A Hilbert Manifold Model for Mapping Spaces

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Let M be an n-dimensional compact manifold and N be an arbitrary manifold. The mapping space $Map(M,N)$ can be equipped with the compact-open topology. This can be seen as an infinite-dimensional manifold modelled on Banach spaces. For many geometric purposes, such Banach manifolds are inconvenient since several of the theorems one knows from the finite-dimensional cases fail in this context. Hilbert manifolds, i.e. manifolds modelled on separable Hilbert spaces, behave much better (see e.g. [Lan]).

Transversality theorems for Hilbert manifolds are used in work by Chataur and also by the author to provide geometric descriptions of string topology. Since for algebraic topology applications finally only the homotopy type of $Map(M,N)$ matters, we are interested in getting a Hilbert manifold homotopy equivalent to $Map(M,N)$. To that end, we use the theory of Sobolev spaces to construct a Hilbert manifold $H^n(M,N)$ which serves our purposes. It should be noted that our main results are known for a long time. It has been stated which a sketched proof, for example, in [E-M]. Furthermore, a great part of our section 2 was proved in greater generality in [Pal2]. In spite of that, the author has the opinion that it is good to have a complete and concrete proof of this fundamental construction written down. Our constructions and argumentation are modelled on the situation of the free loop space as discussed in [Kli], 1.2. There is no claim for originality.

1 A Hilbert Manifold Model

We define $H^n(M,N) \subset Map(M,N)$ as the space of all continuous maps $f\colon M \to N$ such that there is for every $p \in M$ a chart (U,ϕ) around p and a chart (V,ψ) around $f(p)$ such that $\psi f \phi^{-1}\colon \phi(U) \to \psi(V)$ is of Sobolev class H^n[1]. Note that every L^2-map $\phi(U) \to \psi(V)$ of class H^n has a unique continuous representative (see [Alt], 8.13). To construct an atlas for $H^n(M,N)$, we choose Riemannian metrics on M and N. As a preliminary notion, we define the following:

Definition 1.1. Define for $f \in Map(M,N)$ the *energy* $E(f) := \int_M ||T_pf||^2$ and the *length* $L(f) := \int_M ||T_pf||$ of f. Here $||T_pf|| = \max_{v\in T_pM, |v|=1} |T_pf(v)|$.

[1] Some authors denote this also by L^2_n or $W^{2,n}$.

Let $g\colon O \to M$ be a map from a Riemannian manifold O. Then $||T_q(fg)|| \leq ||T_{g(q)}f|| \cdot ||T_q g||$ for $q \in O$. Therefore, by Cauchy-Schwarz we have $L(fg) \leq \sqrt{E(f) \cdot E(g)}$.

Consider some $f \in \mathcal{C}^\infty(M,N)$. Since the image of f is compact, there exists an $\varepsilon > 0$ such that

$$\exp \times \pi : TN \to N \times N$$

induces a diffeomorphism from the ε-neighbourhood $\mathcal{O}_\varepsilon \subset TN$ of the zero section onto an open neighbourhood of the diagonal in $N \times N$. Let $f_*\colon f^*TN \to TN$ be the pullback and $\mathcal{O}_f = (f_*)^{-1}(O_\varepsilon)$. We define

$$\begin{aligned} \exp_f\colon H^n(\mathcal{O}_f) &\to H^n(M,N), \\ \xi &\mapsto (p \mapsto \exp(f_*\xi(p))) \end{aligned}$$

Here $H^n(\mathcal{O}_f)$ denotes the following: all continuous sections ξ of $\pi\colon f^*TN \to M$ with image in $\mathcal{O}_f$ such that there is a chart (U,ϕ) around every point of M and a local trivialization $\pi^{-1}(U) \to U \times \mathbb{R}^k$ such that $\mathrm{pr}_2\, \xi\phi^{-1}\colon \phi(U) \to \mathbb{R}^k$ is of Sobolev class H^n. We see in the next section that $H^n(\mathcal{O}_f) \subset H^n(E)$ is an open subset.

Lemma 1.2. *The map* $\exp_f$ *is injective and its image is the open set*

$$\mathcal{U}_f = \{g \in H^n(M,N)\colon g(p) \in \exp(\mathcal{O}_\varepsilon \cap T_{f(p)}N)\}$$

Proof. The injectivity and that $\operatorname{im}\exp_f \subset \mathcal{U}_f$ are clear. Now we want to show that ξ is of Sobolev class H^n if $\exp_f(\xi)$ is. If $\exp_f(\xi)$ is of Sobolev class H^n, then also $\mathrm{id} \times \exp_f(\xi)\colon M \to M \times N$ is Sobolev class H^n. The map $\mathrm{id} \times \exp_f \circ f_*$ defines a diffeomorphism from $\mathcal{O}_f$ onto an open set in $M \times N$. Reversing this diffeomorphism we get ξ which is therefore also H^n.

To show that $\mathcal{U}_f$ is open, it is enough to show $U_f = \{g \in \mathcal{C}^0(M,N)\colon g(p) \in \exp(\mathcal{O}_\varepsilon \cap T_{f(p)}N)\}$ is open in $\mathcal{C}^0(M,N)$. But U_f is just the ε-ball around f in the maximum metric. □

Theorem 1.3. *The space* $H^n(M,N)$ *is a (smooth) Hilbert manifold.*

Proof. Note that $H^n(\mathcal{O}_f)$ is an open subset of the separable Hilbert space $H^n(f^*TN)$. Therefore, $\exp_f^{-1}\colon \mathcal{U}_f \to H^n(\mathcal{O}_f)$ is a chart for every $f \in \mathcal{C}^\infty(M,N)$. Now we want to show that for every $g \in H^n(\mathcal{O}_f)$ there is a $\mathcal{U}_f$ with $g \in \mathcal{U}_f$. Approximate g by smooth functions f_k in the maximum metric. This can be done locally and then be globalized by a partition of unity. Then choose some $\delta > 0$ such that the closed δ-neighbourhood of $\mathrm{im}(g)$ is compact. Let ε be the minimal injectivity radius on the closed δ-neighbourhood of $\mathrm{im}(g)$. Choose k with $d_\infty(f_k, g) < \min(\delta, \varepsilon)$. Then $g \in \mathcal{U}_{f_k}$. The collection of all $(\mathcal{U}_f, \exp_f^{-1})$ defines now an atlas $\mathcal{A}$ on $H^n(M,N)$. We will deal with the smoothness of $\mathcal{A}$ in the next section.

We still need to show that $H^n(M,N)$ has a countable base of topology. To that end, it suffices to show that our atlas $\mathcal{A}$ has a countable subatlas.

So let $k, c \in \mathcal{N}$ and be natural numbers. Triangulate M with a finite triangulation finer than $\frac{1}{3k\sqrt{c}}$. Let $e_1, \ldots e_s$ be the vertices of this triangulation and denote by P_k a countable set of points p_i in N such that the $\frac{1}{3k}$-balls around the p_i cover N. Choose for every sequence $S = (p_1, \ldots p_s)$ in P_k of cardinality s a function $f_{k,c,S} \in \mathcal{C}^\infty(M,N)$ of energy less than c with $f_{k,c,S}(e_j) = p_j$ for every $p_j \in S$. These are countably many functions.

Now let $g \in H^n(M,N)$ be a function of energy less than c. Choose some compact δ-neighbourhood K of $\mathrm{im}(g)$. Choose furthermore some $k \in \mathcal{N}$ such that the injectivity radius on K is bounded below by $1/k$ and $1/k < \delta$. Triangulate M as above. Then we have for p and q contained in the same simplex:

$$\begin{aligned} d(g(p), g(q)) &\leq L(g\gamma) \leq \sqrt{E(g)}\sqrt{E(\gamma)} = \sqrt{E(g)}L(\gamma) = \sqrt{E(g)}d(p,q) \\ &< \sqrt{c}\frac{1}{3k\sqrt{c}} = \frac{1}{3k} \end{aligned}$$

Here $\gamma\colon [0,1] \to M$ denotes a minimal geodesic connecting p and q. Therefore, every simplex is contained in a $\frac{1}{3k}$-ball. Choose $p_j \in N$ as above such that $g(e_j) \in B_{1/3k}(p_j)$. Let $S = (p_1, \ldots p_s)$ be the sequence defined by these points. Then we claim that $||f_{k,c,S} - g||_\infty < 1/k$ where $|| \bullet ||_\infty$ denotes the supremum norm. Clearly we have $d(f_{k,c,S}(e_j), g(e_j)) < \frac{1}{3k}$. Let Δ be a simplex containing e_j. Then we have by the inequality above $d(f_{k,c,S}(x), f(e_j)) < \frac{1}{3k}$ and $d(g(x), g(e_j)) < \frac{1}{3k}$ for every $x \in \Delta$ by the inequality above. Then the claim follows from the triangle inequality. We see that $g \in \mathcal{U}_{f_{k,c,S}}$. □

2 The Smoothness of the Chosen Atlas

Before we come to the smoothness, we have to study Sobolev spaces of sections of a vector bundle. A good source for this topic is the section III.2 of [L-M]. Although they consider only complex vector bundles, all results carry over to the real case.

So let $\pi\colon E \to M$ be a vector bundle equipped with a connection ∇ and a euclidean metric $|\cdot|$ over a compact base manifold of dimension n. Let the space $\mathcal{C}^k(E)$ be equipped with the norm

$$||\xi||_k^2 = \sum_{i=0}^{k} ||\nabla^i \xi||_0^2.$$

Here the euclidean metric is extended to all forms and $||\cdot||_0$ denotes the L^2-norm. We call the completion of $\mathcal{C}^k(E)$ with respect to this norm $L_k^2(E)$. According to [L-M], we get an equivalent norm for other choices of metric and connection.

Proposition 2.1 ([L-M], III.2.14+2.15)**.** *For each integer $k > n/2$, there is a continuous inclusion $L_k^2(E) \to \mathcal{C}^0(E)$ extending the inclusion of $\mathcal{C}^k(E)$. In particular, there is a constant $K(k)$ with $||u||_k \leq K(k)||u||_\infty$.*

We see that we can identify $L^2_n(E) = H^n(E)$.

Lemma 2.2. *Let $\mathcal{O}$ be an open subset of E. Then $H^n(\mathcal{O}) \subset H^n(E)$ is open.*

Proof. Let $\xi \in H^n(\mathcal{O})$. Since M is compact, there exists an $\varepsilon > 0$ such that for $\eta \in H^n(E)$, $||\eta - \xi||_\infty < \varepsilon$ implies that $\eta(p) \in \mathcal{O}$ for all $p \in M$. Therefore, $||\eta - \xi||_n < \varepsilon/K(n)$ implies $\eta \in H^n(\mathcal{O})$. □

Lemma 2.3. *Let $\mathcal{O}$ be again an open subset of E and $\Phi\colon F \to M$ a second vector bundle. Furthermore, let $f\colon \mathcal{O} \to F$ be a smooth fibre map, i.e. $\Phi \circ f = \pi$. Then the induced map*

$$\tilde{f}\colon H^n(\mathcal{O}) \to H^n(F), \xi \mapsto f \circ \xi$$

is continuous.

Proof. Let $\xi_m \to \eta$ in $H^n(\mathcal{O})$, i.e. $||\eta - \xi_m||_n$ tends to zero. This implies that $||\eta - \xi_m||_\infty \to 0$ and $||\eta - \xi_m||_1 \to 0$.

The tangent bundle TE of E splits into a horizontal and a vertical summand; denote the latter by T_vE and the differential of f restricted to T_vE by D_vf. The tangent bundle of T_vE splits into a horizontal and vertical summand again; denote the latter by T^2_vE and more generally the purely vertical component of the k-th iterated tangent bundle by T^k_vE. The differential D^kf restricted to T^k_vE we denote by D^k_vf.

We want to show that $||\nabla^l(f\circ\eta) - \nabla^l(f\circ\xi_m)||_0$ converges to 0. By induction and the chain and product rule of differentiation, we have that

$$\begin{aligned}\nabla^l(f\circ\eta)(p) - \nabla^l(f\circ\xi_m)(p) = &\sum_{k=1}^{l} D^k_v f(\eta(p)) \cdot \sum_{(j_1,\ldots,j_k)\colon \sum j_i = l} \prod_{1\le i\le k} \nabla^{j_i}\eta(p) \\ &- \sum_{k=1}^{l} D^k_v f(\xi_m(p)) \cdot \sum_{(j_1,\ldots,j_k)\colon \sum j_i = l} \prod_{1\le i\le k} \nabla^{j_i}\xi_m(p).\end{aligned}$$

We sort the terms with respect to the degree of differentiation of f and want to prove zero convergence for each k. To avoid cumbersome notation, we present only the case $k = 2$. The other cases work the same way.

We have

$$\begin{aligned}D^2_vf(\eta(p))\nabla\eta(p)\nabla\eta(p) - D^2_vf(\xi_m(p))\nabla\xi_m(p)\nabla\xi_m(p) = &D^2_vf(\eta(p))(\nabla\eta(p) - \nabla\xi_m(p))\nabla\eta(p) \\ &+ D^2_vf(\eta(p))\nabla\xi_m(p)(\nabla\eta(p) - \nabla\xi_m(p)) \\ &+ (D^2_vf(\eta(p)) - D^2_vf(\xi_m(p)))\nabla\xi_m(p)\nabla\xi_m(p)\end{aligned}$$

and therefore

$$\begin{aligned}||D^2_vf(\eta)\nabla\eta\nabla\eta - D^2_vf(\xi_m)\nabla\xi_m\nabla\xi_m||_0 \le 2(&||D^2_vf(\eta)||_\infty||\nabla\eta - \nabla\xi_m||_0||\nabla\eta||_\infty \\ &+ ||D^2_vf(\eta)||_\infty||\nabla\xi_m||_\infty||\nabla\eta - \nabla\xi_m||_0 \\ &+ ||D^2_vf(\eta) - D^2_vf(\xi_m)||_\infty||\nabla\xi_m||_0||\nabla\xi_m||_\infty).\end{aligned}$$

Since $||\nabla\eta - \nabla\xi_m||_0 \to 0$, the first two summands tend to 0. An ε-neighbourhood U of the image of η in E is compact and therefore $D_v^2 f$ is uniformly continuous on U. For $m >> 0$, $\xi_m(p) \in U$ and hence $||D_v^2 f(\eta) - D_v^2 f(\xi_m)||_\infty$ goes to 0. This shows that all terms tend to zero.

By this, we have shown that $||(f \circ \eta) - (f \circ \xi_m)||_n \to 0$ for $||\eta - \xi_m||_n \to 0$, what is exactly what we needed. □

Lemma 2.4. *Under the same conditions,* $\tilde{f}\colon H^n(\mathcal{O}) \to H^n(F)$ *is smooth.*

Proof. Let η and ξ be in $H^n(\mathcal{O})$. By 2.3, we know that $\tilde{f}$ is continuous. The Taylor formula gives for every $p \in M$ the equation

$$f(\eta(p)) - f(\xi(p)) - D_v f(\xi(p)) \cdot (\eta(p) - \xi(p)) = r(\xi(p), \eta(p)) \cdot (\eta(p) - \xi(p))$$

where

$$r(\xi(p), \eta(p)) = \int_0^1 D_v(\xi(p) + s(\eta(p) - \xi(p)))ds - D_v f(\xi(p))$$

is a fibre map of $\mathcal{O}' \times \mathcal{O}' \subset \mathcal{O} \times \mathcal{O} \subset E \times E$, $\mathcal{O}'$ convex, into the bundle $L(\pi, \Phi)\colon L(E, f) \to M$.

From 2.3 we have that the associated map

$$\tilde{r}\colon H^n(\mathcal{O}' \times \mathcal{O}') \to H^n(L(E, F))$$

is continuous and

$$||\tilde{f}(\eta) - \tilde{f}(\xi) - ((D_v f)\tilde{}(\xi)) \cdot (\eta - \xi)||_n = ||\tilde{r}(\xi, \eta) \cdot (\eta - \xi)||_n \leq ||\tilde{r}(\xi, \eta)||_n ||\eta - \xi||_n.$$

Hence, we have $||\tilde{r}(\xi, \eta)|| \to 0$ for $\xi \to \eta$ since $r(\eta(p), \eta(p)) = 0$. Therefore, $\tilde{f}$ is differentiable with $D\tilde{f} = (D_v f)\tilde{}$. In the same manner, one shows $D^r \tilde{f} = (D_v^r f)\tilde{}$. □

Proposition 2.5. *Let* $f, g \in \mathcal{C}^\infty(M, N)$. *Then*

$$\exp_g^{-1} \circ \exp_f\colon \exp_f^{-1}(\mathcal{U}_f \cap \mathcal{U}_g) \to \exp_g^{-1}(\mathcal{U}_g \cap \mathcal{U}_f).$$

is smooth.

Proof. For every $p \in M$, define

$$\mathcal{O}_{f,g,p} := \mathcal{O}_{f,p} \cap ((\exp f_*)^{-1} \circ (\exp g_*)\mathcal{O}_{g,p})$$

where $\mathcal{O}_{f,p}$ and $\mathcal{O}_{g,p}$ denote the the fibres of $\mathcal{O}_f$ and $\mathcal{O}_g$ over p. Set $\mathcal{O}_{f,g} = \bigcup_p \mathcal{O}_{f,g,p}$ if $\mathcal{O}_{f,g,p} \neq \varnothing$ for all $p \in M$ and $\mathcal{O}_{f,g} = \varnothing$ else.

We have an inclusion of an open subset $\mathcal{O}_{f,g} \subset \mathcal{O}_f$ and

$$H^n(\mathcal{O}_{f,g}) = \exp_f^{-1}(\mathcal{U}_f \cap \mathcal{U}_g).$$

The map

$$\phi_{f,g} := (\exp g_*)^{-1} \circ (\exp f_*)\colon \mathcal{O}_{f,g} \to g^* TM$$

is a fibre map. Now we have the identity $\exp_g^{-1} \circ \exp_f = \widetilde{\phi_{f,g}}$. Therefore, the lemma follows from 2.4. □

Corollary 2.6. *The atlas* $\mathcal{A}$ *is smooth.*

3 The Homotopy Type and a Uniqueness Result

We cite the following theorem of Palais:

Proposition 3.1 ([Pal], Thm 16). *Let X be a Banach space, Y a dense subspace and $U \subset X$ open. Then the inclusion $Y \cap U \hookrightarrow U$ is a homotopy equivalence.*

This allows us to prove the following:

Proposition 3.2. *The inclusion $H^n(M, N) \hookrightarrow \mathcal{C}^0(M, N)$ is a homotopy equivalence.*

Proof. Embed N as a closed submanifold in some euclidean space $\mathbb{R}^m$. Let T be a tubular neighbourhood of N in $\mathbb{R}^m$. Then $H^n(M, N)$ is homotopy equivalent to $H^n(M, T)$ and $\mathcal{C}^0(M, N)$ is homotopy equivalent to $\mathcal{C}^0(M, T)$. Since $\mathcal{C}^0(M, T)$ is an open subset of the Banach space $\mathcal{C}^0(M, \mathbb{R}^m)$ and $H^n(M, \mathbb{R}^m)$ is dense in $\mathcal{C}^0(M, \mathbb{R}^m)$ (already $\mathcal{C}^\infty(M, \mathbb{R}^m)$ is dense), we get our result. □

Although the construction of $H^n(M, N)$ depends a priori on the Riemannian structure of N, actually all choices lead to diffeomorphic spaces as the following result of Eells and Elworthy shows:

Theorem 3.3 ([E-E]). *Every homotopy equivalence between Hilbert manifolds is homotopic to a diffeomorphism.*

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