Calculations for equivariant topological modular forms

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(joint work with David Gepner)

Every even periodic cohomology theory h gives rise to a formal group, which can be defined as the formal spectrum $\mathrm{Spf}\, h^0(\mathbb{CP}^\infty)$. In case of K-theory, this formal group is isomorphic to the formal completion $\widehat{\mathbb{G}}_m$ of the multiplicative group scheme $\mathbb{G}_m = \mathrm{Spec}[t^{\pm 1}]$. In classical language, this corresponds to the formal group law $x + y + xy$.

In general, every commutative group scheme that is smooth of relative dimension 1 gives rise to a formal group by completion at the unit. Besides $\mathbb{G}_m$ essentially the only other examples of such group schemes are the additive group (corresponding to ordinary homology) and elliptic curves. An *elliptic cohomology theory* consists of an even-periodic cohomology theory h, an elliptic curve C over $h^0(\mathrm{pt})$ and an isomorphism of formal groups between $\mathrm{Spf}\, h^0(\mathbb{CP}^\infty)$ and $\widehat{C}$.

A natural demand is to extend elliptic cohomology theories to equivariant theories. Going one step back to K-theory, we observe that S^1-equivariant K-theory of a point is isomorphic to the representation ring $R(S^1) \cong \mathbb{Z}[t^{\pm 1}]$, where t corresponds to the tautological representation of $S^1 = U(1)$. Thus $K^0_{S^1}(X)$ becomes for every S^1-space a module over $\mathbb{Z}[t^{\pm 1}]$. As this is the coordinate ring of $\mathbb{G}_m$, the module $K^0_{S^1}(X)$ defines a quasi-coherent sheaf on $\mathbb{G}_m$.

Thus it becomes natural to expect that S^1-equivariant elliptic cohomology takes values in sheaves on the corresponding elliptic curve, an idea already present in the original work of Grojnowski [2] over the complex numbers and Greenlees over the rationals [1]. An idea of Lurie [3] [4] was to work fully derived and only pass to homotopy groups at the end. This relies heavily on spectral algebraic geometry; in particular Lurie had to define elliptic curves and formal groups over E_∞-rings. We will assume these in the following to represent functors valued in commutative topological groups instead of just E_∞-spaces. The following definition is a derived analogue of the notion of an elliptic cohomology theory:

Definition. An *oriented elliptic curve* consists of an E_∞ ring spectrum R, an elliptic curve C over R and an equivalence over R of formal groups between $\mathrm{Spf} E^{\mathbb{CP}^\infty}$ and $\widehat{C}$.

Given now an oriented elliptic curve (R, C), we want following Lurie to define a contravariant functor

$$R^{\mathrm{shv}}_{S^1} \colon (\mathcal{S}^{\mathrm{fin}}_G)^{op} \to \mathrm{QCoh}(C, \mathcal{O}_C)$$

from the ∞-category $\mathcal{S}^{\mathrm{fin}}_G$ of finite S^1-CW complexes to the ∞-category of quasi-coherent $\mathcal{O}_C$-modules. By Elmendorf's theorem, $\mathcal{S}^{\mathrm{fin}}_G$ embeds into space-valued presheaves on the orbit-category Orb_{S^1} as the sub-∞-category generated by finite colimits from the orbits S^1/H for closed subgroups $H \subset S^1$. Thus it suffices to specify $R^{\mathrm{shv}}_{S^1}$ on these orbits (with appropriate functoriality) if we demand that $R^{\mathrm{shv}}_{S^1}$ sends finite colimits in $\mathcal{S}^{\mathrm{fin}}_G$ to finite limits. We set $R^{\mathrm{shv}}_{S^1}(S^1/S^1)$ to be $\mathcal{O}_C$ and

$R^{\mathrm{shv}}_{S^1/C_n} = (i_n)_*\mathcal{O}_{C[n]}$, where $i_n\colon C[n] \hookrightarrow C$ is the inclusion of the n-torsion. This is again in line with K-theory as $K_{S^1}(S^1/C_n) \cong R(C_n)$ and $\operatorname{Spec} R(C_n) = \mathbb{G}_m[n]$.

There are two methods to obtain more classical invariants from $R^{\mathrm{shv}}_{S^1}$. The first is to apply (sheafified) homotopy groups to obtain sheaves of abelian groups on the underlying classical elliptic curve of C, resulting in a Grojnowski style version of elliptic cohomology. The other is to take global sections, resulting in a functor

$$R_{S^1}\colon (\text{finite } S^1\text{-spaces})^{op} \to \text{ Spectra}$$

Taking homotopy groups results in an S^1-equivariant cohomology theory

$$R^*_{S^1}\colon (\text{finite } S^1\text{-spaces})^{op} \to \text{ graded abelian groups}$$

Actually, this is represented by an S^1-spectrum R with $R^{S^1} = R_{S^1}(\mathrm{pt})$.

This abstract theory leaves the question open how to calculate these objects, which we answer in the simplest case.

Theorem (Gepner–M.)**.** *There is an equivalence $R^{S^1} = R_{S^1}(\mathrm{pt}) \simeq R \oplus \Sigma R$. The map $R \to R^{S^1}$ is given by restriction along $S^1 \to \{e\}$ and $\Sigma R \to R^{S^1}$ by a transfer.*

While there is just one multiplicative groups, there are a lot of elliptic curves, resulting in many elliptic cohomology theories. There is one universal theory, called *topological modular forms TMF* associated with the moduli stack of all elliptic curves. While itself not an elliptic cohomology theory, it maps to all elliptic cohomology theories (associated with an oriented elliptic curve) and supports an equivariant theory as well. The naturality in the previous theorem implies:

Corollary. *We have an equivalence $TMF^{S^1} \simeq TMF \oplus \Sigma TMF$.*

Note that the homotopy groups of TMF are completely known and thus we obtain a complete calculation of $TMF^*_{S^1}(\mathrm{pt})$. Note moreover that (in contrast to K^{S^1}) the TMF-module TMF^{S^1} is dualizable, with dual $\Sigma^{-1}TMF^{S^1}$.

Actually, equivariant TMF can be defined for all compact Lie groups G, in particular resulting in fixed points TMF^G. Our results together with work of Lurie suggest the following conjecture.

Conjecture. *Let G be a compact Lie group and L its adjoint representation. Then TMF^G is a dualizable TMF-module with dual $(\Sigma^{-L}TMF)^G$.*

The case of G finite is a consequence of tempered ambidexterity, one of the main results of [4]. The case $G = S^1$ (and actually $G = (S^1)^r$) follows from the corollary above.

In particular, Lurie's result implies that TMF^{C_n} is a self-dual TMF-module.

Question. *Can one explicitly calculate TMF^{C_n} or at least its homotopy groups?*

Much of the difficulty lies in understanding explicitly the n-torsion points in the universal elliptic curve. This simplifies significantly if we invert n in the basis or even p-complete away from n. The following is one of the main results from [5].

Theorem (M.)**.** *For p not dividing n, the TMF-module TMF^{C_n} splits after p-completion into shifted copies of TMF, $TMF_1(2)$ (if $p = 3$) and $TMF_1(3)$ (if $p = 2$).*

References

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