Modules over Real K-Theory and *TMF*

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Young Topologists Meeting 2012

Spaces

Ho(Spaces)

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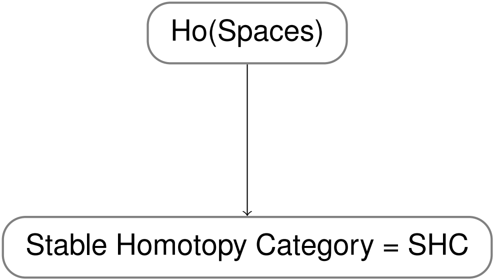
Slogan: If things are difficult, make them easier.

Spaces



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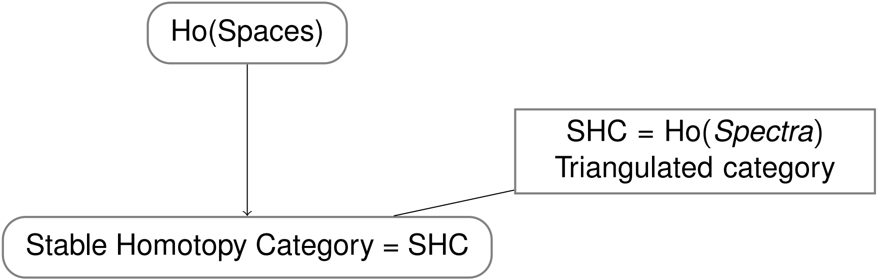
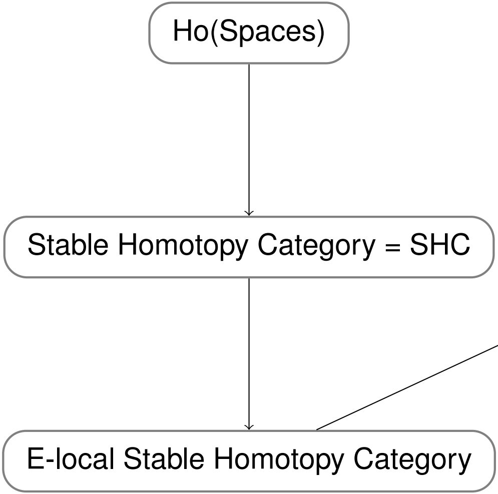

Ho(Spaces)
SHC = Ho(*Spectra*)
Triangulated category
Stable Homotopy Category = SHC



Let E_* be a homology theory.

Make all E_*-equivalences in SHC to isomorphisms.

$\rightsquigarrow$ E-local stable homotopy category.

Examples of Homology Theories

- Rational singular homology $H\mathbb{Q}_*$. For every spectrum X, have maps $X \longrightarrow Y \leftarrow \bigvee \mathbb{S}^{n_i}$ of spectra which are $H\mathbb{Q}_*$-isomorphisms. $\Rightarrow H\mathbb{Q}$-local SHC $\simeq$ graded $\mathbb{Q}$-vector spaces.

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- Set often $E(0) = H\mathbb{Q}$.
- $E(1)$ is a summand of $KU_{(p)}$

$$\Rightarrow E(1)\text{-local SHC} \simeq KU_{(p)}\text{-local SHC}$$

$E(n)$-localizations

Theorem (Chromatic Convergence, Hopkins, Ravenel)

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- $n = 2$ for $p > 3$ (Shimomura–Yabe).

Ring and Module Spectra

- Many familiar spectra admit the structure of a commutative (symmetric) ring spectrum, e.g. all Eilenberg–MacLane spectra HR (for R a ring), KO, KU, the bordism spectra ...

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- $\pi_* M$ gets the structure of a $\pi_* R$-module.
- By Schwede–Shipley get a model structure on R-mod with homotopy category Ho(R-mod).

Approximation by Module Spectra

$E(0)$-local SHC

$E(1)$-local SHC

$E(2)$-local SHC

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$E(0)$-local SHC $\xleftarrow{\simeq}$ Ho($H\mathbb{Q}$-mod)

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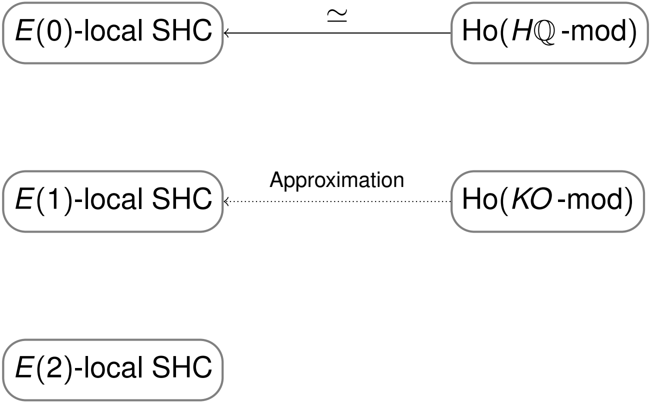
Approximation by Module Spectra



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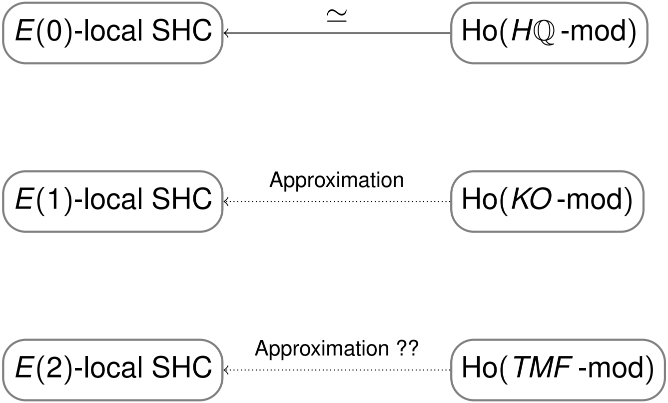

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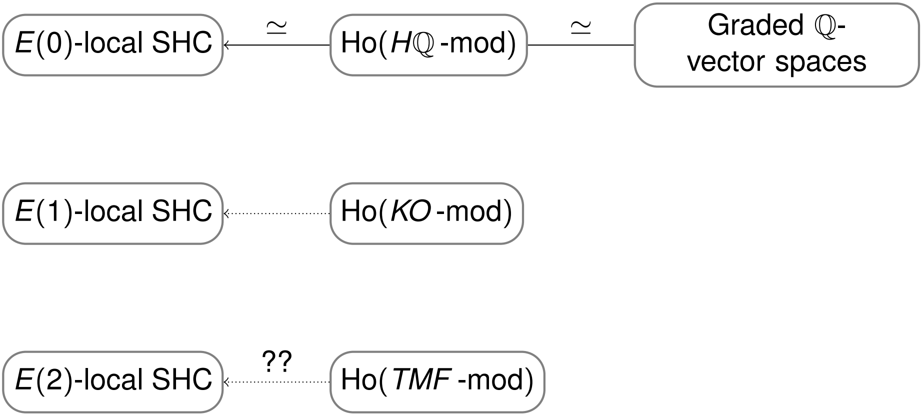

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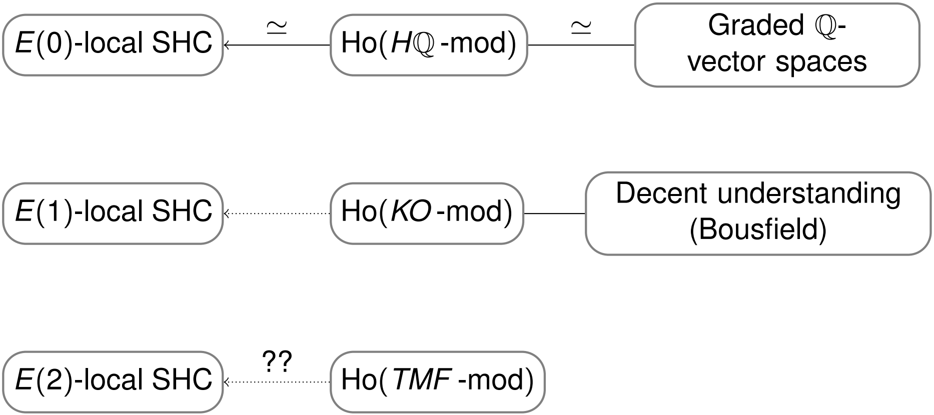

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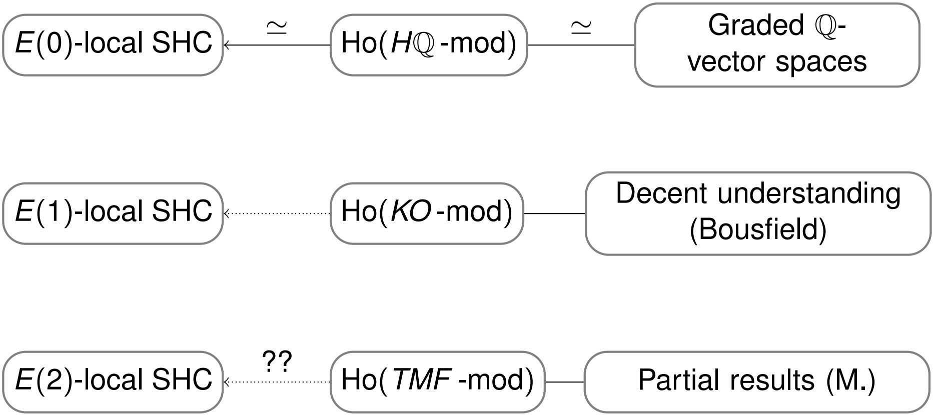

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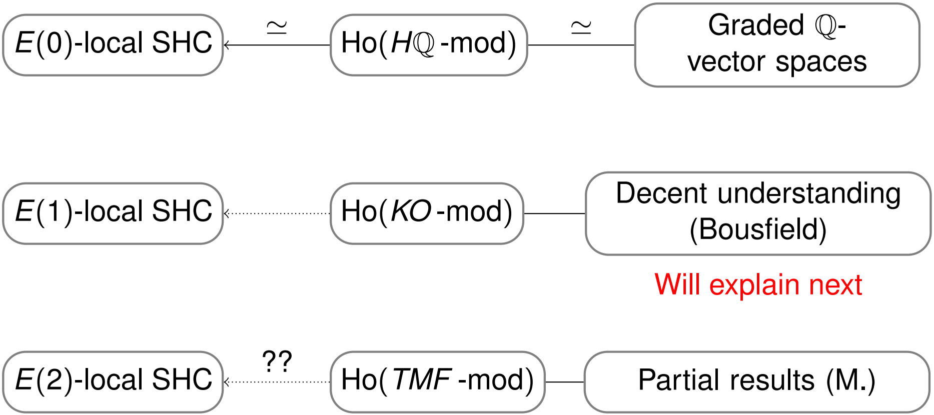

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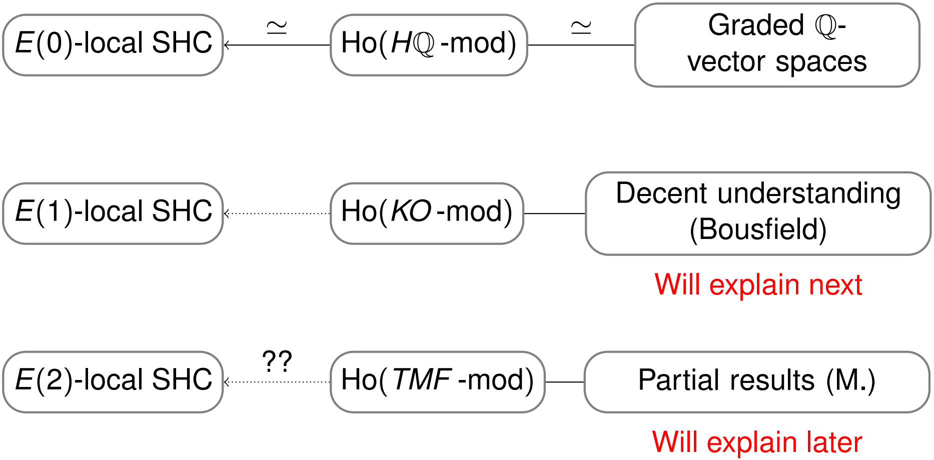

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Definition

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- For every projective module P_0 over $\pi_* R$, there is an R-module P with $\pi_* P \cong P_0$.
- Let P be a projective R-module and M be an arbitrary one. Then every $\pi_* R$-linear map $\pi_* P \longrightarrow \pi_* M$ is induced by a map $P \longrightarrow M$, which is unique up to homotopy.

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Upshot: The homotopy category of projective R-modules is equivalent to the category of projective $\pi_* R$-modules.

Modules over KU

We have $\pi_* KU \cong \mathbb{Z}[B_C, B_C^{-1}]$ with $|B_C| = 2$. So for M a KU-module, we have two-term free resolutions

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \pi_* M \longrightarrow 0.$$

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$\rightsquigarrow$ Understanding of KU-modules:

- $\mathrm{Ho}(KU\text{-mod}) \simeq \mathcal{D}(\pi_* KU\text{-grmod})$
- Two KU-modules are isomorphic in $\mathrm{Ho}(KU\text{-mod})$ iff there homotopy groups agree.

KO-Modules

Question:

How about *KO*-modules?

$$\pi_* KO \cong \mathbb{Z}[\eta, \xi, B_R^{\pm 1}]/(\xi^2 = 4B_R, \eta^3 = 0, \eta\xi = 0, 2\eta = 0)$$

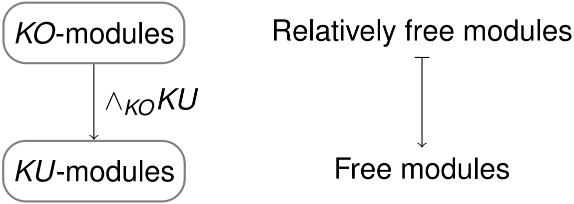
8-periodic

8	$\square$	B_R
7		
6		
5		
4	$\square$	ξ
3		
2	$\bullet$	η^2
1	$\bullet$	η
0	$\square$	1

This has infinite homological dimension, so projective resolutions may be arbitrarily long.

Strategy doesn't work!

Relatively Free Modules



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Easy to see: For every $M \in KO$-mod exists cofiber sequence

$$R_1 \longrightarrow R_0 \longrightarrow M$$

with R_0, R_1 relatively free.

Examples of Relatively Free Modules

Recall: $\eta \in \pi_1 KO$. Multiplication by η gives cofiber sequence

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In general: If we cone off a torsion element from a relatively free module, get a relatively free module again.

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Theorem (Bousfield)

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Crucial:

$$\mathrm{Ho}(KO\text{-mod}) \longrightarrow \text{graded } \mathbb{Z}[\mathbb{Z}/2]\text{-modules}$$

$$M \longmapsto \pi_*(M \wedge_{KO} KU)$$

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- We get *TMF* as the global sections $\mathcal{O}^{top}(\mathcal{M}^{ell})$. This is the spectrum of topological modular forms.
- We have a morphism

$$\pi_* TMF \longrightarrow \underset{\mathbb{Z}[c_4, c_6, \Delta^{\pm 1}]/(c_4^3 - c_6^2 = 1728\Delta).}{\text{Ring of modular forms}}$$

This becomes an isomorphism after inverting 2 and 3.

Homological Dimensions

- For $p > 3$, the ring $\pi_* TMF_{(p)} \cong \mathbb{Z}_{(p)}[c_4, c_6, \Delta^{-1}]$, has homological dimension 2. This can be used to show: Ho($TMF_{(p)}$-mod) $\simeq \mathcal{D}(\pi_* TMF_{(p)})$ (Franke, Patchkoria)

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KU	*TMF*(2)

$$\pi_* TMF(2) \cong \mathbb{Z}_{(3)}[x_2, y_2, \Delta^{-1}]$$

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A *TMF*-module M is relatively projective if $M \wedge_{TMF} TMF(2)$ is a projective $TMF(2)$-module.

For every *TMF*-module M, there exist cofiber sequences

$$N \longrightarrow R_0 \longrightarrow M$$

$$R_2 \longrightarrow R_1 \longrightarrow N$$

such that R_0, R_1 and R_2 are relatively projective.

Types of Relatively Projective Modules

Two moves

- If M is a relatively projective module, $a \in \pi_* M$ torsion, the cone of $\Sigma^{|a|} TMF \xrightarrow{a} M$ is a relatively projective module of one rank higher.
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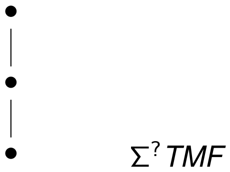

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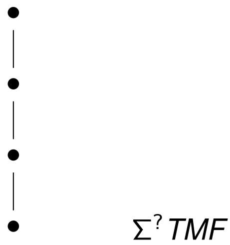

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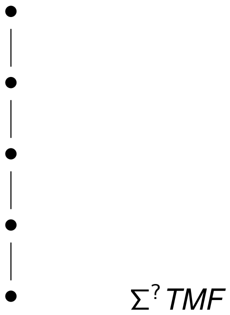

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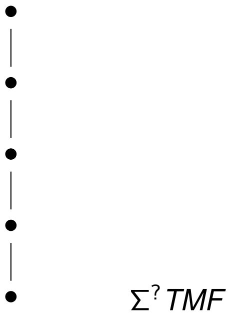

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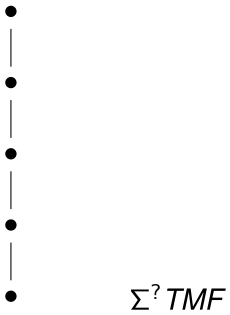

Hook modules

Types of Relatively Projective Modules

Two moves

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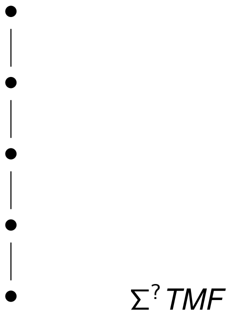

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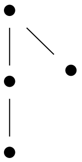

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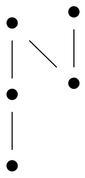

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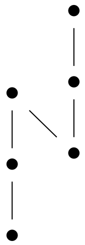

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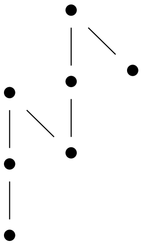

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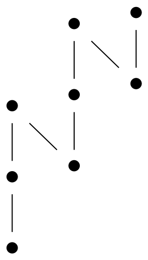

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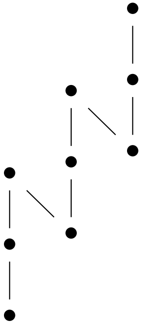


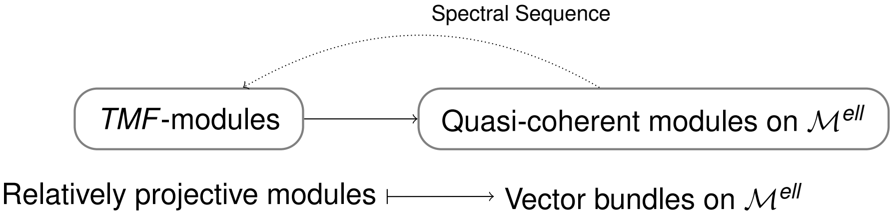
Vector Bundles

Spectral Sequence

TMF-modules $\longrightarrow$ Quasi-coherent modules on $\mathcal{M}^{ell}$

Relatively projective modules $\longmapsto$ Vector bundles on $\mathcal{M}^{ell}$

Vector Bundles



- Here, a quasi-coherent module is called a vector bundle if it is locally free.

Towards a Classification of Relatively Projective Modules?

Theorem (Hook Theorem)

Let M be a relatively projective finite TMF-module. Assume that the vector bundle associated to M is "nice". Then M is a hook module.

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Thank you for your attention!

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Have torsion element $\beta \in \pi_{10} \mathit{TMF} \rightsquigarrow \beta^2, \beta^3, \beta^4 \in \pi_* \mathit{TMF}$.

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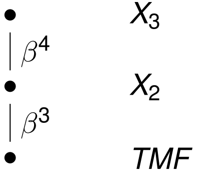

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$\bullet$ X_4

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An Infinite Familiy

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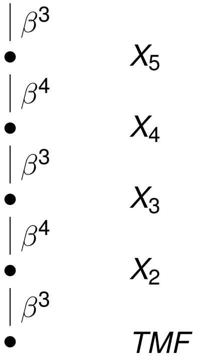

- X_k is not decomposable into modules of smaller rank that one gets by coning off iteratively torsion elements from $\Sigma^? TMF$.
- If X_k is decomposable, then into hook modules.