Invariant prime ideals in the equivariant Lazard ring

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(joint work with Markus Hausmann)

Chromatic homotopy theory is based on the paradigm that the structure of the stable homotopy category is predicted by the moduli stack of formal groups $\mathcal{M}_{FG}$. This correspondence is mediated by complex cobordism MU, whose coefficients π_*MU carry the universal formal group law. More precisely, every spectrum X defines a graded quasi-coherent sheaf $\mathcal{F}_*^X$ on $\mathcal{M}_{FG}$, corresponding to the graded (π_*MU, MU_*MU)-comodule $MU_*(X)$, and the properties of $\mathcal{F}_*^X$ reflect those of X.

We may look, for example, at the support $\operatorname{supp}\mathcal{F}_*^X$ of $\mathcal{F}_*^X$ in the space $|\mathcal{M}_{FG}|$ of points of $\mathcal{M}_{FG}$. The points of $\mathcal{M}_{FG}$ correspond to formal groups over fields and are thus classified by the residue characteristic and the height. By definition of the Balmer spectrum $\operatorname{Spc}(\mathrm{Sp}^{\mathrm{fin}})$, the support theory $\operatorname{supp}\mathcal{F}_*^{(-)}$ defines a continuous map $|\mathcal{M}_{FG}| \to \operatorname{Spc}(\mathrm{Sp}^{\mathrm{fin}})$. By the Hopkins–Smith thick subcategory theorem, this is a homeomorphism.

We claim that analogous statements are true for every compact abelian Lie group G, which we will fix throughout.

Tenet. *Mediated by equivariant complex cobordism MU_G, the moduli stack of G-equivariant formal groups predicts the structure of the G-equivariant genuine stable homotopy category.*

1. The moduli stack of G-equivariant formal groups

We warn that G-equivariant formal groups are *not* the same as formal groups with a G-action. The latter are relevant for theories like $K\mathbb{R}$ or $MU_{\mathbb{R}}$, while the former are relevant for G-equivariantly complex oriented theories like KU_G or MU_G (the universal example). We believe that the notion of equivariant formal groups is not as widely known as it should, and therefore we give both motivation and definition of this notion. The motivation we give is topological, but the notion should also be interesting from the purely algebro-geometric point of view.

If E is a non-equivariant complex-oriented ring spectrum, then we have an isomorphism $E^{2*}(\mathbb{CP}^\infty) \cong E^*[\![x]\!]$ and hence $\operatorname{Spf} E^{2*}(\mathbb{CP}^\infty) \cong \widehat{\mathbb{A}}^1_{E^{2*}}$. If E_G is a G-equivariant complex-oriented ring spectrum, we need to replace $\mathbb{CP}^\infty$ by $\mathbb{CP}^\infty_G$, the G-space of complex lines in the complex complete universe $\mathcal{U} = \bigoplus_{V \in G^*} V^\infty$; here $G^* = \operatorname{Hom}(G, U(1))$ is the set of irreducible complex representations of G. The complex orientation is a class $y \in E^2_G(\mathbb{CP}^\infty_G)$. The group G^* acts by tensoring on $\mathcal{U}$ and hence on $\operatorname{Spf} E^{2*}_G(\mathbb{CP}^\infty_G)$, and the map $\mathbb{CP}^\infty \to \mathbb{CP}^\infty_G$ defines a map $\widehat{\mathbb{A}}^1_{E^{2*}_G} \to \operatorname{Spf} E^{2*}_G(\mathbb{CP}^\infty_G)$. This motivates the notion of a G-equivariant formal group, defined in different language by Cole–Greenlees–Kriz [CGK00].

Definition. *A* G-equivariant formal group *over a commutative ring k consists of*

- *a group object X in formal schemes over k,*
- *a G^*-action on the underlying formal scheme over k,*

- *a map* $\widehat{\mathbb{A}}^1_k \xrightarrow{\varphi} X$*,*

such that

- *the* G^**-translates of* φ *cover* X*,*
- *the coordinate of* $\widehat{\mathbb{A}}^1_k$ *extends to a non-zero divisor* y *on* X*.*

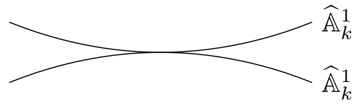

FIGURE 1. A schematic depiction of a C_2-equivariant formal group

Requiring y as part of the data, gives the notion of a G-equivariant formal group *law.* Every G-equivariant complex oriented theory defines such a group law in the manner sketched above. To obtain a notion of G-equivariant formal group that satisfies descent and hence defines a moduli stack $\mathcal{M}^G_{FG}$, we should weaken the definition above to asking for the existence of y (and the coordinate on $\widehat{\mathbb{A}}^1_k$) only Zariski-locally on k.

In the monograph [Str11], Strickland investigated many aspects of equivariant formal groups and showed in particular:

Theorem (Strickland)**.** *The points of* $\mathcal{M}^G_{FG}$ *are classified by the residue characteristic* p*, the height* n *and the "subgroup of definition"* $H \subseteq G$*.*

The importance of the notion of equivariant formal groups to topology was cemented when, extending earlier work of Greenlees and of Hanke–Wiemeler (for $G = C_2$), Hausmann showed in seminal work an analogue of Quillen's theorem:

Theorem (Hausmann, [Hau22])**.** *The coefficients* $\pi^G_* MU$ *carry the universal group law, and the Hopf algebroid* $(\pi^G_* MU, MU^G_* MU)$ *stackifies to* $\mathcal{M}^G_{FG}$*.*

2. The equivariant thick subcategory theorem

Relying on Hausmann's theorem, we can associate to every G-spectrum X a graded quasi-coherent sheaf $\mathcal{F}^X_*$ on $\mathcal{M}^G_{FG}$, corresponding to the $(\pi^G_* MU, MU^G_* MU)$-comodule $MU^G_* X$.

Theorem ([HM23])**.** *The map*

$$\begin{aligned} \textit{finite } G\textit{-spectra} &\to \{\textit{closed subsets of } |\mathcal{M}^G_{FG}|\} \\ X &\mapsto \textit{support of } \mathcal{F}^X_* \end{aligned}$$

is the universal support theory on finite G*-spectra. This induces a homeomorphism* $|\mathcal{M}^G_{FG}| \to \mathrm{Spc}(\mathrm{Sp}^{\mathrm{fin}}_G)$ *to the Balmer spectrum of finite* G*-spectra.*

This theorem has a curious history, as the topological side, namely the Balmer spectrum $\mathrm{Spc}(\mathrm{Sp}^{\mathrm{fin}}_G)$, was calculated first, in work of Strickland, Balmer–Sanders, Barthel–Hausmann–Naumann–Noel–Nikolaus–Stapleton and Barthel–Greenlees–Hausmann [BS17], [BHNNNS19], [BGH20]. In our work, we calculate the algebraic side, namely the topology on $|\mathcal{M}_{FG}^G|$, and establish that the map above is a support theory; this induces the required map $|\mathcal{M}_{FG}^G| \to \mathrm{Spc}(\mathrm{Sp}_G^{\mathrm{fin}})$. Establishing this support theory is harder than in the non-equivariant case: $\pi_* MU$ is known to be coherent, but the analogous result is not known for $\pi_*^G MU$. We conjecture:

Conjecture. *The stacks $\mathcal{M}_{FG}^G$ are coherent in the sense that coherent sheaves on them (corresponding to comodules whose underlying module is finitely presented) form an abelian category.*

3. Further results and the road ahead

The points in $\mathcal{M}_{FG}$ corresponds to the invariant prime ideals $I_{p,n} \subseteq \pi_* MU$. These are generated by the elements $p = v_0, v_1, \ldots, v_{n-1}$, where each v_k is canonically defined modulo $I_{p,k}$. We show that the v_n naturally refine to elements $\mathbf{v}_n$ in $\pi_*^{C_p^n \times U(1)} MU$, canonically defined modulo a certain invariant prime ideal. These allow us to write down generators for many of the invariant prime ideals of $\pi_*^G MU$; these do not form regular sequences in general, however.

The $\mathbf{v}_n$ are crucial in determining the topology on $|\mathcal{M}_{FG}|$ and form, in some sense, the algebraic replacements of the partition complexes used for the determination of the topology on $\mathrm{Spc}(\mathrm{Sp}_G^{\mathrm{fin}})$.

We expect that the $\mathbf{v}_n$ will play a fundamental role in the chromatic picture for G-spectra, especially for equivariant analogues of the periodicity theorem.

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