Fibrancy of (Relative) Categories

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Overview

Goal of the talk is to discuss the homotopy theory of (relative) categories and characterize fibrant objects in the corresponding model structures.

The nerve functor

Let $\mathcal{C}$ be a category. Its nerve is the simplicial set Nerve $\mathcal{C}$ with n-simplices

$$(\text{Nerve } \mathcal{C})_n = \mathrm{Fun}([n], \mathcal{C}),$$

i.e. all chains of n composable morphisms.
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Examples:

- $\text{Nerve}\,[n] = \Delta[n]$
- $|\text{Nerve } G| = BG$ for a group G seen as a category with one object.

Weak equivalences

We call a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ a weak equivalence if

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- Algebraic K-Theory

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A category equipped with a subcategory of weak equivalences (containing all objects) is called a relative category. So Cat gets the structure of a relative category.

Homotopy Category

To a relative category $\mathcal{C}$, we can associate its homotopy category $\mathrm{Ho}(\mathcal{C})$. Its morphisms are given by equivalence classes of zigzags

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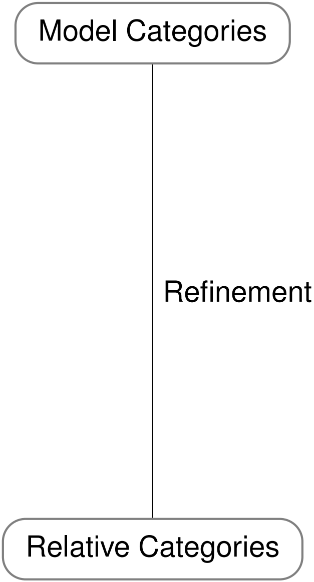
- Arbitrary long zigzags are difficult to work with.
- There may be many non-weak-equivalences that go under the functor $\mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ to isomorphisms.

Model Categories



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3. $\mathcal{W}$, $\mathcal{C}$ and $\mathcal{F}$ are closed under retracts.
4. (Acyclic) fibrations can be characterized by lifting properties.
5. Every morphisms f in $\mathcal{M}$ can (functorially) be factorized as follows:

$$X \overset{\simeq}{\rightarrowtail} X' \twoheadrightarrow Y \qquad (f : X \to Y) \qquad\qquad X \rightarrowtail Y' \overset{\simeq}{\twoheadrightarrow} Y \qquad (f : X \to Y)$$

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$$\mathrm{Ho}(\mathcal{M})(X, Y) = [X, Y] \text{ for } X \text{ cofibrant and } Y \text{ fibrant}$$

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- Categories?

Model Structure on Cat

Naive attempt to get a model structure on Cat: $f : \mathcal{C} \longrightarrow \mathcal{D}$ is a fibration/weak equivalence if Nerve f : Nerve $\mathcal{C} \longrightarrow$ Nerve $\mathcal{D}$ is.

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Alternatively: If Nerve was a right Quillen functor, its left adjoint

$$c : \mathrm{sSet} \longrightarrow \mathrm{Cat}$$

would have to be a homotopy inverse (as every simplicial set is cofibrant). But Nerve $cX \not\simeq X$ in general.

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Denote by

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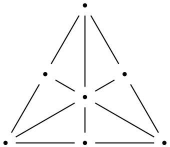

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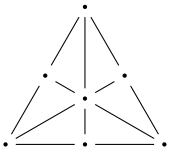

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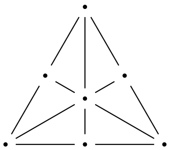

There is a natural weak equivalence $X \longrightarrow \mathrm{Ex}\, X$ and if X was fibrant, $\mathrm{Ex}\, X$ is as well. The functor Ex makes more things fibrant and $\mathrm{Ex}^{\infty} X$ is always fibrant.

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Still does not work: $(\mathrm{Ex}\,\mathrm{Nerve}\,)(c\,\mathrm{Sd})X \not\simeq X$ in general.

$$\mathrm{Cat} \underset{c\,\mathrm{Sd}}{\overset{\mathrm{Ex}\,\mathrm{Nerve}}{\rightleftarrows}} \mathrm{sSet}$$

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This determines a model structure on Cat that is Quillen equivalent to sSet via

$$\mathrm{Ex}^2\,\mathrm{Nerve}\colon\ \mathrm{Cat} \longrightarrow \mathrm{sSet}$$

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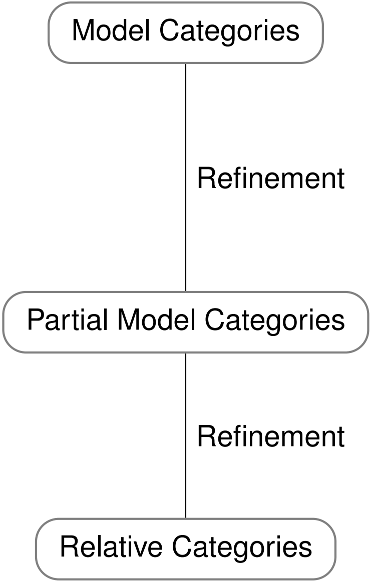
- Ex^2 Nerve $\mathcal{C}$ fibrant $\Leftrightarrow$???

Partial Model Categories



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This is enough to control $\mathcal{M} \longrightarrow \text{Ho}(\mathcal{M})$: Existence of 3-arrow calculus.

Main theorem

Theorem (M.-Ozornova)

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If $(\mathcal{M}, \mathcal{W})$ is a partial model category, then $\mathcal{W}$ is fibrant in the Thomason model structure on Cat.

A category $\mathcal{W}$ is the category of weak equivalences of partial model category if there are subclasses $\mathcal{C}, \mathcal{F} \subset \mathcal{W}$ (called cofibrations and fibrations, respectively) such that

1. Pushouts of cofibrations exist and are again cofibrations
2. Pullbacks of fibrations exist and are again fibrations
3. Every map can be functorially factorized into a cofibration and a fibration.

Examples

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- Many more...

Proof

A category $\mathcal{C}$ is fibrant iff it has the right lifting property with respect to all maps $c\,\mathrm{Sd}^2\,\Lambda^k[n] \longrightarrow c\,\mathrm{Sd}^2\,\Delta[n]$.

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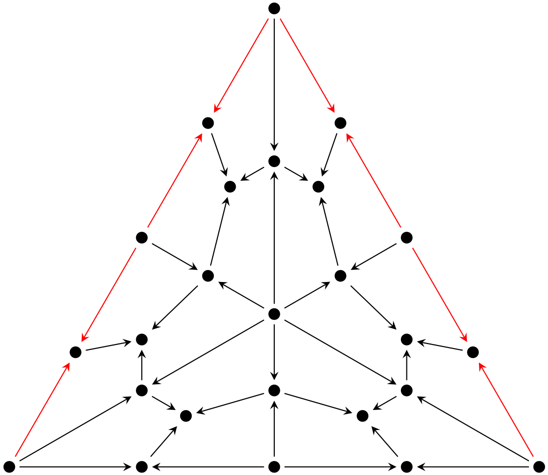

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A relative functor $f : \mathcal{C} \longrightarrow \mathcal{D}$ is a weak equivalence if it induces an equivalence

$$\mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{D})$$

and weak (homotopy) equivalences of all mapping spaces.

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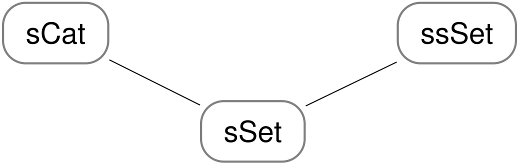

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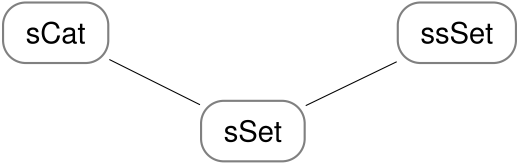

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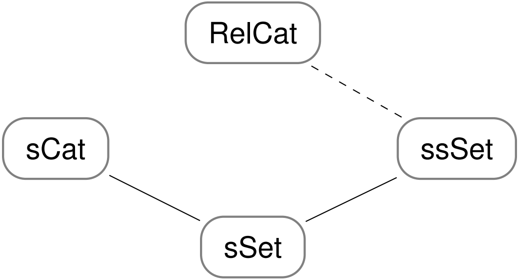

The classifying diagram functor

A relative functor $f : \mathcal{C} \longrightarrow \mathcal{D}$ is a weak equivalence iff Nf is one in the Rezk model structure for the classifying diagram functor

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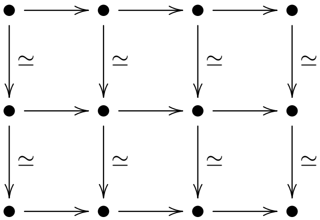
$((N\mathcal{C})_4)_2$



Barwick–Kan model structure

Barwick and Kan define a functor

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(analogous to Ex^2 Nerve) and define $f : \mathcal{C} \longrightarrow \mathcal{D}$ to be a fibration if $N_\xi f$ is. This defines a model structure on RelCat, Quillen equivalent to the Rezk model structure on ssSet.

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Partial answer: Barwick and Kan show: If $\mathcal{M}$ is a partial model category, a Reedy fibrant replacement of $N_\xi \mathcal{M}$ is fibrant is a complete Segal space.

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(Pre-)theorem (M.)

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Barwick and Kan construct for every relative category $\mathcal{C}$ an equivalence $\mathcal{C} \longrightarrow \mathcal{D}$ to a homotopically full subcategory of simplicial sets.

Model categories are fibrant

(Pre-)theorem (M.)

(Homotopically full subcategories of) model categories are fibrant in the Barwick–Kan model structure.

A full subcategory $\mathcal{D} \subset \mathcal{M}$ is called homotopically full if for $c \in \mathcal{D}$ and $c \simeq d$ also $d \in \mathcal{D}$.

Barwick and Kan construct for every relative category $\mathcal{C}$ an equivalence $\mathcal{C} \longrightarrow \mathcal{D}$ to a homotopically full subcategory of simplicial sets.

Corollary

This defines a fibrant replacement functor for RelCat.

Thank you!