Real spectra and their duals

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Real K-theory

- *Real K-theory* $K\mathbb{R}$ is a C_2-spectrum invented by Atiyah. Its underlying spectrum is complex K-theory KU and its C_2-fixed points are KO.
- To a C_2-spectrum X, we can associate its $RO(C_2)$-graded homotopy groups $\pi_\bigstar^{C_2} X$.
- By Real Bott periodity, $\pi_\bigstar^{C_2} K\mathbb{R}$ is ρ-periodic for $\rho = \rho_{C_2}$ the regular representation and we denote the periodicity class by $\overline{v} \in \pi_\rho^{C_2} K\mathbb{R}$.
- Other important class: $a\colon S^0 \to S^\sigma$ (for σ sign representation) defines element $a \in \pi_{-\sigma}^{C_2} K\mathbb{R}$.
- Will consider also the connective cover $k\mathbb{R}$ of $K\mathbb{R}$ with underlying spectrum ku and fixed points ko.

Tate square

One of the most important tools in C_2-equivariant homotopy is the Tate square. Let X be a C_2-spectrum and set

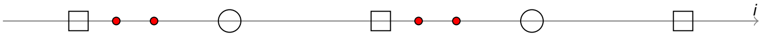
- $X^h = X^{(EC_2)_+}$. We have: $(X^h)^{C_2} = X^{hC_2}$ (homotopy fixed points)
- $X^\Phi = X[a^{-1}]$. We have: $(X^\Phi)^{C_2} = X^{\Phi C_2}$ (geometric fixed points)
- $X^t = X^h[a^{-1}]$. We have $(X^t)^{C_2} = X^{tC_2}$ (Tate construction)

We have a cartesian square:

$$\begin{array}{ccc} k\mathbb{R} & \longrightarrow & k\mathbb{R}^\Phi \\ \downarrow & & \downarrow \\ k\mathbb{R}^h & \longrightarrow & k\mathbb{R}^t \end{array}$$

Real K-theory

$\pi_i^{C_2} K\mathbb{R} = \pi_i KO$



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$\pi_i^{C_2} K\mathbb{R} = \pi_i KO$
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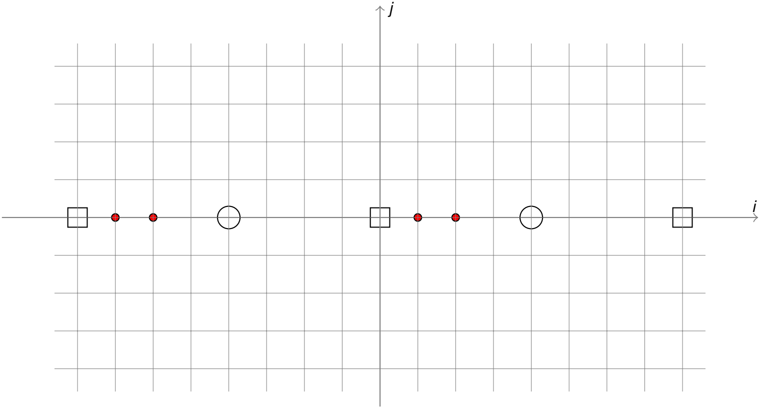

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$\pi_i^{C_2} K\mathbb{R} = \pi_i KO$, $\quad \overline{v} = \overline{v}_1$ is invertible on $K\mathbb{R}$, thus $K\mathbb{R}$ is ρ-periodic

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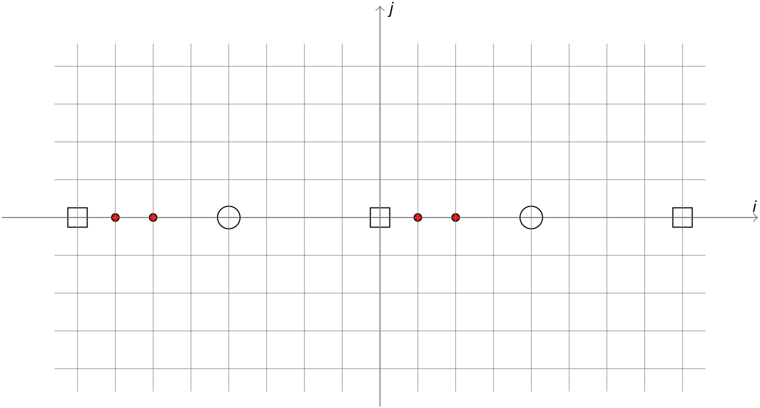

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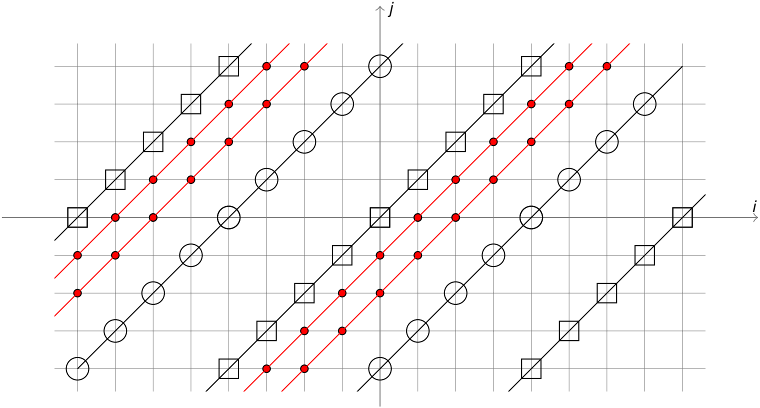

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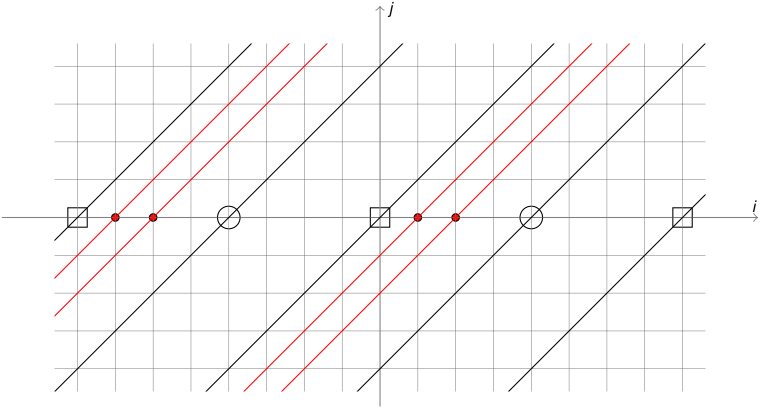

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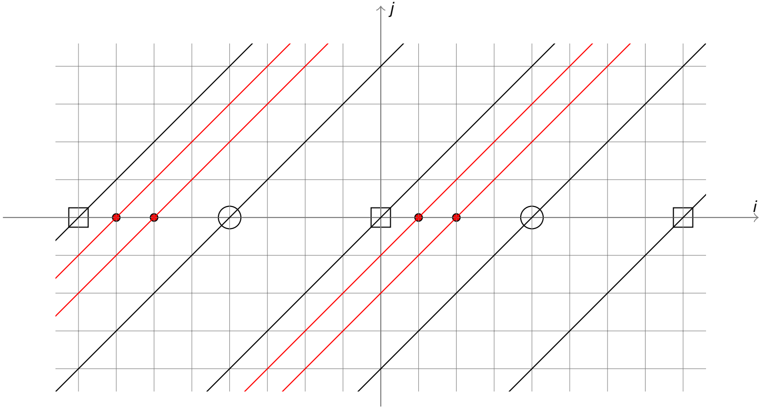

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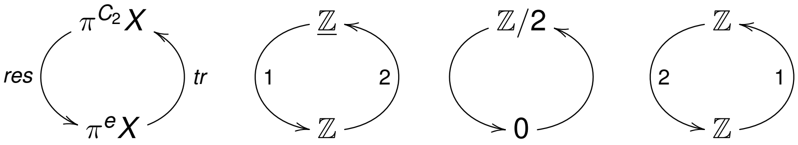

$\underline{\pi}^{C_2}(X)$ $\qquad$ $\underline{\mathbb{Z}}$ $\qquad$ $\mathbb{Z}/2$ $\qquad$ $\underline{\mathbb{Z}}^*$

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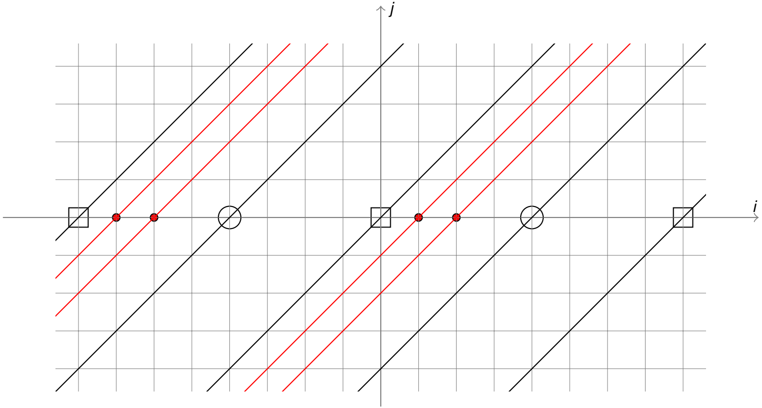

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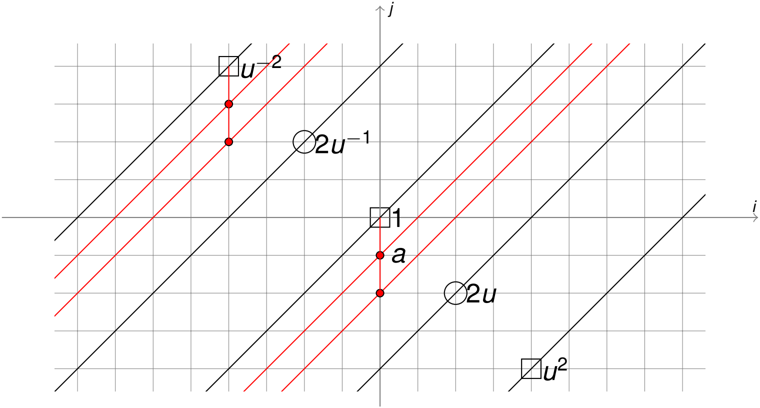

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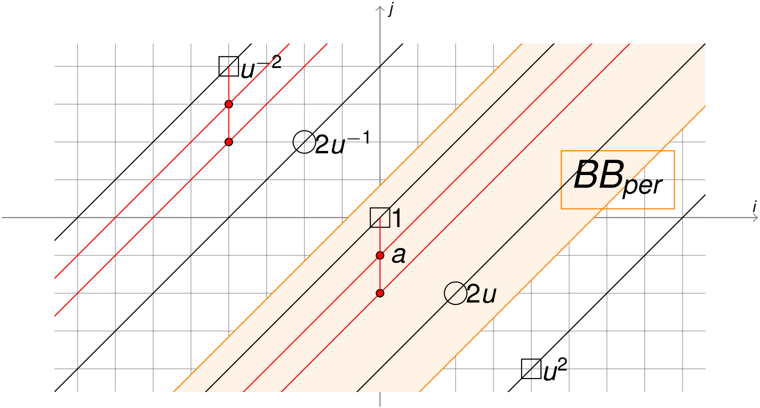
Real K-theory

$$\pi_\bigstar^{C_2} K\mathbb{R} = \bigoplus_{n\in\mathbb{Z}} BB_{per} \cdot u^{2k}, \text{ where}$$
$$BB_{per} = \mathbb{Z}[\overline{v}^{\pm 1}, a]/(2a, \overline{v}a^3) \oplus \mathbb{Z}[\overline{v}^{\pm 1}] \cdot \{2u\}$$



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Connective Real K-theory

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- By definition, $\pi_i^{C_2} k\mathbb{R} = \pi_i^{C_2} K\mathbb{R}$ for $i \geq 0$ and 0 else, but what about $\pi_{i+j\sigma}^{C_2} k\mathbb{R}$?

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- Easy to see: $\pi_\star^{C_2} k\mathbb{R}^h = \bigoplus_{n\in\mathbb{Z}} BB \cdot u^{2n}$ with

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A picture of *BB*

We have $\pi_{\star}^{C_2} k\mathbb{R}^h = \bigoplus_{n\in\mathbb{Z}} BB \cdot u^{2n}$ with

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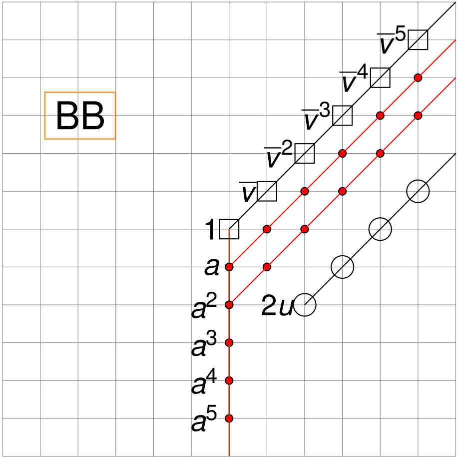

The Tate square

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$$\bigoplus_{n\in\mathbb{Z}} BB \cdot u^{2n} \longrightarrow \bigoplus_{n\in\mathbb{Z}} \mathbb{Z}/2[a^{\pm 1}] \cdot u^{2n}$$

$$BB = \mathbb{Z}[\overline{v}, a]/(2a, \overline{v}a^3) \oplus \mathbb{Z}[\overline{v}] \cdot \{2u\}$$

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$$\begin{array}{ccc} \begin{array}{c}\bigoplus_{n<0} NB \cdot u^{2n} \oplus \\ \bigoplus_{n\geq 0} BB \cdot u^{2n}\end{array} & \longrightarrow & \bigoplus_{n\geq 0} \mathbb{Z}/2[a^{\pm 1}] \cdot u^{2n} \\ \downarrow & & \downarrow \\ \bigoplus_{n\in\mathbb{Z}} BB \cdot u^{2n} & \longrightarrow & \bigoplus_{n\in\mathbb{Z}} \mathbb{Z}/2[a^{\pm 1}] \cdot u^{2n} \\ \downarrow & & \downarrow \\ \bigoplus_{n<0} \mathbb{Z}/2[a^{\pm 1}] \cdot u^{2n} & \xrightarrow{=} & \bigoplus_{n<0} \mathbb{Z}/2[a^{\pm 1}] \cdot u^{2n} \end{array}$$

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Picture of *NB*

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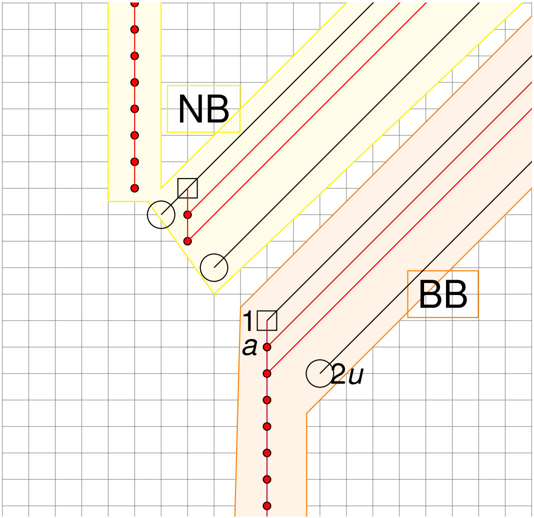

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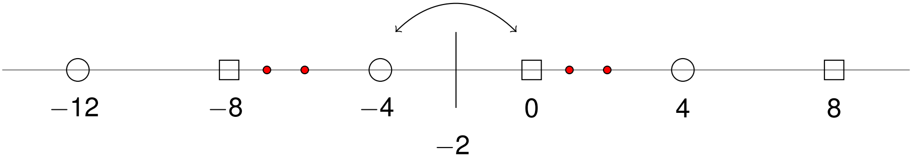

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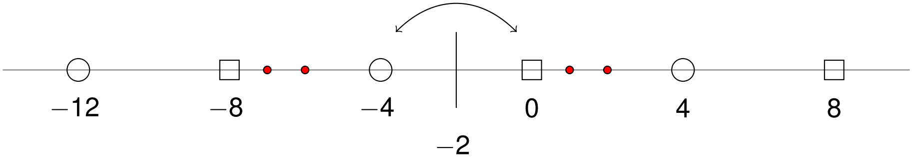

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$$0 \to \mathrm{Ext}_{\mathbb{Z}}(\pi_{(-2-*)-1}KO, \mathbb{Z}) \to \pi_{-2+*}KO \to \mathrm{Hom}_{\mathbb{Z}}(\pi_{-2-*}KO, \mathbb{Z}) \to 0$$

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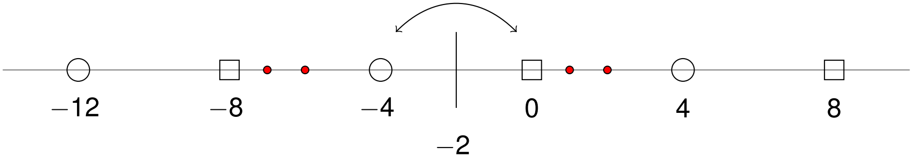

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Anderson duality

- For every spectrum E one can define its *Anderson dual* $\mathbb{Z}^E$ with the following property:

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- Get usual universal coefficient sequence by $\mathbb{Z}^{H\mathbb{Z}} \simeq H\mathbb{Z}$.

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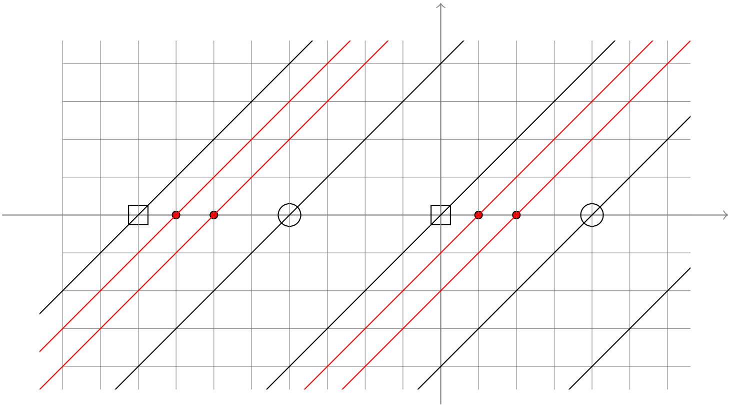
- What about $\mathbb{Z}^{K\mathbb{R}}$ and $\mathbb{Z}^{k\mathbb{R}}$?

Anderson duality for $K\mathbb{R}$



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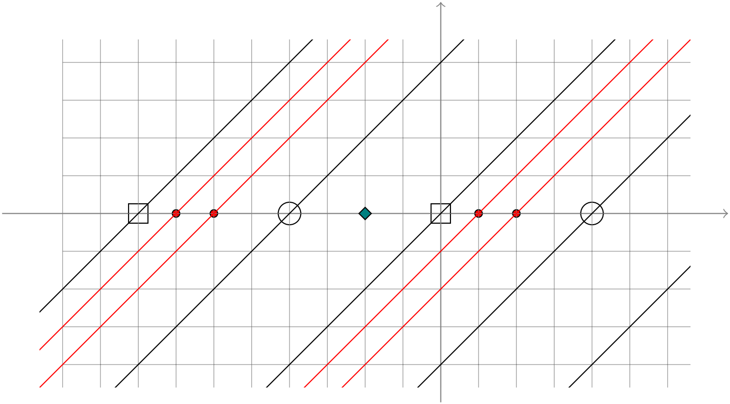

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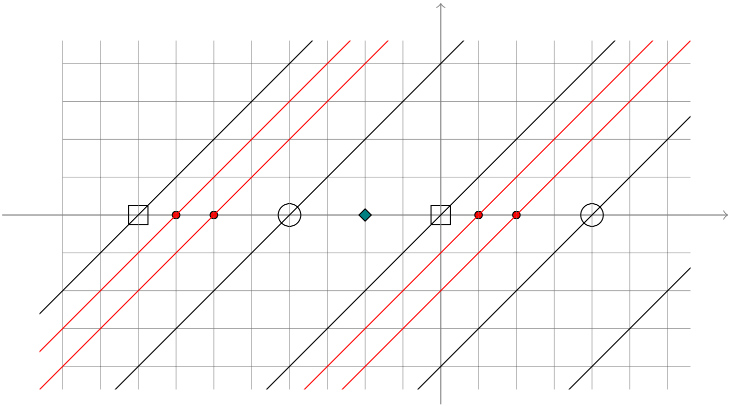

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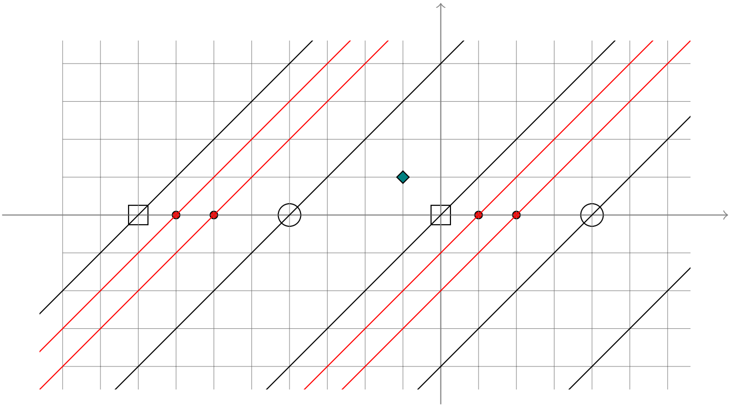

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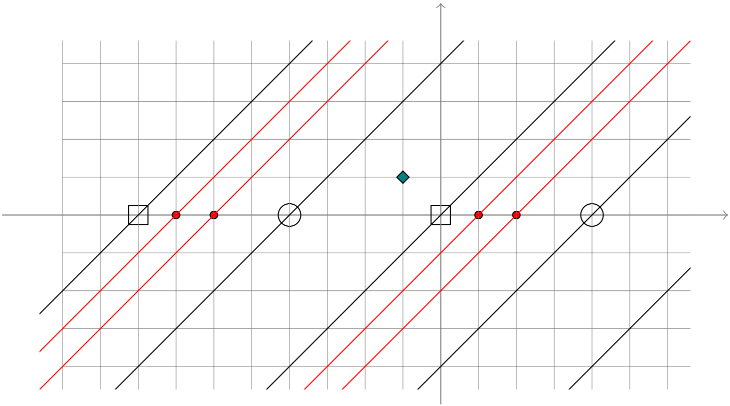
Anderson duality for $K\mathbb{R}$



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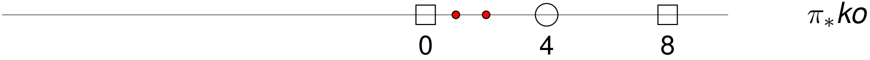
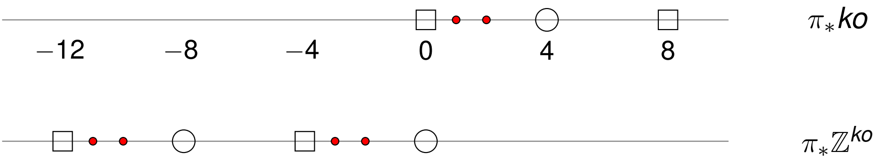
(Heard, Stojanoska)

Anderson duality for *ko*



Anderson duality for *ko*



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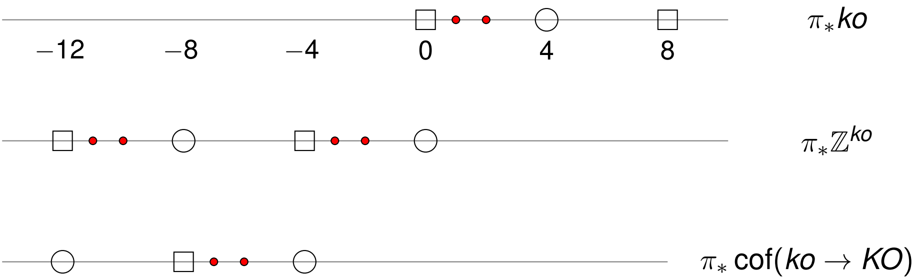

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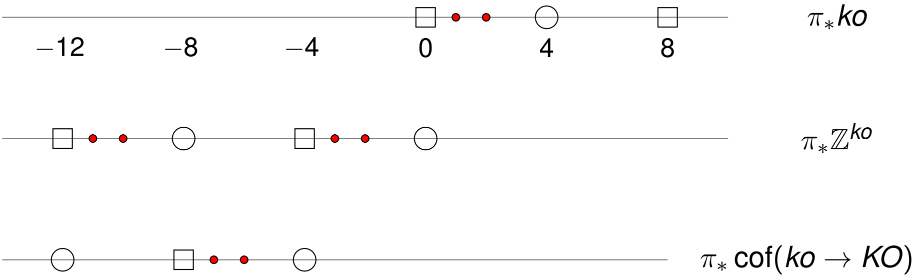
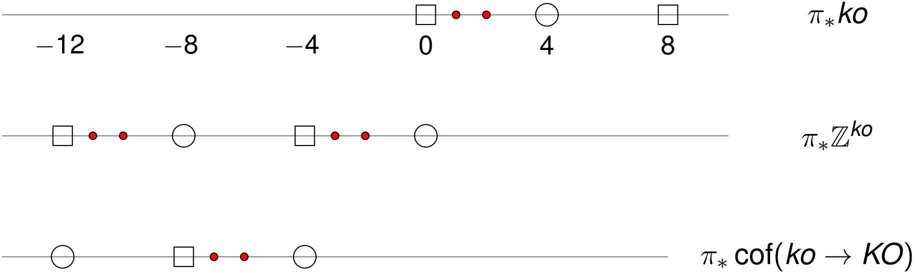


$$\mathbb{Z}^{ko} \simeq \Sigma^4 \operatorname{cof}(ko \to KO)$$

(Greenlees–Stojanoska, Greenlees–M.)

Anderson duality for *ko*



$$\mathbb{Z}^{ko} \simeq \Sigma^4 \operatorname{cof}(ko \to KO) \simeq \Sigma^5 \operatorname{fib}(ko \to KO)$$

(Greenlees–Stojanoska, Greenlees–M.)

Anderson duality for $k\mathbb{R}$ – Preliminaries

Want to compute $\mathrm{fib}(k\mathbb{R} \to K\mathbb{R}) = \mathrm{fib}(k\mathbb{R} \to k\mathbb{R}[\overline{v}^{-1}]) =: \Gamma_{\overline{v}} k\mathbb{R}$.

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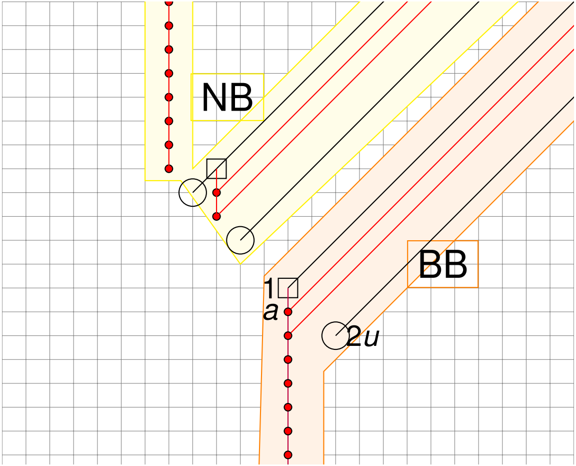

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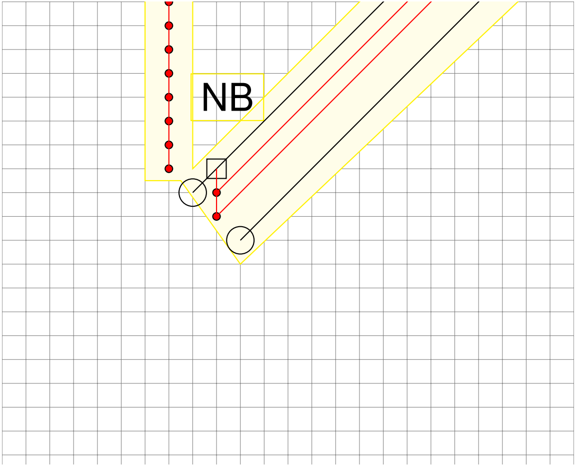

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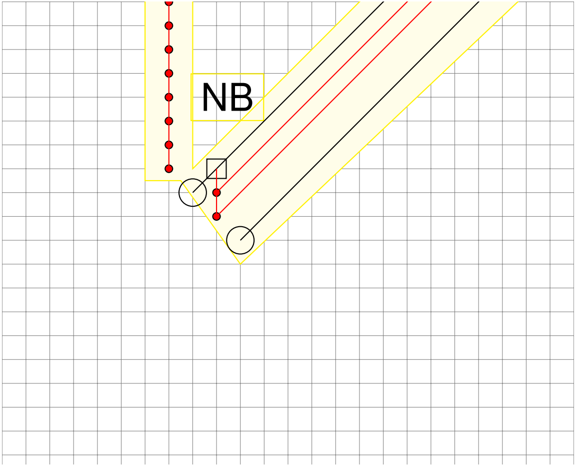
$$\Sigma^{-1}NB \to \Sigma^{-1}NB[\overline{v}^{-1}] \to \Gamma_{\overline{v}}NB \to NB \to NB[\overline{v}^{-1}] \overset{\text{shift}}{=} BB_{per}$$



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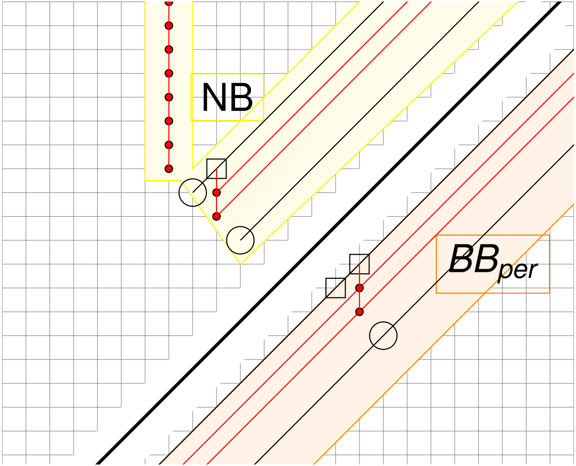

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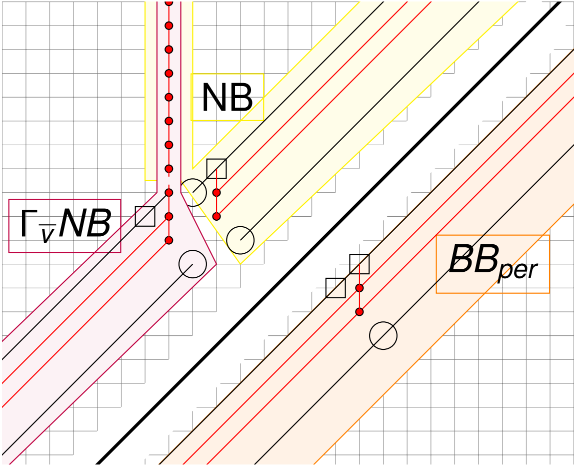
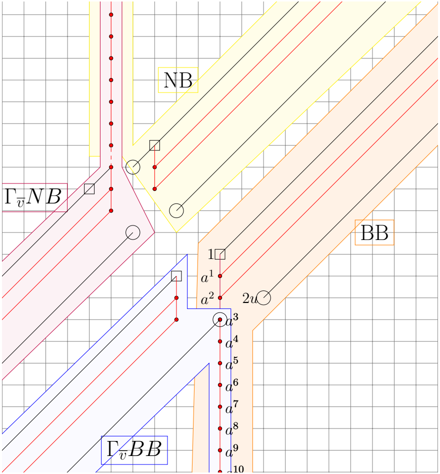


The duality picture



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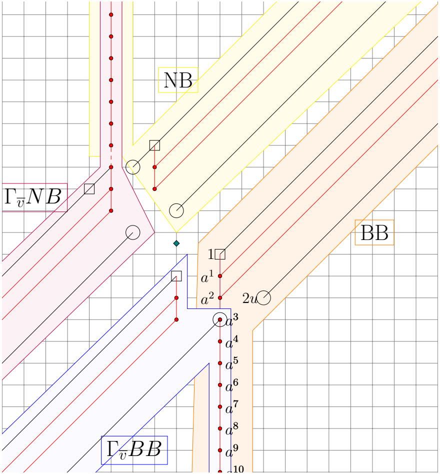

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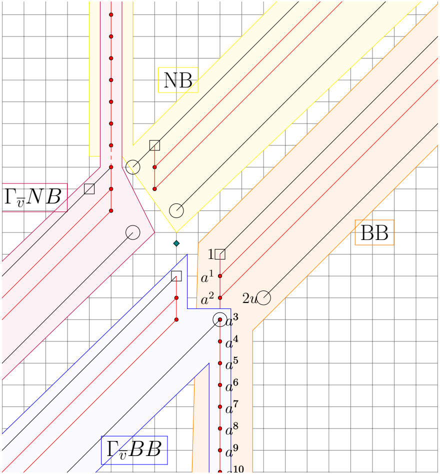

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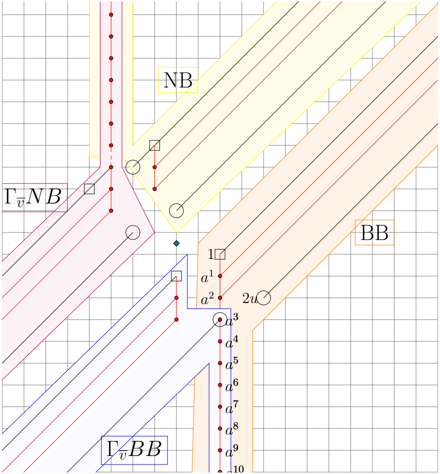

$$\mathbb{Z}^{k\mathbb{R}} \simeq \Sigma^{4-\sigma}\Gamma_{\overline{v}}k\mathbb{R} \simeq \Sigma^{2-2\sigma+|\overline{v}|+1}\Gamma_{\overline{v}}k\mathbb{R}$$ (Greenlees–M.)