Integral equivariant elliptic cohomology

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(joint work with David Gepner)

1. Motivation

The most classical equivariant cohomology theory is *Borel equivariant cohomology*, sending a space X with a G-action to $H^*(EG \times_G X; \mathbb{Z})$. For us, G will always be a compact Lie group. For $X = \mathrm{pt}$, we obtain $H^*(BG; \mathbb{Z})$. As shown by Swan [6], $H^*(BG; \mathbb{Z})$ is nonzero in infinitely many degrees for every non-trivial G and thus $H^*(BG; \mathbb{Z})$ will never be finitely generated as an abelian group.

Equivariant K-theory enjoys better finiteness properties. Assuming from now on X to be a finite G-CW-complex (e.g. any smooth compact G-manifold), $K^0_G(X)$ is defined as the Grothendieck group of G-equivariant complex vector bundles on X. Thus, $K^0_G(\mathrm{pt})$ is the representation ring $R(G)$ of G, which coincides as an abelian group with the free abelian group on the set of irreducible complex representations of G. Thus, $K^0_G(\mathrm{pt})$ is finitely generated for every finite G. But for $G = S^1$, we obtain $K^0_{S^1}(\mathrm{pt}) \cong \mathbb{Z}[t^{\pm 1}]$, being of infinite rank.

Question. *What are examples of equivariant cohomology theories having good finiteness properties for all compact Lie groups?*

Our answer will be *equivariant elliptic cohomology.* To motivate it, let us first reinterpret equivariant K-theory in algebro-geometric terms. For every G, we can consider the scheme $\operatorname{Spec} R(G)$. The $R(G)$-algebra structure on $K^0_G(X)$ corresponds to considering $K^0_G(X)$ as a sheaf of quasi-coherent algebras on $\operatorname{Spec} R(G)$. For $G = S^1$, we obtain $\operatorname{Spec} R(S^1) \cong \operatorname{Spec} \mathbb{Z}[t^{\pm 1}] = \mathbb{G}_m$, the multiplicative group – this represents the functor sending a commutative ring to its group of units. The group structure is actually induced by the multiplication map $S^1 \times S^1 \to S^1$. We could also paint a similar picture for Borel equivariant cohomology, where we see $\operatorname{Spec} H^*(BS^1) \cong \operatorname{Spec} \mathbb{Z}[x] \cong \mathbb{G}_a$, the additive group. The lack of finiteness in these examples corresponds to the fact that $\mathbb{G}_a$ and $\mathbb{G}_m$ are not *proper.* But there is a third family of one-dimensional proper group schemes, namely elliptic curves.

2. Equivariant Elliptic Cohomology

The aim of equivariant elliptic cohomology is repeat the above story for equivariant K-theory, replacing $\mathbb{G}_m$ by a fixed elliptic curve E. In particular, the S^1-equivariant theory takes values in quasi-coherent sheaves on E. The original motivations for constructions such theories came both from the theory of elliptic genera (Miller, Rosu) and from geometric representation theory. Motivated by the latter, Grojnowksi gave in [3] the first construction of equivariant elliptic cohomology for elliptic curves *over the complex numbers* and for *connected* compact Lie groups of equivariance. Much more recently, Berwick-Evans and Tripathy [1] developed a coherent theory for all compact Lie groups, but still over the complex numbers. In [4], Lurie gave a sketch how to obtain a theory without restricting to complex coefficients. Our work follows the same outline Lurie gives and is also heavily based on Lurie's foundational work on spectral algebraic geometry.

We will present the general form of our theory in a way that is inspired by the axiomatics from Ginzburg–Kapranov–Vasserot [2]. As they already mention, there are many advantages to work in the derived context, which means here in the context of spectral algebraic geometry. A *spectral scheme* is a topological space X together with a sheaf $\mathcal{O}_X$ of E_∞-ring spectra such that $(X, \pi_0\mathcal{O}_X)$ is a usual scheme, plus two more technical conditions. Like in usual algebraic geometry, we can talk about quasi-coherent sheaves on a spectral scheme. We refer to [5] for further details.

Using this language, let us explain the outline of our theory. Let S be a spectral base scheme such that $\mathcal{O}_S$ is even-periodic and let E be a spectral elliptic curve over S, plus a futher piece of datum, called an orientation. Our construction gives us:

(1) for every G, a spectral S-scheme X_G;
(2) for every G, a functor $\mathcal{E}ll_G\colon$ finite G-CW complexes$^{\mathrm{op}} \to \mathrm{QCoh}(X_G)$, sending G-homotopy equivalences to equivalences;
(3) for each group homomorphism $\varphi\colon G \to H$ an affine morphism

$$X_\phi\colon X_G \to X_h$$

such that $(X_\varphi)_*\mathcal{E}ll_G(Z) \simeq \mathcal{E}ll_H(G \times_H Z)$.

In the case of K-theory, X_G corresponds to $\operatorname{Spec} K_G(\mathrm{pt})$ and the functor $\mathcal{E}ll_G$ corresponds to viewing $K_G(X)$ as a $K_G(\mathrm{pt})$-module and hence a quasi-coherent sheaf on $\operatorname{Spec} K_G(\mathrm{pt})$.

In our case, the spectral schemes X_G are in general hard to describe explicitely. For G abelian, however, we have $X_G \simeq \mathrm{Hom}(\widehat{G}, E)$, where $\widehat{G}$ is the Pontryagin dual. As the Pontryagin dual of S^1 is $\mathbb{Z}$, this gives us in particular $X_{S^1} \simeq E$. In the case $G = U_n$ one can identify X_G with the Hilbert scheme of length n divisors on E. In general, we have the structural result that X_G is always proper over S, which gives strong finiteness results.

Theorem (Gepner–M.)**.** *Assume that $S = \operatorname{Spec} R$ is affine. Then for every finite G-CW-complex X, the global sections $\Gamma(\mathcal{E}ll_G(X))$ are a finite R-module.*

The relevance of these global sections is that the composite functor

$$\text{finite } G\text{-CW complexes}^{\mathrm{op}} \xrightarrow{\mathcal{E}ll_G} \mathrm{QCoh}(X) \xrightarrow{\Gamma} \mathrm{Mod}_R \xrightarrow{\pi_{-n}} \mathrm{AbelianGroups}$$

is an equivariant cohomology theory in the classical sense, having the finiteness properties we asked for.

By results of Lurie, the most canonical spectral elliptic curve with an orientation is a spectral refinement of the universal elliptic curve. While the base is not an (affine) spectral scheme, the results above apply, mutatis mutandis. In particular, this yields a genuine equivariant refinement of the spectrum TMF of topological modular forms such that all of its G-fixed points are finite TMF-modules.

References

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