Higher-algebraic Localizations of Algebraic K-Theory

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- This has, e.g., helped to prove the redshift conjecture and disprove the telescope conjecture.
- This disproof has both profound implications for our global picture of algebraic topology and concrete applications on the growth of homotopy groups of spheres.

What is (algebraic) K-theory and what is it good for?

What is K-theory? Start topological!

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Definition (Atiyah–Hirzebruch, 1959)

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- $\mathrm{Vect}(X) \to \mathrm{Vect}(X)^{\mathrm{gp}}$ is generally not injective (e.g. $TS^n \oplus \underline{1} \cong \underline{n} \oplus \underline{1}$ and thus $[TS^n] = \underline{n}$ in K-theory)
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- Origins of this idea come from algebraic geometry, with vector bundles on algebraic varieties
- Important ingredient in Grothendieck–Riemann–Roch (1957)

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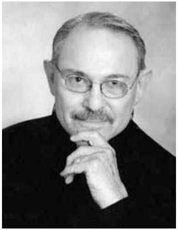
- X affine variety: $K(X) = K_0(\text{Ring of functions})$

Higher Algebraic K-theory: 60s and 70s

- $K_0(R)$ is just the first in a series $K_i(R)$ of groups (Bass, Milnor, Swan, Quillen)



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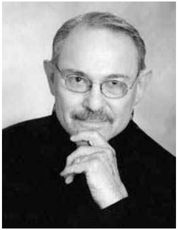

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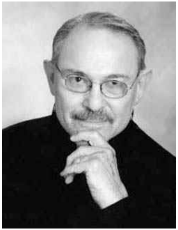

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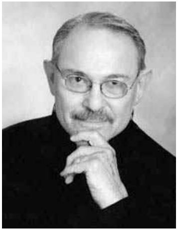
- Input R can be generalized to schemes, exact categories, ...



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Higher Algebraic K-theory: 70s and beyond

- Applications in algebraic geometry:
  Bloch–Quillen: K_i of schemes $\leftrightarrow$ Chow groups (intersection theory),
  Thomasons's proof of absolute cohomological purity, etc.
- Applications in number theory: Borel, Tate, Lichtenbaum, ..., Voevodsky: $K_i(\mathcal{O}_F)$ encode arithmetic properties: reciprocity laws, zeta values etc.
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- Relation to p-adic geometry: Bhatt, Morrow, Scholze,...
- Tradition in the Netherlands: Strooker, van der Kallen, Keune, Stienstra, Maazen, Vorst, ..., de Jeu, Sagave, Klang, ...

How to define higher algebraic K-theory?
Group completion in higher algebra

Step 1: incorporate automorphisms

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Group completing $\mathcal{P}roj(R) \quad \rightsquigarrow \quad$ groupoid $\mathcal{P}roj(R)^{\mathrm{gp}}$ such that

- isomorphism classes of objects $\cong K_0(R)$
- automorphism group of each object is $K_1(R)$.

But where are the higher $K_i(R)$?

Step 2: going to space

The classifying space functor

$$B\colon \text{groupoids} \longrightarrow \text{(topological) spaces}$$

satisfies

- $\pi_0 B\mathcal{G}$ = isomorphism classes of objects
- $\pi_1(B\mathcal{G}, x) \cong \mathrm{Aut}_{\mathcal{G}}(x)$.
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Thus,
$$\pi_i B(\mathcal{P}roj^{\text{gp}}(R)) \cong \begin{cases} K_i(R) & \text{for } i \leq 1 \\ 0 & \text{for } i > 1 \end{cases}$$

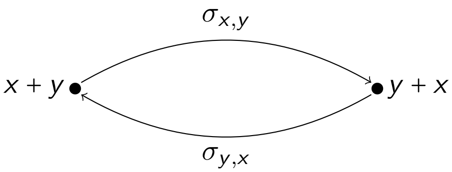
Higher algebra

$B\mathcal{P}roj(R)$ is an example of an E_∞-monoid.

Definition

An E_∞-monoid is a space X with a multiplication $X \times X \xrightarrow{+} X$ that is is associative and commutative up to coherent homotopy.

Thus, for points $x, y \in X$, there are paths $\sigma_{x,y} : x + y \rightsquigarrow y + x$, homotopies $H_{x,y} : \sigma_{x,y} * \sigma_{y,x} \simeq \mathrm{const}_{x+y}$, a homotopy of homotopies ...



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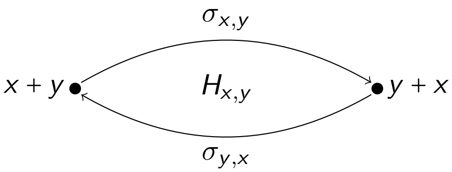

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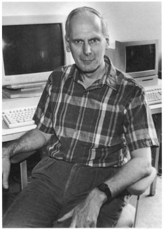
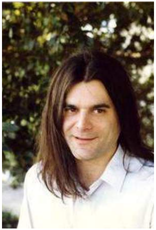
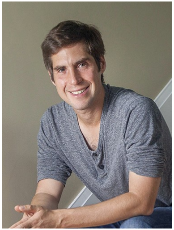


Mike Boardman

Rainer Vogt



Graeme Segal



Jacob Lurie

The K-theory space and higher K-groups

Definition (Quillen–Segal, ~1970)

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  $"K(\mathbb{F}_1)" = (B\mathcal{P}roj(\mathbb{F}_1))^{gp} =: \mathbb{S}$ has as homotopy groups the stable homotopy groups of spheres
- $\mathbb{S}$ plays in E_∞-groups the same role as $\mathbb{Z}$ in abelian groups; often called the sphere spectrum

Going higher algebra all the way

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In an E_∞-ring, this multiplication is also commutative up to coherent homotopy.

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- Higher algebra instrumental in many advances in higher algebra of the last decades (growing field in the Netherlands).
- K-theory only completely computable in a few examples like $\mathbb{F}_q$
- Need approximations with more convenient properties.

(Higher algebraic) localizations of K-theory

Rational K-theory

- Borel (1974) computed $K_i(\mathcal{O}_F) \otimes \mathbb{Q}$ for the ring of integers in a number field, e.g.

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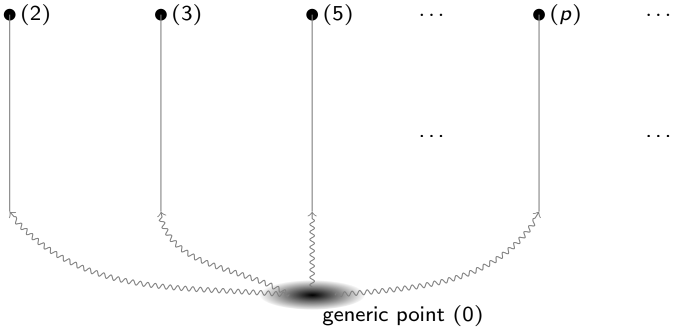
- Example: $\mathbb{S} \to \mathbb{Z}$ is a rational equivalence inducing an isomorphism on π_0. Thus,

$$K_i(\mathbb{S}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } i = 0, 5, 9, 13, \ldots \\ 0 & \text{else} \end{cases}$$

(Higher-algebraic/chromatic) localizations



The prime ideals of $\mathbb{Z}$

$$\text{Abelian group } A \rightsquigarrow \begin{cases} \text{Rationalization } A_{(0)} = A \otimes \mathbb{Q} \\ p\text{-completions } \widehat{A}_p \end{cases}$$

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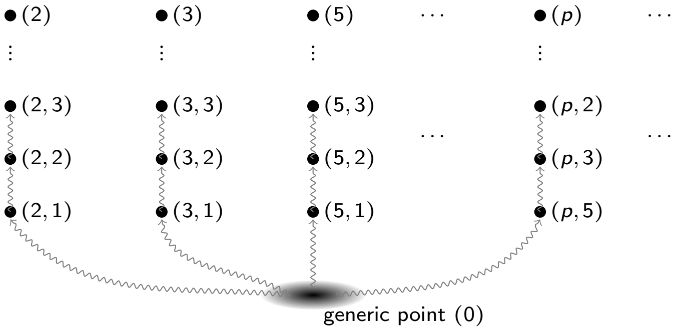

"Prime ideals" of $\mathbb{S}$

E_∞-group $A \rightsquigarrow$ Localizations/completions $\widehat{A}_{(p,n)} = L_{T(n)}A$

By convention, $L_{T(0)}A = A[\frac{1}{p}]$.

The purity theorem

The following generalizes Waldhausen's theorem ($n = 0$):
$K(A)[\frac{1}{p}]$ only depends on $A[\frac{1}{p}]$.

Theorem (Purity theorem, Land–Mathew–M.–Tamme, CMNN)

For an A_∞-ring R, the localization $L_{T(n)}K(R)$ only depends on $L_{T(n)}R$ and $L_{T(n-1)}R$.

Context: How well do the $L_{T(n)}$ approximate?

- These completions can be very destructive: if A is a discrete E_∞-group, then $L_{T(n)}A = 0$ for all $n \geq 1$.

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Theorem (Thomason '80, Voevodsky, Clausen–Mathew '19)

Let R be a decent commutative ring with $\frac{1}{p} \in R$. Up to p-completion, $\pi_i L_{T(1)} K(R)$ agrees with étale K-theory of R and also with usual $K_i(R)$ if $i \gg 0$. (E.g. for $i > 0$ for $R = \mathbb{Z}[\frac{1}{p}]$.)

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- $L_{T(n)}K(-)$ enjoys many nice properties, in particular étale descent. (Thomason, Clausen–Mathew–Naumann–Noel, Land–Mathew–M.-Tamme)

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Conjecture (A la Rognes, early 2000s)

If R is an E_∞-ring or nice A_∞-ring, then

$$L_{T(n+1)}K(R) \neq 0 \Leftrightarrow 0 \text{ iff } L_{T(n)}R \neq 0.$$

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Theorem (Burklund, Clausen, Hahn, Land, Mathew, M., Naumann, Noel, Schlank, Tamme, Wilson, Yuan, 2020-22)

This is true.

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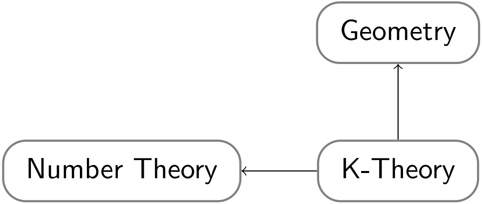
Theorem (Burklund–Hahn–Levy–Schlank, 2023)

This is not true for $n \geq 2$. First counterexample: $K(L_{T(1)}\mathbb{S})$.

- Has changed our global understanding of algebraic topology.
- Was used to show that homotopy groups of spheres grow fast.

Summary



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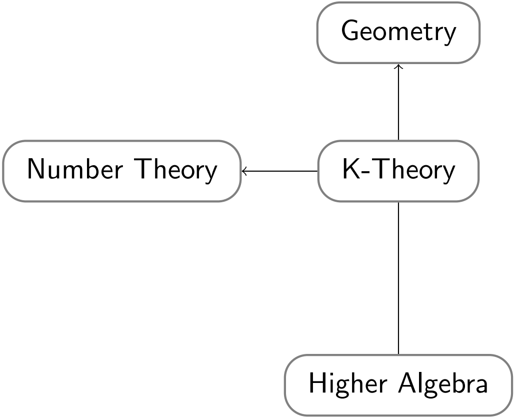

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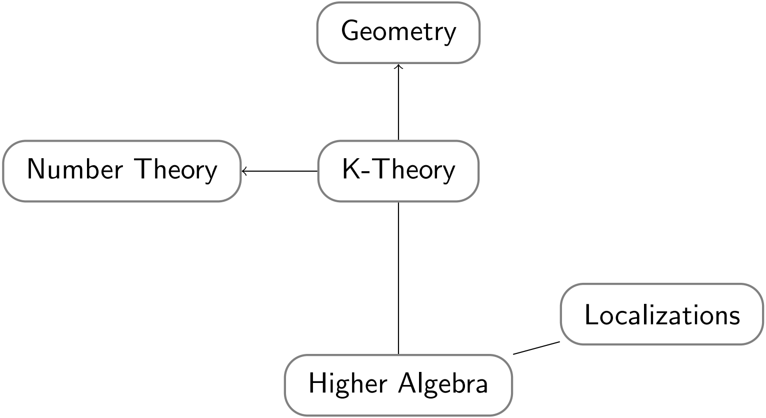
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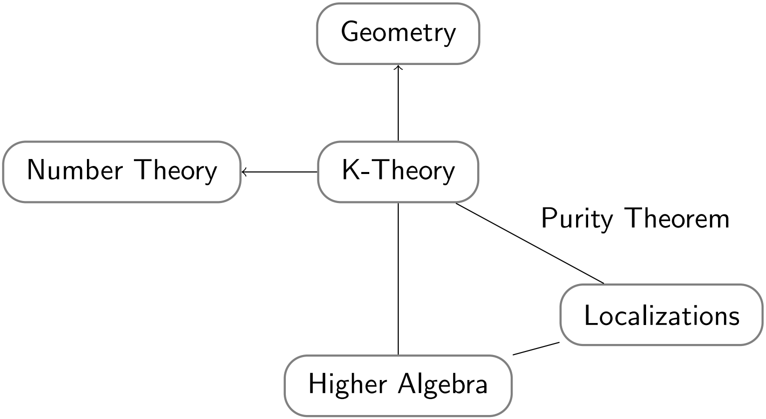
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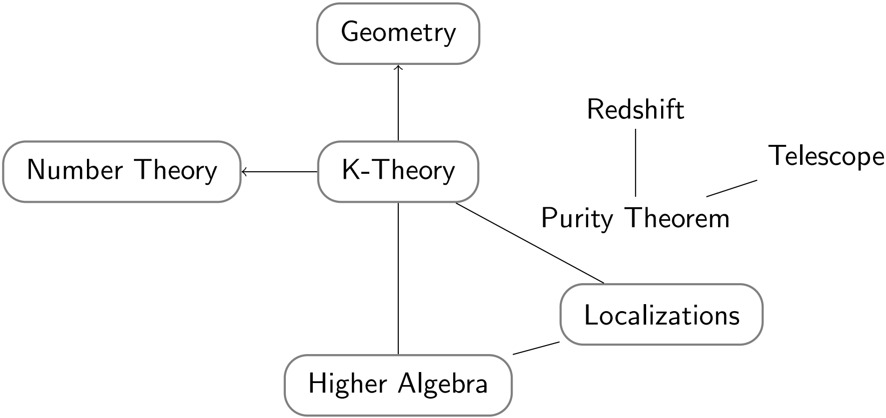
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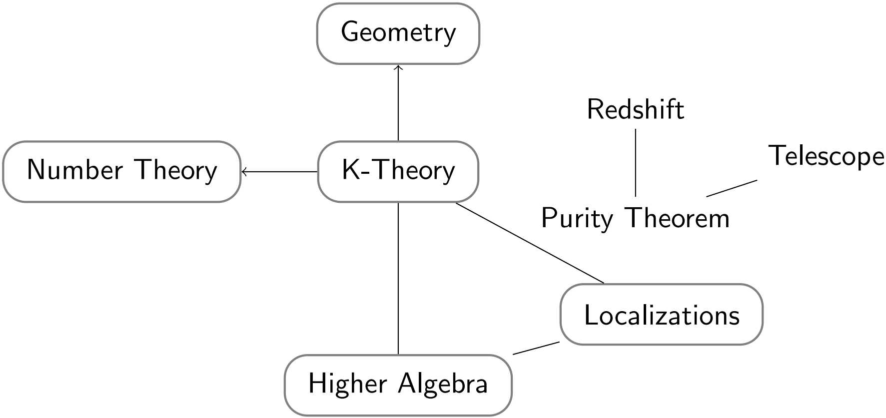

Thank you!