

Notes on localizations

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Regarding Lemma 4: $\mathcal{L}: (\mathbb{Q}SF)^{Eu3} \rightleftarrows \mathbb{Q}S((\mathbb{Q})^{Eu3}_oF) : r$

$$lr = id \implies id$$

$$rl \left(\begin{array}{c} i_0 \xrightarrow{f_0} \dots \xrightarrow{f_{n-1}} i_n \\ x_j \in F(i_j) \end{array} \right) = \left(\begin{array}{c} i_n \rightarrow \dots \rightarrow i_n \\ F(f_0 \dots f_{n-1})(x_0) \rightarrow \dots \rightarrow x_n \end{array} \right)$$

$id \rightarrow rl$ defined by

$$\begin{array}{ccc} i_0 & \rightarrow & \dots & \rightarrow & i_n \\ \downarrow & & & & \downarrow \\ i_n & \rightarrow & \dots & \rightarrow & i_n \end{array}$$

Easy to see that this defines an adjunction.

Regarding Lemma 8

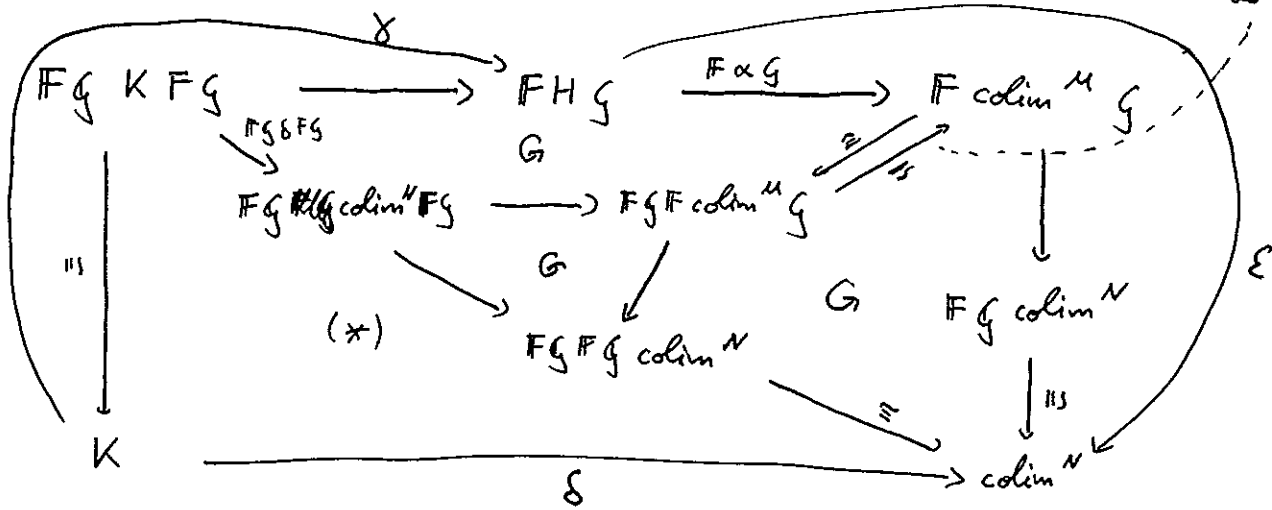
Assume

$$FG \cong Q_1 \cong Q_2 \cong \dots \cong Q_n \cong id_N \dots \text{ inducing the natural iso } FG \cong id_{HoN}$$
~~$$(GF \cong R_1 \cong R_2 \cong \dots \cong R_n \cong id_M)$$~~

are natural zig-zags of weak equivalences.

1) We want to show the commutativity of

we use here that $F \rightarrow FG \rightarrow F$ is id_F as F, G are adjoints.



$$\begin{array}{ccccc}
 (*) & FGKFG & \longrightarrow & FG \text{ colim}^N FG & \longrightarrow & FG FG \text{ colim}^N \\
 \cong \downarrow & & \cong & \downarrow & (*) & \downarrow \cong \\
 Q_1 K Q_1 & \longrightarrow & Q_1 \text{ colim}^N Q_1 & \longrightarrow & Q_1 Q_1 \text{ colim}^N \\
 \cong \downarrow & & \downarrow & & \downarrow \cong \\
 \vdots & & \vdots & & \vdots \\
 \cong \downarrow & & \downarrow & & \downarrow \cong \\
 Q_n K Q_n & \longrightarrow & Q_n \text{ colim}^N Q_n & \longrightarrow & Q_n Q_n \text{ colim}^N \\
 \cong \downarrow & & \downarrow & & \downarrow \cong \\
 K & \longrightarrow & \text{colim}^N & \xrightarrow{\cong} & \text{colim}^N
 \end{array}$$

(**) In general, if $F_1, F_2 : N \rightarrow N$ are functors with nat. trafo $\alpha : F_1 \Rightarrow F_2$, the square

$$\begin{array}{ccc}
 \text{colim } F_1 & \longrightarrow & F_1 \text{ colim} \\
 \downarrow & & \downarrow \\
 \text{colim } F_2 & \longrightarrow & F_2 \text{ colim}
 \end{array}$$

commutes

For this, we have only for a given diagram to precompose with $F_1(X(d)) \rightarrow \text{colim } F_1 X$

$X: \mathcal{D} \rightarrow \mathcal{N}$, $d \in \mathcal{D}$, \square to get

$$\begin{array}{ccc} F_1 X(d) & \longrightarrow & F_1 \text{colim } X \\ \downarrow & \cong & \downarrow \\ F_2 X(d) & \longrightarrow & F_2 \text{colim } X \end{array}$$

which commutes by the definition of a natural transformation.

2) We now have to show uniqueness of γ .

$$\begin{array}{ccccc} K & \xrightarrow{\gamma, \gamma'} & FHG & \xrightarrow{\epsilon} & \text{colim } \mathcal{N} \\ & \searrow \cong & & & \\ GK & \xrightarrow{\gamma \circ F, \gamma' \circ F} & GFHG & \xrightarrow{\gamma \circ \epsilon} & G \text{colim } \mathcal{N} \\ & \searrow \cong & & & \downarrow \cong \\ & & H & \xrightarrow{\alpha} & \text{colim } \mathcal{M} \end{array}$$

(xxx) \cong (yys) \cong

(yys) \cong

(yys) \cong

(yys) \cong

(xxx) Recall: $G \circ F \xrightarrow{\cong} \text{id}_{\text{Ho}(\mathcal{M}^{\mathcal{D}})}$ is constructed via $FG \xrightarrow{\cong} \text{id}_{\text{Ho}(\mathcal{M})}$: $X \rightarrow GF X$
 $\cong FX \rightarrow FGF X$
 inverse to $(FG)FX \rightarrow FX$

Likewise we get $G \circ F \xrightarrow{\cong} \text{id}_{\text{Ho}(\mathcal{M}^{\mathcal{D}})}$.

As H factors over $\text{Ho}(\mathcal{M}^{\mathcal{D}})$, we get the iso.

(*) ~~square commutes if it does after precomp with G and postcomp with F .~~

(*):

$$\begin{array}{ccccccc} GFHG & \xrightarrow{\alpha} & GF \text{colim } \mathcal{M} GF & \longrightarrow & GF \circ \text{colim } \mathcal{N} F & \longrightarrow & G \text{colim } \mathcal{N} F \\ \cong \downarrow & \cong \downarrow & \cong \downarrow & & \cong \downarrow & & \downarrow \cong \\ HGF & \xrightarrow{\alpha} & \text{colim } \mathcal{M} GF & \longrightarrow & & & GF \text{colim } \mathcal{M} \\ \cong \downarrow & \cong \downarrow & \cong \downarrow & & \cong \downarrow & & \downarrow \cong \\ H & \xrightarrow{\alpha} & \text{colim } \mathcal{M} & \longrightarrow & & & \text{colim } \mathcal{M} \end{array}$$

As in (*).
 [For the argument in (*) you do not need that $\text{id} \circ GF$ is actually induced by $\gamma \circ \epsilon$ of w.e., you only need this for every object]

$\Rightarrow G \circ \gamma \circ F = G \circ \gamma' \circ F \Rightarrow \gamma \circ F = \gamma' \circ F \Rightarrow \gamma = \gamma'$ as $K, FHG: \text{Ho}(\mathcal{N}^{\mathcal{D}}) \rightarrow \text{Ho}(\mathcal{N})$.

Notes on $\text{hocolim}_{\text{BK}}$

Given a functor $H: \mathcal{D} \rightarrow \text{sSet}$, its BK-hocolim is usually defined as the geometric realization of

$$\coprod_{i_0} H(i_0) \hookrightarrow \coprod_{i_0 \leftarrow i_1} H(i_1) \hookrightarrow \coprod_{i_0 \leftarrow i_1 \leftarrow i_2} H(i_2) \hookrightarrow \dots$$

But the diagonal is isomorphic to the geom. realization.

Given $G: \mathcal{D} \rightarrow \text{Cat}$, it follows that $\text{hocolim}_{\text{BK}} \text{nerve } G$ is the diagonal of the bisimplicial set X whose p - q -simplices X_{pq} are given by a p -chain $i_p \xrightarrow{f_p} \dots \xrightarrow{f_1} i_0$ in \mathcal{D} and a q -chain $x_q \rightarrow \dots \rightarrow x_0$ in $G(i_p)$.

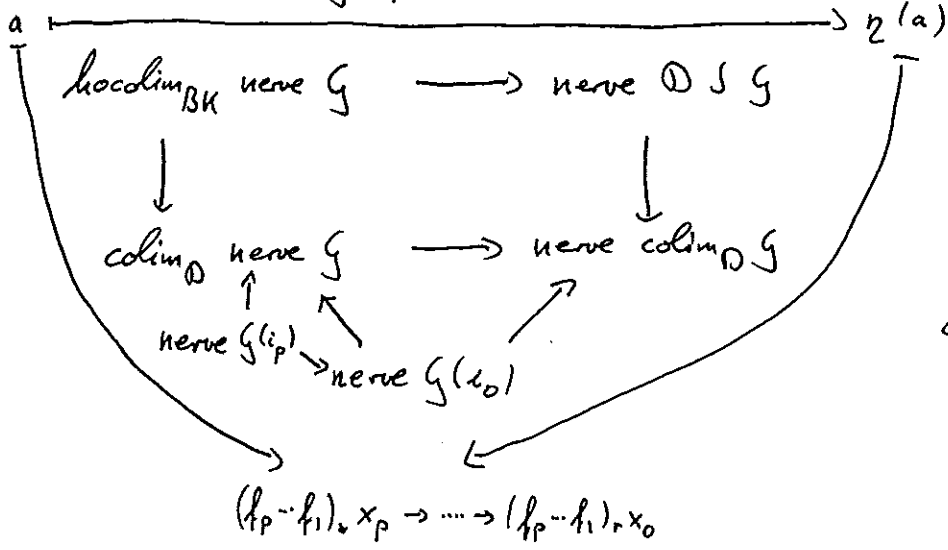
$$\eta: \text{hocolim}_{\text{BK}} \text{nerve } G \longrightarrow \text{nerve } \mathcal{D} \mathcal{S} G$$

$$a = \begin{pmatrix} i_p \rightarrow \dots \rightarrow i_0 \\ x_{pp} \rightarrow \dots \rightarrow x_0 \end{pmatrix} \longmapsto (i_p, x_p) \rightarrow (i_{p-1}, (f_p)_* x_p) \rightarrow \dots \rightarrow (i_0, (f_1 \dots f_p)_* x_0)$$

The ~~functor~~ map $(\text{hocolim}_{\text{BK}} \text{nerve } G)_p \longrightarrow \text{colim}_{\mathcal{D}} \text{nerve } G$ is the obvious one.

$$\begin{matrix} \text{hocolim}_{\text{BK}} \text{nerve } G & \longrightarrow & \text{colim}_{\mathcal{D}} \text{nerve } G \\ \parallel & & \parallel \\ X_{pp} & \longrightarrow & \text{colim}_{q \in \Delta} X_{qp} \end{matrix}$$

We have to show commutativity of

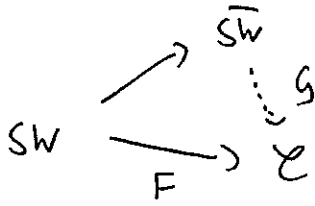


Thus, η is compatible with the natural transf. to the colim.

$$(f_p \dots f_1)_* x_p \rightarrow \dots \rightarrow (f_p \dots f_1)_* x_0$$

Spanier-Whitehead Category

Want to show that for



F sending \cong to \cong , there is a unique functor G making the diagram commute.

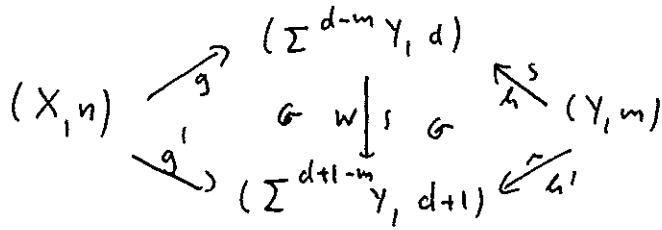
$$ob\ SW = ob\ \overline{SW} \quad \rightsquigarrow \quad G(X, n) = F(X, n)$$

$$G([\mathbb{Z}f: \sum_{d \geq m, n}^{d-n} X \rightarrow \sum^{d-m} Y]) = F(h)^{-1} F(g).$$

$$(X, n) \xrightarrow{g} (\sum^{d-m} Y, d) \xleftarrow[h]{\sim} (Y, m)$$

Well-definedness: \cong : Same argument as for $Ho(Top)$

$$Stab: G([\mathbb{Z}f]) = F(h')^{-1} F(g') = F(h)^{-1} F(w)^{-1} F(w) F(g) = G([\mathbb{Z}f])$$



Functor: Key:

$$\begin{array}{ccccccc}
 & & & G & & & \\
 & & & \downarrow & & & \\
 (X, n) & \rightarrow & (\sum^{d-m} Y, d) & \xleftarrow{\sim} & (Y, m) & \rightarrow & (\sum^{d-k} Z, d) \xleftarrow{\sim} (Z, k) \\
 & & & G & & &
 \end{array}$$