Notes on hocolims

Regarding Lemma 4: $\qquad l : (\mathbb{1} \int F)^{[n]} \rightleftarrows \mathbb{1} \int (()^{[n]} \circ F) : r$

$lr = \mathrm{id} \xrightarrow{=} \mathrm{id}$

$$rl \begin{pmatrix} i_0 \xrightarrow{f_0} \cdots \xrightarrow{f_{n-1}} i_n \\ x_j \in F(i_j) \end{pmatrix} = \begin{pmatrix} i_n \to \cdots \to i_n \\ F(f_0 \cdots f_{n-1})(x_0) \to \cdots \to x_n \end{pmatrix}$$

$\mathrm{id} \to rl$ defined by

$$\begin{tikzcd} i_0 \arrow[r] \arrow[d] & \cdots \arrow[r] & i_n \arrow[d, equal] \\ i_n \arrow[r] & \cdots \arrow[r] & i_n \end{tikzcd}$$

Easy to see that this defines an adjunction.

Regarding Lemma #8

Assume

$$\mathbb{F}\mathcal{G} \overset{\simeq}{\text{——}} Q_1 \overset{\simeq}{\text{——}} Q_2 \overset{\simeq}{\text{——}} \dots \overset{\simeq}{\text{——}} Q_n \overset{\simeq}{\text{——}} \mathrm{id}_{\mathcal{N}}$$ ⟵ inducing the natural iso $\mathbb{F}\mathcal{G} \xrightarrow{\cong} \mathrm{id}_{\mathrm{Ho}\mathcal{N}}$.

$$\cancel{(\mathcal{G}F \overset{\simeq}{\text{——}} R_1 \overset{\simeq}{\text{——}} R_2 \overset{\simeq}{\text{——}} \dots \overset{\simeq}{\text{——}} R_n \overset{\simeq}{\text{——}} \mathrm{id}_{\mathcal{M}})}$$

are natural zig-zags of weak equivalences.

1) We want to show the commutativity of

we use here that $\mathbb{F} \to \mathbb{F}\mathcal{G}\mathbb{F} \to \mathbb{F}$ is $\mathrm{id}_{\mathbb{F}}$ as $\mathbb{F}, \mathcal{G}$ are adjoints.

$$\begin{tikzcd}
\mathbb{F}\mathcal{G}KF\mathcal{G} \arrow[rr] \arrow[rr, bend left, "\gamma"] \arrow[dr, "\mathbb{F}\mathcal{G}\delta F\mathcal{G}"] \arrow[ddd, "\cong"'] & & FH\mathcal{G} \arrow[rr, "\mathbb{F}\alpha\mathcal{G}"] \arrow[ddrr, bend left=60, "\varepsilon"] & & \mathbb{F}\,\mathrm{colim}^{\mathcal{M}}\mathcal{G} \arrow[dd] \arrow[dl, shift left, "\simeq"] \\
& F\mathcal{G}\,\mathrm{colim}^{\mathcal{N}}\mathbb{F}\mathcal{G} \arrow[rr] \arrow[dr] \arrow[d, phantom, "(*)"] & & F\mathcal{G}\mathbb{F}\,\mathrm{colim}^{\mathcal{M}}\mathcal{G} \arrow[ur, shift left, "\simeq"] \arrow[dl] & \\
& & F\mathcal{G}\mathbb{F}\mathcal{G}\,\mathrm{colim}^{\mathcal{N}} \arrow[drr, "\simeq"] & & F\mathcal{G}\,\mathrm{colim}^{\mathcal{N}} \arrow[d, "\cong"] \\
K \arrow[rrrr, "\delta"'] & & & & \mathrm{colim}^{\mathcal{N}}
\end{tikzcd}$$

$$(*)\quad \begin{tikzcd}
\mathbb{F}\mathcal{G}KF\mathcal{G} \arrow[r] \arrow[d, dash, "\cong"'] & F\mathcal{G}\,\mathrm{colim}^{\mathcal{N}}F\mathcal{G} \arrow[r] \arrow[d, dash] \arrow[r, phantom, shift right=4ex, "(**)"] & \mathbb{F}\mathcal{G}F\mathcal{G}\,\mathrm{colim}^{\mathcal{N}} \arrow[d, dash, "\cong"] \\
Q_1KQ_1 \arrow[r] \arrow[d, dash, "\cong"'] & Q_1\,\mathrm{colim}^{\mathcal{N}}Q_1 \arrow[r] \arrow[d, dash] & Q_1Q_1\,\mathrm{colim}^{\mathcal{N}} \arrow[d, dash, "\cong"] \\
\vdots \arrow[d, dash, "\cong"'] & \vdots \arrow[d, dash] & \vdots \arrow[d, dash, "\cong"] \\
Q_nKQ_n \arrow[r] \arrow[d, dash, "\cong"'] & Q_n\,\mathrm{colim}^{\mathcal{N}}Q_n \arrow[r] \arrow[d, dash] & Q_nQ_n\,\mathrm{colim}^{\mathcal{N}} \arrow[d, dash, "\cong"] \\
K \arrow[r] & \mathrm{colim}^{\mathcal{N}} \arrow[r, "\simeq"] & \mathrm{colim}^{\mathcal{N}}
\end{tikzcd}$$

(**) In general, if $F_1, F_2 : \mathcal{N} \to \mathcal{N}$ are functors with nat. trafo $\alpha : F_1 \Rightarrow F_2$, the square

$$\begin{tikzcd}
\mathrm{colim}\, F_1 \arrow[r] \arrow[d] & F_1\, \mathrm{colim} \arrow[d] \\
\mathrm{colim}\, F_2 \arrow[r] & F_2\, \mathrm{colim}
\end{tikzcd}$$

commutes

For this, we have only for a given diagram $X : \mathbb{D} \to \mathcal{N}$, $d \in \mathbb{D}$, to precompose with $F_1(X(d)) \to \operatorname{colim} F_1 X$ to get

$$\begin{tikzcd}
F_1 X(d) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & F_1 \operatorname{colim} X \arrow[d] \\
F_2 X(d) \arrow[r] & F_2 \operatorname{colim} X
\end{tikzcd}$$

, which commutes by the definition of a natural transformation.

2) We now have to show uniqueness of γ.

$$\begin{tikzcd}
K \arrow[r, "{\gamma, \gamma'}"] \arrow[rr, bend right, "\delta"'] & \mathbb{F}H\mathcal{G} \arrow[r, "\varepsilon"] \arrow[d, phantom, "\circlearrowleft"] & \operatorname{colim}^{\mathcal{N}} \\
 & {} &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{G}K\mathbb{F} \arrow[r, "{\mathcal{G}\gamma\mathbb{F},\ \mathcal{G}\gamma'\mathbb{F}}"] \arrow[drr, phantom, "\circlearrowleft"] \arrow[ddr] & \mathcal{G}\mathbb{F}H\mathcal{G}\mathbb{F} \arrow[r, "\mathcal{G}\varepsilon\mathbb{F}"] \arrow[dd, "(***)\ \cong"] \arrow[ddr, phantom, "(*_5)\ \circlearrowleft"] & \mathcal{G}\operatorname{colim}^{\mathcal{N}}\mathbb{F} \arrow[d] \\
 & & \mathcal{G}\mathbb{F}\operatorname{colim}^{\mathcal{M}} \\
 & H \arrow[r, "\alpha"] & \operatorname{colim}^{\mathcal{M}} \arrow[u, "\cong"']
\end{tikzcd}$$

(***) Recall: $\mathcal{G}\mathbb{F} \overset{\simeq}{\Leftarrow} \mathrm{id}_{\mathrm{Ho}(\mathcal{M})}$ is constructed via $\mathbb{F}\mathcal{G} \overset{\simeq}{\Rightarrow} \mathrm{id}_{\mathrm{Ho}(\mathcal{N})}$: $X \to \mathcal{G}\mathbb{F}X$ $\hat{=}$ $\mathbb{F}X \to \mathbb{F}\mathcal{G}\mathbb{F}X$ inverse to $(\mathbb{F}\mathcal{G}\mathbb{F}X \to \mathbb{F}X)$

Likewise we get $\mathcal{G}\mathbb{F} \overset{\simeq}{\Leftarrow} \mathrm{id}_{\mathrm{Ho}(\mathcal{M}^{\mathbb{D}})}$.

As H factors over $\mathrm{Ho}(\mathcal{M}^{\mathbb{D}})$, we get the iso.

~~$(*_4)$: square triangle commutes if it does after precomp with $\mathcal{G}$ and postcomp with $\mathbb{F}$.~~

$(*_5)$:

$$\begin{tikzcd}
\mathcal{G}\mathbb{F}H\mathcal{G}\mathbb{F} \arrow[r, "\alpha"] \arrow[dd, "\cong"'] \arrow[d, phantom, "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{G}\mathbb{F}\operatorname{colim}^{\mathcal{M}}\mathcal{G}\mathbb{F} \arrow[r] \arrow[d, "\cong"] \arrow[drr, phantom, "\circlearrowleft"] & \mathcal{G}\mathbb{F}\mathcal{G}\operatorname{colim}^{\mathcal{N}}\mathbb{F} \arrow[r] & \mathcal{G}\operatorname{colim}^{\mathcal{N}}\mathbb{F} \arrow[d] \\
H\mathcal{G}\mathbb{F} \arrow[r, "\alpha"] \arrow[d, "\cong"] & \operatorname{colim}^{\mathcal{M}}\mathcal{G}\mathbb{F} \arrow[rr] \arrow[d, phantom, "\circlearrowleft"] & & \mathcal{G}\mathbb{F}\operatorname{colim}^{\mathcal{M}} \arrow[d, "\cong"] \\
H \arrow[rrr, "\alpha"] & {} & & \operatorname{colim}^{\mathcal{M}}
\end{tikzcd}$$

As in $(*)$.

[For the argument in $(*)$ you do not need that $\mathrm{id} \overset{\simeq}{\Rightarrow} \mathcal{G}\mathbb{F}$ is actually induced by a zig zag of w.e., you only need this for every object]

$\Rightarrow \mathcal{G}\gamma\mathbb{F} = \mathcal{G}\gamma'\mathbb{F} \Rightarrow \gamma\mathbb{F} = \gamma'\mathbb{F} \Rightarrow \gamma = \gamma'$ as $K, \mathbb{F}H\mathcal{G} : \mathrm{Ho}(\mathcal{N}^{\mathbb{D}}) \to \mathrm{Ho}(\mathcal{N})$.

Notes on $\mathrm{hocolims}_{BK}$

Given a functor $H: \mathcal{D} \to sSet$, its BK-hocolim is usually defined as the geometric realization of

$$\coprod_{i_0} H(i_0) \leftleftarrows \coprod_{i_0 \leftarrow i_1} H(i_1) \;\overset{\leftarrow}{\leftleftarrows}\; \coprod_{i_0 \leftarrow i_1 \leftarrow i_2} H(i_2) \;\cdots\; \ldots$$

But the diagonal is isomorphic to the geom. realization.

Given $\mathcal{G}: \mathcal{D} \to Cat$, it follows that $\mathrm{hocolim}_{BK}\,\mathrm{nerve}\,\mathcal{G}$ is the diagonal of the bisimplicial set X ~~with~~ whose p-q-simplices X_{pq} are given by a p-chain $i_p \xrightarrow{f_p} \cdots \xrightarrow{f_1} i_0$ in $\mathcal{D}$ and a q-chain $x_q \to \cdots x_0$ in $\mathcal{G}(i_p)$.

$$\eta: \mathrm{hocolim}_{BK}\,\mathrm{nerve}\,\mathcal{G} \longrightarrow \mathrm{nerve}\,\mathcal{D}\textstyle\int\mathcal{G}$$

$$a = \begin{pmatrix} i_p \to \cdots \to i_0 \\ x_p \to \cdots \to x_0 \end{pmatrix} \longmapsto (i_p, x_p) \to (i_{p-1}, (f_p)_* x_{p-1}) \to \cdots \to (i_0, (f_1 \cdots f_p)_* x_0)$$

The ~~functor~~ map

$$\begin{array}{ccc} (\mathrm{hocolim}_{BK}\,\mathrm{nerve}\,\mathcal{G})_p & \longrightarrow & \mathrm{colim}_{\mathcal{D}}\,\mathrm{nerve}\,\mathcal{G} \\ \| & & \| \\ X_{pp} & \longrightarrow & \mathrm{colim}_{q \in \Delta} X_{qp} \end{array}$$

is the obvious one.

We have to show commutativity of

$$\begin{tikzcd}
a \arrow[rrr, mapsto] \arrow[ddddr, bend right, mapsto] & & & \eta(a) \arrow[ddddll, bend left, mapsto] \\
& \mathrm{hocolim}_{BK}\,\mathrm{nerve}\,\mathcal{G} \arrow[rr] \arrow[d] & & \mathrm{nerve}\,\mathcal{D}\int\mathcal{G} \arrow[d] \\
& \mathrm{colim}_{\mathcal{D}}\,\mathrm{nerve}\,\mathcal{G} \arrow[rr] & & \mathrm{nerve}\,\mathrm{colim}_{\mathcal{D}}\,\mathcal{G} \\
& \mathrm{nerve}\,\mathcal{G}(i_p) \arrow[u] \arrow[r] & \mathrm{nerve}\,\mathcal{G}(i_0) \arrow[ul] \arrow[ur] & \\
& (f_p \cdots f_1)_* x_p \to \cdots \to (f_p \cdots f_1)_* x_0 & &
\end{tikzcd}$$

Thus, η is compatible with the natural transf. to the colim.

//

Spanier-Whitehead Category

Want to show that for

$$\begin{tikzcd} & \overline{SW} \arrow[d, dotted, "G"] \\ SW \arrow[ur] \arrow[r, "F"'] & \mathcal{C} \end{tikzcd}$$

F sending $\simeq$ to $\cong$, there is a unique functor G making the diagram commute.

$\mathrm{ob}\, SW = \mathrm{ob}\, \overline{SW} \quad \rightsquigarrow \quad G(X,n) = F(X,n)$

$$G([f: \Sigma^{d-n}X \underset{d \geq m,n}{\longrightarrow} \Sigma^{d-m}Y]) = F(h)^{-1}F(g).$$

$$(X,n) \xrightarrow{g} (\Sigma^{d-m}Y, d) \underset{h}{\xleftarrow{\simeq}} (Y,m)$$

Well-definedness: $\simeq$: Same argument as for $Ho(Top)$

Stab: $G([\Sigma f]) = F(h')^{-1}F(g') = F(h)^{-1}F(w)^{-1}F(w)F(g) = G([f])$

$$\begin{tikzcd} & (\Sigma^{d-m}Y, d) \arrow[dd, "w"', "s"] & \\ (X,n) \arrow[ur, "g"] \arrow[dr, "g'"'] & \circlearrowleft \qquad \circlearrowleft & (Y,m) \arrow[ul, "s"', "h"] \arrow[dl, "\simeq", "h'"'] \\ & (\Sigma^{d+1-m}Y, d+1) & \end{tikzcd}$$

Functor: Key:

$$\begin{tikzcd} (X,n) \arrow[r] \arrow[rrr, bend left, "G"] & (\Sigma^{d-m}Y, d) \arrow[rr, bend right, "G"'] & (Y,m) \arrow[l, "\simeq"'] \arrow[r] & (\Sigma^{d-k}Z, d) & (Z,k) \arrow[l, "\simeq"'] \end{tikzcd}$$