Introduction to stable homotopy theory

(Rough notes - Use at your own risk)

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These are notes accompanying a lecture course given at the GQT school 2019 in Groesbeek. I have tried to keep things rather classical and concrete. For a different kind of introduction to stable homotopy theory, see e.g. [Mal11].

1 Stable phenomena and spectra

1.1 Homotopy groups

Recall that the fundamental group $\pi_1(X, x_0)$ of space X with chosen base point x_0 consists of homotopy classes of loops based at x_0. We will often leave the choice of base points in our notation implicit. In any case, we can equivalently write $\pi_1(X) = [S^1, X]^\bullet$, where this notation denotes *pointed homotopy classes*, i.e. equivalence classes of base point preserving maps under the relation of homotopy that leaves the base points fixed. Here, we can choose any base point of S^1, but the traditional choice is to write $S^1 = [0,1]/0 \sim 1$ and taking the base point $[0]$.

More generally, we can define the *homotopy groups* $\pi_n(X) = [S^n, X]^\bullet$ for any $n \geq 0$. As cell complexes are built out of disks and spheres, these are fundamental to compute $[Y, X]^\bullet$ for any cell complex Y and most spaces of interest are homeomorphic or at least homotopy equivalent to a cell complex (e.g. every smooth manifold). Note that $\pi_0(X)$ is the pointed set of path-components, $\pi_n(X)$ is a group for any $n \geq 1$ and this is actually abelian for $n \geq 2$.

The groups $\pi_n(X)$ are in general *very hard* to compute. In a few lucky cases, $\pi_n(X)$ vanishes for all $n \geq 2$ (e.g. for $X = S^1$ or a surface of genus at least 1). But there is no finite complex, where we can compute all of its homotopy groups if they are not vanishing for $n \geq 2$. In particular, we do not know all the homotopy groups of S^2!

There a some things we know though. For example:

- $\pi_k(S^n) = 0$ if $k < n$. More generally, $\pi_k(X) = 0$ if X is a cell complex with one 0-cell and no other cells of dimension $\leq k$.
- The morphism $\pi_n(S^n) \xrightarrow{\deg} \mathbb{Z}$ is an isomorphism.

Let us record some low-dimensional information about homotopy groups of spheres in the following table.

	π_2	π_3	π_4	π_5	π_6	π_7
S^2	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$
S^3	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$
S^4	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z} \times \mathbb{Z}/12$
S^5	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

One can observe a few things from this table.

- $\pi_k(S^2) \cong \pi_k(S^3)$ for $k \geq 3$. Indeed, there is the Hopf fibration $S^3 \to S^2$ with fiber S^1 and the long exact sequence of homotopy groups shows this isomorphism.
- Most homotopy groups of spheres are finite. It is indeed true that $\pi_k(S^n)$ is finite unless $k = n$ or n is even and $k = 2n - 1$ (Serre)
- The groups seems to (eventually) stabilize along the diagonal.

This last point is the *Freudenthal suspension theorem.*

Theorem 1.1 (Freudenthal)**.** *The suspension map*[1] *induces an isomorphism*

$$\pi_{n+k}(S^n) \to \pi_{n+1+k}(S^{n+1})$$

for $k \leq n-2$ and a surjection for $k = n-1$.

More generally, let X be a pointed CW complex with one 0-cell and all other cells in dimensions at least n. Then

$$\pi_{n+k}(X) \to \pi_{n+1+k}(\Sigma X)$$

for $k \leq n-2$ and a surjection for $k = n-1$, where ΣX is the suspension of X.

This means that for every connected pointed CW-complex X, the system of groups

$$\pi_k X \to \pi_{k+1} \Sigma X \to \pi_{k+2} \Sigma^2 X \to \cdots$$

will eventually stabilize (as $\Sigma^n X$ becomes more and more connected). This eventually stable guy is called the *k-th stable homotopy group* $\pi_k^{st} X$ of X.

It seems useful to collect the stable information of X into one object.

Definition 1.2. A *spectrum* $E = (E_n)_{n \geq 0}$ is a sequence of pointed spaces E_n together with pointed maps $\sigma_n \colon S^1 \wedge E_n = \Sigma E_n \to E_{n+1}$.

We define

$$\pi_k E = \operatorname{colim}_n \pi_{k+n} E_n.$$

Example 1.3. For a pointed space X, we can define its *suspension spectrum* $\Sigma^\infty X$, whose n-th space is $\Sigma^n X$.

We directly see that if X is a connected CW-complex, then $\pi_k \Sigma^\infty X = \pi_k^{st} X$.

[1] The suspension ΣX of a space X is defined to be the quotient of $[-1,1] \times X$, where we collapse $\{-1\} \times X$ to a poin and $\{1\} \times X$ to another point. For example, $\Sigma S^n \cong S^{n+1}$. Strictly speaking, there is also another suspension construction for a pointed space (X, x_0), where we additionally collapse $[-1,1] \times \{x_0\}$ to a point, but under very mild assumptions this is homotopy equivalent to the usual suspension construction and thus we usually confuse these two notions. More precisely, they are equivalent if X is *well-pointed*, i.e. if the inclusion of the base point is a cofibration. We will assume from now on that all our spaces are well-pointed. If X is a CW-complex or a manifold, every choice of base-point makes X into a well-pointed space.

1.2 Stability for vector bundles

Given a space X, denote by $\mathrm{Vect}_n^{\mathbb{R}}(X)$ and $\mathrm{Vect}_n^{\mathbb{C}}(X)$ the set of isomorphism classes of n-dimensional vector real/complex vector bundles on X. We denote by $\underline{\mathbb{R}}$ the trivial one-dimensional real vector bundle and use $\underline{\mathbb{C}}$ analogously. We obtain a map

$$\underline{\mathbb{R}}\oplus\colon \mathrm{Vect}_n^{\mathbb{R}}(X) \to \mathrm{Vect}_{n+1}^{\mathbb{R}}(X). \tag{1.4}$$

Proposition 1.5. *This map is a surjection if X is a CW-complex of dimension at most n and an isomorphism if X is a CW-complex of dimension less than n.*

Proof. There is a fiber sequence

$$GL_n(\mathbb{R}) \to GL_{n+1}(\mathbb{R}) \to S^n.$$

This implies that $GL_n(\mathbb{R}) \to GL_{n+1}(\mathbb{R})$ is an $(n-1)$-equivalence, i.e. an isomorphism on π_i for $i > n-1$ and a surjection for $i = n-1$.

We have further fiber sequences

$$GL_n(\mathbb{R}) \to EGL_n(\mathbb{R}) \to BGL_n(\mathbb{R}),$$

where $BGL_n(\mathbb{R})$ is the Grassmannian of n-dimensional subspaces of $\mathbb{R}^\infty$ and $EGL_n(\mathbb{R})$ is the corresponding Stiefel manifold of embeddings of $\mathbb{R}^n$ into $\mathbb{R}^\infty$. The latter is contractible. This implies that $\pi_k BGL_n(\mathbb{R}) \cong \pi_{k+1} GL_n \mathbb{R}$. We see that $BGL_n(\mathbb{R}) \to BGL_{n+1}(\mathbb{R})$ is an n-equivalence. As CW-complexes are built from spheres, this implies that

$$[X, BGL_n(\mathbb{R})] \to [X, BGL_{n+1}(\mathbb{R})]$$

is a surjection if X is of dimension at most n and an isomorphism if X is of dimension less than n. As $BGL_n(\mathbb{R})$ represents $\mathrm{Vect}_n(\mathbb{R})$, this map is actually isomorphic to the one in (1.4). □

There is a similar result for complex vector bundles, where $\mathrm{Vect}_n^{\mathbb{C}}(X) \to \mathrm{Vect}_{n+1}^{\mathbb{C}}(X)$ is an isomorphism if X is a CW-complex of dimension at most $2n$.

Example 1.6. The tangent bundle TS^n is in the kernel of $\mathrm{Vect}_n^{\mathbb{R}}(S^n) \to \mathrm{Vect}_{n+1}^{\mathbb{R}}(S^n)$ (just add the normal bundle), but is only trivial for $n = 0, 1, 3, 7$. The latter is a hard theorem, but it is much easier to see that TS^n is non-trivial for n even (hairy ball theorem/Euler class).

Definition 1.7. If X is a finite connected CW-complex, we define $\widetilde{K}(X) = \mathrm{colim}_n \mathrm{Vect}_n^{\mathbb{C}}(X)$ and $\widetilde{KO}(X) = \mathrm{colim}_n^{\mathbb{R}}(X)$. These theories are called *reduced K-theory*.

Remark 1.8. There are also unreduced K-theory groups $K(X)$ and $KO(X)$. Here, $K(X)$ is defined as the Grothendieck construction on the set of isomorphism classes of complex vector bundles (of arbitrary, not necessarily constant dimension) on X. Choosing a base point $x \in X$ defines a map $K(X) \to K(\mathrm{pt}) = \mathbb{Z}$ and one can identify $\widetilde{K}(X)$ with the kernel. Moreover, $K(X) \cong \widetilde{K}(X) \oplus \mathbb{Z}$, where the isomorphism depends on the path-component of the base-point. Note that $K(X)$ is naturall defined as well for non-connected finite CW-complexes and by this procedure, we can define $\widetilde{K}(X)$ as well for non-connected CW-complexes.

The results above imply that $\widetilde{KO}(X) \cong \mathrm{Vect}_n^{\mathbb{R}}(X)$ if n is greater than the dimension of X.

One of the most important (and amazing!) results in K-theory is *Bott periodicity.*

Theorem 1.9 (Bott). *There are natural isomorphisms*

$$\widetilde{K}(X) \cong \widetilde{K}(\Sigma^2 X)$$

and

$$\widetilde{KO}(X) \cong \widetilde{KO}(\Sigma^8 X).$$

1.3 (Co)homology theories

For a pointed space (X, x), we can define $\widetilde{H}^n(X) = \ker(H^n(X) \to H^n(\mathrm{pt}))$. This differs only from usual singular cohomology for $n = 0$, where killed the $\mathbb{Z}$-summand that is present for all spaces. This satisfies $\widetilde{H}^{n+1}(\Sigma X) \cong \widetilde{H}^n(X)$. Moreover, it behaves well with respect to *mapping cones*: Given a map $f\colon X \to Y$, we define its mapping cone Cf as $CX \cup_X Y$, where $CX = X \times [0,1]/X \times \{1\}$. More precisely, $\widetilde{H}^n$ satisfies the axioms of a reduced cohomology theory:[2]

Definition 1.10. A *reduced cohomology theory* is a sequence of contravariant functors $\widetilde{h}^n$: $\mathrm{Top}_* \to \mathrm{Ab}$, $n \in \mathbb{Z}$ together with natural isomorphisms $\sigma_n : \widetilde{h}_n \circ \Sigma \to \widetilde{h}^{n-1}$ fulfilling the following axioms:

1. (Pointedly) homotopic maps induce the same map in $\widetilde{h}^n$.
2. For any map $f : X \to Y$ of pointed topological spaces and $n \in \mathbb{Z}$, the sequence

$$\widetilde{h}^n(Cf) \to \widetilde{h}^n(Y) \to \widetilde{h}^n(X)$$

   is exact.
3. For a collection X_i of pointed spaces, the map

$$\widetilde{h}^n(\bigvee X_i) \to \prod_i \widetilde{h}^n(X_i)$$

   is an isomorphism for all i. (Additivity axiom)

Note that the additivity axiom for two (and hence finitely many) wedge summands follows already from the cone sequence: The mapping cone of $X \to X \vee Y$ is homotopy equivalent to Y (using that our spaces are well-pointed and thus $X \to X \vee Y$ is a cofibration). Thus we obtain a *split* exact sequence

$$\widetilde{h}^n(Y) \to \widetilde{h}^n(X \vee Y) \to \widetilde{h}^n(X),$$

as we wanted to.

[2]Note that $\widetilde{H}^n(Cf) \cong H^n(Y, X)$ if $f\colon X \to Y$ an inclusion.

Remark 1.11. The mapping cone of the inclusion $Y \to Cf$ is homotopy equivalent to ΣX. Using the exact sequence for the cone and the suspension isomorphism in conjuction thus produces for a map $f\colon X \to Y$ and a reduced cohomology theory a *long exact sequence*

$$\cdots \to \widetilde{h}^{n-1}(X) \to \widetilde{h}^{n}(Cf) \to \widetilde{h}^{n}(Y) \to \widetilde{h}^{n}(X) \to \widetilde{h}^{n+1}(Cf) \to \cdots$$

If we set $h^n(X,Y) = \widetilde{h}^n(Cf)$, we obtain the familiar long exact sequence in cohomology.

Can we make $\widetilde{K}$ into a reduced cohomology theory as well? For a finite CW-complex, we set $\widetilde{K}^0(X) = \widetilde{K}(X)$ and $\widetilde{K}^{-1}(X) = \widetilde{K}(\Sigma X)$. Now we simply use Bott periodicity by setting $\widetilde{K}^{2n-i}(X) = \widetilde{K}^{-i}(X)$ for $i = 0, 1$. Similarly for $\widetilde{KO}$. What about the two axioms?

Lemma 1.12. *Let (Z,z) be a pointed space and $f\colon X \to Y$ be a pointed map. Then the sequence*

$$[Cf, Z]^\bullet \to [Y, Z]^\bullet \to [X, Z]^\bullet$$

is exact.

Lemma 1.13. *Set $BGL(\mathbb{R}) = \operatorname{colim}_n BGL_n(\mathbb{R})$ and $BGL(\mathbb{C}) = \operatorname{colim}_n BGL(\mathbb{C})$. Let X be a pointed space. Then we have natural isomorphisms*

$$[X, BGL(\mathbb{R}) \times \mathbb{Z}]^\bullet \cong \widetilde{KO}(X)$$

and

$$[X, BGL(\mathbb{C}) \times \mathbb{Z}]^\bullet \cong \widetilde{K}(X).$$

The sharper form of Bott periodicity says that $\Omega BGL(\mathbb{C}) \times \mathbb{Z} \simeq GL(\mathbb{C})$ (you can also leave out the $\mathbb{Z}$ here) and $\Omega GL(\mathbb{C}) \simeq BGL(\mathbb{C}) \times \mathbb{Z}$. Similarly, $\Omega^8 BGL(\mathbb{R}) \simeq BGL(\mathbb{R})$.

Definition 1.14. An Ω-*spectrum* $\underline{Z}$ is a sequence of pointed spaces Z_n with weak homotopy equivalences $Z_n \to \Omega Z_{n+1}$, where Ω denotes the loop space.[3]

Remark 1.15. The functors Σ and Ω are adjoint. Hence, an Ω-spectrum also defines maps $\Sigma Z_n \to Z_{n+1}$ and thus an Ω-spectrum is in particular a spectrum.

Proposition 1.16. *Let $\underline{Z}$ be an Ω-spectrum. Then the functors*

$$\underline{Z}^n\colon \mathrm{Top}_* \to \mathrm{Ab}, \qquad X \mapsto [X, Z_n]^\bullet$$

form a reduced cohomology theory on CW-complexes. If the maps in $\underline{Z}$ are actual homotopy equivalences, it is even a reduced cohomology theory on all pointed spaces.

Example 1.17. The functors $\widetilde{K}^n$ and $\widetilde{KO}^n$ are part of reduced cohomology theories.

What is more surprising is that the converse also holds.

Theorem 1.18 (Brown representability)**.** *Every reduced cohomology is on the category of pointed CW-complex representable by an Ω-spectrum.*

[3]Sometimes, we will underline spectra for emphasis, but usually we won't to simplify our notation.

Remark 1.19. For every cohomology theory defined on CW-complexes, there is a canonical way to extend it to all spaces. Indeed, for every space X, there is a weak homotopy equivalence[4] $f\colon X' \to X$ from a CW-complex X'. Idea: We build X' inductively.

- The 0-cells of X' are exactly $\pi_0 X$ and we map the 0-cells x_i to corresponding points $f(x_i)$.
- For every generator of $\pi_1(X, f(x_i))$ we add a loop at x_i.
- For every relation in $\pi_1(X, f(x_i))$ we add a corresponding two-cell in X'. Moreover, we add for every generator of $\pi_2(X, f(x_i))$ as 2-cell.
- $\cdots$

For details see [Hat02, Proposition 4.13]. We can set now $\widetilde{h}^n(X) = \widetilde{h}^n(X')$. One can show that every other CW-complex that is weakly homotopy equivalent to X is (canonically) homotopy equivalent to X', basically because a weak homotopy equivalence between CW-complexes is a homotopy equivalence.

2 Basic properties of spectra and homology theories

There is also a corresponding story for homology theories. The definition of a *reduced homology theory* is entirely analogous to that of a reduced cohomology theory, only replacing contravariant functors by covariant functors and $\prod$ by $\bigoplus$.

In the moment, we do not have many examples of homology theories except for singular homology. But actually, every spectrum defines a homology theory.

Construction 2.1. Let $\underline{E}$ be a spectrum and X be a pointed space. We define a new spectrum $\underline{E} \wedge X$ to have n-th space $E_n \wedge X$ with the obvious structure maps.

Theorem 2.2. *Given a spectrum $\underline{E}$, the functor*

$$X \mapsto \pi_* \underline{E} \wedge X$$

is a reduced homology theory, which we will denote by $\underline{\widetilde{E}}_*(X)$. The corresponding unreduced homology theory will be denoted by $\underline{E}_*(X)$, i.e. for $\underline{\widetilde{E}}_*(X \sqcup \mathrm{pt})$.

We will deduce the four necessary properties and structures (suspension isomorphism, homotopy axiom, mapping cone axiom and additivity) from a list of more general results. For this, we start with the following definition.

Definition 2.3. Let E and F be spectra. A *morphism* or *map* $f : E \to F$ is a family $f_n : E_n \to F_n$ of pointed maps such that the diagrams

$$\begin{array}{ccc} E_n & \xrightarrow{f_n} & F_n \\ \downarrow{\scriptstyle \sigma_n^E} & & \downarrow{\scriptstyle \sigma_n^F} \\ E_{n+1} & \xrightarrow{f_{n+1}} & F_{n+1} \end{array}$$

commute for all n. We denote the category of spectra by Sp.

[4]Recall that a map $f\colon X \to Y$ is a *weak homotopy equivalence* if f induces isomorphisms $\pi_i(X,x) \to \pi_i Y, f(x)$ for every $i \geq 0$ and $x \in X$.

Definition 2.4. Let $f, g : E \to F$ be two morphisms. A *homotopy between f and g* is a map $E \wedge I_+ \to F$, restricting on the boundary of I to f and g. Equivalently, It consists of pointed homotopies between f_n and g_n, compatible with the suspension maps.

Lemma 2.5. *Two homotopic maps $f, g\colon E \to F$ induce the same map $\pi_* E \to \pi_* F$.*

Proof. This is true as the maps $f_*, g_*\colon \pi_{n+k}E_n \to \pi_{n+k}F_n$ are the same for all k, n. □

The suspension of a spectrum E is defined levelwise: $(\Sigma E)_n = E_n \wedge S^1$.

Proposition 2.6. *Let E be a spectrum. Then there is a canonical isomorphism $\pi_n E \cong \pi_{n+1}\Sigma E$.*

Proof. We compute:

$$\begin{aligned}
\pi_k E &\cong \operatorname{colim}\left(\pi_k E_0 \to \pi_{k+1}\Sigma E_0 \to \pi_{k+1}E_1 \to \Sigma E_0 \cdots\right) \\
&\cong \operatorname{colim}\left(\pi_{k+1}\Sigma E_0 \to \pi_{k+2}\Sigma E_1 \to \cdots\right) \\
&\cong \operatorname{colim}\left(\pi_{k+1}E_0 \wedge S^1 \to \pi_{k+2}E_1 \wedge S^1 \to \cdots\right) \\
&\cong \pi_{k+1}E \wedge S^1
\end{aligned}$$

While the first and the last isomorphism hold by definition and the second one just by leaving out terms, the third one is a bit more subtle. The reason is that while all the groups in the colimit system are isomorphic (as $\Sigma E_i = S^1 \wedge E_i \cong E_i \wedge S^1$ by interchanging the factors), the maps do *not* agree under this isomorphism.

Indeed: The morphism $\pi_{k+1}S^1 \wedge E_0 \to \pi_{k+2}S^1 \wedge E_1$ is a composite: First we apply the suspension map to $\pi_{k+2}S^1 \wedge S^1 \wedge E_0$ and then we apply $\mathrm{id}_{S^1} \wedge \sigma_0$.

The morphism $\pi_{k+1}E_0 \wedge S^1 \to \pi_{k+2}E_1 \wedge S^1$ can also be factored in first applying the suspension $\pi_{k+1}E_0 \wedge S^1 \to \pi_{k+2}S^1 \wedge E_1 \wedge S^1$ and then $\sigma_0 \wedge \mathrm{id}_{S^1}$.

The issue is: Interchanging S^1 and E_0, then applying $\sigma_0 \wedge \mathrm{id}_{S^1}$ and then interchanging E_1 and S^1 produces the map $S^1 \wedge S^1 \wedge E_0 \to S^1 \wedge E_1$ that is the composite of $\mathrm{tw} \wedge \mathrm{id}_{E_0}$ and $\mathrm{id}_{S^1} \wedge \sigma_0$, where $\mathrm{tw}\colon S^1 \wedge S^1 \to S^1 \wedge S^1$ interchanges the two factors. But as $\mathrm{tw}\colon S^2 \to S^2$ has degree -1, we see that the maps in our system differ by the sign -1.

This is still enough to obtain an isomorphism in the colimit. Indeed, we can go two steps at a time and the maps exactly correspond to each other. □

Given a map $f\colon E \to F$, we can also define its *mapping cone* Cf levelwise by $(Cf)_n = Cf_n$.

Proposition 2.7. *Given a map $f : X \to Y$ of spectra, we have a long exact sequence of homotopy groups*

$$\cdots \xrightarrow{\partial} \pi_n X \xrightarrow{f_*} \pi_n Y \xrightarrow{i_*} \pi_n Cf \xrightarrow{\partial} \pi_{n-1}X \xrightarrow{f_*} \cdots .$$

Here, ∂ is defined as the composition $\pi_n Cf \to \pi_n \Sigma X \cong \pi_{n-1}X$, where the isomorphism is from the proof of Proposition 2.6.

Proof. Exactness at $\pi_n Y$: Twofold composition zero is clear. Assume that $\alpha : S^{n+k} \to Y_k$ is a pointed map representing an element in the kernel of i_*. After possibily enlargen k, we can assume that the composition $S^{n+k} \to Y_k \xrightarrow{i_k} Cf_k$ is nullhomotopic. Thus, we can a map of pairs $(D^{n+k+1}, S^{n+k}) \to (Cf_k, Y_k)$. Collapsing gives a map $\beta : S^{n+k+1} \to \Sigma X_k$. Exercise: Check that the composition $S^{n+k+1} \to \Sigma X_k \to \Sigma Y_k$ is homotopic to $\Sigma\alpha$. Thus, $\sigma_X \circ \beta$ is a preimage of α.

The exactness at the other points follows since the mapping cone of i is level homotopy equivalent to the suspension of X. □

Given a collection E^i of spectra, we define its wedge $\bigvee_i E^i$ levelwise.

Corollary 2.8. *The map $\pi_*E \oplus \pi_*F \to \pi_*E \vee F$, induced by the two wedge inclusions, is an isomorphism.*

Proof. This uses that the mapping cone of $E \to E \vee F$ is levelwise homotopy equivalent to F. Thus, we obtain a *split* exact sequence

$$\pi_*E \to \pi_*E \vee F \to \pi_*F$$

□

We want to define the *homotopy colimit* or *mapping telescope* of a sequence of spaces

$$X_0 \xrightarrow{f_0} X_1 \to X_2 \xrightarrow{f_1} \cdots$$

as

$$X_0 \times [0,1] \cup_{(x,1)\sim(f_0(x),1)} X_1 \times [1,2] \cup_{(x,2)\sim(f_1(x),2)} X_2 \times [2,3] \cup \cdots$$

We denote it by $\operatorname{hocolim} X_i$. While the notation seems to indicate a countable sequence, this works as well for other ordinals. There is also a pointed version collapsing $\{x_i\} \times [i, i+1]$ to a point, where x_i is the base point of X_i. Under our well-pointedness assumption, this is equivalent to the unpointed construction. The following lemma is extremely useful.

Lemma 2.9. *Let K be compact, $X_0 \to X_1 \to X_2 \to \cdots$ be a sequence of closed inclusions of T_1-spaces and X be the union $\bigcup X_i$. Then:*

(a) Every map $f : K \to X$ factors over some X_i.

(b) The canonical map $\operatorname{colim}_i[K, X_i] \to [K, X]$ is a bijection. If all inclusion are pointed, the corresponding statement is also true for pointed homotopy classes.

Proof. As $K \times I$ is also compact, (b) follows directly from (a). For (a): Suppose not so. Then we can choose $x_i \in K$ with $f(x_i) \in X_i/X_{i-1}$ (after possibly taking a subsequence of X_i). A subset of $\{f(x_i)\}_{i\in\mathbb{N}_0}$ is closed if and only if its intersection with each X_i is closed. But this intersection is always finite and all finite subsets of X_i are closed (as it is T_1). Thus $\{f(x_i)\}$ is discrete. But it is also compact as a closed subset of $f(K)$. This is a contradiction. □

This implies the first half of the following lemma (at least for T_1-spaces).

Lemma 2.10. *There is a canonical isomorphism $\operatorname{colim} \pi_k X_i \cong \pi_k \operatorname{hocolim} X_i$. Moreover, if the maps f_i are cofibrations, $\operatorname{colim} X_i \simeq \operatorname{hocolim} X_i$.*

We can similarly define the homotopy colimit of a sequence of maps of spectra levelwise. Likewise, we define all colimits of spectra to be levelwise. Moreover, we call a map $E \to F$ of spectra a *level cofibration* if all the map $E_n \to F_n$ are cofibrations.

Lemma 2.11. *Let*

$$E^0 \xrightarrow{f_0} E^1 \xrightarrow{f_1} \cdots$$

be sequence of spectra. There is a canonical isomorphism $\operatorname{colim} \pi_k E^i \cong \pi_k \operatorname{hocolim} E^i$. *Moreover, if the maps f_i are level cofibrations,* $\operatorname{colim} E^i \simeq \operatorname{hocolim} E^i$.

Finally, we are ready to prove Theorem 2.2.

Proof of Theorem 2.2. The suspension isomorphism follows from Proposition 2.6 as

$$E \wedge \Sigma X \cong \Sigma(E \wedge X).$$

The homotopy axiom follows from Lemma 2.5. The cone sequence follows from Proposition 2.7. Lastly, the additivity axiom follows from Corollary 2.8 and Lemma 2.11 as $E \wedge \bigvee_{i \in I} X_i$ is a directed colimit (over an ordinal with the same cardinality as I) and we can show via transfinite induction that $\pi_* E \wedge \bigvee_{i \in I} \cong \bigoplus_{i \in I} E \wedge X_i$. The suspension isomorphism is also not hard: □

Example 2.12. Consider the sphere spectrum $\mathbb{S} = \Sigma^\infty S^0$. Then $\mathbb{S} \wedge X \cong \Sigma^\infty X$. We see that the homology theory defined by $\mathbb{S}$ is precisely the functor $X \mapsto \pi_*^{st}(X)$. Note that usual homotopy groups are very far from being a homology theory, but *stable* homotopy groups are one. This could have also been deduced by contemplating the Blakers–Massey theorem, but our proof does not use ingredients of this difficulty and is more formal.

Remark 2.13. From the previous example, we obtain in particular that $\pi_k^{st}(X \vee Y) \cong \pi_k^{st}(X) \oplus \pi_k^{st}(Y)$. The corresponding statement for unstable homotopy groups is *not* true. Indeed, $S^2 \times S^2$ has a cell structure with one 2-skeleton $S^2 \vee S^2$ and one 4-cell. Let $f \colon S^3 \to S^2 \vee S^2$ be the corresponding attaching map. Then $[f] \in \pi_3(S^2 \vee S^2)$ is not in the image of $\pi_3(S^2) \oplus \pi_3(S^2)$. Indeed, by a change of basis it would be in the image from one of the wedge summands, say of some $[g] \in \pi_3(S^2)$. But this would imply that $S^2 \times S^2 \simeq Cf \simeq Cg \vee S^2$. The cup product structure on $H^*(S^2 \times S^2)$ shows that this is impossible.

Example 2.14. Say, we want to compute some low-dimensional stable homotopy groups of $\mathbb{CP}^2$. We obtain $\mathbb{CP}^2$ by attaching a 4-cell to S^2 along the Hopf map $\eta \colon S^3 \to S^2$. Contemplating the associated long exact sequence of stable homotopy groups of $\mathbb{CP}^2$ yields the following table for $\pi_k^{st}\mathbb{CP}^2$.

0	1	2	3	4	5	6	7
0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}/12$	0	$\mathbb{Z}/24$

Example 2.15. The spectrum KU representing K-theory also defines a homology theory, called *K-homology.* The geometric interpretation is a bit more tricky than for K-cohomology (aka K-theory), but quite important in index theory.

3 Thom spectra and bordism theories

3.1 Definitions and motivation

Definition 3.1. Let $\mathfrak{N}_n$ be the set of equivalence classes of n-dimension smooth closed manifolds, where two manifolds M and N are equivalent if they are *bordant*, i.e. if there is a compact $(n+1)$-dimensional manifold W with $\partial W \cong M \sqcup N$.[5]

More generally, we define $\mathfrak{N}_n(X)$ for a space X to be the set of bordism classes of maps $f\colon M \to X$ from closed n-dimensional manifolds M. Two maps $f\colon M \to X$ and $g\colon N \to X$ are *bordant* if there is a map $H\colon W \to X$ from a compact manifold such that H restricts to f and g under a diffeomorphism $\partial W \cong M \sqcup N$.

Following Thom, we will interpret $\mathfrak{N}_n$ as the homotopy groups of a spectrum MO. More generally, it will be true that $\mathfrak{N}_n(X) \cong MO_n(X)$.

How can we associate with a closed n-manifold M an element of a homotopy group, i.e. a map from a sphere? Embed M into a sphere S^{n+k}. We would like to define a map $S^{n+k} \to M_+$ by just collapsing everything outside of M to one point; but this clearly not continuous. We can consider instead a tubular neighboorhod of M, i.e. an injective map $\phi : \nu_i \to S^{n+k}$ from the normal bundle of M identifying $M \subset S^{n+k}$ with the zero section of ν_i. Note that the metric on S^{n+k} induces a metric on ν_i.

Definition 3.2. Given a vector bundle $\xi : E \to B$ on a space B, we define the *Thom space* $\mathrm{Th}(\xi)$ to be $E \sqcup \infty$ as a set (where ∞ is a point)and with the following topology: For every point in E, a system of fundamental neighborhoods is given by those in E. A neighborhood of ∞ is given by $U \ni \mathrm{pt}$ with $\xi^{-1}(x) - U \cap \xi^{-1}(x)$ compact for every $x \in B$.

Given an Euclidean metric on ξ, there is an alternative description: Let $D\xi \to B$ be the *disk bundle*, i.e. the subbundle of all vectors of length ≤ 1 and $S\xi \to B$ be the *sphere bundle*, i.e. the subbundle of all vectors of length exactly 1. Then $\mathrm{Th}(\xi) \cong D\xi/S\xi$.

We get now a map $g(M, i, \phi) : S^{n+k} \to \mathrm{Th}(\nu_i)$ as follows: For $x = \phi(y) \in \phi(D\xi)$, choose $(g(M, i, \phi))(x)$ to be the image of $y = \phi^{-1}(x)$ in $\mathrm{Th}(\nu_i) = D\nu_i/S\nu_i$. For x outside of $\phi(D\xi)$, send x to the point ∞. This is clearly continuous.

We would like to show that $g(M, i, \phi)$ is independent of the choices of i and ϕ. But this does not even make sense as the target depends on i. Trick: Map into the Thom space of the universal vector bundle. So let γ_k be the universal bundle over the Grassmannian $BO(k)$. As we can describe ν_i as a pullback of γ_k, we get a map of total spaces between these bundles, so a map of Thom spaces $\mathrm{Th}(\nu_i) \to MO_k := \mathrm{Th}(\gamma_k)$. Set $f(M, i, \phi) : S^{n+k} \to MO_k$ postcomposed by this map.

Is $f(M, i, \phi)$ independent of the choice of i and ϕ (up to homotopy)? For ϕ, the answer is yes by the following theorem:

Theorem 3.3. *Given two tubular neighborhoods $\phi_1, \phi_2 : \nu_i \to S^{n+k}$, there is an isotopy $\Phi : \nu_i \times I \to S^{n+k}$ restricting to ϕ_1, ϕ_2 on the boundary.*

For two such tubular neighborhoods, define now a homotopy $H : S^{n+k} \times I \to \mathrm{Th}(\nu_i)$ on (x, t) as follows. If $x \in \Phi(\bullet, t)(D\nu_i)$, set $H(x, t)$ to be the image of $\Phi(\bullet, t)^{-1}(x)$ in $\mathrm{Th}(\nu_i)$. If $x \notin \Phi(\bullet, t)(D\nu_i)$, set $H(x, t) = \infty$. This is continuous.

[5] This is a set (as opposed to a proper class) as diffeomorphic manifolds are bordant (with W a cylinder) and the Whitney embedding theorem shows that there are only set many diffeomorphism classes of manifolds.

In general, $f(M,i,\phi)$ will be not be independent of i. We have two problems:

1. Not all embeddings are isotopic.
2. If we embed into S^{n+k+1} instead of S^{n+k}, source and target of the maps will not even be the same!

Solution: Consider the bundle $\gamma_k \oplus \epsilon_{\mathbb{R}}$ on $BO(k)$. As this is $(k+1)$-dimensional, we get a map into the total space of γ_{k+1} on $BO(k+1)$ and thus a map $\mathrm{Th}(\gamma_k \oplus \epsilon_{\mathbb{R}}) \to \mathrm{Th}(\gamma_{k+1}) = MO_{k+1}$. In general, we have $\mathrm{Th}(\xi \times \nu) \cong \mathrm{Th}(\xi) \wedge \mathrm{Th}(\nu)$. In particular $\mathrm{Th}(\xi \oplus \epsilon_{\mathbb{R}}) \cong \mathrm{Th}(\xi) \wedge S^1 = \Sigma\,\mathrm{Th}(\xi)$. Thus, we get a map $\sigma_k : \Sigma MO_k \to MO_{k+1}$.

Definition 3.4. The (unoriented) *Thom spectrum* MO is defined by the spaces MO_k (with base point ∞) and the maps $\Sigma MO_k \to MO_{k+1}$ above.

Theorem 3.5 (Thom)**.** *We obtain a well-defined map $P_n : \mathfrak{N}_n \to \pi_n MO$, which is an isomorphism. This is called the Pontryagin–Thom isomorphism*

More generally, one can construct an isomorphism $\mathfrak{N}_n(X) \cong MO_n(X_+)$.

Corollary 3.6. *The functor $X \mapsto \mathfrak{N}_*(X)$ is part of a homology theory.*

In the next subsection, we will sketch a proof of Theorem 3.5.

3.2 Proof of Theorem 3.5

We want to sketch a proof of Theorem 3.5. Recall that we associated with a closed n-manifold M the class $[f(M,i,\phi)] \in \pi_n MO$ and want to show that this is independent of the choice of i.

First observe: If we postcompose our embedding i with the inclusion $\iota : S^{n+k} \to S^{n+k+1}$ the class $[f(M,i,\phi)]$ does not change. Indeed, we can choose a product tubular neighborhood of M in $S^{n+k} \times I \subset S^{n+k+1}$. The map

$$S^{n+k+1} \cong \Sigma_u S^{n+k} \to S^{n+k} \wedge S^1 \to \mathrm{Th}(\nu) \wedge S^1 \cong \mathrm{Th}(\nu \oplus \epsilon)$$

is not the identity on the second factor, but contracts an interval (the things outside the tubular neighborhood of M); this is homotopic to the identity. Thus, we can identify $[f(M, \iota \circ i, \phi \oplus \epsilon)]$ with $\sigma_k \circ \Sigma f(M,i,\phi)$.

Next we use the theorem:

Theorem 3.7. *For k big enough, any two embeddings of M into S^{n+k} are isotopic.*

Now assume that $i : M \to S^{n+k}$ and $j : S^{n+k}$ are isotopic. This gives us an embedding $H : M \times I \to S^{n+k} \times I$ (compatible with the projection onto the second coordinate). We can choose a "product tubular neighborhood" $\Phi : \nu \oplus \epsilon \to S^{n+k} \times I$ for H in the sense that if we intersect it with $S^{n+k} \times \{t\}$, we get again a tubular neighborhood. [We can do this as follows: Choose on $S^{n+k} \times I$ the product metric. Choose a radius ϵ and the standard Riemannian tubular neighborhood.] Then clearly $f(M, i, \Phi \cap S^{n+k} \times 0) \simeq f(M, j, \Phi \cap S^{n+k} \times 1)$.

So, we have shown independence of tubular neighborhood and embedding. But more is true:

Lemma 3.8. *If M and N are bordant closed n-dimensional, then $[M] = [N] \in \pi_n MO$.*

Proof. By a form of the Whitney embedding theorem, we can embed a bordism W between M and N into $S^{n+k} \times I$ such that the intersections with $S^{n+k} \times 0$ and $S^{n+k} \times 1$ ar exactly M and N. As above, we get a homotopy. □

So, we get a map $P_n : \mathfrak{N}_n \to \pi_n MO$ is called the *Pontryagin–Thom homomorphism.*

Lemma 3.9. *This map is a homomorphism.*

Proof sketch: If we consider $M \coprod N$, we embed M and N into different hemispheres of S^{n+k}. Then we can view them separately. □

It remains to show the following:

Theorem 3.10 (Thom)**.** *The maps $P_n : \mathfrak{N}_n \to \pi_n MO$ are bijections.*

We want to construct a map $T : \pi_n MO \to \mathfrak{N}_n$ that is an inverse. So let $[f] \in \pi_{n+k} MO_k$ with $f : S^{n+k} \to MO_k$.

Step 1: Tranversality

Definition 3.11. Let $f : X \to M$ be a differentiable map of manifolds and $N \subset M$ a closed submanifold. Then f is called *transverse* to N (written: $f \pitchfork N$) if for every $x \in f^{-1}(N)$, we have $\operatorname{im}(T_x f) + T_{f(x)} N = T_{f(x)} M$. Equivalently, for every $x \in f^1(N)$ the projection of $T_x X$ onto the normal bundle part of $T_{f(x)} M$ is surjective.

Example 3.12. Let $x \in \mathbb{R}$. Then $f : M \to \mathbb{R}$ is transverse to $\{x\} \in \mathbb{R}$ iff x is a regular value of f.

Proposition 3.13. *Let $f : X \to M$ be a differentiable map of manifolds, $N \subset M$ a closed submanifold and f transverse to N. Then $f^{-1}(N)$ is a submanifold of X. If X is a manifold with boundary and M is without boundary, then $\partial f^{-1}(N) = \partial M \cap f^{-1}(N)$.*

Theorem 3.14 (Transversality theorem and differentiable approximation)**.** *Let $f : X \to M$ be a map of manifolds and $N \subset M$ a closed submanifold. Assume that there is a closed subset $A \subset X$ such that there is a neighborhood U of A such that $f|_U$ is differentiable and transverse to N. Then there exists a $g : X \to M$ and a homotopy from f to g that is constant on A such that g is differentiable and transverse to N.*

The space MO_k is the colimit $\operatorname{colim}_n \operatorname{Th}(\gamma_{k,m})$ of the Thom spaces of the tautological bundles $\gamma_{k,m}$ on $\operatorname{Gr}_k(\mathbb{R}^m)$. As S^{n+k} is compact, f factors over some map $S^{n+k} \to \operatorname{Th}(\gamma_{k,m})$, which we also call f. We would like to make f differentiable and transverse to the zero-section $\operatorname{Gr}_k(\mathbb{R}^m)$, but $\operatorname{Th}(\gamma_{k,m})$ is not a manifold. Consider $E = \operatorname{Th}(\gamma_{k,m}) - \infty$, which is a manifold. Now choose a closed codimension-0-submanifold $V \subset S^{n+k}$ with $V \subset f^{-1}(E)$ and $f^{-1}(\operatorname{Gr}_k(\mathbb{R}^m) \subset V^\circ$. This can be achieved as follows: Choose an Urysohn function $u : S^{n+k} \to I$ such that $u(f^{-1}(\operatorname{Gr}_k(\mathbb{R}^m)) = 0$ and $u(S^{n+k} - f^{-1}(E)) = 1$. We can assume that u is smooth and has a regular value $t \in (0,1)$. We can take $V = u^{-1}([0,t])$. Set $W = u^{-1}[t,1]$.

Choose a homotopy $H : \partial V \times I \to \gamma_{k,m} - \operatorname{Gr}_k(\mathbb{R}^m)$ with $H|_{\partial V \times 0} = f|_{\partial V}$ and $H|_{\partial V \times 1}$ smooth. By choosing a (bi)collar, we can view S^{n+k} as $W \cup \partial V \times [-1,1] \cup V$. We define now a new map $f_2 : S^{n+k} \to \mathrm{Th}(\gamma_{k,m})$ as $f|_W$ on W, as $f|_V$ on V and as H respectively the inverse of H on $\partial V \times [-1, 0]$ and $\partial V \times [0, 1]$. Note also that $f_2|_{\partial V \times 0}$ is transverse to $\mathrm{Gr}_k(\mathbb{R}^m)$. So, we can find a homotopy from $f_2|_{\partial V \times [0,1] \cup V} : \partial V \times [0, 1] \cup V \to \gamma_{k,m}$ to a map that is smooth and transverse to $\mathrm{Gr}_k(\mathbb{R}^m)$ and the homotopy is constant on $\partial V \times 0$. So, we get a map $g : S^{n+k} \to \mathrm{Gr}_k(\mathbb{R}^m)$, homotopic to f such that there is a codimension-0-submanifold $N \subset S^{n+k}$ such that $g^{-1}(\mathrm{Gr}_k(\mathbb{R}^m)) \subset N \subset g^{-1}(\gamma_{k,m})$ and $g|_N$ is differentiable and transverse to $\mathrm{Gr}_k(\mathbb{R}^m)$.

Step 2: Well-definedness The choice of m obviously does not matter.

Consider a homotopy $H : S^{n+k} \times I \to MO_k$. By similar arguments as above, it factors over some $\mathrm{Th}(\gamma_{k,m})$, and we can homotope it relative boundary so that it is differentiable in a neighborhood of the preimage of the zero section and transverse to it. Then $H^{-1}(\mathrm{Gr}_k(\mathbb{R}^m))$ is a bordism between $H|^{-1}_{S^{n+k}\times 0}(\mathrm{Gr}_k(\mathbb{R}^m))$ and $H|^{-1}_{S^{n+k}\times 1}(\mathrm{Gr}_k(\mathbb{R}^m))$.

Now consider $f : S^{n+k} \to MO_k$ and the corresponding map $\sigma_k \circ \Sigma f : S^{n+k+1} \to MO_{k+1}$. We can choose f so that it is transverse to the Grassmannian (i.e. zero section) inside MO_k. The preimage of this zero section lies completely in S^{n+k}. So $T([f]) \in \mathfrak{N}_n$ is well-defined.

Step 3: Inverseness $T(P(M)) = M$ is easy: ransversality follows as the induced map of normal bundles of M in S^{n+k} and that of $BO(k)$ in γ_k is an isomorphism.

As $T(f)$, we get a closed submanifold $M \subset S^{n+k}$ together with a map $g : M \to BO(k)$ such that $g^*\gamma_k$ is the normal bundle of M. [This tubular neighborhood does not need to be Riemannian; need uniqueness up to isotopy.] Then $P(T(f))$ is homotopic to f since we can push f out to ∞.

This proves the theorem.

3.3 Complex bordism

There are many variants of Thom spectra. We will just mention one.

Definition 3.15. Define a spectrum MU as follows: Let γ_n be the universal bundle over the Grassmannian $\mathrm{Gr}_n(\mathbb{C}^\infty)$. We set $MU_{2n} = \mathrm{Th}(\gamma_n)$ and $MU_{2n+1} = \Sigma MU_{2n}$. As the pullback of γ_{n+1} to $\mathrm{Gr}_n(\mathbb{C}^\infty)$ is isomorphic to $\gamma_n \oplus \mathbb{C}$, we obtain maps

$$S^2 \wedge MU_{2n} \cong \mathrm{Th}(\gamma_n \oplus \underline{\mathbb{C}}) \to \mathrm{Th}(\gamma_{n+1}) = MU_{2n+2}.$$

This defines the structure maps for MU.

The geometric meaning is a bit less transparent than for MO. There is still a Pontryagin–Thom isomorphism, but now $\pi_n MU$ classifies closed n-manifolds M together with a *stable almost complex structure* up to bordism. This is essentially a choice of complex structure on the normal bundle of an embedding $M \hookrightarrow S^{n+k}$ for k large.

4 The stable homotopy category

4.1 Homotopy categories

We would like to define the *stable homotopy category* to be category of spectra, where every π_*-isomorphism becomes an isomorphism. The abstract categorical framework is given by the following definition.

Definition 4.1. Let $\mathcal{C}$ be a category and $\mathcal{W}$ be a collection of maps in $\mathcal{C}$. Then a *localization* of $\mathcal{C}$ at $\mathcal{W}$ is a functor F from $\mathcal{C}$ to a category $\mathcal{W}^{-1}\mathcal{C}$ with the following universal property: Given a functor $G : \mathcal{C} \to \mathcal{D}$ such that $G(f)$ is an isomorphism for all $f \in \mathcal{W}$, then there exists a unique functor $H : \mathcal{W}^{-1}\mathcal{C} \to \mathcal{D}$ such that $G = H \circ F$.

The localization, if it exists, is unique up to unique isomorphism. Furthermore, it can be shown to exist for all small categories, i.e. all categories that have only a set of objects. As a first example we take Ho(Top), i.e. the category with all topological spaces as objects and *homotopy classes of maps* as morphisms.

Proposition 4.2. *The functor* $F : \mathrm{Top} \to \mathrm{Ho(Top)}$ *defines a localization at the class of all homotopy equivalences.*

Proof. Let $G : \mathrm{Top} \to \mathcal{D}$ be a functor such that $G(h)$ is an isomorphism for all homotopy equivalences h. We want that $G(f) = G(g)$ for homotopic maps $f, g : X \to Y$. This produces then a (unique) factorization of G through Ho(Top).

Consider the cylinder $X \times I$, the two inclusion $i_0, i_1 : X \to X \times I$ and the projection $r : X \times I \to X$. Then there exists a map $H : X \times I \to Y$ such that $f = Hi_0$ and $g = Hi_1$. Postcomposing with r gives $G(f)\circ G(r) = G(H)\circ G(i_0\circ r)$ and $G(g)\circ G(r) = G(H)\circ G(i_1\circ r)$. Note now that $G(r)$ is an isomorphism and $G(r) \circ G(i_0) = \mathrm{id} = G(r) \circ G(i_1)$. Thus,

$$G(i_0 \circ r) = \mathrm{id} = G(i_1 \circ r).$$

Thus, $G(f) \circ G(r) = G(g) \circ G(r)$ and thus $G(f) = G(g)$. $\square$

What happens if we localize the category of spaces at the class $\mathcal{W}$ of weak homotopy equivalences? A refinement of the statement that every space is weakly equivalent to a CW-complex and the Whitehead theorem shows:

Proposition 4.3. *The localization* $\mathcal{W}^{-1}\,\mathrm{Top}$ *of* Top *at the class* $\mathcal{W}$ *exists and is equivalent to* $\mathrm{Ho}(CW)$*, the homotopy category of CW-complexes.*

4.2 The stable homotopy category

Recall that we denote the category of spectra by Sp. Denote by $\mathcal{W}$ the class of morphisms inducing isomorphisms on π_*, i.e. the π_*-isomorphisms.

We can define the naive homotopy category $\mathrm{Ho(Sp)}^{\mathrm{naiv}}$ as having the same objects as Sp and setting $[X, Y]^{\mathrm{naiv}}$ to be the homotopy classes of morphisms of spectra. Likewise, we can define $\mathrm{Ho}(\mathcal{C})^{\mathrm{naiv}}$ for every subcategory $\mathcal{C}$ of Sp.

Our aim is to describe the localization $\mathcal{W}^{-1}\,\mathrm{Sp}$ as $\mathrm{Ho}(\mathcal{C})^{\mathrm{naiv}}$ of a suitable subcategory.

Definition 4.4. A spectrum E is called a *CW-spectrum* if E_0 is a CW-complex and each structure map $\Sigma E_k \to E_{k+1}$ is the inclusion of a relative CW-complex.

As before, one can show that one can replace every spectrum up to level equivalence by a CW-spectrum. Here, we say that a morphism $f\colon X \to Y$ of spectra is a *level equivalence* if every $f_n\colon X_n \to Y_n$ is a weak homotopy equivalence. Moreover, every level equivalence between CW-spectra is already a homotopy equivalence. This suggests the following proposition.

Proposition 4.5. *The localization of* Sp *at the class of level equivalences exists and is equivalent to* $\mathrm{Ho}(CW)^{\mathrm{naiv}}$*, where* $CW \subset \mathrm{Sp}$ *denotes the subcategory of CW-spectra.*

Every level equivalence is a π_*-isomorphism, but not every π_*-isomorphism is a level equivalence. These notions are only equivalent if source and target are Ω*-spectra*. Moreover, one can show that every spectrum is π_*-isomorphic to an Ω-spectrum. This suggests the following theorem.

Theorem 4.6. *The localization of* Sp *at* $\mathcal{W}$ *exists and is equivalent to the naive homotopy category of all* Ω*-CW spectra.*

Definition 4.7. We call the localization $\mathcal{W}^{-1}\,\mathrm{Sp}$ the *stable homotopy category* and denote it by SHC. The morphism set between two spectra X, Y in SHC is denoted by $[X, Y]$.

One can calculate the morphism with the following proposition, which essentially is a refinement of the theorem above.

Proposition 4.8. *Let* X *be a CW-spectrum and* Y *be arbitrary. Then* $[X, QY]^{\mathrm{naiv}} \cong [X, Y]$ *for any* Ω*-spectrum* QY *with a* π_**-isomorphism* $f: Y \to QY$*.*

Remark 4.9. This is essentially a special case of the general philosophy of model categories. Model categories $\mathcal{M}$ have a class of *weak equivalences* $\mathcal{W}$ and subcategories of *cofibrant* and *fibrant* objects satisfying certain axioms. Quillen has shown that the localization $\mathcal{W}^{-1}\mathcal{M}$ always exists and morphisms in this localization can be computed as morphisms in a naive kind of homotopy category after replacing the source cofibrantly and target fibrantly. There is a model structure on Sp with $\mathcal{W}$ the π_*-isomorphism, the Ω-spectra as fibrant objects and the CW-spectra as a subclass of the cofibrant objects. See for example [MMSS01], where also model structures on several variants of spectra are constructed (and our spectra are called prespectra).

At least if all the spaces in the spectrum Y are Hausdorff, one can construct an example of an Ω-spectrum QY with a π_*-isomorphism $Y \to QY$ as follows: Set $(QY)_n = \mathrm{hocolim}\,\Omega^k Y_{k+n}$. We have a structure map

$$\Sigma(QY)_n = \Sigma\,\mathrm{hocolim}\,\Omega^k Y_{k+n} \cong \mathrm{hocolim}\,\Sigma\Omega^k Y_{k+n} \to \mathrm{hocolim}\,\Omega^{k-1} Y_{(k-1)+(n+1)} = (QY)_{n+1}$$

induced by the adjunction $\Sigma\Omega^k \to \Omega^{k-1}$. It is not hard to check that this is an Ω-spectrum as Ω commutes (under very mild assumptions) with directed homotopy colimits.

Corollary 4.10. *Let* X *be a finite CW-complex and* Y *be a spectrum. Then* $[\Sigma^\infty X, Y] \cong \mathrm{colim}_n[\Sigma^n X, Y_n]^\bullet$*. In particular,* $\pi_k X = [\Sigma^k \mathbb{S}, X]$ *for* $\mathbb{S} = \Sigma^\infty S^0$*.*

Proof. Clearly, $\Sigma^\infty X$ is a CW-spectrum (as the suspension spectrum of every CW-spectrum). Thus $[\Sigma^\infty X, Y] = [\Sigma^\infty X, QY]^{naive}$, i.e. an element consists of compatible maps $\Sigma^n X \to \mathrm{hocolim}_k\,\Omega^k Y_{k+n}$. The compactness of X allows us to show that

$$[\Sigma^n X, \mathrm{hocolim}_k\,\Omega^k Y_{k+n}] \cong \mathrm{colim}_k[\Sigma^n X, \Omega^k Y_{k+n}] \cong \mathrm{colim}_k[\Sigma^{k+n} X, Y_{k+n}].$$

This implies the result. □

Corollary 4.11. *For* X, Y *finite CW-complex* $[\Sigma^\infty X, \Sigma^\infty Y]$ *agrees with* $[\Sigma^k X, \Sigma^k Y]^\bullet$ *for* k *large.*

Proof. Freudenthal suspension. Note: $[\Sigma^k X, \Sigma^k Y]^\bullet \cong \pi_k \operatorname{Map}_\bullet(X, \Sigma^k Y)$. □

Proposition 4.12. *The hom-sets in SHC have natural structures of abelian groups.*

Proof. A variant of the argument in Proposition 2.6 shows that the map $Y \to \Omega\Sigma Y$ is a π_*-isomorphism for all Y, where $\Omega\Sigma X$ has n-th space $\Omega(Y_n \wedge S^1)$. The same is true for the map $\Sigma\Omega X \to X$. We obtain that every spectrum X is isomorphic to $\Sigma^2 X'$ for another spectrum X' in SHC. Replacing X' by a CW-spectrum, we can assume that $\Sigma^2 X'$ is a CW-spectrum as well. Thus, $[X, Y] = [\Sigma^2 X', QY]^{\mathrm{naiv}}]$. Naive homotopy classes out of double suspensions form abelian groups, by the same argument that π_k of spaces are abelian groups for $k \geq 2$. □

Example 4.13. There are many examples of spaces that are not weakly equivalent, but their suspension spectra become equivalent in SHC. For example, let T be the 2-torus. One can easily show that $\Sigma T \simeq S^2 \vee S^2 \vee S^3$ and thus $\Sigma^\infty T \cong \Sigma^\infty(S^1 \vee S^1 \vee S^2)$ in SHC.

4.3 Homology- and cohomology of spectra

Let E and Z be two spectra. We define the *n-th E-cohomology of Z* to be $[Z, \Sigma^n E]$. This generalizes our earlier definition if X if $Z = \Sigma^\infty X$.

Actually, there is also a *smash product* of spectra. The smash product $E \wedge Z$ of two CW-spectra can be defined via $(E \wedge Z)_{2n} = E_n \wedge Z_n$ And $(E \wedge Z)_{2n+1} = E_{n+1} \wedge Z_n$. Quite remarkably this asymmetric definition induces a symmetric monoidal structure on SHC (at least if we restrict our definition to CW-spectra). This is a quite non-trivial fact. (See e.g. the treatment in [Swi75] of the smash product or the more modern approach in [MMSS01, Section 11].)

This allows us to define the *n-th E-homology of Z $E_n(Z)$* to be $\pi_n E \wedge Z$. Again this is generalizing our earlier definition in case of $Z = \Sigma^\infty X$. More precisely, $E_n(\Sigma^\infty X) = \widetilde{E}_n(X)$. Concretely, one can calculate $E_n(Z)$ as $\operatorname{colim}_k E_{n+k} Z_k$.

Lemma 4.14 (Yoneda lemma)**.** *Let $\mathcal{C}$ be a category and*

$$h_X, h_Y \colon \mathcal{C}^{op} \to \mathrm{Set}$$

functors represented by $X, Y \in \mathcal{C}$. Then the map

$$\mathcal{C}(X, Y) \to \mathrm{Nat}(h_X, h_Y)$$

to the set of natural transformation from h_X to h_Y is a bijection.

Specializing to SHC we obtain:

Lemma 4.15. *Let E, F be spectra. The set of natural transformations of the represented cohomology theories on* all spectra *is in bijection with* $[E, F]$.

Remark 4.16. We will later see that this set does not need to agree with the set of stable natural transformations between cohomology theories just defined on spaces.[6] Let's denote

[6]There are many unstable operations between cohomology theories. For example, the cup square defines a natural transformation $H^n(-, \mathbb{Z}) \to H^{2n}(-, \mathbb{Z})$. As this is not compatible with suspension, this has no chance of being induced by a map of spectra. All unstable operations are by the Yoneda induced by a map of the corresponding spaces. For example, the set of natural transformation $H^n(-, \mathbb{Z}) \to H^k(-, \mathbb{Z})$ is in bijection with $[K(\mathbb{Z}, n), K(\mathbb{Z}, k)] \cong H^k(K(\mathbb{Z}, n); \mathbb{Z})$.

the category of the latter by Coh. Brown's representability theorem (in a strong form) says that the functor

$$\mathrm{SHC} \to \mathrm{Coh}$$

is essentially surjective and full. We will later see that it does not need to be faithful.

Example 4.17. The algebra $[H\mathbb{Z}/2, \Sigma^* H\mathbb{Z}/2]$ is called the (2-primary) *Steenrod algebra* $\mathcal{A}$. It is generated by the Steenrod squares Sq^n.

The $\mathbb{Z}/2$-cohomology of every spectrum becomes a module over $\mathcal{A}$.

Theorem 4.18. *The cohomology $(H\mathbb{Z}/2)^* MO$ is free as an $\mathcal{A}$-module in the graded sense, i.e. isomorphic to $\bigoplus_i \mathcal{A}[n_i]$, where n_i denotes the shift. Moreover, there are only finitely many generators g_i in every degree.*

By definition, the generators g_i define maps $MO \to \Sigma^{n_i} H\mathbb{Z}/2$ and thus a map

$$g\colon MO \to \prod_i \Sigma^{n_i} H\mathbb{Z}/2.$$[7]

As there are only finitely many generators in every degree, the map

$$H := \bigvee_i \Sigma^{n_i} H\mathbb{Z}/2 \to \prod_i \Sigma^{n_i} H\mathbb{Z}/2$$

turns out to be a π_*-isomorphism and thus an isomorphism in SHC. This relies on the fact that π_* of a wedge is a direct sum, while π_* of a product is a product; in this case, we have only finitely many summands/factors in each degree and so they agree. Essentially by construction, the map g is an *isomorphism* in $\mathbb{Z}/2$-cohomology.

Proposition 4.19 (Hurewicz). *A map of* connective *spectra is a π_*-isomorphism if it is an equivalence in $\mathbb{Z}$-homology. Here, a spectrum X is connective if $\pi_i X = 0$ for $i << 0$.*

The spectra MO and H are connective (for MO you can see it as there are no non-empty manifolds of negative dimension).

Lemma 4.20. *The integral homologies $(H\mathbb{Z})_* MO$ and $(H\mathbb{Z})_* H$ are 2-torsion.*

Proof. We know that $MO_*(X) \cong \mathfrak{N}_*(X)$ is 2-torsion for all spaces X. Indeed, for a map $M \to X$, its double $M \sqcup M \to X$ is null-bordant via the cyclinder $M \times I$. By definition, we have

$$(H\mathbb{Z})_n MO \cong MO_n(H\mathbb{Z}) \cong \operatorname{colim}_k MO_{n+k} K(\mathbb{Z}, k).$$

As these groups are all 2-torsion, the colimit is 2-torsion as well.

For H, it suffices to show that $(H\mathbb{Z})_* H\mathbb{Z}/2$ is 2-torsion. But even the identity map in $[H\mathbb{Z}/2, H\mathbb{Z}/2] = (H\mathbb{Z}/2)^0(H\mathbb{Z}/2) = \mathbb{Z}/2$ is 2-torsion. □

Moreover, the $\mathbb{Z}$-homology of both MO and H is finitely generated in every degree (as can be seen by CW-models for MO and H). Playing with the universal coefficient sequence (which is also true for spectra) shows that g being an isomorphism in $\mathbb{Z}/2$-cohomology also implies that g is an isomorphism in $\mathbb{Z}$-homology and hence a π_*-isomorphism.

[7] The product of spectra is defined levelwise. It turns out to define not only a product in Sp but in SHC as well.

Theorem 4.21 (Thom)**.** *The map $g\colon MO \to H$ above is a π_*-isomorphism. In particular, there is an isomorphism*

$$\mathfrak{N}_*(X) \cong MO_*(X_+) \cong H_*(X_+) \cong H_*(X;\mathbb{Z}/2) \otimes_{\mathbb{Z}/2} \pi_* MO.$$

Moreover, looking more closely at $(H\mathbb{Z}/2)^*MO$, one can read of the dimensions of the generators g_i and obtains:

Theorem 4.22 (Thom)**.** *We have*

$$\mathfrak{N}_* \cong \pi_* MO \cong \mathbb{Z}/2[x_2, x_4, x_5, x_6, x_8, \dots],$$

with generators in all dimensions not of the form $2^i - 1$. The elements x_i can be represented by $\mathbb{RP}^i$ for i even.

4.4 Rationalizing

We have the following remarkable theorem.

Theorem 4.23 (Serre)**.** *The groups $\pi_*\mathbb{S}\otimes\mathbb{Q}$ are concentrated in degree 0 with $\pi_0\mathbb{S}\otimes\mathbb{Q} = \mathbb{Q}$.*

Actually, $(H\mathbb{Q})^0(\mathbb{S}) \cong \operatorname{colim}_k H^k(S^k;\mathbb{Q}) \cong \mathbb{Q}$. Picking the generator 1, this produces a map $\mathbb{S} \to H\mathbb{Q}$, which induces by the theorem by Serre above an isomorphism on rational homotopy groups. This can be used to deduce the following theorem.

Theorem 4.24. *Define $\mathrm{SHC}_{\mathbb{Q}}$ to have the same objects as Sp with $\mathrm{SHC}_{\mathbb{Q}}(X,Y) = [X,Y]\otimes\mathbb{Q}$. Then*

$$\pi_* \otimes \mathbb{Q}\colon \mathrm{SHC}_{\mathbb{Q}} \to \textit{ graded } \mathbb{Q}\textit{-vector spaces}$$

is an equivalence of categories.

This theorem is very remarkable again. The category SHC is extremly complicated, but after rationalizing, almost all difficulties go away. We obtain for example that $\pi_*^{st}(X)\otimes\mathbb{Q} \cong \widetilde{H}_*(X;\mathbb{Q})$ in general etc.

5 Complex-oriented cohomology theories and the non-existence of an integral Chern character

Recall that there exists a Chern character $K^*(X) \to \prod_i H^{2i}(X;\mathbb{Q})$. Restricting to degree 0, we have in particular a morphism $K \to H\mathbb{Q}$. Is there a similar transformation $K \to H\mathbb{Z}$? This is a question we will answer in this section in a rather roundabout way.

5.1 Ring spectra and cup product

Definition 5.1. A *ring spectrum E* is a monoid in SHC with respect to the smash product. More concretely, it has a unit map $\iota\colon \mathbb{S} \to E$ and a multiplication map $\mu\colon E\wedge E \to E$ satisfying unitality and associativity. If we have additional commutativity, we call E a *commutative ring spectrum*.

We also denote the image of ι under the isomorphism $[\mathbb{S}, E] = E^0(\mathbb{S}) \cong \pi_0\mathbb{E}$ by 1.

Let E be a ring spectrum and X be a space. We obtain a cup product by the following construction. We first obtain a map:

$$E^k(X) \otimes E^l(X) \cong [\Sigma^\infty X_+, \Sigma^k E] \otimes [\Sigma^\infty X_+, \Sigma^l E] \xrightarrow{\wedge} [\Sigma^\infty X_+ \wedge \Sigma^\infty_+ X, \Sigma^{k+l} E \wedge E].$$

The smash product in the source is isomorphic to $\Sigma^\infty(X \times X)_+$. Using the map $E \wedge E \to E$ and the diagonal map $X \to X \times X$, we can map further into $[\Sigma^\infty X, \Sigma^{k+l}E] = E^{k+l}(X)$. Ths is the *cup product* for E-cohomology. It is associative and unital. If E is a commutative ring spectrum, the cup product will also be graded commutative.

Examples 5.2. All the spectra considered so far $(\mathbb{S}, H\mathbb{Z}, H\mathbb{Q}, H\mathbb{F}_2, K, KO, MO, MU, \dots)$ are commutative ring spectra.

Remark 5.3. There are stricter notion of commutativity that do not just work in the stable homotopy category, but in some variant of spectra (e.g. symmetric spectra or orthogonal spectra) that supports a good notion of smash product before passing to a homotopy category. Actually, all the examples above also admit the stricter notion, but this is a bit more difficult to show, especially for K and KO.

5.2 Complex orientations

Say we want to set up a theory of Chern classes for our favorite cohomology theory E (represented by a ring spectrum with the same name) and say we are very modest and we just want to have the first Chern class c_1 of line bundles. For a paracompact space X, there is a one-to-one correspondence between isomorphism classes of line bundles on X and $[X, \mathbb{CP}^\infty]$, given by pulling back the tautological line bundle γ over $\mathbb{CP}^\infty$. Thus, it suffices to define $c_1(\gamma) \in \widetilde{E}^2(\mathbb{CP}^\infty)$. The usual normalization condition says that $c_1(L) \in \widetilde{E}^2(\mathbb{CP}^1)$ corresponds to 1 under the isomorphism $\widetilde{E}^2(\mathbb{CP}^1) \cong \widetilde{E}^2(S^2) \cong \widetilde{E}^0(S^0)$ for L the tautological line bundle on $\mathbb{CP}^1$. This translates into the following notion.

Definition 5.4. A *complex orientation* is a class $x \in E^2(\mathbb{CP}^\infty)$ restricting to $1 \in E^2(\mathbb{CP}^1)$.

Example 5.5. The spectrum $H\mathbb{Z}$ has a complex orientation, just taking the standard generator of $H^2(\mathbb{CP}^\infty; \mathbb{Z})$.

Example 5.6. The spectrum K has a complex orientation. Indeed, $u \in K^2(S^0)$ be the Bott periodicity element. Then $([\gamma] - 1)u \in K^2(\mathbb{CP}^\infty)$. One can check that this indeed a complex orientation.

Some other spectra like $\mathbb{S}$ and KO do not have complex orientations.

Theorem 5.7. *There are isomorphisms*

$$E^*(\mathbb{CP}^\infty) \cong E^*(\mathrm{pt})[\![x]\!]$$

more generally

$$E^*((\mathbb{CP}^\infty)^n) \cong E^*(\mathrm{pt})[\![x_1, \dots, x_n]\!].$$

This has a remarkable consequence. Observe that $\mathbb{CP}^\infty$ has the structure of a commutative topological monoid by identifying $\mathbb{C}^\infty$ with the polynomial ring $\mathbb{C}[t]$ and using multiplication of polynomials. The morphism $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ induces a morphism

$$E^*(\mathrm{pt})[\![x]\!] \to E^*(\mathrm{pt})[\![x_1, x_2]\!].$$

The image of x is a power series $F(x_1, x_2)$. The space $\mathbb{CP}^\infty$ being a commutative topological monoid implies that $F(x_1, x_2)$ is *(graded) formal group law* over $E^*(\mathrm{pt})$ in the following sense.

Definition 5.8. A *formal group law* over a ring R is a power series $F \in R[\![x_1, x_2]\!]$ satisfying

1. $F(x_1, x_2) = F(x_2, x_1)$,
2. $F(x_1, 0) = x_1$ and $F(0, x_2) = x_2$,
3. $F(x_1, F(x_2, x_3)) = F(F(x_1, x_2), x_3)$.

If R is graded and the coefficients in front of $x_1^i x_2^j$ has degree $-2i - 2j - 2$.

Examples 5.9. The formal group law for $H\mathbb{Z}$ is $x_1 + x_2$. This is called the *additive formal group law* $\widehat{\mathbb{G}}_a$.

For K, the formal group law is $x_1 + x_2 + ux_1x_2$.

5.3 Formal group laws

Definition 5.10. Let F, G be two formal group laws over a ring R. A power series $f \in R[\![x]\!]$ with no constant term is called an *homomorphism* from $F \to G$ if

$$F(f(x_1), f(x_2)) = f(G(x_1, x_2)).$$

An invertible homomorphism is called an *isomorphism*.

Note that a homomorphism f is an isomorphism if and only if the linear coefficient is invertible.

Example 5.11. Consider the multiplicative formal group law $\hat{G}_m(x, y) = x + y + xy$. It satistifes $1 + \hat{G}_m(x, y) = (1 + x)(1 + y)$. Recall the classical logarithm series

$$f(x) = \log(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots .$$

We from calculus that $\log((1 + x)(1 + y)) = \log(1 + x) + \log(1 + y)$. That is $f(x) + f(y) = f(\hat{G}_m(x, y))$. Thus, f is an isomorhism between $\widehat{\mathbb{G}}_a$ and $\widehat{\mathbb{G}}_m$ over $\mathbb{Q}$.

This already suggests that we need denominators in R if the additive formal group law should be isomorphic to a multiplicative one. This is indeed so. Before we prove this, we need some notation.

Definition 5.12. Let F be a formal group law over a ring R. Define inductively $[n]_F(x)$ by $[1]_F(x) = x$ and $[n]_F(x) = F([n - 1]_F(x), x)$. I.e. we "multiply by n".

Examples 5.13. For $F(x,y) = x+y$, we obtain $[n]_F(x) = nx$.
For $F(x,y) = x+y+uxy$, we obtain $[n]_F(x) = \frac{(1+ux)^n-1}{u}$.

Lemma 5.14. *Fix a ring R and an invertible element $u \in R$. The additive formal group law $\hat{\mathbb{G}}_a(x,y) = x+y$ is only isomorphic to $F(x,y) = x+y+uxy$ over R if R is a $\mathbb{Q}$-algebra.*

Proof. Let f be such an isomorphism $\hat{\mathbb{G}}_a \to F$. Then

$$0 = f([n]_{\mathbb{G}_a})(f^{-1}(x)) = [n]_F(x) = \frac{(1+ux)^n}{u} - 1$$

over R/n. Thus $u^{n-1} = 0$ in R/n and thus $R/n = 0$ as u is invertible. As this is true for every n, the ring R must be a $\mathbb{Q}$-algebra. □

5.4 (Co)homology of K-theory

Lemma 5.15. *Let $x, x' \in E^2(\mathbb{CP}^\infty)$ correspond to two different complex orientations of a ring spectrum E. Then the resulting formal group laws F and F' are isomorphic. More precisely, we can express x' as a power series $f(x)$ as $E^*(\mathbb{CP}^\infty) \cong E^*[\![x]\!]$. Then*

$$f(F(x_1,x_2)) = F'(x_1', x_2') = F'(f(x_1), f(x_2)).$$

Corollary 5.16. *The integral homology $H_*(K;\mathbb{Z})$ is a rational vector space. In particular, $H_*(K) \cong \mathbb{Q}[u^{\pm 1}]$ and $H_*(K;\mathbb{F}_p) \cong H_*(K/p;\mathbb{Z}) = 0$, where K/p denotes the mapping cone of $p \cdot \mathrm{id}_K \colon K \to K$.*

Proof. By definition $H_*(K;\mathbb{Z}) \cong \pi_* H\mathbb{Z} \wedge K$. The spectrum $H\mathbb{Z} \wedge K$ admits ring spectra maps from $H\mathbb{Z}$ and K and thus carries both the additive and a multiplicative formal group law from the two different complex orientations. These are isomorphic by the last lemma. By Lemma 5.14, it follows that $\pi_* H\mathbb{Z} \wedge K$ is a $\mathbb{Q}$-algebra.

The rational Hurewicz map

$$\pi_* K \otimes \mathbb{Q} \to H_*(K;\mathbb{Q}) \cong H_*(K;\mathbb{Z})$$

is (as always) an isomorphism and $\pi_* K \cong \mathbb{Z}[u^{\pm 1}]$.

The last part follows by the Bockstein sequence

$$\cdots \to H_*(K;\mathbb{Z}) \xrightarrow{p} H_*(K;\mathbb{Z}) \to H_*(K;\mathbb{F}_p) \to \cdots$$ □

This shows that the Hurewicz theorem only holds for connective spectra (as it is not true for K/p). More interestingly, one can now apply the universal coefficient sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}((H\mathbb{Z})_{i-1}(K), \mathbb{Z}) \to (H\mathbb{Z})^i(K) \to \mathrm{Hom}_{\mathbb{Z}}((H\mathbb{Z})_i(K), \mathbb{Z}) \to 0$$

to obtain:

Proposition 5.17. *We have*

$$[K, H\mathbb{Z}] = (H\mathbb{Z})^0(K) = 0$$

and

$$[K, \Sigma H\mathbb{Z}] = (H\mathbb{Z})^1(K) = \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}).$$

As SHC $\to$ Coh is full, this shows that there is no nonzero transformation of cohomology (of spaces or spectra) from K to $H\mathbb{Z}$. On the other hand, $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$ is uncountable! In contrast, we have:

Proposition 5.18. *There is no nonzero natural transformation of the cohomology theories represented by K to that for $\Sigma H\mathbb{Z}$* on spaces. *In particular,* SHC $\to$ Coh *is not faithful.*

Proof. It suffices to show that all natural transformations from K^{2n} to $H^{2n+1}(-, \mathbb{Z})$ are zero on all spaces. As the former is represented by $BU \times \mathbb{Z}$, these natural transformations are in one-to-one correspondence with $H^{2n+1}(BU \times \mathbb{Z}; \mathbb{Z})$, which turns out to be zero. Indeed, $H^*(BU; \mathbb{Z}) \cong \mathbb{Z}[c_1, c_2, \dots]$ with $|c_i| = 2i$. $\square$

6 Exercises

Exercise 6.1. Show the following homeomorphism/homotopy equivalence for pointed spaces X and Y.

(a) $\Sigma X \cong X \wedge S^1$

(b) $\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$ (if X, Y are CW-complexes)

Exercise 6.2. In this exercise, you compute two colimits.

(a) Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of spaces. Show that the colimit over

$$X_1 \to X_1 \vee X_2 \to X_1 \vee X_2 \vee X_3 \to \cdots$$

is isomorphic to $\bigvee_{i \in \mathbb{N}} X_i$.

(b) Compute the abelian group $\mathrm{colim}(\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \cdots)$

Exercise 6.3. Use the classification of vector bundles on S^1 to compute $\pi_1 K \cong \widetilde{K}(S^1) = 0$ and $\pi_1 KO \cong \widetilde{KO}(S^1) \cong \mathbb{Z}/2$.

Exercise 6.4. Fill in the details in the proof of the long exact sequence of homotopy groups associated with a map $f \colon X \to Y$ of spectra.

Exercise 6.5. Compute the first few stable homotopy groups of $S^2 \vee S^4$, $S^2 \times S^2$, $\mathbb{RP}^2$ and $\mathbb{CP}^2$.

Exercise 6.6. In this exercise we will demonstrate the different behavior of unstable and stable homotopy groups.

(a) Let $\eta \colon S^3 \to S^2$ be the Hopf map sending $(z_1, z_2) \in S^3 \subset \mathbb{C}^2$ to $\frac{z_1}{z_2} \in \mathbb{C} \cup \{\infty\} \cong S^2$. Show that the postcomposition with the complex conjugation map is homotopic to η, but that η is not homotopic to $-\eta$ (which is *precomposition* with a map of degree -1). [Hint: Use that η is a fibration with fiber S^1]

(b) Show that in general for a map $f \colon S^n \to S^k$ the precomposition with a degree (-1) map $S^{n+1} \to S^{n+1}$ of $\Sigma f \colon S^{n+k+1} \to S^{k+1}$ is homotopic to to the postcomposition with a degree (-1) map $S^{k+1} \to S^{k+1}$. Deduce that indeed $2[\Sigma\eta] \in \pi_4 S^3 \cong \pi_1^{st} S^0$ is zero. If you know Steenrod operations, use the Freudenthal suspension theorem to deduce further that $\pi_1^{st} S^0 \cong \mathbb{Z}/2$.

Exercise 6.7. This exercise is about the bordism relation.

(a) Show that bordism is an equivalence relation.

(b) Show that $M\#N$ is bordant to $M\coprod N$ and deduce that $[\mathbb{RP}^2]$ generates $\mathfrak{N}_2$.

Exercise 6.8. Given a spectrum X, define a new spectrum X' by $X'_n = X_n$ for $n \geq 10$ and $X'_n = \mathrm{pt}$ for $n < 10$. Show that X and X' become isomorphic in SHC.

Exercise 6.9. Compute the localization of the category of abelian groups at the class of morphisms $f\colon A \to B$ that induce an isomorphism $A \otimes \mathbb{Q} \to B \otimes \mathbb{Q}$.

Exercise 6.10. If you are familiar with derived categories, compute for the category Ch_R of nonnegatively graded chain complexes over a ring R the localization at

(a) the class of chain homotopy equivalences, and

(b) the class of homology-isomorphisms.[8]

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[8]You should observe how this is analogous to spaces, where chain complexes of projectives play now the role of CW-complexes.