) \to Map(Z,O))[(-o,-m)]$$

can be equipped with the structure of a module spectral sequence over $E(L^n M \to L^n N \to L^n O)[(-o,-m)]$ which converges to the module structure on $\boldsymbol{h}_*(Map(Z,N)$. Furthermore the induced module structure on the E^2-term

$$H_{p+o}(Map(Z,O);\mathfrak{h}_{q+m}(Map(Z,M))$$

coincides with the module structure described in 3.4.

Proof. As in the multiplicative case. □

4.5 Examples

In this subsection, we will do two different things, in object and method. First, we want to widen our knowledge about ordinary homology of free loop spaces to the case of certain sphere bundles. Secondly, we want to compute some extraordinary homologies of free loop spaces, namely complex cobordism, complex K-Theory and oriented bordism. We will study the Atiyah-Hirzebruch spectral sequence associated to spheres and (complex) projective space and show that it degenerates on E^2. In some cases, we will furthermore be able to show that all filtrations are trivial extensions.

4.5.1 Sphere Bundles

We want to study the homology of the free loop space of sphere bundles. While the integral homology of free loop spaces is usually hard to compute, there are more efficient tools for the rational homology, namely rational homotopy theory. Rational homotopy theory associates functorially to every simply-connected space X a commutative differential graded algebra

[3] A similar spectral sequence in the case of singular homology was already considered in [K-S].

over $\mathbb{Q}$ of the form ΛV. Here V is a graded rational vector space, i.e. a tensor product of polynomial rings for the basis elements of V of even degree and exterior algebras for the odd parts as an algebra. This is called the *minimal model* $\mathcal{M}(X)$ of X. The cohomology of ΛV is isomorphic to the rational cohomology of X (see [FHT]).

We will need the following two facts of rational homotopy theory:

1. The vector space V is naturally isomorphic to the dual of $\pi_*(X;\mathbb{Q}) := \pi_*(X) \otimes \mathbb{Q}$ (see [FHT], Thm 15.11).

2. The minimal model of LX depends only on the minimal model of X. This can be seen by the explicit formulas of [V-S].

While the minimal model of LX only gives information about the rational cohomology, we want to use rational homotopy theory in combination with the Serre spectral sequence to do integral computations for the free loop space LE of a fibre bundle $S^k \to E \to S^n$.

First assume $k > 1, n > 1$ odd. The odd dimensional spheres have only one nontrivial rational homotopy group, namely $\pi_k(S^k;\mathbb{Q}) = \mathbb{Q}$. Hence $\pi_i(E;\mathbb{Q}) = \mathbb{Q}$ for $i = k, n$ and 0 else by the long exact sequence of homotopy groups. So we have $\mathcal{M}(E) = \Lambda(x_k) \otimes \Lambda(x_n)$ with $|x_k| = k$ and $|x_n| = n$. For dimension reasons, there are no differentials. Thus we have $\mathcal{M}(E) \cong \mathcal{M}(S^k \times S^n)$ as differential graded algebras. We conclude $H_*(LE;\mathbb{Q}) \cong H_*(L(S^k \times S^n);\mathbb{Q})$. Consider the E^2-term of the Serre spectral sequence associated to $LS^k \to LE \to LS^n$. Every occuring group is torsionfree. Therefore, our rational computation shows that the spectral sequence degenerates at E^2 and we have $\mathbb{H}_*(LE) \cong \mathbb{H}_*(LS^k) \otimes \mathbb{H}_*(LS^n) \cong \Lambda(a_k, a_n) \otimes \mathbb{Z}[u_k, u_n]$ with $|a_k| = -k$, $|a_n| = -n$, $|u_k| = k-1$ and $|u_n| = n-1$ (for the grading conventions, see 3.2). Note that all extension are trivial in the sense that

$$0 \to F^{n-1} \to F^n \to F^n/F^{n-1} \to 0$$

splits since all occuring groups in E^∞ are torsionfree. To show that the isomorphism holds also multiplicatively, we use the following proposition for the Serre spectral sequence associated to $LM \to LN \to LO$: [4]

Proposition 4.18. *Let E be a multiplicative convergent spectral sequence isomorphic to the Serre spectral sequence of a fibre bundle $F \to E \to B$ with connected base. Assume the local system to be trivial and that $E^2 = E^\infty \cong H_*(B) \otimes h_*(F)$. Furthermore, require that all extension are trivial and $E^2_{*0} = \mathbb{Z}[\overline{x}_0, \overline{x}_1, \dots \overline{x_n}] \otimes \Lambda(\overline{x}_{n+1}, \overline{x}_{n+2}, \dots)$. Then $h_k(E) \cong \bigoplus_{p+q=k} E^\infty_{pq}(E,h)$ holds multiplicatively.*

Proof. Denote the filtration of $h_q(E)$ by F^*_q. As $E^\infty_{p0} = F^p_p/F^{p-1}_p$ and $F^p_p = h_p(E)$, we have a surjective map $h_p(E) \to E^\infty_{p0} = E^2_{p0}$. Lift the $\overline{x_j}$ to x_j in $h_p(E)$. Since the multiplication on $E^\infty = F^p/F^{p-1}$ is induced by that on $h_*(E)$, we have that $x_j x_i$ is a lift for $\overline{x_j x_i}$. The homology groups $h_*(F)$ acts on $h_*(E)$ by multiplication (we have an injection $h_*(F) \to h_*(E)$) and on E^∞ (since $h_*(F) = E^\infty_{0*}$) in a compatible way. Define a map $L\colon E^\infty_{**} \to \mathbf{h}_*(E)$ via $y \cdot \Pi x_i^{k_i} \mapsto y \cdot \Pi \overline{x_i}^{k_i}$ where $y \in h_*(F)$. This map is clearly a map of algebras. It is also clear that it is surjective onto F^0. Assume inductively that it is surjective onto F^p. The products

[4] Surprisingly enough, in general the graded abelian group a multiplicatively convergent spectral sequence is converging to is not multiplicatively isomorphic to E^∞ of this spectral sequence, even if all filtrations extensions are trivial.

$\Pi\overline{x_i}^{k_i}$ with $\sum k_i|x_i| = p+1$ form a $h_*(F)$-basis for F^{p+1}/F^p and are images of L. Therefore, we see that L is surjective onto F^{p+1} and conclude by induction that it is surjective onto the whole of $h_*(E)$. Since $E^\infty_{**} \cong h_*(E)$ additively and both are finitely generated abelian groups in every degree, L is an isomorphism (of algebras). □

Now assume $k > 1$ odd, $n > 2$ even and $k \neq n \pm 1$ and that $n-1$ is no multiple of $k-1$. Even dimensional spheres S^n have two non-zero rational homotopy groups, namely $\pi_n(S^n;\mathbb{Q}) \cong \mathbb{Q}$ and $\pi_{2n-1}(S^n,\mathbb{Q}) \cong \mathbb{Q}$. By the long exact homotopy sequence, we have $\pi_i(E;\mathbb{Q}) = \mathbb{Q}$ for $i = k, n, 2k-1$ and 0 else. So we get $\mathcal{M}(E) \cong \Lambda(x_k) \otimes \mathbb{Z}[x_n] \otimes \Lambda(y_{2n-1})$ with $|x_k| = k$, $|x_n| = n$ and $|y_{2n-1}| = 2n-1$. Since the Serre spectral sequence associated to $S^k \to E \to S^n$ degenerates at E^2, we have $d(x_k) = d(x_n) = 0$ and $d(y_{2n-1})$ must be a non-zero multiple of x_n^2. This is isomorphic to the minimal model $\mathcal{M}(S^k \times S^n)$. Therefore, we have $H_*(LE;\mathbb{Q}) \cong H_*(LS^k \times LS^n;\mathbb{Q})$. Consider the Serre spectral sequence associated to $LS^k \to LE \to LS^n$. A differential $d_i(x)$ can only be non-zero, if x and $d_i(x)$ are torsion. The only torsion elements of $\mathbb{H}_*(LS^n)$ are the av^j for $j \geq 1$ (see 3.5 for notation). Hence, we have $d(1 \otimes a_{LS^k}) = d(1 \otimes u_{LS^k}) = d(a_{LS^n} \otimes 1) = d(b_{LS^n} \otimes 1) = d(v_{LS^n} \otimes 1) = 0$ since all these generators are non-torsion. The E^2-term of the Serre spectral sequence is isomorphic to $\mathbb{H}_*(LS^n) \otimes \mathbb{H}_*(LS^k)$. By multiplicativity, the spectral sequence degenerates at E^2. Because filtration issues may come up, we cannot deduce the concrete structure of the homology.

We want to emphasize that it is somewhat surprising that we are able to control even torsion phenomena by these rational methods. One easy concrete example for n odd is the bundle $S^3 = Sp(1) \to Sp(2) \to S^7$. By the results above we get $\mathbb{H}_*(LSp(2)) \cong \mathbb{Z}[u_1, u_2] \otimes \Lambda(a_1, a_2)$ (additively) where $|u_1| = 2, |u_2| = 6, |a_1| = -3$ and $|a_2| = -7$.

4.5.2 Complex Cobordism

We denote the bordisms groups of compact stably almost complex manifolds mapping into a space X by $MU_*(X)$. Since every almost complex manifold is oriented, we get a natural transformation $\tau\colon MU_*(X) \to MSO_*(X)$, where $MSO_*(X)$ denotes the bordism group of compact oriented manifolds mapping into X. Furthermore, we have a natural transformation $\mu\colon MSO_*(X) \to H_*(X)$, sending each manifold to its fundamental class. We define $\nu\colon MU_*(X) \to H_*(X)$ to be the composition $\mu \circ \tau$. We can characterize μ and ν also via the Atiyah-Hirzebruch spectral sequence:

Lemma 4.19. *The edge homomorphism*

$$MSO_n(X) \to E^\infty_{n0}(X, MSO) \to E^2_{n0}(X, MSO) = H_n(X)$$

equals μ. The edge homomorphism

$$MU_n(X) \to E^\infty_{n0}(X, MU) \to E^2_{n0}(X, MU) = H_n(X)$$

equals ν.

Proof. The first statement is proven in [C-F], lemma 7.2. The second statement follows since τ induces a morphism of convergent spectral sequences. □

We have now the following proposition:

Proposition 4.20. *If X is (homotopy equivalent to) a CW-complex, then the MU spectral sequence degenerates at E^2 if and only if $\nu\colon MU_n(X) \to H_n(X)$ is an epimorphism for all $n \geq 0$.*

Proof. By definition, the spectral sequence collapses if and only if $d^r\colon E^r_{pq} \to E^r_{p-r,q+r}$ is trivial for all $r \geq 2$. It is clear that, if the spectral sequence collapses, then $\nu\colon MU_n(X) \to H_n(X)$ is an epimorphism for all $n \geq 0$.

Let us therefore assume that ν is an epimorphism. Then $d_r\colon E^r_{n0} \to E^r_{n-r,r+q}$ is trivial for all $r \geq 2$ and all n. Consider the operation of MU_* on the spectral sequence. Since MU_* is torsionfree, we get an isomorphism

$$H_p(X) \otimes MU_q \stackrel{\cong}{\to} E^2_{pq}.$$

Since the operation of MU_* commutes with differentials, all differentials vanish. □

To see the degeneration of the Atiyah-Hirzebruch spectral sequence, we have to find stably almost complex structures on our generators of 3.5 and 3.6. Since S^n has trivial normal bundle in $\mathbb{R}^{n+1}$ (because the first cohomology of S^n vanishes and line bundles are classified by $H^1(S^n;\mathbb{Z}/2)$ if you like) we have that $TS^n \oplus \varepsilon$ is trivial, where ε is the trivial line bundle. Since trivial bundles are stably complex, the sphere is stably almost complex. Recall, we denoted the sphere subbundle of the tangent bundle by STS^n. We have that $STS^n \subset S(TS^n \oplus \epsilon) \cong S^n \times S^n$ has trivial normal bundle. Since the tangent bundle of $S^n \times S^n$ is stably trivial, the tangent bundle of STS^n is stably trivial, too, and therefore stably complex. This finishes the case for the sphere.

The manifolds $\mathbb{CP}^n$ and $\mathbb{CP}^n \times S^1$ are clearly almost complex. It remains to show that STS^{2n+1}/S^1 is stably almost complex, where S^1 acts via complex multiplication and its derivative. As in the paragraph above, it suffices to consider $(S^{2n+1} \times S^{2n+1})/S^1$, where S^1 acts via the diagonal action. Embed $S^{2n+1} \times S^{2n+1}$ into $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$. This gives an embedding $(S^{2n+1} \times S^{2n+1})/S^1 \hookrightarrow (\mathbb{C}^{n+1} \times \mathbb{C}^{n+1})/\mathbb{C}^* \cong \mathbb{CP}^{2n+1}$ of codimension 1. Since the latter is complex and the normal bundle is trivial (note that $(S^{2n+1} \times S^{2n+1})/S^1$ is simply connected), we are done.

For the odd dimensional spheres, we have in addition that all filtration extension are trivial, i.e.

$$0 \to F^{n-1} \to F^n \to F^n/F^{n-1} \to 0$$

splits, since all occuring groups in E^∞ are torsionfree (MU_* is torsionfree by [Mil2]). Therefore, we have additively:

$$MU_*(LS^{2k+1}) \cong H_*(LS^{2k+1}) \otimes MU_*$$

We have $E^2(LM, MU)_{pq} = H_p(LM) \otimes MU_*(pt)$. By 4.17, the above isomorphism

$$MU_*(LS^{2k+1}) \cong H_*(LS^{2k+1}) \otimes MU_*$$

holds now also multiplicatively.

4.5.3 Complex K-Theory

We want to use that complex K-Theory is determined by complex cobordism. More concretly we have the following:

Theorem 4.21. *For every space X homotopy equivalent to a CW-complex, we have $K_*(X) \cong MU_*(X) \otimes_{MU_*} K_*$.*

For this result, see [C-F2], 10.2, or [Rav], p. 116. The map $MU_* \to K_*$ is given by the Todd genus $Td : MU_{2n} \to \mathbb{Z}$. Since the Todd genus of $\mathbb{CP}^n$ equals 1, the natural transformation $MU_*(X) \to K_*(X), M \to M \otimes T(M)$ is surjective. Therefore, the K-Theory Atiyah-Hirzebruch spectral sequence collapses whenever the complex cobordism AHSS collapses, e. g. for free loop spaces of spheres and complex projective spaces. For odd dimensional spheres, we get (even multiplicatively):

$$K_*(LS^{2k+1}) \cong H_*(LS^{2k+1}) \otimes K_*$$

4.5.4 Oriented Bordism

As in 4.5.2, we have

Proposition 4.22 ([C-F], 15.1)**.** *If X is (homotopy equivalent to) a CW-complex then the Atiyah-Hirzebruch spectral sequence for oriented bordism degenerates at E^2 if and only if $\mu\colon MSO_n(X) \to H_n(X)$ is surjective for all $n \geq 0$.*

As we have described concrete manifold generators in 3.5 and 3.6, we get degeneration for free loop spaces of spheres and (complex and quaternionic) projective spaces. But we can prove even more in some cases:

Theorem 4.23 ([C-F], 15.2)**.** *If X is (homotopy equivalent to) a CW-complex for which each $H_n(X)$ ist finitely generated and has no odd torsion, then the bordim spectral sequence degenerates at E^2. Moreover*

$$MSO_n(X) \cong \bigoplus_{p+q=n} H_p(X; MSO_q)$$

We can apply this theorem to these free loop spaces of spheres and complex or quaternionic projective spaces which have no odd torsion in homology. Therefore, we have additive isomorphisms

$$\begin{aligned} MSO_*(LS^{2k}) &\cong H_*(LS^{2k}; MSO_*(pt)) \\ MSO_*(LS^{2k+1}) &\cong H_*(LS^{2k+1}) \otimes MSO_*(pt) \\ MSO_*(\mathbb{CP}^{2^k-1}) &\cong H_*(L\mathbb{CP}^{2^k-1}; MSO_*(pt)) \\ MSO_*(\mathbb{HP}^{2^k-1}) &\cong H_*(L\mathbb{HP}^{2^k-1}; MSO_*(pt)) \end{aligned}$$

Sadly enough, MSO_* is not torsionfree, but has also 2-torsion (a complete determination can be found in [Wal]). Therefore, we cannot decide by this method whether these isomorphisms hold also for the multiplicative structure except in the case of odd-dimensional spheres.

A Generalized Spaces and Spectral Sequences

In this appendix we are going to give a treatment of a certain generalization of topological spaces and how to extend homology theories to these. I hoped these considerations would lead to a new description of the Serre spectral sequence which is functorial on E^1, but I was unable to prove that the spectral sequence described below is isomorphic to the Serre spectral sequence, although it converges to the homology of the total space of a fibration (for bordism groups). In spite of this, I hope that some definitions and thoughts might be interesting for other applications or for a more succesfull attempt towards the same application.

A.1 Definitions

What is an object in a category? An algebraic geometer would say: the functor it represents. This makes sense, because by the Yoneda lemma the functor uniquely specifies the object representing it. A natural generalization of an object in a category is then to consider (certain classes of) (set-valued) functors on the category. We will take another route and identify an object with the class of morphisms into this object. Now a natural generalization of an object is a class of morphisms in the category satisfying some axioms:

Definition A.1. Let $\mathcal{C}$ be a category with coproducts and $\mathcal{V}$ be a class of morphisms in $\mathcal{C}$. Then $\mathcal{V}$ is called a *generalized object* of $\mathcal{C}$ iff the following axioms hold:

1. If $f\colon X \to Y$ is a morphism in $\mathcal{C}$ and $g\colon Y \to Z$ is in $\mathcal{V}$, then $g \circ f\colon X \to Z$ is in $\mathcal{V}$.

2. For every $X \in Ob(\mathcal{C})$ the class of morphisms in $\mathcal{V}$ with source X is a set.

A *pair of generalized objects* $(\mathcal{X}, \mathcal{A})$ consists of two generalized objects $\mathcal{A}$ and $\mathcal{X}$ such that $\mathcal{A} \subset \mathcal{X}$. A generalized object in Top is called *generalized space*.

These axioms are chosen in a way that a homology theory can be extended to generalized spaces (see below). It is not known to the author if some similiar axioms were considered by other people.

Example A.2. Let Y be a space and $\mathcal{S}$ be a class of spaces. Maps of the form $f\colon X \to Y$ which factor over some space in $\mathcal{S}$ form a generalized space.

Denote by $s\colon Mor(\mathcal{C}) \to Ob(\mathcal{C})$ the *source map*.

Definition A.3. A *morphism* between two generalized objects $\mathcal{V}$ and $\mathcal{W}$ consists of a map $F\colon \mathcal{V} \to \mathcal{W}$ with the following two properties:

1. $s(F(f)) = s(f)$ for a morphism $f \in \mathcal{V}$.

2. $F(f \circ g) = F(f) \circ g$ for two composable morphisms f and g with $f \in \mathcal{V}$.

We call the category of all generalized object $\mathcal{GC}$. The category of pairs of generalized objects is denoted by $\mathcal{GC}^2$

We get a functor from $\mathcal{C}$ to the $\mathcal{GC}$ by sending an object X to the class $\mathcal{V}_X$ of morphisms into X. This is a full embedding. Indeed, take a morphism $F\colon \mathcal{V}_X \to \mathcal{V}_Y$. Then by property 2 of morphisms we get $F(f) = F(\mathrm{id} \circ f) = F(\mathrm{id}) \circ f$ for a morphism with target X. So F is induced by $F(\mathrm{id})$.

A.2 Extensions of Homology Theories

We will concentrate on generalized spaces with one extra property, the so called *gluing axiom*:

Definition A.4. A generalized space $\mathcal{X}$ is called a *generalized space with gluing* if the following holds: Let M and N be n-manifolds with boundary and $\partial M = \partial N$. Let $f\colon M \to X$ and $g\colon M \to X$ be maps in $\mathcal{X}$ with $f|_{\partial M} = g|_{\partial N}$. Then the pushout map $M \cup_{\partial M} N \to X$ is in $\mathcal{X}$, too.

Let h_* be a homology theory on spaces. As in 2.2, one can think of $h_*(X, A)$ as bordism classes of triples (P, a, f) where P is a compact h_*-oriented manifold with boundary, a is a class in $h^*(P)$ and $f\colon (P, \partial P) \to (X, A)$ is a map (+one extra relation). For $(\mathcal{X}, \mathcal{A})$ a pair of generalized spaces with gluing we make the following definition:

Definition A.5. A *geometric cycle* is a triple (P, a, f) where P is a compact h_*-oriented manifold with boundary, a is a class in $h^*(P)$ and $f \in \mathcal{X}$ with $f|_{\partial P} \in \mathcal{A}$. We define an equivalence relation generated by:

1. (Bordism relation) We call two triples (P, a, f) and (P', a', f') bordant, if there is a (W, b, g) with $g \in \mathcal{X}$, such that $P \coprod (-P') \subset \partial W$ is a regularly embedded submanifold of codimension 0 which inherits the h^*-orientation of W. We require further that $b|_P = a, b|_{P'} = a'$, $g|_P = f, g|_{P'} = f'$ and $g|_{(\partial W - P \coprod P')} \in \mathcal{A}$. Two bordant cycles are defined to be equivalent.

2. (Vector bundle modification) Let (P, a, f) be a geometric cycle and consider a smooth h^*-oriented d-dimensional vector bundle $\pi\colon E \to P$, take the unit sphere bundle $S(E \oplus 1)$ of the Whitney sum of E with a copy of the trivial line bundle over P. The bundle $S(E \oplus 1)$ admits a section s, by $s_!\colon h^*(P) \to h^{*+d}(S(E \oplus 1))$ we denote the Gysin morphism in cohomology associated to this section. Then we impose: $(P, a, f) \sim (S(E \oplus 1), s_!(a), fp)$. (note $fp \in \mathcal{X}$)

The *dimension* of geometric cycle is defined as the difference of the dimension of P and the dimension of a. We denote by $h_k(\mathcal{X}, \mathcal{A})$ the class of geometric cycles of dimension k modulo the above relations.

Note that $h_k(\mathcal{X}, \mathcal{A})$ is a set, because the category of manifolds has a small skeleton and for every P there are only set-many morphisms in $\mathcal{X}$ with source P. We get a boundary map $h_n(\mathcal{X}, \mathcal{A}) \to h_{n-1}(\mathcal{A})$ by taking the boundary of the representing cycle. Furthermore, morphisms of pairs of generalized spaces induce homomorphisms on homology groups. As in the case of topological spaces one can prove many properties for these homology groups. We will be content with the long exact sequence of the pair since this is enough for our applications.

Lemma A.6. *Let P be a closed manifold with a cohomology class $a \in h^*(P)$, Q be a regular submanifold with boundary and furthermore $(\mathcal{X}, \mathcal{A})$ be a pair of generalized spaces. If $f\colon M \to X$ is in $\mathcal{X}$ and $f|_{P-int(Q)} \in \mathcal{A}$, then $[P, a, f] = [Q, a|_Q, f|_Q]$ in $h_n(\mathcal{X}, \mathcal{A})$.*

Proof. Let $F\colon P \times I \to X$ be defined by $f \circ \mathrm{pr}_2$. Now $\partial(I \times P) = \partial I \times P$ and $Q \times 1 \coprod -P \times 0$ is a regular submanifold of the boundary. Since $F|_{(P-Int(Q))\times 1)} \in \mathcal{A}$ we get a bordism $(P \times I, \mathrm{pr}_1^*(a), F)$ and have $[P, a, f] = [Q, a|_Q, f|_Q]$. $\square$

Proposition A.7. *Let $(\mathcal{X}, \mathcal{A})$ be a pair of generalized spaces with glueing. Then we have a long exact sequence of homology groups:*

$$\cdots \to h_n(\mathcal{A}) \overset{i_*}{\to} h_n(\mathcal{X}) \overset{j_*}{\to} h_n(\mathcal{X}, \mathcal{A}) \overset{\partial}{\to} h_n(\mathcal{A}) \to \cdots$$

Here $i\colon \mathcal{A} \to \mathcal{X}$ and $j\colon (\mathcal{X}, \varnothing) \to (\mathcal{X}, \mathcal{A})$ are the inclusions.

Proof. Then proof is completely analogous to the classical case (see e. g. [C-F], 5.6). It is clear that $\partial j_* = 0$ and also $i_*\partial = 0$ with the obvious zero bordism. To show $j_*i_* = 0$ consider $[P, a, f] \in h_n(\mathcal{A})$. Then apply the preceding lemma with $Q = \varnothing$.

Next consider $[P, a, f] \in h_n(\mathcal{X}, \mathcal{A})$ which is in the kernel of ∂. By definition then there is a triple (Q, b, g) which is a bordism for $(\partial P, a|_{\partial P}), f|_{\partial P}$, i.e. $\partial Q = \partial P$, $b|_{\partial Q} = a|_{\partial P}$, $g|_{\partial Q} = f|_{\partial Q}$ with $g \in \mathcal{A}$. By Mayer-Vietoris and the glueing axiom we get a a class $[R, c, h] = [P \cup_{\partial P} Q, a \cup_{a|_{\partial P}} b, f \cup g] \in h_n(\mathcal{X})$. By the lemma we get that $j_*[R, c, h] = [P, a, f]$.

Now consider an element $[P, a, f] \in h_n(\mathcal{X})$ in the kernel of j_*. We have a bordism (Q, b, g) with $P \subset \partial Q$ a component of the boundary and $g|_{\partial Q - P} \in A$. We have $i_*[\partial Q - P, b|_{\partial Q-P}, g|_{\partial Q-P}] = [P, a, f]$.

Last consider $[P, a, f] \in h_n(\mathcal{A})$ in the kernel of i_*. We get a zero bordism (Q, b, g) with $\partial Q = P$ and $g \in \mathcal{X}$. This (Q, b, g) defines now a class in $h_{n+1}(\mathcal{X}, \mathcal{A})$ which is mapped to $[P, a, f]$ under ∂. □

If we pause a moment and look back which instances of axiom 1 of generalized spaces we haved use in the proof of exactness, these are only three:

1. f is the inclusion of the boundary into a manifold
2. f is the projection map of a fibre bundle (in the case of sphere bundles and $P \times I \to P$)
3. $(f\colon P \coprod Q \to X) \in \mathcal{X}$ if $(P \to X) \in \mathcal{X}$ and $(Q \to X) \in \mathcal{X}$ (this is only an instance of axiom 1 in the case if $Q \subset P$ and $f|_Q = (f|_P)|_Q$ which is the only one used)

We call a class satisfying these three conditions instead of axiom 1 a *weak generalized space.*

A.3 The Serre Spectral Sequence - Revisited

Let $\xi = (F \to E \to B)$ be a fibre bundle which projection map π. Let $\mathcal{P}_n(\xi)$ be the class of maps $f\colon P \to E$ where P is a manifold which can be glued together of finitely many manifolds P_i along their boundary components (and each P_i is glued to each P_j in at most one boundary component) and $\pi f|_{P_i}$ factors over an n-dimensional p-stratifold (or n-dimensional simplicial complex/n-dimensional CW-complex) for each i. Clearly $\mathcal{P}_n(\xi)$ is a weak generalized space with gluings. Therefore, we get an exact couple Δ:

$$\begin{array}{ccc} \bigoplus_{p,q} h_{p+q}(\mathcal{P}_p(\xi)) & \xrightarrow{\quad i \quad} & \bigoplus_{p,q} h_{p+q}(\mathcal{P}_p(\xi)) \\ & {}_{k}\nwarrow \qquad \swarrow_{j} & \\ & \bigoplus_{p,q} h_{p+q}(\mathcal{P}_p(\xi), \mathcal{P}_{p-1}(\xi)) & \end{array}$$

Now assume, B is triangulated. For example, every smooth manifold can be triangulated. Then we get from the exact couple of the classical description of the Serre spectral sequence C (see 4.2):

$$\begin{array}{ccc} \bigoplus_{p,q} h_{p+q}(E^{(p)}) & \xrightarrow{\quad i \quad} & \bigoplus_{p,q} h_{p+q}(E^{(p)}) \\ & {\scriptstyle k}\nwarrow \qquad \swarrow{\scriptstyle j} & \\ & \bigoplus_{p,q} h_{p+q}(E^{(p)}, E^{(p-1)}) & \end{array}$$

Since every simplicial complex can be given the structure of a p-stratifold, we get a morphism $L\colon C \to \Delta$ of exact pairs. In general, this is probably not an isomorphism at any finite level. But observe that for h a bordism theory the spectral sequence is convergent and converges to $h_*(E)$, since every manifold is a p-stratifold.

The reader sees that the problem is the following: While every map $f\colon P \to B$ from a manifold factoring over an n-dimensional p-stratifold can be homotoped to the n-skeleton of B, this is no longer true for a manifold glued together from manifolds which factor over n-dimensional p-stratifolds (at least the author suspects this). The problem with the former ansatz (i.e. to consider manifolds which factor over p-stratifolds) is that it is not clear whether there is a long exact sequence since this class forms a generalized space without gluings.

B Zusammenfassung

Die String-Topologie beschäftigt sich mit algebraischen Strukturen auf der Homologie von Abbildungsräumen zwischen Mannigfaltigkeiten, insbesondere auf dem freien Schleifenraum. Dieser ist der Raum aller geschlossenen Wege auf einer Mannigfaltigkeit.

Historisch gesehen war die erste Struktur auf der Homologie eines Raums das Schnittprodukt auf der Homologie einer Mannigfaltigkeit M, das als Poincare-Duales des Cup-Produkts gesehen werden kann. Auf Räumen ohne Poincare-Dualität gibt es allerdings im Allgemeinen kein Produkt auf der Homologie. Daher war es durchaus überraschend, als 1999 Chas und Sullivan ein Produkt auf der Homologie des freien Schleifenraums LM einer Mannigfaltigkeit definierten, obwohl es hier aufgrund der Unendlichdimensionalität von LM keine Poincare-Dualität geben kann. Dieses Produkt wird heutzutage als das *Chas-Sullivan-Produkt* bezeichnet und ist der Form $H_p(LM) \otimes H_q(LM) \to H_{p+q-d}(LM)$, wobei d die Dimension von M ist.

Falls zwei Homologieklassen auf M durch Abbildungen von Mannigfaltigkeiten repräsentiert werden, kann man das Schnittprodukt als ihren transversalen Schnitt beschreiben. Chas und Sullivan ahmten eine ähnliche Definition auf der Ebene des freien Schleifenraums nach. Allerdings waren ihre Definiionen und Beweise nicht immer klar und präzise. Cohen und Jones ([C-J]) fanden einen Weg, das Chas-Sullivan-Produkt mittels Homotopietheorie und dem Thom-Isomorphismus zu beschrieben; dies erlaubte auch, diese algebraische Struktur auf beliebige Homologietheorien zu verallgemeinern.

Wenn man jede Homologieklasse auf M durch Mannigfaltigkeiten repräsentieren könnte, wäre es einfach, eine (präzise und) geometrische Beschreibung des Chas-Sullivan-Produktes zu geben. Dies ist allerdings nicht der Fall wie Thom zeigte. Chataur ([Cha]) fand einen Weg, dieses Problem zu umgehen, indem er M. Jakobs Theorie der geometrischen Homologie benutzte; diese stattet die Mannigfaltigkeiten mit der Zusatzstruktur einer Kohomologieklasse aus, wodurch dann jede Homologieklasse (in einer beliebigen Homologietheorie) repräsentierbar wird. Wir werden M. Krecks Theorie der Stratifolds benutzen, um eine alternative Beschreibung anzugeben. Diese sind eine etwaig singulär Variante von Mannigfaltigkeiten und bieten den Vorteil, eine noch geometrischere Beschreibung des Chas-Sullivan-Produktes zu geben als die Chataurs. So sind wir in der Lage, eine komplett endlich-dimensionale Definition des Produkts zu geben, in der wir keine unendlich-dimensionalen Räume benutzen müssen.

Um geometrische Methoden anwenden zu können, müssen wir ein Homotopie-Modell des freien Schleifenraums benutzen, das eine Hilbertmannigfaltigkeit ist; diese sind Mannigfaltigkeiten, die statt an einem $\mathbb{R}^n$ an einem (unendlich-dimensionalen) Hilbertraum modelliert sind. In diesem Kontext konstruieren wir sogenannte *Gysin-Abbildungen*. Während Homologie üblicherweise kovariant ist, sind sie für eine Unterhilbertmannigfaltigkeit $L \subset X$ von endlicher Kodimension d Homomorphismen $h_*(X) \to h_{*-d}(L)$. Auf dem Level von Stratifolds kann man sich Gysin-Abbildungen als transversalen Schnitt vorstellen. Wir geben auch Beschreibungen mit Hilfe des Thomisomorphismus und mit Jakobs Theorie der geometrischen Homologie und zeigen, das alle diese Definitionen in unserem Kontext äquivalent sind. Die Gysin-Abbildungen erlauben uns dann eine einfache Definition des Chas-Sullivan-Produkts und auch von weiteren algebraischen Strukturen.

Nun stellt sich die Frage: sind diese algebraischen Strukturen auch berechenbar? Die Lieblingsberechnungsmaschinen der algebraischen Topologen sind Spektralsequenzen. Zu jeder Faserung gibt es eine Spektralsequenz, die (zumindest im Prinzip) aus der Homologie der Faserung und der Basis die Homologie des Totalraums ausrechnet, die sogenannte *Serre-Spektralsequenz*. Im Falle der Faserung $\Omega M \to LM \to M$ und singulärer Homologie waren Cohen, Jones und Yan in [CJY] in der Lage, die Serre-Spektralsequenz mit einer multiplikativen Struktur auszustatten, die auf dem E^2-Term $H_*(M; H_*(\Omega M))$ durch das Schnittprodukt gegeben ist, wobei die Ringstruktur auf $H_*(\Omega M)$ durch die Komposition von Schleifen gegeben ist. Da die Homologie des (punktierten) Schleifenraums zugänglicher ist als die des freien, konnten sie so die Produktstrukturen auf $H_*(LS^n)$ und $H_*(L\mathbb{CP}^n)$ berechnen.

Wir nehmen diesen Artikel als Startpunkt, um ihre Ergebnisse in drei Richtungen zu vertiefen und zu erweitern. Erstens wollen wir sie konkretisieren, indem wir explizite Mannigfaltigkeitenerzeuger für die Homologie von den freien Schleifenräumen von Sphären und projektiven Räumen angeben. Zweitens verallgemeinern wir ihre Spektralsequenz auf allgemeine Homologietheorien, allgemeinere algebraische Strukturen und statten auch die Serre-Spektralsequenz zu $LM \to LN \to LO$ (wobei $M \to N \to O$ ein Faserbündel ist) mit einer multiplikativen Struktur aus. Wir erreichen dies durch die Konstruktion von Gysin-Abbildungen von Spektralsequenzen.

Die dritte Richtung besteht darin, dass wir versuchen, konkret das Chas-Sullivan-Produkt von weiteren Räumen, nämlich Sphärenbündeln, in singulärer Homologie und von Sphären und projektiven Räumen in verallgemeinerten Homologietheorien (nämlich komplexer K-Theorie und orientiertem und komplexem Bordismus) auszurechnen. Dies erreichen wir, indem wir mit Hilfe von rationaler Homotopietheorie und unseren Beschreibungen von Erzeugern in singulärer Homologie Degeneration von bestimmten Spektralsequenzen beweisen.

Wir wollen hier kurz die Ergebnisse zusammenfassen, die wir im einfachsten Fall, nämlich dem der ungeraddimensionalen Sphären, erhalten. Sei also d ungerade. Dann gilt nach [CJY]:

$$H_{*+d}(LS^d) \cong \mathbb{Z}[u] \otimes \Lambda(a)$$

Die Grade der Erzeuger sind gegeben durch $|u| = d-1$ und $|a| = -d$. Der Indexshift auf der linken Seite ist nötig, weil beim Chas-Sullivan-Produkt selbst ein Indexshift auftritt. Der Erzeuger a wird durch einen Punkt repräsentiert. Der Erzeuger u kann durch eine Abbildung vom Einheitssphärenbündel STS^d repräsentiert werden. Diese Beschreibung hilft, Degeneration der Atiyah-Hirzebruch-Spektralsequenzen für komplexen und orientierten Bordismus zu beweisen und wir bekommen:

$$\begin{aligned} K_{*+d}(LS^d) &\cong \mathbb{Z}[u] \otimes \Lambda(a) \otimes K_* \\ MU_{*+d}(LS^d) &\cong \mathbb{Z}[u] \otimes \Lambda(a) \otimes MU_* \\ MSO_{*+d}(LS^d) &\cong \mathbb{Z}[u] \otimes \Lambda(a) \otimes MSO_* \end{aligned}$$

Alle diese Isomorphismen sind multiplikativ. Die Grade sind die gleichen wie oben. Für ein Bündel $S^d \to E \to S^e$ mit d, u ungerade, erhalten wir

$$H_{*+(d+e)}(E) \cong \mathbb{Z}[u_1, u_2] \otimes \Lambda(a_1, a_2),$$

wobei $|u_1| = d-1$, $|u_2| = e-1$, $|a_1| = -d$ und $|a_2| = -e$. Additiv lassen sich ähnliche Ergebnisse auch für Sphärenbündel mit entweder geraddimensionaler Basis oder geraddimensionaler Faser erzielen.

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