Homotopy Colimits of Relative Categories (Preliminary Version)

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Grothendieck constructions of model categories have recently received some attention as in [HP14]. We will show that the Grothendieck construction of (more general) relative categories is actually a homotopy colimit in the model category RelCat. This will be proven, after some definitions and lemmas, as Theorem 10. At the end, we will give an example involving the classical Spanier–Whitehead category.

We begin by reviewing the definition of a relative category.

Definition 1. A *relative category* $\mathcal{M}$ is a category $\mathcal{M}$ together with a subcategory $\mathrm{we}\,\mathcal{M}$ containing all objects of $\mathcal{M}$. The morphisms in $\mathrm{we}\,\mathcal{M}$ are usually called *weak equivalences*. A *relative functor* between relative categories $\mathcal{M}$ and $\mathcal{M}'$ is a functor $F\colon \mathcal{M} \to \mathcal{M}'$ with $F(\mathrm{we}\,\mathcal{M}) \subset \mathrm{we}\,\mathcal{M}'$. We denote the category of small relative categories with relative functors between them by RelCat.

For a category $\mathcal{D}$, define a functor

$$()^{\mathcal{D}}\colon\ \mathrm{RelCat} \to \mathrm{RelCat}, \quad \mathcal{C} \mapsto \mathcal{C}^{\mathcal{D}},$$

where $\mathrm{we}\,\mathcal{C}^{\mathcal{D}}$ has as morphisms those natural transformations that are objectwise weak equivalences.

For a relative category $\mathcal{C}$, denote by $N(\mathcal{C})$ the Rezk classifying diagram, a simplicial space whose p-th space $N(\mathcal{C})_p$ is given by $\mathrm{nerve}(\mathrm{we}(\mathcal{C}^{[p]}))$. Here, $[p]$ the category of p composable morphisms. In [BK12b], Barwick and Kan construct a model structure on RelCat whose weak equivalences are detected by

$$N\colon\ \mathrm{RelCat} \to s\mathcal{S},$$

where the category of simplicial spaces $s\mathcal{S}$ carries the Rezk model structure (see [Rez01]). This is a certain localization of the Reedy model structure. Barwick and Kan give in [BK12a] also an alternative characterization of the weak equivalences. They show that $f\colon \mathcal{M} \to \mathcal{N}$ is a weak equivalence in RelCat if and only if it induces a Dwyer–Kan equivalence of the hammock localizations

$$L^H\mathcal{M} \to L^H\mathcal{N},$$

i.e. an equivalence of homotopy categories $\mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$ and weak equivalences of mapping spaces. Here, the homotopy category $\mathrm{Ho}(\mathcal{M})$ is defined be the localization of $\mathcal{M}$ at the class of weak equivalences we$\mathcal{M}$.

We review now the Grothendieck construction.

Definition 2. Let $\mathcal{D}$ be a (small) relative category and $F\colon \mathcal{D} \to \mathrm{RelCat}$ be a (not necessarily relative) functor. Define its *Grothendieck construction* $\mathcal{D} \int F$ to be a relative category with objects pairs $(i \in \mathcal{D}, x \in F(\mathcal{D}))$ and morphisms $(i,x) \to (j,y)$ given by a pair $(f\colon i \to j,\ g\colon (F(f))(x) \to y)$. We declare such a morphism to be a weak equivalence if f and g are.

The Grothendieck construction comes with a canonical map $\alpha_F\colon \mathcal{D} \int F \to \operatorname{colim}_{\mathcal{D}} F$ defined as follows: Given an object $i \in \mathcal{D}$, $x \in F(i)$, we set $\alpha_F(x)$ to be the image of x under the canonical map $F(i) \to \operatorname{colim}_{\mathcal{D}} F$. Given a morphism $(f,g)\colon (i,x) \to (j,y)$ defined by a pair $(f\colon i \to j,\ g\colon (F(f))(x) \to y)$, define $\alpha(f,g)$ to be the image of g under the canonical map $F(i) \to \operatorname{colim}_{\mathcal{D}} F$ where one uses that $F((f))(x)$ and x become identified in the colimit.

If $\mathcal{D}$ is an ordinary category, we will view it for this definition as a relative category with the maximal relative structure, where every morphism is a weak equivalence.

Remark 3. Harpaz and Prasma consider in [HP14] instead a functor $F\colon \mathcal{M} \to \mathrm{ModCat}$ from a model category to the category of model categories with left Quillen functors (with chosen adjoint) between them. They equip the Grothendieck construction $\mathcal{M} \int F$ with notions of weak equivalences and (co)fibrations, which under certan conditions define a model structure. As a left Quillen functor is not necessarily a relative functor, this does not fall under the scope of our definition. But denote by $\widetilde{F}\colon \mathcal{M}^{cof} \to \mathrm{RelCat}^b$ the postcomposition of the restriction of F to the full subcategory of cofibrant objects with the functor

$$\mathrm{Mod\,Cat} \to \mathrm{RelCat}^b, \quad \mathcal{C} \mapsto \mathcal{C}^{cof},$$

where RelCat^b denotes the (large) category of not-necessarily small relative categories. Then one can check that the relative category $(\mathcal{M} \int F)^{cof}$ of cofibrant objects agrees with $\mathcal{M}^{cof} \int \widetilde{F}$.

In the following, let $\mathcal{D}$ always be a fixed (ordinary) category.

Lemma 4. *Let $F\colon \mathcal{D} \to \mathrm{RelCat}$ be functor. Then there is an adjunction*

$$(\mathcal{D} \textstyle\int F)^{[n]} \rightleftarrows \mathcal{D} \textstyle\int (()^{[n]} \circ F)$$

that restricts to an adjunction between its categories of weak equivalences. Furthermore, the right adjoint is natural both in $F \in \mathcal{M}^{\mathcal{D}}$ and in $[n] \in \Delta$.

Proof. An object of $\mathcal{D} \int (()^{[n]} \circ F)$ consists of $i \in \mathcal{D}$ and a sequence of composable morphisms $x_0 \to \cdots \to x_n$ in $F(i)$. An object in $(\mathcal{D} \int F)^{[n]}$ consists of a sequence of composable morphisms $i_0 \xrightarrow{f_0} \cdots \xrightarrow{f_{n-1}} i_n$ in $\mathcal{D}$, objects $x_j \in F(i_j)$ and morphisms $(F(f_j))(x_j) \to x_{j+1}$ in $F(i_{j+1})$.

The left adjoint sends an object in $(\mathcal{D} \int F)^{[n]}$ as above to $i_n \in \mathcal{D}$ and to the sequence

$$(F(f_{n-1} \circ \cdots \circ f_0))(x_0) \to (F(f_{n-1} \circ \cdots \circ f_1))(x_1) \to \cdots \to x_n$$

in $F(i_n)$.

The right adjoint sends an object in $\mathcal{D} \int (()^{[n]} \circ F)$ as above to $i \xrightarrow{\mathrm{id}_i} \cdots \xrightarrow{\mathrm{id}_i} i$ and to the sequence $x_0 \to x_1 \to \cdots \to x_n$ in $F(i)$. □

Lemma 5. *Denote by* $\mathrm{we}\colon \mathrm{RelCat} \to \mathrm{Cat}$ *the functor that sends* $\mathcal{M}$ *to* $\mathrm{we}\,\mathcal{M}$*. Then there is a natural isomorphism* $\mathrm{we}(\mathcal{D} \textstyle\int F) \cong (\mathcal{D} \textstyle\int (\mathrm{we} \circ F))$.

Proof. Clear. □

We recall the definition of a homotopy colimit from [CS02]:

Definition 6. Given an indexing category $\mathcal{D}$ and a relative category $\mathcal{M}$ admitting $\mathcal{D}$-shaped colimits, a *homotopy colimit* is a *terminal homotopical approximation of* colim, i.e. a functor $H\colon \mathcal{M}^{\mathcal{D}} \to \mathrm{Ho}(\mathcal{M})$ together with a natural transformation $\alpha\colon H \Rightarrow \mathrm{colim}$, having the following two properties:

1. H is *homotopical* in the sense that it sends objectwise weak equivalences to isomorphisms.
2. If $K\colon \mathcal{M}^{\mathcal{D}} \to \mathrm{Ho}(\mathcal{M})$ is another homotopical functor with a natural transformation $\delta\colon K \Rightarrow \mathrm{colim}$, then there is a unique natural transformation $\gamma\colon K \to H$ with $\alpha\gamma = \delta$.

Here and in the following, we view colim also as a functor $\mathcal{M}^{\mathcal{D}} \to \mathrm{Ho}(\mathcal{M})$.

Remark 7. It is clear that all homotopy colimit functors are unique up to unique isomorphism. Note also that this definition agrees with the definition of a total left derived functor of colim in the sense of [Rie14, Definition 2.1.16].

Lemma 8. *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be relative categories admitting* $\mathcal{D}$*-shaped colimits and homotopy categories and let*

$$G\colon \mathcal{N} \to \mathcal{M}$$

be a homotopy equivalence *between them, which means that there is a relative functor* $F\colon \mathcal{M} \to \mathcal{N}$ *and zig-zags of natural weak equivalences between* FG *and* $\mathrm{id}_{\mathcal{N}}$ *and between* GF *and* $\mathrm{id}_{\mathcal{M}}$*. Then* G *detects homotopy colimit functors in the following sense:*

Denote by $\mathbb{F}$ *and* $\mathbb{G}$ *the induced equivalences between* $\mathrm{Ho}(\mathcal{M})$ *and* $\mathrm{Ho}(\mathcal{N})$*. Let furthermore*

$$(H\colon \mathcal{M}^{\mathcal{D}} \to \mathrm{Ho}(\mathcal{M}),\ \alpha\colon H \Rightarrow \mathrm{colim}^{\mathcal{M}})$$

be a homotopy colimit functor and

$$(J\colon \mathcal{N}^{\mathcal{D}} \to \mathcal{N},\ \beta\colon J \Rightarrow \mathrm{colim}^{\mathcal{N}})$$

be another pair of a functor and a natural transformation. Assume there is an isomorphism $f\colon HG \xrightarrow{\cong} \mathbb{G}J$ *such that*

$$\begin{array}{ccc} HG & \xrightarrow{\alpha G} & \mathrm{colim}^{\mathcal{M}} G \\ \downarrow f & & \downarrow \\ \mathbb{G}J & \xrightarrow{\mathbb{G}\beta} & \mathbb{G}\,\mathrm{colim}^{\mathcal{N}} \end{array}$$

commutes. Then J *is a homotopy colimit functor as well.*

Proof. First note that there are natural isomorphisms $\mathbb{F}\mathbb{G} \cong \mathrm{id}_{Ho(\mathcal{N})}$ and $\mathbb{G}\mathbb{F} \cong \mathrm{id}_{Ho(\mathcal{M})}$. These can be chosen such that they define an adjunction between $\mathbb{F}$ and $\mathbb{G}$ and we will fix such a choice.

The assumptions imply that $\beta\colon J \Rightarrow \mathrm{colim}^{\mathcal{N}}$ is isomorphic to

$$\epsilon\colon \mathbb{F}HG \xRightarrow{\mathbb{F}\alpha G} \mathbb{F}\,\mathrm{colim}^{\mathcal{M}} G \Rightarrow \mathbb{F}\mathbb{G}\,\mathrm{colim}^{\mathcal{N}} \Rightarrow \mathrm{colim}^{\mathcal{N}}.$$

Clearly, $\mathbb{F}HG$ is homotopical. Let $\delta\colon K \Rightarrow \operatorname{colim}^{\mathcal{N}}$ be a natural transformation. Then the natural transformation

$$\mathbb{G}KF \Rightarrow \mathbb{G}\operatorname{colim}^{\mathcal{N}} F \Rightarrow \mathbb{G}\mathbb{F}\operatorname{colim}^{\mathcal{M}} \Rightarrow \operatorname{colim}^{\mathcal{M}}$$

induces a unique natural transformation $\mathbb{G}KF \Rightarrow H$ and hence $\gamma\colon K \cong \mathbb{F}\mathbb{G}KFG \Rightarrow \mathbb{F}HG$. Here we use that K factors over $\operatorname{Ho}(\mathcal{N}^{\mathcal{D}})$ so that KFG is naturally isomorphic to K. It is a tedious, but routine check that we have $\epsilon\gamma = \delta$ and that γ is unique with this property. □

Remark 9. The derived functors of every Quillen equivalence between model categories with functorial factorization define a homotopy equivalence of the underlying relative categories.

Theorem 10. *The Grothendieck construction*

$$\int\colon \mathrm{RelCat}^{\mathcal{D}} \to \mathrm{RelCat}, \quad F \mapsto \mathcal{D}\int F$$

together with the canonical natural transformation $\alpha\colon \int \to \operatorname{colim}$ *is a homotopy colimit functor.*

Proof. As $s\mathcal{S}$ is a simplicial model category in which every object is cofibrant, by [Rie14, Theorems 5.1.1 and 2.2.8] the Bousfield–Kan homotopy colimit $\operatorname{hocolim}_{BK}$ defines a homotopy colimit functor $s\mathcal{S}^{\mathcal{D}} \to \operatorname{Ho}(s\mathcal{S})$. We use the convention that for K a simplicial set and $X \in s\mathcal{S}$, we have $(K \otimes X)_p = K \times X_p$.

The functor $N\colon \mathrm{RelCat} \to s\mathcal{S}$ is a homotopy equivalence. This follows from the natural equivalence $N \simeq N_\xi$ to a left Quillen functor shown in [BK12b], but actually Barwick and Kan present in [BK13] an easier proof (that also works for n-relative categories). It follows from Lemma 8 that it suffices now to show that $N\int \simeq \operatorname{hocolim}_{BK} N$.

Our main ingredient is that by [Tho79], there is for every functor $G\colon \mathcal{D} \to \mathrm{Cat}$ a natural weak equivalence $\operatorname{hocolim}_{BK} \operatorname{nerve} G \to \operatorname{nerve}(\mathcal{D}\int G)$. Fix now a functor $F\colon \mathcal{D} \to \mathrm{RelCat}$. We have

$$(\operatorname{hocolim}_{BK} NF)_p \cong \operatorname{hocolim}_{BK}(NF)_p = \operatorname{hocolim}_{BK} \operatorname{nerve}(\mathrm{we} \circ ()^{[p]} \circ F)$$

and compose this isomorphism with the natural weak equivalence

$$\operatorname{hocolim}_{BK} \operatorname{nerve}(\mathrm{we} \circ ()^{[p]} \circ F) \xrightarrow{\simeq} \operatorname{nerve} \mathcal{D}\int (\mathrm{we} \circ ()^{[p]} \circ F)$$

by Thomason. The latter is by Lemma 5 isomorphic to $\operatorname{nerve} \mathrm{we}(\mathcal{D}\int(()^{[p]} \circ F))$ and we have a weak equivalence

$$\operatorname{nerve} \mathrm{we}\, \mathcal{D}\int (()^{[p]} \circ F) \xrightarrow{\simeq} \operatorname{nerve} \mathrm{we}(\mathcal{D}\int F)^{[p]}$$

induced by Lemma 4, both natural in p and F. The target is equal to $N(\mathcal{D}\int F)_p$.

Composing gives a zig zag of weak equivalences between $(\operatorname{hocolim}_{BK} NF)_p$ and $N(\mathcal{D}\int F)_p$ that is natural both in p and in F. This induces a zig zag of weak equivalence between $\operatorname{hocolim}_{BK} NF$ and $N(\mathcal{D}\int F)$. It is easy to check that this is compatible with the natural transformation to the colimit. □

Example 11. Denote by CW^{fin} the relative category of pointed spaces that are (pointed) homotopy equivalent to a finite CW-complex with homotopy equivalences as weak equivalences. Then the homotopy colimit of

$$CW^{fin} \xrightarrow{\Sigma} CW^{fin} \xrightarrow{\Sigma} \cdots$$

as constructed above is a relative category SW, whose objects are given by pairs (X, n) with $X \in CW^{fin}$ and n a natural number. If $m \geq n$, a morphism from $(X, n) \to (Y, m)$ is given by a morphism $\Sigma^{m-n} X \to Y$ in $\mathrm{Top}_\bullet$. Its homotopy category agrees with the classical Spanier–Whitehead category $\overline{SW}$. Recall that $\overline{SW}$ has the same objects and

$$\overline{SW}((X, n), (Y, m)) = \mathrm{colim}_d \, [\Sigma^{d-n} X, \Sigma^{d-m} Y]^\bullet.$$

The key idea for showing that $SW \to \overline{SW}$ is a localization at $\mathrm{we}\, SW$ is to associate with $[f\colon \Sigma^{d-n} X \to \Sigma^{d-m} Y]$ the zig zag $(X, n) \to (\Sigma^{d-m} Y, d) \xleftarrow{\simeq} (Y, m)$.

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