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# Erratum for *Gorenstein duality for real spectra*

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In this erratum we report a mistake in the authors' article [2] and record a corrected statement. Throughout this erratum we will use the notation of [2] and all numbering of lemmas and theorems refers to this source.

Theorem 1.4 is not correct as stated for $n \geq 2$, even on underlying non-equivariant spectra. Indeed, the homotopy groups of $\Gamma_{J_{n-1}} E\mathbb{R}(n)$ are countable in every degree and this is not true for $\mathbb{Z}_{(2)}^{E\mathbb{R}(n)}$. Let us explain where the mistake in the proof lies and then how to correct the statement.

The mistake is in Lemma 5.8. Consider for example $B = BP\mathbb{R}\langle 2\rangle$, whose underlying homotopy groups are $\mathbb{Z}_{(2)}[v_1, v_2]$. Contrary to the statement of Lemma 5.8, the $\mathbb{Z}[v_2]$-module $\mathbb{Z}_{(2)}[v_1^{\pm 1}, v_2]$ does not have bounded $v_2$-divisibility. Indeed, for every $k$ the element $v_2^k v_1^{-3k}$ is of degree 0 and divisible by $v_2^k$. Thus, the proof of Theorem 5.9 breaks down and the theorem is indeed wrong as stated. Note though that Example 5.10 (recovering a result of Ricka) remains unaffected.

What happens indeed is that $\mathbb{Z}_{(2)}^{E\mathbb{R}(n)}$ is a kind of completion of a shift of $\Gamma_{J_{n-1}} E\mathbb{R}(n)$, at least on the level of underlying homotopy groups. Instead of making this precise, we show that Theorems 1.4 and 5.9 are true after a suitable cellularization. For an $M\mathbb{R}$-module $K$, we say that there is a *$K$-cellular equivalence* between two $M\mathbb{R}$-modules if there is an $M\mathbb{R}$-linear equivalence between their $\mathbb{R}$-cellularizations with respect to $K$ in the sense of Section 2.B and Proposition 3.8.

**Theorem** (Corrected form of Theorem 5.9) *Let the notation be as in Theorem 5.1 and assume for simplicity that only finitely many $m_i$ are zero and that $m_n = 0$. Let $K$ be the $M\mathbb{R}$-module $M\mathbb{R}/(\underline{\overline{v}} \setminus \overline{v}_n)$, where $\underline{\overline{v}} \setminus \overline{v}_n$ denotes the sequence of all $\overline{v}_i$ such that $m_i = 0$ and $i \neq n$. Then there is a $K$-cellular equivalence*

$$\mathbb{Z}_{(2)}^{M[\overline{v}_n^{-1}]} \simeq_K \Sigma^{-m' + |\underline{\overline{v}}| + (k-1) + 4 - 2\rho} M.$$

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**Proof** Modifying the proof of Theorem 5.9, it suffices to show that there is a $K$-cellular equivalence between $\operatorname{holim}\left(\cdots \xrightarrow{\overline{v}_n} \Gamma_{\overline{v}_n}(\Gamma_{\underline{\overline{v}}\setminus\overline{v}_n}M)\right)$ and $\Sigma^{-1}M[\overline{v}_n^{-1}]$. For this, we claim first that the natural map

$$\operatorname{holim}\left(\cdots \xrightarrow{\overline{v}_n} \Gamma_{\overline{v}_n}(\Gamma_{\underline{\overline{v}}\setminus\overline{v}_n}M)\right) \to \operatorname{holim}\left(\cdots \xrightarrow{\overline{v}_n} \Gamma_{\overline{v}_n}M)\right) \tag{1}$$

is a $K$-cellular equivalence. We have indeed a chain of equivalences

$$\begin{aligned}\operatorname{Hom}_{M\mathbb{R}}\left(K, \operatorname{holim}\left(\cdots \xrightarrow{\overline{v}_n} \Gamma_{\overline{v}_n}(\Gamma_{\underline{\overline{v}}\setminus\overline{v}_n}M)\right)\right) &\simeq \operatorname{holim}\operatorname{Hom}_{M\mathbb{R}}\left(K, \Gamma_{\overline{v}_n}(\Gamma_{\underline{\overline{v}}\setminus\overline{v}_n}M)\right)\\ &\simeq \operatorname{holim}\operatorname{Hom}_{M\mathbb{R}}\left(K, \Gamma_{\overline{v}_n}M)\right)\\ &\simeq \operatorname{Hom}_{M\mathbb{R}}\left(K, \operatorname{holim}\left(\cdots \xrightarrow{\overline{v}_n} \Gamma_{\overline{v}_n}M\right)\right).\end{aligned}$$

Thus, the map in (1) is a $K$-cellular equivalence. Moreover,

$$\operatorname{holim}\left(\cdots \xrightarrow{\overline{v}_n} \Gamma_{\overline{v}_n}M)\right) \simeq \Sigma^{-1}M[\overline{v}_n^{-1}]$$

by Lemma 5.7. Combining this equivalence and the $K$-cellular equivalence (1) gives the result. □

In particular, we obtain:

**Theorem** (Corrected form of Theorem 1.4) *For each $n \geq 1$ set $K = M\mathbb{R}/(\overline{v}_1, \ldots, \overline{v}_{n-1})$. Then we have a $K$-cellular equivalence*

$$\mathbb{Z}_{(2)}^{E\mathbb{R}(n)} \simeq_K \Sigma^{D_n\rho+(n-1)+2(1-\sigma)}E\mathbb{R}(n).$$

**Remark** We explain how classical algebra shows that cellularization should be expected in these kinds of examples.

Most basically, let $R$ be classical local Noetherian $k$-algebra $R$ with residue field $k$. The $R$-module $\operatorname{Hom}_k(R,k)$ plays the role of an Anderson dual, but it is not $k$-cellular. Its $k$-cellularization agrees with the injective hull $I(k)$ of $k$; see Dwyer, Greenlees and Iyengar [1, 7.1].

Dropping the $k$-algebra assumption, for a commutative Gorenstein local ring $R$ with residue field $k$ and Krull dimension $d$, we have $\Gamma_{\mathfrak{m}}R \simeq \Sigma^{-d}I(k)$. The statement corresponding to the corrected form of 1.4 is $R \simeq_k \Sigma^{-d}I(k)$. The subscript $k$ refers to the fact that the equivalence is only true after (derived) completion at the maximal ideal or after $k$-cellularization.

## References


[1] **W G Dwyer**, **J P C Greenlees**, **S Iyengar**, *Duality in algebra and topology*, Adv. Math. 200 (2006) 357–402

[2] **J P C Greenlees**, **L Meier**, *Gorenstein duality for real spectra*, Algebr. Geom. Topol. 17 (2017) 3547–3619



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