RELATIVELY FREE TMF-MODULES

LENNART MEIER

Abstract. This paper is devoted to the study of TMF-modules or, equivalently, to quasi-coherent sheaves on the derived stack $(\mathcal{M}_{ell}, \mathcal{O}^{top})$. Every TMF-module can be resolved by relatively free ones, i.e. those that correspond to vector bundles on $(\mathcal{M}_{ell}, \mathcal{O}^{top})$. Every relatively free $TMF_{(l)}$-module for $l > 3$ is already free and therefore we concentrate on the situation at the prime 3. Here, we obtain a partial classification of relative free $TMF_{(3)}$-modules as so-called hook modules.

Contents

1. Introduction

The spectrum TMF of topological modular forms was constructed by Goerss, Hopkins and Miller. As it is an E_∞-ring spectrum, we can consider its ∞-category of modules. We might be interested in its general structure or in specific examples of TMF-modules. Both will be the topic of the present article.

Denote by $\mathcal{M}_{ell}$ the moduli stack of elliptic curves. The spectrum TMF arises as the global sections of a sheaf of E_∞-ring spectra $\mathcal{O}^{top}$ on $\mathcal{M}_{ell}$. We denote the ∞-category of quasi-coherent sheaves on the resulting derived stack $(\mathcal{M}_{ell}, \mathcal{O}^{top})$ by $\mathrm{QCoh}(\mathcal{M}_{ell}, \mathcal{O}^{top})$. By [MM15], the global sections functor

$$\Gamma\colon\ \mathrm{QCoh}(\mathcal{M}_{ell}, \mathcal{O}^{top}) \to TMF\text{-mod}$$

is an equivalence of ∞-categories. One of the advantages of the left-hand side is that the algebraic geometry of $\mathcal{M}_{ell}$ is much better behaved than the commutative algebra of π_*TMF. In particular, we can consider locally free $\mathcal{O}^{top}$-modules (i.e. derived vector bundles on $\mathcal{M}_{ell}$ – these are exactly those $\mathcal{O}^{top}$-modules $\mathcal{F}$ such that $\pi_i\mathcal{F}$ is a (classical) vector bundle on $\mathcal{M}_{ell}$ for every i. We call the TMF-modules whose corresponding quasi-coherent $\mathcal{O}^{top}$-module $\mathcal{F}_M$ is locally free *relatively free TMF-modules.* The name is inspired by the following fact: If we localize, say, at $p = 3$, then a TMF-module M is relatively free iff $M \wedge_{TMF} TMF_0(2)$ is a free $TMF_0(2)$-module.

We will prove that every TMF-module has a 2-step resolution by relatively free TMF-modules. Thus, if we want to compute, say, the algebraic K-theory $K_0(TMF)$, we can concentrate on the relatively free TMF-modules. Another reason to care about relatively free TMF-modules is the abundance of examples. All variants of TMF with level structure, $TMF(n)$, $TMF_0(n)$ and $TMF_1(n)$, are relatively free $TMF[\frac{1}{n}]$-modules. Furthermore, the G-fixed points TMF^G of Lurie's G-equivariant TMF are relatively free for G an abelian compact Lie group (and possibly also for non-abelian ones). These are the reasons we will concentrate on the relatively free TMF-modules in the present work.

Besides the structural examples above, there are also more constructive and inductive ways to build relatively free TMF-modules. For example, suppose that M is a relatively free TMF-module of rank n (in the sense that its corresponding derived vector bundle has rank n). Let $x \in \pi_k M$ be a torsion element. Then the cofiber M' in the cofiber sequence

$$\Sigma^k TMF \xrightarrow{x} M \to M'$$

is relatively free of rank $n+1$. We call this *coning off a torsion element.* Modules that are built from TMF via iteratively coning off torsion elements are called *standard modules.* For a standard module M, the vector bundles $\pi_i\mathcal{F}_M$ have the property that they are iterated extensions of line bundles. We call such vector bundles *standard* as well. Presently it is not known whether all vector bundles on $\mathcal{M}_{ell}$ are standard.

Another way to get a new relatively free module from a given relatively free TMF-module M of rank n is *to kill a generator*. This means to take a non-torsion element $x \in \pi_k M$ such that the cofiber M' in the cofiber sequence

$$\Sigma^k TMF \to M \to M'$$

is relatively free again, necessarily of rank $n-1$.

The first question we pose is whether we can obtain every relatively free TMF-module via these two procedures from TMF itself. It is often wise to consider first the corresponding question for real K-theory KO. Here, we call a KO-module M relatively free if $M \wedge_{KO} KU$ is a free KU-module of finite rank. Then we have the following theorem:

Theorem 1.1. *Every relatively free KO-module can be obtained from a suspension of KO by coning off torsion elements. Consequently, every relatively free KO-module is a sum of suspensions of KO, $KO \wedge \mathrm{Cone}(\eta) \simeq KU$ and $KO \wedge \mathrm{Cone}(\eta^2) \simeq KT$.*

This result is implicitly contained in [Bou90], but we will reprove in Section 2 using different techniques. For the corresponding question for TMF, we will always invert 2. In this case, the answer is that the only obstruction is algebraic.

Theorem 1.2. *Let M be a relatively free $TMF[\frac{1}{2}]$-module such that $\pi_i\mathcal{F}_M$ is a standard vector bundle for all $i \in \mathbb{Z}$. Then we can obtain M from a suspension of $TMF[\frac{1}{2}]$ by coning off torsion elements and killing generators. If M has rank n, then we have to cone off at most $2n-4$ torsion elements and kill at most $n-3$ generators to obtain M from a suspension of TMF.*

More precise statements can be found in Section 5.2. At least in the case that $\pi_i\mathcal{F}_M$ are standard, this allows in principle to classify relatively free $TMF[\frac{1}{2}]$-modules up to given rank. This is feasible at least up to rank 3, but in contrast to the KO-case the situation becomes quickly quite complicated. Presently, it is unknown whether there are finitely or infinitely many indecomposable relatively free TMF-modules, but our examples (see e.g. Section 7.3) strongly suggest that there infinitely many.

If X is a finite complex with cells only in even dimensions, $X \wedge TMF$ is evidently a standard relatively free TMF-module. In the case of $X = \mathbb{CP}^n$, we will prove the following theorem:

Theorem 1.3. *The $TMF[\frac{1}{2}]$-module $TMF \wedge \mathbb{CP}^n$ decomposes into suspensions of TMF, $TMF \wedge \mathrm{Cone}(\nu)$ and $TMF_0(2)$.*

A more precise statement can be found in Section 7.2. It allows easily to deduce the following:

Corollary 1.4. *A $TMF[\frac{1}{2}]$-algebra R is complex oriented if $\nu = 0$ in $\pi_3 R$.*

A further result of interest may be the following spectral sequence result:

Theorem 1.5. *For a TMF-module M, the MU-based Adams–Novikov spectral sequence is isomorphic to the descent spectral sequence for the sheaf of spectra $\mathcal{F}_M$ on $\mathcal{M}_{ell}$.*

This generalizes to every 0-affine derived stack (in the sense of [MM15]) instead of $(\mathcal{M}_{ell}, \mathcal{O}^{top})$.

We give an overview over the structure of this article. Section 2 will prove the classification of relatively free KO-modules and deduce the value of the algebraic K-theory $K_0(KO)$. Section 3 discusses the moduli stack of elliptic curves (with level structure) on vector bundles on it. Section 4 containes various topological preliminaries: We discuss what TMF is, how to build the descent spectral sequence and compare it to the Adams–Novikov spectral sequence and then we talk about TMF with level structures and other TMF-modules. We will introduce and study the notion of relatively free TMF-modules in 5. In particular, we will discuss how to build relatively free TMF-modules via coning off torsion elements and killing generators and introduce the notions of standard and hook-standard modules. We will then present our main results and do some preliminary reductions and low-rank cases. Section 6 contains the proof of the main technical result. In the last section 7, we give several examples, in particular discusses equivariant TMF and $TMF \wedge \mathbb{CP}^\infty$. The table of π_*TMF, π_*TMF_α and other low-rank examples might be helpful.

Acknowledgments. Most of the present article is based on my thesis written under the supervision of Stefan Schwede; to him belongs my gratitude for suggesting the topic and for his guidance. I had helpful discussions with Bob Bruner, Paul Goerss, Mike Hopkins, Niko Naumann, Irakli Patchkoria, Doug Ravenel, Vesna Stojanoska and many people else.

In talks in the year 2011 I claimed a substantially stronger result than Theorem 1.2. This was based on two errors in the proof. Special thanks belong to Viktoriya Ozornova for catching these two mistakes and for her extensive comments on earlier versions of my thesis.

2. Warm-up: The Case of K-Theory

Denote by KO real K-theory and by KU complex K-theory. Define a KO-module M to be *relatively free* if $M_{KU} = M \wedge_{KO} KU$ is a free KU-module of finite rank. It is easy to see that every finite KO-module N has a length-1 resolution by relatively free ones in the sense that there is a cofiber sequence

$$P_1 \to P_0 \to N$$

of KO-modules with P_0 and P_1 relatively free. The aim of this section is to classify relatively free modules KO-modules. More precisely, we will prove the following theorem:

Theorem 2.1. *Every relatively free KO-module is a direct sum of shifts of KO, KU and KT.*

Here, KT denotes K-theory with self-conjugation. While it has also a geometric interpretation, for our purposes, we can define it as the KO-module $KO \wedge C(\eta^2)$ for η the Hopf map and C the cone. The theorem is essentially equivalent to the finite rank case of [Bou90, Theorem 3.2]. Although not new, we will present a proof anyhow since our proof is different from Bousfield's and hopefully provides a helpful warm-up for the much more subtle case of TMF. Note the following corollary.

Corollary 2.2. *The morphism $\mathbb{Z} \to K_0(KO)$ given by $n \mapsto KO^n$ is an isomorphism.*

Proof. We first want to show that $\mathbb{Z} \to K_0(KO)$ is injective. One way to detect this is the trace map. There is an E_∞-maps

$$K(KO) \to THH(KO).$$

As KO is an E_∞-ring spectrum, it splits off $THH(KO)$ and thus 1 is non-torsion in $\pi_0 THH(KO)$ and thus the same is true in $\pi_0 K(KO)$.

Surjectivity is clear by the last theorem as $[\Sigma^n KO] = (-1)^n$, $[KU] = 2$ and $[KT] = 0$. □

We will both take as input and as inspiration the following well-known algebraic statement:

Proposition 2.3. *Every finitely generated $\mathbb{Z}[C_2]$-module whose underlying abelian group is free is a direct sum of (several copies of) the trivial representation $\mathbb{Z}$, the sign representation $\mathbb{Z}^-$ and $\mathbb{Z}[C_2]$ itself.*

Proof. A proof can be found in many sources, but we give a sketch. Let M be a free $\mathbb{Z}$-module of finite rank with an automorphism t of order 2. Given a nonzero $x \in M$, then $t(x) = -x$ or $x + t(x)$ is nonzero and invariant. By dividing x or $x + t(x)$ by a suitable integer, we obtain a split injection $\mathbb{Z}^- \to M$ or $\mathbb{Z} \to M$. By induction we can assume that the cokernel M' is of the required form. Then we are reduced to classify all extensions of M' and $\mathbb{Z}$ or $\mathbb{Z}'$, which is easy. □

Likewise, we will show in the next subsection for a relatively free KO-module M that there is a map $\Sigma^j KO \to M$ that induces a split injection $\Sigma^j KU \to M_{KU}$. Then we are reduced to classify extensions, which will be done in Subsection 2.2.

2.1. **The KO-Extension Theorem.** Before we state and proof the KO-extension theorem, we start with some general observations. We have (geometrically defined) maps $c\colon KO \to KU$ and $r\colon KU \to KO$, *complexification* and *realification*. The first is a morphism of ring spectra and gives KU the structure of a KO-module. Complex conjugation induces an involution τ on KU, which acts as a KO-algebra map. We have $cr = \mathrm{id} + \tau$ and $rc = 2$. Furthermore, we have $KU^{hC_2} \simeq KO$ by [Rog08, 5.3] with the equivalence induced by c.

By the theorem of Wood, $KU \simeq KO \wedge C\eta$ (see [Mat13, Thm 3.2] or [Ati66, 3.4] for proofs). Thus, we have an induced cofiber sequence

$$\Sigma M \xrightarrow{\eta} M \xrightarrow{c} M_{KU} \xrightarrow{\rho} \Sigma^2 M.$$

The aim of this section is to prove the following proposition:

Proposition 2.4. *Let M be a nonzero relatively free KO-module. Then there is a map $f\colon \Sigma^j KO \to M$ such that the map*

$$(f \wedge_{KO} KU)_*\colon \pi_*\Sigma^j KU \to \pi_* M_{KU}$$

is split injective (equivalently as map of abelian groups in every degree or as a map of π_*KU-modules).

Remark 2.5. Since maps between free modules are determined by their effect on homotopy groups, $f \wedge_{KO} KU\colon \Sigma^j KU \to M_{KU}$ splits for M_{KU} free iff

$$(f \wedge_{KO} KU)_*\colon \pi_*\Sigma^j KU \to \pi_* M_{KU}$$

splits.

Corollary 2.6. *Every nonzero relatively free KO-module M can be obtained by iteratively coning off torsion elements from a suspension of KO. We call such modules* standard KO-modules*.*

Proof. For a relatively free M, the dual $D_{KO}M$ is also relatively free since

$$\mathrm{Hom}_{KO}(M, KO) \wedge_{KO} KU \simeq \mathrm{Hom}_{KU}(M_{KU}, KU).$$

Thus, using the proposition, we can choose an $f\colon \Sigma^j KO \to D_{KO}M$ splitting off a direct summand after smashing with KU and call the Spanier–Whitehead dual of the cofiber N. Note that this is relatively free of one rank less than M.

After dualizing f, we get a cofiber sequence $\Sigma^{-j-1}KO \xrightarrow{g} N \to M \to \Sigma^{-j}KO$. As $M_{KU} \to \Sigma^{-j}KU$ has a section, $g_{KU}\colon \Sigma^{-j-1}KU \to N_{KU}$ is zero. Therefore, the corresponding element $x = g(1) \in \pi_{-j-1}N$ satisfies $c_*(x) = 0$. Hence, x is in the image of η and therefore torsion.

All in all, we get that we can obtain M from a relatively free module of smaller rank by coning off a torsion element. Now, we can assume inductively that every relatively free module of smaller rank than M (e.g., N) is standard and get that M is standard. Note that we use as an induction start that $M_{KU} = 0$ implies $M = 0$. Indeed, $M_{KU} = 0$ implies that $\eta\colon \Sigma M \to M$ is an isomorphism of KO-modules, but $\eta^3 = 0$. □

Let now M be a KO-module. Then by [Mat13, Corollary 3.5], we have $(M_{KU})^{hC_2} \simeq M$. The homotopy fixed point spectral sequence

$$E_2^{**}(M) = H^*(C_2, \pi_* M_{KU}) \Rightarrow \pi_* M,$$

for $M_{KU}^{hC_2}$ is a module spectral sequence over the homotopy fixed points spectral sequence for KU^{hC_2}.

Lemma 2.7. *Let M be a relatively free KO-module. Let $x \in \pi_* M$ be a torsion element. Then $x = \eta^k y$ for some $k \in \{1, 2\}$ and some $y \in \pi_* M$ with $c_*(y)$ an indivisible element in $\pi_* M_{KU}$. Here y is called indivisible if $l \cdot y' = y$ for $l \in \mathbb{Z}$ implies $l = \pm 1$.*

Proof. Let $x \in \pi_* M$ be a torsion element. Every torsion element is divisible by η since its image in $\pi_* M_{KU}$ is torsion, hence zero. Therefore, we can write $x = \eta^k y$, for y non-torsion and $k \in \{1, 2\}$ maximal (since $\eta^3 = 0$). We claim that y is detected by a nonzero element $\overline{y}$ in the 0-line of the homotopy fixed point spectral sequence. Indeed, $\overline{y}$ can be identified with $c_*(y)$ and $c_*(y) = 0$ would imply that y is in the image of η. Asssume first (for contradiction) that $\overline{y} \in 2H^0(C_2; \pi_* M_{KU})$.

Denote by $\widetilde{KU[C_2]}$ the direct sum $KU \oplus KU$ with C_2-action both acting on KU and interchanging the factors. As $KU \wedge_{KO} KU \simeq \widetilde{KU[C_2]}$ by [Rog08, Prop 5.3.1], we see

$$(M_{KU})_{KU} \simeq M_{KU} \wedge_{KU} \widetilde{KU[C_2]} \simeq \widetilde{KU[C_2]}^n.$$

Thus, the homotopy fixed point spectral sequence for $(M_{KU})_{KU}^{hC_2}$ is concentrated in the zero line, where it equals $\pi_* M_{KU}$. The map $r\colon M_{KU} \to M$ induces a map of spectral sequences, which induces in the 0-line the map

$$\pi_* M_{KU} \to H^0(C_2; \pi_* M_{KU}) \subset \pi_* M_{KU}$$

given by $x \mapsto x + \tau x$ (since $cr = 1 + \tau$). Clearly, $2H^0(C_2; \pi_* M_{KU})$ is in the image. Thus, there is a $y' \in \mathrm{im}(r_*) \subset \pi_* M$ such that $y - y'$ is of filtration at least 1 and $\eta y' = 0$. Hence, $\eta^k(y - y') = x$ and $y - y'$ is torsion, which is a contradiction to the maximality of k.

Therefore, y projects non-trivially to $H^0(C_2, \pi_* M_{KU})/2$. The edge morphism

$$\pi_* M \to H^0(C_2, \pi_* M_{KU}) \subset \pi_* M_{KU}$$

converges to c_*. Thus, $c_*(y)$ is not divisible by 2. Thus, it must be of the form $c_*(y) = lz$ for an odd number l and an indecomposable element $z \in (\pi_* M_{KU})^{C_2}$. Set $\tilde{y} = y - \frac{l-1}{2} r_*(z)$. Then $c_*(\tilde{y}) = z$ and $\eta^k \tilde{y} = \eta^k y = x$. □

Proof of proposition: Let M be a nonzero relatively free KO-module. Consider the morphisms $c\colon M = M_{KU}^{hC_2} \to M_{KU}$. It is enough to find an indivisible element $e \in \pi_* M_{KU}$ which is in the image of c_* (since every indivisible element in a free abelian group generates a direct summand). By the last lemma, we just have to show that $\pi_* M$ contains torsion.

Assume $\pi_* M$ has no torsion. As $k\rho_*(x) = \rho_*(kx) = 0$ implies $\rho_*(x) = 0$ for $x \in M_{KU}$, we see that kx in the image of c_* iff x is. If the image of c_* contains no indivisible element, c_* must be zero. On the other hand, the image of η in $\pi_* M$ is also zero as $\pi_* M$ is torsion free. Thus, $\pi_* M = 0$, which is a contradiction. □

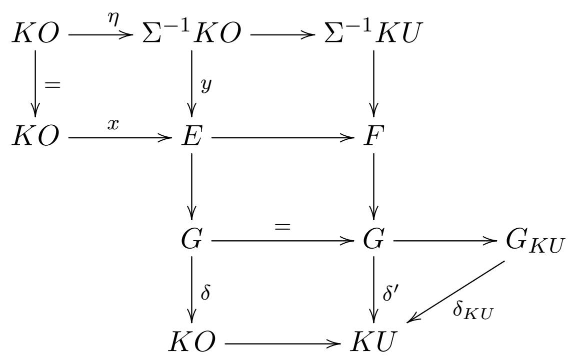
2.2. **Classification of KO-Standard Modules.** Our goal in this section is the classification of relatively free KO-modules, recovering [Bou90, Theorem 3.2] in the finite rank case. We will need the computation of the homotopy groups of KT, which is easily deduced from the equivalence $KT \simeq KO \wedge C(\eta^2)$:

$$\pi_i KT \cong \begin{cases} \mathbb{Z}/2 & \text{for } i \equiv 1 \mod 4 \\ \mathbb{Z} & \text{for } i \equiv 0, 3 \mod 4 \\ 0 & \text{else} \end{cases}$$

Theorem 2.8. *Every relatively free KO-module is a direct sum of shifts of KO, KU and KT.*

Proof. Assume for induction that all standard modules of rank $< n$ are a direct sum of shifts of KO, KU and KT. Here, the rank of a relatively free module M is defined to be the rank of $\pi_* M_{KU}$ as a $\pi_* KU$-module.

By Corollary 2.6, after a shift every relatively free module F of rank $n > 0$ sits in an exact triangle of the form $KO \to E \to F$ with $\operatorname{rk} E = n - 1$ and $KO \to E$ corresponding to a torsion-element $x \in \pi_* E$. We can assume x to be non-zero. Denote by $c\colon E \to E_{KU}$ the map induced by $KO \to KU$. Every torsion element in $\pi_* E$ is divisible by η (as its image under c_* is zero) and we choose a $y \in \pi_* E$ with $\eta y = x$. Then we have by the octahedral axiom a diagram of the form



where the two columns and the upper two rows are triangles. Assume first that x is not divisible by η^2. By Lemma 2.7, we can choose y in a way such that $c(y)$ is an indivisible vector in $\pi_* E_{KU} \cong \mathbb{Z}^{n-1}$. Therefore, the map $\Sigma^{-1} KU \xrightarrow{c_*(y)} E_{KU}$ has a section and G_{KU} a direct summand of E_{KU} of rank $n-2$ (and therefore a direct sum of shifts of KO, KU and KT by induction). In particular, $\delta_{KU}\colon G_{KU} \to KU$ must be zero (since it is zero on homotopy groups and the source is a free module). Since $\delta'\colon G \to KU$ factors over δ_{KU}, it has also to be zero. Therefore, $F \cong G \oplus \Sigma^{-1} KU$.

If x is divisible by η^2, we can assume $E \cong \bigoplus \Sigma^{-2} KO$ since only in these summands there is a π_0-element divisible by η. Thus, $\pi_0 E \cong \mathbb{F}_2^k$ and we can lift $x \in \mathbb{F}_2^k$ to an indivisible vector x' in $\mathbb{Z}^k$. We can choose a matrix $A \in GL_k(\mathbb{Z})$ with x' as first column. Its inverse defines an automorphism of E sending x to $(\eta^2, 0, \ldots, 0)$. After this change of basis, it is immediate that $F \cong \Sigma^{-2} KT \oplus \bigoplus \Sigma^{-2} KO$. $\square$

3. Algebraic Preliminaries

3.1. **The moduli stack of elliptic curves with level structure.** There are several variations of moduli stacks of elliptic curve $\mathcal{M} = \mathcal{M}_{ell}$, based on the notion of a *level* *structure*. We will give the definition and a few simple properties and investigate then the moduli stacks of elliptic curves with level-2-structure in detail.

An elliptic curve E over S is, in particular, an abelian group scheme over S and we can consider for a given n the finite sub group scheme $E[n]$ of n-torsion points.

Definition 3.1. Let E/S be an elliptic curve with n invertible on S. A *level-n-structure* is an isomorphism $S \times (\mathbb{Z}/n)^2 \to E[n]$. The moduli stack of elliptic curves with level-n-structure is denoted by $\mathcal{M}(n)$.

A $\Gamma_1(n)$-structure is an injection $S \times \mathbb{Z}/n \to E[n]$. We denote the moduli stack of elliptic curves w ith $\Gamma_1(n)$-structure by $\mathcal{M}_1(n)$. For $n = 2$, the notion of a $\Gamma_1(2)$-structure coincides with that of a $\Gamma_0(2)$-structure; for this reason, we will use mostly the notation $\mathcal{M}_0(2)$ for $\mathcal{M}_1(2)$.

We have maps $\mathcal{M}(n) \to \mathcal{M}$ and $f\colon \mathcal{M}_1(n) \to \mathcal{M}$, which are étale and surjective if we invert n. The surjectivity can be seen by the well-known fact that over an algebraically closed field of characteristic not dividing n, the n-torsion of an elliptic curve is isomorphic to $(\mathbb{Z}/n)^2$. We will denote the map $\mathcal{M}(2) \to \mathcal{M}[\frac{1}{2}]$ by p and the map $\mathcal{M}_0(2) \to \mathcal{M}[\frac{1}{2}]$ by f.

To every elliptic curve $\pi\colon E \to S$ with identity section $s\colon S \to E$, we can associate the line bundle $\pi_*\Omega^1_{E/S} \cong s^*\Omega^1_{E/S}$, the direct image of the differentials. This defines a line bundle ω on $\mathcal{M}$. There is an isomorphism

$$\mathcal{O}_{\mathcal{M}} \xrightarrow{\cong} \omega^{\otimes 12}$$

defined by $\Delta \in \Gamma(\omega^{\otimes 12})$. We will often also denote the pullback of ω to $\mathcal{M}(n)$ or $\mathcal{M}_1(n)$ by ω for simplicity of notation.

Notation 3.2. We set for an $\mathcal{O}_{\mathcal{M}}$-module $\mathcal{F}$

$$H^i_j(\mathcal{M};\mathcal{F}) = H^i(\mathcal{M};\mathcal{F} \otimes \omega^{\otimes j})$$

and

$$\Gamma_j(\mathcal{F}) = \Gamma(\mathcal{F} \otimes \omega^{\otimes j}.$$

We will use similar notation for module sheaves on $\mathcal{M}(2)$ or $\mathcal{M}_0(2)$ with respect to the pullback of ω.

An alternative interpretation of these groups uses the stack $\mathcal{M}^1$ of elliptic curves with chosen invariant differential; we denote the projection $\mathcal{M}^1 \to \mathcal{M}$ by π. This stack has a $\mathbb{G}_m$-action via multiplication on the invariant differential and $\mathcal{M}^1/\mathbb{G}_m \simeq \mathcal{M}$. It is easy to see that $\mathcal{M}^1$ is the relative Spec of $\bigoplus_{i\in\mathbb{Z}} \omega^{\otimes i}$ and that the latter sheaf is isomorphic to $\pi_*\pi^*\mathcal{O}_{\mathcal{M}}$ with the grading corresponding to the $\mathbb{G}_m$-action. More generally, we have $\Gamma_*(\mathcal{F}) \cong \Gamma(\pi^*\mathcal{F})$.

We get a similar picture using $\mathcal{M}^1(n) = \mathcal{M}^1 \times_{\mathcal{M}} \mathcal{M}(n)$ and $\mathcal{M}^1_1(n) = \mathcal{M}^1 \times_{\mathcal{M}} \mathcal{M}_1(n)$.

We will now discuss the cases $\mathcal{M}(2)$ and $\mathcal{M}_0(2) = \mathcal{M}_1(2)$ in more detail. Let R be a ring which contains $\frac{1}{2}$. Then every elliptic curve can be represented by an equation of the form

$$y^2 = 4x^3 + b_2x^2 + 2b_4x + b_6 \tag{3.3}$$

(in $\mathbb{P}^2_R$). A point of exact order 2 corresponds to a point with $y = 0$ (see also [Beh06], 1.3.2). Therefore, a level-2-structure gives a splitting

$$4x^3 + b_2x^2 + 2b_4x + b_6 = 4(x - e_1)(x - e_2)(x - e_3).$$

By a coordinate change $x \mapsto x + e_3$, we get an equivalent form

$$y^2 = 4(x - (e_1 - e_3))(x - (e_2 - e_3))x.$$

Set $x_2 := e_1 - e_3$ and $y_2 := e_2 - e_3$. One can see that these two values are determined by the elliptic curve with level-2-structure and a chosen invariant differential uniquely – therefore, we get that

$$\mathcal{M}^1(2) \simeq \operatorname{Spec} \mathbb{Z}[\frac{1}{2}][x_2, y_2, \Delta^{-1}]$$

and hence

$$\mathcal{M}(2) \simeq \operatorname{Spec} \mathbb{Z}[\frac{1}{2}][x_2, y_2, \Delta^{-1}]/\mathbb{G}_m,$$

where Δ is the image of $\Delta \in H^0_*(\mathcal{M}; \mathcal{O}_{\mathcal{M}})$ under the map $H^0_*(\mathcal{M}; \mathcal{O}) \to H^0_*(\mathcal{M}(2); \mathcal{O}_{\mathcal{M}(2)})$ (see [Sto12, Section 7] for more details). The grading is given by $|x_2| = |y_2| = 2$, i.e. $x_2, y_2 \in H^0(\mathcal{M}(2); \omega^{\otimes 2})$. We will denote the ring $\operatorname{Spec} \mathbb{Z}[\frac{1}{2}][x_2, y_2, \Delta^{-1}]$ by $TMF(2)_{2*}$ for reasons that will become apparent later.

By [Beh06, Section 1.3.2], one gets similarly

$$\mathcal{M}_0(2) \simeq \operatorname{Spec} \mathbb{Z}[\frac{1}{2}][b_2, b_4, \Delta^{-1}]//\mathbb{G}_m$$

with $|b_2| = 2$ and $|b_4| = 4$.

The importance of $\mathcal{M}(2)$ for us lies in its concrete algebraic description and the fact that many question about $\mathcal{M}[\frac{1}{2}]$ can be reduced to $\mathcal{M}(2)$ by the following lemma:

Lemma 3.4. *The map $p\colon \mathcal{M}(2) \to \mathcal{M}[\frac{1}{2}]$ is an S_3-Galois covering. Thus, there is an equivalence between quasi-coherent sheaves on $\mathcal{M}[\frac{1}{2}]$ and S_3-equivariant graded $TMF(2)_{2*}$-modules, giving by*

$$\mathcal{F} \mapsto \Gamma_*(p^*\mathcal{F}).$$

For a quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{M}[\frac{1}{2}]$, the adjunction unit $\mathcal{F} \to p_*p^*\mathcal{F}$ induces on graded global sections Γ_* the morphism

$$(\Gamma_*(p^*\mathcal{F}))^{S_3} \to \Gamma_*(p^*\mathcal{F}).$$

Proof. In general, $\mathcal{M}(n) \to \mathcal{M}[\frac{1}{n}]$ is a $GL_2(\mathbb{Z}/n)$-Galois cover with $GL_2(\mathbb{Z}/n)$ acting on the trivialization of the n-torsion (see [DR73, Section 2.3]). We have $GL_2(\mathbb{Z}/2) \cong S_3$. The rest follows by standard fpqc-descent theory. □

Explicitly, the S_3-action on $TMF(2)_{2*}$ can be described as follows: The S_3-action on $\mathcal{M}(2)$ permutes the e_1, e_2 and e_3. Thus we get the following formulas for the group action:

$$\begin{aligned}
(1\ 2)\colon &\quad x_2 \mapsto y_2, && y_2 \mapsto x_2\\
(1\ 3)\colon &\quad x_2 \mapsto -x_2, && y_2 \mapsto y_2 - x_2\\
(2\ 3)\colon &\quad x_2 \mapsto x_2 - y_2, && y_2 \mapsto -y_2\\
(1\ 2\ 3)\colon &\quad x_2 \mapsto y_2 - x_2, && y_2 \mapsto -x_2\\
(1\ 3\ 2)\colon &\quad x_2 \mapsto -y_2, && y_2 \mapsto x_2 - y_2
\end{aligned}$$

As before, the map $p\colon \mathcal{M}(2) \to \mathcal{M}$ induces

$$H^0_*(\mathcal{M}; \mathcal{O}_{\mathcal{M}}) \to H^0_*(\mathcal{M}(2); \mathcal{O}_{\mathcal{M}(2)})$$

The source is *the ring of (integral mereomorphic) modular forms* and is multiplicatively generated by c_4,c_6 and $\Delta^{\pm 1}$ with the relation $1728\Delta = c_4^3 - c_6^2$. The target is, as indicated above, isomorphic to $\mathbb{Z}[\frac{1}{2}][x_2, y_2, \Delta^{-1}]$. We want to compute the images of c_4 and c_6.

For an elliptic curve in Weierstrass form 3.3, there is an associated trivialization of ω on $\operatorname{Spec} R$; thus c_4 and c_6 actually define elements in the ring R. The universal formulas for these are given as follows [Sil09, III.1]:

$$\begin{aligned} c_4 &= b_2^2 - 24b_4 \\ c_6 &= -b_2^3 + 36b_2b_4 - 216b_6 \end{aligned}$$

Given a level-2-structure, we get the following formulas:

$$\begin{aligned} b_2 &= -4(x_2+y_2) \\ b_4 &= 2x_2y_2 \\ b_6 &= 0 \end{aligned}$$

Therefore, we get

$$\begin{aligned} c_4 &= 16(x_2+y_2)^2 - 48x_2y_2 = 16(x_2^2+y_2^2-x_2y_2) \\ c_6 &= 64(x_2+y_2)^3 - 288x_2y_2(x_2+y_2) = 64(x_2^3+y_2^3) - 96(x_2^2y_2+x_2y_2^2) \end{aligned}$$

Here, we denote the images of c_4 and c_6 in $H^0_*(\mathcal{M}(2);\mathcal{O})$ by the same name. If we reduce modulo 3, the formulas become much simpler and we have:

$$\begin{aligned} c_4 &\equiv (x_2+y_2)^2 \mod 3 \\ c_6 &\equiv (x_2+y_2)^3 \mod 3 \end{aligned}$$

In general, we have the following formula:

$$\Delta = -27b_6^2 + (9b_2b_4 - \frac{1}{4}b_2^3)b_6 - 8b_4^3 + \frac{1}{4}b_2^2b_4^2 \text{ [Sil09, III.1]}$$

This gives in terms of x_2 and y_2:

$$\Delta = \frac{1}{4}b_4^2(b_2^2 - 32b_4) = x_2^2y_2^2(16(x_2+y_2)^2 - 64x_2y_2) = 16x_2^2y_2^2(x_2-y_2)^2$$

These formulas will be used in some way in Sections 3 and 5.4. We will also need the following lemma:

Lemma 3.5. *Let $n \geq 2$. Then the cohomology groups $H^i(\mathcal{M}[\frac{1}{n}]; h_*\mathcal{F})$ vanish for $i > 0$ for every quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{M}_1(n)$ or $\mathcal{M}(n)$ and h the projection from $\mathcal{M}(n)$ or $\mathcal{M}_1(n)$ to $\mathcal{M}[\frac{1}{n}]$.*

Proof. This follows easily from the Leray spectral sequence and the fact that $\mathcal{M}_1^1(n)$ and $\mathcal{M}^1(n)$ are representable by affine schemes for $n \geq 2$. See [Mei15, Lem 4.7] for details in the case $n = 2$ and [HM17, Lem 4.2] for the general case of $\mathcal{M}_1^1(n)$; the case of $\mathcal{M}^1(n)$ is similar. □

3.2. **Standard vector bundles and their cohomology.** Let A be a set of primes. By localization at A, we mean localization at the multiplicative subset of integers not divisible by any p for $p \in A$. Now assume that $2 \notin A$. but $3 \in A$.

As explained in [Mei15, Section 4.1], Bauer computes in [Bau08] that

$$H^1(\mathcal{M}_{(A)};\omega^{\otimes i}) = \begin{cases} \mathbb{Z}/3\mathbb{Z} & \text{if } i \equiv 2 \mod 12, \\ 0 & \text{else,} \end{cases}$$

$$H^2(\mathcal{M}_{(A)};\omega^{\otimes i}) = \begin{cases} \mathbb{Z}/3\mathbb{Z} & \text{if } i \equiv 6 \mod 12, \\ 0 & \text{else.} \end{cases}$$

Chosen generators of $H^1(\mathcal{M}_{(A)};\omega^{\otimes 2})$ and $H^2(\mathcal{M}_{(A)};\omega^{\otimes 6})$ are denoted by α and β. The algebra $H^*(\mathcal{M}_{(A)};\omega^{\otimes *})$ is for cohomological degree > 0 generated over $\mathbb{Z}/3$ by α, β and $\Delta^{\pm 1}$ with only relation $\alpha^2 = 0$.

For brevity, we denote the structure sheaf $\mathcal{O}_{\mathcal{M}_{(A)}}$ by $\mathcal{O}$ and all Ext-groups will be in the category of $\mathcal{O}$-modules. The class $\alpha \in H^1(\mathcal{M}_{(A)};\omega^2) \cong \operatorname{Ext}^1(\omega^{-2},\mathcal{O})$ classifies an extension

$$0 \to \mathcal{O} \to E_\alpha \to \omega^{-2} \to 0. \tag{3.6}$$

The following results were stated in [Mei15] only for $A = \{3\}$, but the proofs actually work for all A not containing 2.

Proposition 3.7 ([Mei15], Proposition 4.1)**.** *We have*

$$\operatorname{Ext}^1(\omega^j, E_\alpha) = \begin{cases} \mathbb{Z}/3\mathbb{Z} & \text{if } j \equiv -4 \mod 12, \\ 0 & \text{else,} \end{cases}$$

$$\operatorname{Ext}^2(\omega^j, E_\alpha) = \begin{cases} \mathbb{Z}/3\mathbb{Z} & \text{if } j \equiv -6 \mod 12, \\ 0 & \text{else.} \end{cases}$$

Furthermore, left multiplication with β defines isomorphisms $\operatorname{Ext}^i(\omega^j, E_\alpha) \cong \operatorname{Ext}^{i+2}(\omega^j, E_\alpha)$.

We denote the element in $\operatorname{Ext}^1(\omega^{-4}, E_\alpha)$ corresponding to α under the isomorphism

$$\operatorname{Ext}^1(\omega^{-4}, E_\alpha) \cong \operatorname{Ext}^1(\omega^{-2}, \omega^2 \otimes E_\alpha) \xrightarrow{\cong} \operatorname{Ext}^1(\omega^{-2}, \mathcal{O})$$

by $\widetilde{\alpha}$.

The class $\widetilde{\alpha}$ defines an extension

$$0 \to E_\alpha \otimes \omega^4 \to E_{\alpha,\widetilde{\alpha}} \to \mathcal{O} \to 0 \tag{3.8}$$

As shown at the end of Section 4.3 of [Mei15], there is an isomorphism $f_*f^*\mathcal{O} \cong E_{\alpha,\widetilde{\alpha}}$, where f denotes still the map $\mathcal{M}_0(2)_{(A)} \to \mathcal{M}_{(A)}$.

Next, we recall from [Mei15] that sums of the vector bundles we just constructed actually form a large class of vector bundles on $\mathcal{M}_{(A)}$. By [FO10], every line bundle on $\mathcal{M}_{(A)}$ (even for $2 \in A$) is of the form $\omega^{\otimes i}$. Let us consider the class of vector bundles that can be built from line bundles by iterative extensions.

Definition 3.9. We define the notion of a *standard vector bundle* for a set of primes A (possibly including 2) inductively: Every line bundle on $\mathcal{M}_{(A)}$ is called *standard*. Furthermore, a vector bundle $\mathcal{E}$ on $\mathcal{M}_{(A)}$ is called *standard* if there is an injection $\mathcal{L} \hookrightarrow \mathcal{E}$ from a line bundle on $\mathcal{M}_{(A)}$ such that the cokernel is a standard vector bundle.

Theorem 3.10 ([Mei15])**.** *For $2 \notin A$, every standard vector bundle over $\mathcal{M}_{(A)}$ is isomorphic to a sum of copies of the vector bundles $\mathcal{O}$, E_α or $E_{\alpha,\widetilde{\alpha}}$ or tensor products of line bundles with them. If $A = \{p\}$ for $p > 3$, then every vector bundle is a sum of line bundles.*

Assume again that $3 \in A$, but $2 \notin A$. Which standard vector bundle we have can essentially be detected on the supersingular elliptic curve $E\colon y^2 = x^3 - 3$ at the prime 3. The group $C_3 = \langle s \rangle$ acts on E via the automorphism

$$s\colon \quad x \mapsto x + 1, \quad y \mapsto y.$$

Evaluating a vector bundle at the classifying map $\operatorname{Spec} \mathbb{F}_3/C_3 \to \mathcal{M}_{(A)}$ gives a finite-dimensional C_3-representation over $\mathbb{F}_3$. Denote the 1-dimensional (trivial) representation by J_1, the 2-dimensional representation given by the Jordan block $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ by J_2 and the

3-dimensional representation by the Jordan block $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ by J_3. These are actually all the indecomposable C_3-representations over C_3.

Proposition 3.11 ([Mei15], Section 4.3)**.** *The C_3-representations associated to $\mathcal{O}$, E_α and $f_*f^*\mathcal{O}$ are J_1, J_2 and J_3. If a standard vector bundle $\mathcal{E}$ has associated C_3-representation J_3, then it is of the form $f_*f^*\mathcal{O} \otimes \omega^{\otimes i}$.*

3.3. **Adjoints.** We will work for the next paragraphs more generally with an arbitrary étale map $p\colon \mathcal{X} \to \mathcal{Y}$ since we do not gain by specializing at this point. Let $\mathcal{F}$ be quasi-coherent sheaf on $\mathcal{Y}$. The adjunction unit $\mathcal{F} \to p_*p^*\mathcal{F}$ induces a map

$$c_{alg}\colon \Gamma(\mathcal{F}) \to \Gamma(p_*p^*\mathcal{F}) = \Gamma(p^*\mathcal{F}).$$

The following lemma is well-known, but I was unable to find a complete and elementary proof in the literature.

Lemma 3.12. *For any étale map $p\colon \mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks, the functor*

$$p^*\colon \mathcal{O}_{\mathcal{X}}\text{-mod} \to \mathcal{O}_{\mathcal{Y}}\text{-mod}$$

has a left adjoint $p_!$.

Proof. We will begin by describe a left adjoint of p^* on the level of presheaves. For $\mathcal{F}$ a presheaf of $\mathcal{O}_{\mathcal{X}}$-modules, a presheaf $p_?\mathcal{F}$ of $\mathcal{O}_{\mathcal{Y}}$-modules is defined as follows: For $f\colon U \to \mathcal{Y}$ an étale map, $p_?\mathcal{F}(U, f) := \oplus_s \mathcal{F}(U, s)$, where the direct sum ranges over all maps $s\colon U \to \mathcal{X}$ such that $ps = f$. We want to prove that $p_?$ is left adjoint to p^* at the level of presheaves. For $\mathcal{G}$ a presheaf of $\mathcal{O}_{\mathcal{Y}}$-modules, define the counit $p_?p^*\mathcal{G} \to \mathcal{G}$ on an $f\colon U \to \mathcal{Y}$ by the summing map

$$\bigoplus_{s \text{ lifting of } f} \mathcal{G}(U, ps) \to \mathcal{G}(U, f)$$

(note that $ps = f$ by definition). For $\mathcal{F}$ a presheaf of $\mathcal{O}_{\mathcal{X}}$, define the unit $\mathcal{F} \to p^*p_?\mathcal{F}$ on a $t\colon U \to \mathcal{X}$ by the inclusion of the t-summand $\mathcal{F}(U, t) \to \bigoplus_{s \text{ lifting of } p\circ t} \mathcal{F}(U, s)$. It is easy to check that the transformations $p_? \to p_?p^*p_? \to p_?$ and $p^* \to p^*p_?p^* \to p^*$ are identity.

Denote the "forgetful" functor $\mathcal{O}_{\mathcal{X}}\text{-mod} \to \mathrm{Pre}_{\mathcal{X}}$ from $\mathcal{O}_{\mathcal{X}}$-modules to presheaves of $\mathcal{O}_{\mathcal{X}}$-modules by u and the sheafification by S and likewise for $\mathcal{Y}$. Define $p_!\mathcal{F}$ as $S(p_?(u\mathcal{F}))$. Moreover, we have that $u(p^*\mathcal{G}) = p^*(u\mathcal{G})$ by definition. Since sheafification is left adjoint to u, we get that $p_!$ is left adjoint to p^*:

$$\begin{aligned} \mathcal{O}_{\mathcal{Y}}\text{-mod}(p_!\mathcal{F}, \mathcal{G}) &= \mathcal{O}_{\mathcal{Y}}\text{-mod}(S(p_?(u\mathcal{F})), \mathcal{G}) \\ &\cong \mathrm{Pre}_{\mathcal{Y}}(p_?(u\mathcal{F}), u\mathcal{G}) \\ &\cong \mathrm{Pre}_{\mathcal{X}}(u\mathcal{F}, p^*u\mathcal{G}) \\ &= \mathrm{Pre}_{\mathcal{X}}(u\mathcal{F}, up^*\mathcal{G}) \\ &= \mathcal{O}_{\mathcal{X}}\text{-mod}(\mathcal{F}, p^*\mathcal{G}) \end{aligned}$$

□

Note that a lifting $U \to \mathcal{X}$ is equivalent to a section of $U \times_{\mathcal{Y}} \mathcal{X} \to U$. Let p now be a finite etale map. By [Aut, 04HN], there exists a basis of the etale topology $\{U_i\}_{i \in I}$ such that $U_i \times_{\mathcal{Y}} \mathcal{X} \cong \coprod_{\{1,\ldots,n\}} U_i$. On these U_i, the presheaf $p_?$ agrees with the sheaf p_*; as the sheafification of a presheaf can be computed on a basis of topology, there is an isomorphism $p_!\mathcal{F} \to p_*\mathcal{F}$. In particular, we get a map

$$r_{alg}\colon \Gamma(p^*\mathcal{F}) \cong \Gamma(p_*p^*\mathcal{F}) \cong \Gamma(p_!p^*\mathcal{F}) \to \Gamma(\mathcal{F}).$$

Clearly, r_{alg} is natural with respect to maps of sheaves since the counit map is a natural transformation. For the rest of this section, we abbreviate r_{alg} and c_{alg} to r and c for ease of notation.

Lemma 3.13. *Let p be a G-Galois cover. Then we have the identities $rc = |G|$ and $cr = \Sigma_{g\in G} g$. Furthermore, r is surjective as a sheaf map.*

Proof. It is enough to show these statements locally since both r and c are induced by morphisms of sheaves. So we may assume that p is trivial, i.e., $\mathcal{X} = \coprod_G \mathcal{Y}$. Hence, we have $\Gamma(p^*\mathcal{F}) \cong \prod_G \Gamma(\mathcal{F})$. For every $g \in G$, the map $ps_g\colon \mathcal{Y} \to \mathcal{X} \to \mathcal{Y}$ is the identity, where s_g is the section corresponding to the element g. Therefore, the map $c\colon \Gamma(\mathcal{F}) \to \prod_G \Gamma(\mathcal{F})$ is the diagonal. Since $ps = \mathrm{id}$ for all sections $s\colon \mathcal{Y} \to \mathcal{X}$, we have that

$$r\colon \prod_G \Gamma(\mathcal{F}) \cong \bigoplus_G \Gamma(\mathcal{F}) \to \Gamma(\mathcal{F})$$

is the summing map (by the definition of the counit) and hence surjective. Therefore an element

$$x = (0, \dots, 0, a, 0, \dots, 0) \in \prod_G \Gamma(\mathcal{F})$$

is sent to $(a, \dots, a) = \Sigma_{g\in G} gx$ by cr. On the other hand, an element $a \in \Gamma(\mathcal{F})$ is sent to $\Sigma_{g\in G} a = |G|a$. □

Now, we come back to the specific situation of $p\colon \mathcal{M}(2) \to \mathcal{M}$ and $G = S_3$, where we localize everything implicitly at a set of primes A with $2 \notin A$. Note that we can view r for a quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{M}$ also as a map $\Gamma_*(p^*\mathcal{F}) \to \Gamma_*(\mathcal{F})$ by considering one degree at a time. For the proof of the following proposition we need a well-known lemma:

Lemma 3.14 ([ML63], II.9.1)**.** *Let*

$$0 \to K \to B \to C \to 0$$

be an extension in an abelian category $\mathcal{A}$ (with enough injectives or projectives), corresponding to the Ext-class $x \in \mathrm{Ext}^1(C, K)$. The boundary map $\mathrm{Ext}^k(T, C) \to \mathrm{Ext}^{k+1}(T, K)$ of the long exact sequence for Ext-groups out of T equals right multiplication by x. Similarly, the boundary map $\mathrm{Ext}^k(K, T) \to \mathrm{Ext}^{k+1}(C, T)$ of the sequence for Ext-groups into T equals left multiplication by x.

Proposition 3.15. *Let E be a standard vector bundle on $\mathcal{M}$. Let furthermore $x \in \Gamma_*(E)$ be an element not in the image of $r\colon \Gamma_*(p^*E) \to \Gamma_*(E)$. Then there is a $z \in \Gamma_*(p^*E)$ such that $c(r(z) + x)$ is a generator of a direct summand of $\Gamma_*(p^*E)$ over $TMF(2)_*$.*

Proof. First, suppose we have shown the proposition for two vector bundles E_1 and E_2. Let now $E = E_1 \oplus E_2$ and $x \in \Gamma_*(E)$ outside $\mathrm{im}(r)$. We can write $x = (x_1, x_2)$ and get that $c(r(z_1) + x_1) = y_1$ or $c(r(z_2) + x_2) = y_2$ is a generator of a direct summand of $\Gamma(p^*E_1)$ and $\Gamma(p^*E_2)$ respectively for some $z_i \in \Gamma(p^*E_i)$. Hence, $(y_1, y_2) = c(r(z_1, z_2) + (x_1, x_2))$ is a generator of a direct summand of $\Gamma(p^*E)$ as well. Therefore, we can assume E in our proposition to be indecomposable.

According to Theorem 3.10, every standard vector bundle on $\mathcal{M}$ is a direct sum of (indecomposable) vector bundles of the form $\mathcal{O}$, E_α and $f_*f^*\mathcal{O}$ and twists of these by ω^j. Here E_α denotes the extension

$$0 \to \mathcal{O} \to E_\alpha \to \omega^{-2} \to 0$$

classified by $\alpha \in H^1(\mathcal{M}; \omega^2)$ and $f\colon \mathcal{M}_0(2) \to \mathcal{M}$ is the usual projection map. It suffices to prove the proposition for each of the listed standard indecomposables.

- Consider the case $E = \mathcal{O}$: The image of r contains the ideal I in $\Gamma_*(\mathcal{O}) \cong \mathbb{Z}_{(A)}[c_4, c_6, \Delta^{\pm 1}]$ generated by 3, c_4 and c_6. Indeed, $cr(\frac{1}{2}) = 3$, $cr(4x_2^2) = c_4$ and $cr(-32x_2^2y_2) = c_6$ by the formulas for the action of S_3 on $\Gamma(p^*\mathcal{O}) \cong \mathbb{Z}_{(A)}[x_2, y_2, \Delta^{-1}]$ in Section 3.1. It follows that the $\pm\Delta^i$ form a set of representatives for the non-zero elements in $\Gamma_*(\mathcal{O})/I$. Since $\pm\Delta^i$ is a unit in $TMF(2)_*$ and hence generates a direct summand, the result follows.
- Consider the case $E = f_*f^*\mathcal{O}$: The stack $\mathcal{M}(2) \times_{\mathcal{M}} \mathcal{M}_0(2)$ classifies elliptic curves with level-2-structure and choice of one point of exact order 2 and is hence equivalent to $\coprod^3 \mathcal{M}(2)$. This implies that the vector bundle p^*E has rank 3 and S_3 operates by interchanging the 3 factors simultaneously with the action on each factor. Since $c\colon \Gamma_*(E) \to \Gamma_*(p^*E)$ is an embedding with image $\Gamma_*(p^*E)^{S_3}$, every element in $\operatorname{im}(c)$ is of the form (a, ta, t^2a) (with respect to the above decomposition) with $t = (2\ 3\ 1) \in S_3$ and $a \in \Gamma_*(p^*\mathcal{O})^{C_2}$ (with respect to the C_2-action given by the involution $(1\ 3\ 2)$). Since the morphism $\mathcal{M}(2) \to \mathcal{M}_0(2)$ (corresponding to the choice of the first point of exact order 2) is C_2-Galois, $\Gamma_*(f_*f^*\mathcal{O}) \cong \Gamma_*(p^*\mathcal{O})^{C_2}$ and we can view a as an element in $\Gamma_*(E)$. Because $cr(\frac{1}{2}a, 0, 0) = (a, ta, t^2a)$ for $a \in \Gamma_*(f^*\ OO)$, the image of c is contained in the image of cr and r is surjective. Thus, an $x \notin \operatorname{im}(r)$ as in the statement of the proposition does not exist.
- Consider the case $E = E_\alpha$: The short exact sequence

$$0 \to \mathcal{O} \to f_*f^*\mathcal{O} \xrightarrow{\sigma} E_\alpha \otimes \omega^{-2} \to 0 \tag{3.16}$$

induces a diagram of the form

$$\begin{array}{ccccc}
H^0_*(\mathcal{M}; f_*f^*\mathcal{O}) & \xrightarrow{\sigma} & H^0_*(\mathcal{M}; E_\alpha \otimes \omega^{-2}) & \xrightarrow{\partial} & H^1_*(\mathcal{M}; \mathcal{O}) \\
\uparrow{\scriptstyle r^{(1)}} & & \uparrow{\scriptstyle r^{(2)}} & & \\
H^0_*(\mathcal{M}(2); p^*f_*f^*\mathcal{O}) & \longrightarrow & H^0_*(\mathcal{M}(2); p^*E_\alpha \otimes \omega^{-2}) & \longrightarrow & H^1_*(\mathcal{M}(2); p^*\mathcal{O}) = 0
\end{array}$$

First observe that $\operatorname{im} r^{(2)} = \operatorname{im}\sigma$ since both $r^{(1)}$ and the lower horizontal map are surjective. By Lemma 3.14, ∂ equals multiplication with the element ${}^t\widetilde{\alpha} \in \operatorname{Ext}^1(E_\alpha \otimes \omega^{-2}, \mathcal{O})$ classifying (3.16). Because $\operatorname{Ext}^2(\omega^{-4}, \mathcal{O}) = 0$, the map

$$\operatorname{Ext}^1(E_\alpha \otimes \omega^{-2}, \mathcal{O}) \to \operatorname{Ext}^1(\omega^{-2}, \mathcal{O})$$

must be surjective; this means that the image of ${}^t\widetilde{\alpha}$ must equal $\pm\alpha \in H^1(\mathcal{M}; \omega^{-2})$. Thus, $\partial(u\Delta^i) = \pm u\alpha\Delta^i$ for $u \in \{0, 1, 2\}$, where we use the convention that we denote an element in $H^*_*(\mathcal{M}, \mathcal{O})$ and its image under the map in $H^*_*(\mathcal{M}, E_\alpha)$ induced by the defining map $\mathcal{O} \to E_\alpha$ by the same letter. Hence, the $u\Delta^i$ are a representing set for $\operatorname{coker}(\sigma) \cong H^0(\mathcal{M}; E_\alpha \otimes \omega^{-2})/\operatorname{im}(r^{(2)})$. Thus, for every $x \in \Gamma_*(E)$ not in $\operatorname{im}(r^{(2)})$, we can find an $r^{(2)}(z)$ such that $x + r^{(2)}(z) = u\Delta^i$ with u a unit. We have an exact sequence

$$0 \to \Gamma_*(p^*\mathcal{O}) \to \Gamma_*(p^*E_\alpha) \to \Gamma_*(p^*\omega^{-2}) \to 0$$

since $H^1_*(\mathcal{M}; p^*\mathcal{O}) = 0$ and it splits since $\Gamma_*(p^*\omega^{-2})$ is free over $TMF(2)_*$. Thus, $u\Delta^i$ is a generator of a direct summand of $\Gamma_*(p^*E_\alpha)$. This implies the proposition.

□

Scholium 3.17. *For $E = \mathcal{O}$ or E_α, the cokernel of $r\colon \Gamma_*(p^*E) \to \Gamma_*(E)$ is an $\mathbb{F}_3$-vector space and the elements Δ^i, $i \in \mathbb{Z}$, form a basis. For $E = E_{\alpha,\widetilde{\alpha}}$, this cokernel is 0.*

Proof. Since $rc = 6$, we have $3\Gamma_*(E) \subset \mathrm{im}(r)$ and $\mathrm{coker}(r)$ is an $\mathbb{F}_3$-vector space. That the elements Δ^i generate $\mathrm{coker}(r)$ follows from the proof above. To show that the Δ^i are non-zero observe that $\Delta^i \in \Gamma_*(\mathcal{O})$ cannot be in $\mathrm{im}(r)$ since $\beta \in H^2_*(\mathcal{M};\mathcal{O})$ operates non-trivially on it and for the same reason $\Delta^i \in \Gamma_*(E_\alpha)$ cannot be in $\mathrm{im}(r)$. The surjectivity of r in the case $E = f_*f^*\mathcal{O}$ is also contained in the proof above. □

Lemma 3.18. *Consider the map*

$$\sigma_\alpha\colon\ \Gamma_*(f_*f^*\mathcal{O} \otimes E_\alpha) \to \Gamma_*(\omega^{-2} \otimes E_\alpha \otimes E_\alpha).$$

By [Mei15, Prop 4.13]*, we have $E_\alpha \otimes E_\alpha \cong f_*f^*\mathcal{O} \oplus \omega^{-2}$. Using this identification, the cokernel of σ_α is an $\mathbb{F}_3$-vector space with basis $(0, \Delta^i)_{i\in\mathbb{Z}}$.*

Proof. The $(0, \Delta^i) \in \Gamma_*(f_*f^*\mathcal{O} \oplus \omega^{-2})$ span a representing set for $\Gamma_*(E_\alpha \otimes E_\alpha)/\ker(\alpha)$. Furthermore, α operates injectively on $H^1_*(\mathcal{M}, E_\alpha)$. Indeed, the extension

$$0 \to E_\alpha \to f_*f^*\mathcal{O} \to \omega^{-4} \to 0$$

is classified by $\widetilde{\alpha} \in H^1(\mathcal{M}; E_\alpha \otimes \omega^4)$. Since $H^1_*(\mathcal{M}; f_*f^*\mathcal{O}) = 0$, multiplication by $\widetilde{\alpha}$ acts injectively on α and, thus, α injectively on $\widetilde{\alpha}$.

As multiplication by α commutes with δ, we have $\ker(\alpha) \subset \ker(\partial) = \mathrm{im}(\sigma_\alpha)$ for the boundary map

$$\partial\colon\ H^0_*(\mathcal{M}; \omega^{-2} \otimes E_\alpha \otimes E_\alpha) \to H^1_*(\mathcal{M}; E_\alpha).$$

Since the restriction of $\alpha\cdot\colon\ H^0_*(\mathcal{M};\mathcal{O}) \to H^1_*(\mathcal{M};\mathcal{O})$ to the span of the Δ^i is surjective, the $(0, \Delta^i)$ generate therefore the cokernel of σ_α (as an $\mathbb{F}_3$-vector space). Since the next term $H^1(\mathcal{M}; f_*f^*\mathcal{O} \otimes E_\alpha)$ in the sequence is zero, ∂ is surjective. Therefore, $\mathrm{coker}(\sigma_\alpha)$ has the same dimension as an $\mathbb{F}_3$-vector space as the span of the Δ^i. Therefore, the Δ^i form a basis for $\mathrm{coker}(\sigma_\alpha)$. □

4. Topological preliminaries

4.1. TMF **and sheaves of modules.** By a theorem of Goerss, Hopkins and Miller there is sheaf $\mathcal{O}^{top}$ of E_∞-ring spectra on the etale site of $\mathcal{M}_{ell}$ with $\pi_{2n}\mathcal{O}^{top} \cong \omega^{\otimes n}$ and $\pi_{2n-1}\mathcal{O}^{top} = 0$ for alle $n \in \mathbb{Z}$ [DFHH14]. Define TMF as the global sections $\Gamma(\mathcal{O}^{top})$. Likewise, we define $TMF(n) = \mathcal{O}^{top}(\mathcal{M}(n))$ and $TMF_0(2) = \mathcal{O}^{top}(\mathcal{M}_0(2))$.

The pair $(\mathcal{M}_{ell}, \mathcal{O}^{top})$ is an example of nonconnective spectral Deligne–Mumford stack in the sense of Lurie. For us, this will mean an ordinary Deligne–Mumford stack $\mathcal{X}$ with a (hypercomplete) sheaf of E_∞-ring spectra $\mathcal{O}^{top}$ on the etale site of $\mathcal{X}$ such that $\pi_i\mathcal{O}^{top}$ is quasi-coherent for all $i \in \mathbb{Z}$. Here and in the following, $\pi_i\mathcal{F}$ for a sheaf $\mathcal{F}$ of spectra always denotes the *sheafification* of the naive presheaf of homotopy groups.

Let now $(\mathcal{X}, \mathcal{O}^{top})$ be a spectral Deligne–Mumford stack. Denote the global sections of $\mathcal{O}^{top}$ by R. In this situation, the global sections functor

$$\Gamma\colon\ \mathrm{QCoh}(\mathcal{X}, \mathcal{O}^{top}) \to R\text{-mod}$$

from the ∞-category of quasi-coherent sheaves has a (symmetric monoidal) left adjoint

$$M \mapsto \mathcal{F}_M = \mathcal{O}^{top} \wedge_R M.$$

This left adjoint is the unique colimit-preserving functor from R-mod to $\mathrm{QCoh}(\mathcal{X}, \mathcal{O}^{top})$ that sends R to $\mathcal{O}^{top}$. The adjunction can also be seen as the (π^*, π_*)-adjunction for $\pi\colon (\mathcal{X}, \mathcal{O}^{top}) \to \operatorname{Spec} R$. This perspective is one way to show that for every etale map $U \to \mathcal{X}$ from an *affine* scheme, we have $\mathcal{F}_M(U) \simeq \mathcal{O}^{top}(U) \wedge_R M$. A detailed account of the theory of spectral Deligne–Mumford stacks and quasi-coherent sheaves on them can be found in [Lur17].

In the case of $(\mathcal{M}_{ell}, \mathcal{O}^{top})$, it was shown in [MM15] that

$$\Gamma\colon \operatorname{QCoh}(\mathcal{M}_{ell}, \mathcal{O}^{top}) \to TMF\text{-mod}$$

is a monoidal equivalence of ∞-categories.

The homotopy groups of connective tmf were computed in [Bau08]. We obtain the homotopy groups of TMF by inverting a power of Δ that is a permanent cycle. After inverting 2 the smallest such power is Δ^3 so that $\pi_*TMF[\frac{1}{2}]$ is 72-periodic. A schematic picture of $\pi_*TMF[\frac{1}{2}]$ can be found in Section 7.4.

The following two lemmas are often useful:

Lemma 4.1. *Let $(\mathcal{X}, \mathcal{O}^{top})$ be a (nonconnective) spectral Deligne–Mumford stack and M and N be $R = \mathcal{O}^{top}(\mathcal{X})$-modules such that $\mathcal{F}_M$ is locally free. Then there is an isomorphism of sheaves*

$$\pi_*\mathcal{F}_{M\wedge_R N} \cong (\pi_*\mathcal{F}_M) \otimes_{\pi_*\mathcal{O}^{top}} (\pi_*\mathcal{F}_N).$$

Proof. There is a natural map

$$(\pi_*\mathcal{F}_M) \otimes_{\pi_*\mathcal{O}^{top}} (\pi_*\mathcal{F}_N) \to \pi_*(\mathcal{F}_M \wedge_{\mathcal{O}^{top}} \mathcal{F}_N).$$

As $\mathcal{F}_M$ is locally free, this map is locally and hence globally an isomorphism. As the left adjoint of Γ is symmetric monoidal, the result follows. □

Lemma 4.2. *Let $(\mathcal{X}, \mathcal{O}^{top})$ be a site equipped with a sheaf of ring spectra and let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}^{top}$-modules. Then the presheaf of spectra defined by $\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) := \operatorname{Hom}_{\mathcal{O}^{top}|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ is already a sheaf.*

Proof. We claim that it is enough to show this for the Hom spaces instead of the Hom-spectra. Indeed, a diagram in spectra is a homotopy limit diagram iff the diagram of $\tau_{\geq n}$-truncations is a homotopy limit diagram in n-connective spectra for all $n \leq 0$ (as can be seen on homotopy groups). A diagram in n-connective spectra is a limit diagram iff the diagram of n-th spaces of the corresponding Ω-spectra is a homotopy limit diagram of spaces (as equivalences between n-connective spectra are detected on the n-th space). Now note that the n-th space of the Ω-spectrum of $\operatorname{Hom}_{\mathcal{O}^{top}|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ is the homomorphism mapping space $\operatorname{Map}_{\mathcal{O}^{top}|_U}(\Sigma^{-n}\mathcal{F}|_U, \mathcal{G}|_U)$.

By [Lur09a, Remark 2.1.11], the construction $(U \in \mathcal{X}) \mapsto \mathcal{O}^{top}|_U$-mod defines a sheaf on $\mathcal{X}$ with values in the ∞-category of ∞-categories. Analogously to [Lur09c, 1.2.13.8], the forgetful functor from ∞-categories under $\Delta^0 \sqcup \Delta^0$ to ∞-categories detects limits. Let I be the ∞-category under $\Delta^0 \sqcup \Delta^0$ given by the inclusion $\Delta^0 \sqcup \Delta^0 \hookrightarrow \Delta^1$ of end points. Then, for an arbitrary ∞-category $\mathcal{C}$ together with a morphism $\Delta^0 \sqcup \Delta^0 \xrightarrow{(X,Y)} \mathcal{C}$, the space of morphisms $I \to \mathcal{C}$ under $\Delta^0 \sqcup \Delta^0$ is equivalent to the space of morphisms from X to Y in $\mathcal{C}$. Thus,

$$\operatorname{Map}_{\mathcal{O}^{top}|_U}(\mathcal{F}|_U, \mathcal{G}|_U) \simeq \operatorname{Map}_{\Delta^0 \sqcup \Delta^0}(I, \mathcal{O}^{top}|_U\text{-mod}).$$

defines a sheaf. □

4.2. **Descent spectral sequence.** Let $(\mathcal{X}, \mathcal{O}^{top})$ be a (nonconnective) spectral Deligne–Mumford stack with $R = \Gamma(\mathcal{O}^{top})$. Assume throughout that $\mathcal{X}$ is quasi-compact and has an affine diagonal (which is equivalent to $\mathcal{X}$ be separated). Given a quasi-coherent $\mathcal{O}^{top}$-module $\mathcal{F}$, we construct a descent spectral sequence

$$H^p(\mathcal{X}, \pi_q\mathcal{F}) \Rightarrow \pi_{q-p}\Gamma(\mathcal{F})$$

as follows: Choose an etale cover $U = \operatorname{Spec} A \to \mathcal{X}$. Define a cosimplicial object $D^\bullet$ by $D^n = \mathcal{F}(U^{\times_{\mathcal{X}} n+1})$. The *descent spectral sequence* DSS($\mathcal{F}$) is the associated Bousfield–Kan spectral sequence and converges to

$$\pi_* \operatorname{holim}_\Delta M^\bullet \cong \pi_*\Gamma(\mathcal{F})$$

as $\mathcal{F}$ is a sheaf. The E_2-term can be identified as in the article by Douglas in [DFHH14]. Our goal is to identify this descent spectral sequence with an Adams–Novikov spectral sequence in certain cases. Much of the following is known to experts.

First, let us recall something about Landweber exact spectra. Essentially by Quillen, the stack $\mathcal{M}_{FG}$ has a presentation by the Hopf algebroid (L, W), where L is the Lazard ring and $W \cong L[u^{\pm 1}, b_1, b_2, \dots]$. If we define a homology theory MUP by $MUP_*(X) = \mathbb{Z}[u^{\pm 1}] \otimes MU_*(X)$ with $|u| = 2$, then $L = MUP_0$ and $W = MUP_0MUP$. Thus, the MUP_*-homology of a spectrum E gets the structure of a graded (L, W)-comodule and hence an (even-periodic) graded quasi-coherent sheaf on $\mathcal{M}_{FG}$, called $(\mathcal{G}_E)_*$. If E is an even homotopy commutative ring spectrum, this a quasi-coherent sheaf of algebras, whose Spec is an algebraic stack $\mathcal{M}_E$ with an affine map to $\mathcal{M}_{FG}$.

Let now $f : \operatorname{Spec} A \to \mathcal{M}_{FG}$ be a flat map into the moduli stack of formal groups. Then the functor h^f_* from spectra to abelian groups given by

$$X \mapsto (\mathcal{G}_X)_*(\operatorname{Spec} A)$$

is a homology theory. The reason is that $X \mapsto (\mathcal{G}_X)_*$ is a homology theory with values in $\operatorname{QCoh}(\mathcal{M}_{FG})$ and f^* preserves exact sequences since f is flat. We call such a homology theory h^f_* and its representing spectrum *Landweber exact*.

Lemma 4.3. *For F Landweber exact associated to $f : \operatorname{Spec} A \to \mathcal{M}_{FG}$ and E any spectrum, we have $\pi_*(F \wedge E) \cong (\mathcal{G}_E)_*(\operatorname{Spec} A)$.*

Proof. This is true by definition. □

Assume in the following that $(\mathcal{X}, \mathcal{O}^{top})$ satisfies the following conditions:

1. $\mathcal{X}$ is noetherian and separated,
2. there is an affine and flat map $f : \mathcal{X} \to \mathcal{M}_{FG}$,
3. the sheaf $\mathcal{O}^{top}$ is a refinement of the induced presheaf of Landweber exact homology theories on the affine etale site of $\mathcal{X}$.

By the main result of [MM15], this implies that

$$\Gamma\colon\ \operatorname{QCoh}(\mathcal{X}, \mathcal{O}^{top}) \to R\text{-mod}$$

and its left adjoint $M \mapsto \mathcal{F}_M$ are adjoint equivalences of symmetric monoidal ∞-categories, i.e. that $(\mathcal{X}, \mathcal{O}^{top})$ is 0-afine in the terminology of [MM15]. As checked in [MM15], $(\mathcal{M}_{ell}, \mathcal{O}^{top})$ is an example of such a stack $(\mathcal{X}, \mathcal{O}^{top})$ and likewise all the derived moduli stacks of elliptic curves with level structure.

Proposition 4.4. *Let $\operatorname{Spec} A \to \mathcal{M}_{FG}$ be a flat morphism and E be the associated Landweber exact spectrum. Assume that $\mathcal{Y} = \mathcal{X} \times_{\mathcal{M}_{FG}} \operatorname{Spec} A$ is affine. Then we have a natural isomorphism*

$$\pi_*(N \wedge E) \cong (\pi_*\mathcal{F}_N)(\mathcal{Y})$$

for every R-module N.

Proof. We denote the map $\mathcal{Y} \to \mathcal{X}$ by h. We will first consider the case $\mathcal{F}_N = \mathcal{O}^{top}$. By [MM15, Proposition 2.14], we have for each etale map $\operatorname{Spec} B \to \mathcal{X}$ a natural isomorphism

$$\pi_*\mathcal{O}^{top}(\operatorname{Spec} B) \wedge E \cong h_*h^*(\pi_*\mathcal{O}^{top})(\operatorname{Spec} B).$$

Thus, $\pi_*(\mathcal{O}^{top} \wedge E) \cong h_*h^*(\pi_*\mathcal{O}^{top})$.

In the general case,

$$\begin{aligned}\pi_*(\mathcal{F}_N \wedge E) &\cong \pi_*(\mathcal{F}_N \wedge_{\mathcal{O}^{top}} (\mathcal{O}^{top} \wedge E)) \\ &\cong \pi_*(\mathcal{F}_N) \otimes_{\pi_*\mathcal{O}^{top}} h_*h^*\pi_*\mathcal{O}^{top} \\ &\cong h_*h^*\pi_*(\mathcal{F}_N)\end{aligned}$$

Here, we use that $h_*h^*\pi_*\mathcal{O}^{top}$ is a flat $\pi_*\mathcal{O}^{top}$-module and the projection formula.

As Γ and $M \mapsto \mathcal{F}_M$ are inverse equivalences,

$$\Gamma(\mathcal{F}_N \wedge E) \simeq \Gamma(\mathcal{F}_{N \wedge E}) \simeq N \wedge E.$$

Thus, we have a descent spectral sequence

$$H^p(\mathcal{X}; h_*h^*\pi_q\mathcal{F}_N) \Rightarrow \pi_{q-p}N \wedge_R E.$$

Because the diagonal of $\mathcal{M}_{FG}$ is affine, $\operatorname{Spec} A \to \mathcal{M}_{FG}$ is affine and hence also h. By a degenerate Leray spectral spectral sequence

$$H^p(\mathcal{X}; h_*h^*\pi_*\mathcal{F}_N) \cong H^p(\mathcal{Y}; h^*\pi_*\mathcal{F}_N).$$

Because $\mathcal{Y}$ is affine, this is concentrated in degree $p = 0$ and agrees there with $(\pi_*\mathcal{F}_N)(\mathcal{Y})$. □

Example 4.5. We have

$$\mathcal{M}_{MUP} \simeq \operatorname{Spec} L$$

and more generally

$$\mathcal{M}_{MUP^{\wedge n}} \simeq \operatorname{Spec} L^{\times_{\mathcal{M}_{FG}} n}.$$

As these are affine schemes, we get

$$\pi_*N \wedge MUP \cong (\pi_*\mathcal{F}_N)(\operatorname{Spec} L \times_{\mathcal{M}_{FG}} \mathcal{X})$$

and

$$\pi_*N \wedge MUP^{\wedge n} \cong (\pi_*\mathcal{F}_N)(\operatorname{Spec} L^{\times_{\mathcal{M}_{FG}} n} \times_{\mathcal{M}_{FG}} \mathcal{X})$$

for every R-module N.

We have the following useful identification (with similar results in [Mat13]):

Theorem 4.6. *Let $(\mathcal{X}, \mathcal{O}^{top})$ and $R = \Gamma(\mathcal{O}^{top})$ be as above and let M be an R-module. Then the descent spectral sequence for $\mathcal{F}_M$ is isomorphic to the MU-Adams–Novikov spectral sequence (ANSS) for M.*

Proof. The MU-based ANSS is isomorphic to the MUP-based ANSS because the graded Hopf algebroids (MU_*, MU_*MU) and (MUP_*, MUP_*MUP) define equivalent stacks (namely $\mathcal{M}_{FG}$). To explain the idea of the proof, set

$$V = \mathcal{X} \times_{\mathcal{M}_{FG}} \operatorname{Spec} L$$

where L denotes the Lazard ring. The MUP-Adams–Novikov spectral sequence looks like a descent spectral sequence for the fpqc cover $V \to \mathcal{X}$, but we have to express the non-existent $\mathcal{O}^{top}(V)$ by $R \wedge MUP$. Choose now an etale cover $U \to \mathcal{X}$ by an affine scheme U.

Define a cosimplicial objects

- $A^\bullet$ by $A^n = M \wedge MUP^{\wedge n+1}$,
- $D^\bullet$ by $D^n = \mathcal{F}_M(U^{\times}_{\mathcal{X}n+1}) \simeq \mathcal{O}^{top}(U^{\times_{\mathcal{X}} n+1}) \wedge_R M$
- $AD^\bullet$ by $AD^n = D^n \wedge MUP^{\wedge n+1}$.

In the language above, $A^\bullet$ corresponds to the cover $V \to \mathcal{X}$ and $AD^\bullet$ to the cover $U \times_{\mathcal{X}} V \to \mathcal{X}$. There is a map $D^\bullet \to AD^\bullet$ induced by the unit maps $\mathbb{S} \to MUP^{\wedge n+1}$. Furthermore, there is a map $A^\bullet \to AD^\bullet$ induced by the map $\mathrm{const}_M \to D^\bullet$, which in turn is induced by the maps $U^{\times_{\mathcal{X}} n+1} \to \mathcal{X}$. We have to show that the maps $D^\bullet \to AD^\bullet$ and $A^\bullet \to AD^\bullet$ induce isomorphisms on the E_2-terms of the associated Bousfield–Kan spectral sequences.

By Example 4.5,

$$\pi_* A^n \cong (\pi_*\mathcal{F}_M)(\mathrm{Spec}\, L^{\times_{\mathcal{M}_{FG}} n+1} \times_{\mathcal{M}_{FG}} \mathcal{X}) \cong (\pi_*\mathcal{F}_M)(V^{\times_{\mathcal{X}} n+1}).$$

The E_2-term of the associated Bousfield-Kan spectral sequence is isomorphic to $H^p(\mathcal{X};\pi_q\mathcal{F}_M)$ because we can compute the cohomology via the fpqc cover $V \to \mathcal{X}$.

Likewise,

$$\pi_* AD^n \cong (\pi_*\mathcal{F}_M)(\mathrm{Spec}\, L^{\times_{\mathcal{M}_{FG}} n+1} \times_{\mathcal{M}_{FG}} U^{\times_{\mathcal{X}} n+1}) \cong (\pi_*\mathcal{F}_M)((V \times_{\mathcal{X}} U)^{\times_{\mathcal{X}} n+1}).$$

The E_2-term of the associated Bousfield-Kan spectral sequence is isomorphic to $H^p(\mathcal{X};\pi_q\mathcal{F}_M)$ because we can compute the cohomology via the fpqc cover $V \times_{\mathcal{X}} U \to \mathcal{X}$.

It is easy to check that the morphisms actually induce isomorphisms of E_2-terms. □

Theorem 4.7. *With notation as above, let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module. Then $DSS(\mathcal{F})$ possesses the structure of a module spectral sequence over $DSS(\mathcal{O}^{top})$ which induces the canonical module action of the E^2-terms.*

Proof. This follows from Theorem 4.6 and [Rav86, Thm 2.3.3]. □

Theorem 4.8. *With notation as above, let*

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} \Sigma W$$

be a cofiber sequence of finite R-modules. Assume that the induced map $h\colon \pi_*\mathcal{F}_Y \to \pi_{*-1}\mathcal{F}_W$ is zero (π_* denotes here again the sheafified homotopy groups). Then we have a map of spectral sequences $DSS(\mathcal{F}_Y) \to DSS(\mathcal{F}_W)$ (raising filtration by 1) which converges to h and induces multiplication by the class in $\mathrm{Ext}^1_{\pi_*\mathcal{O}^{top}}(\pi_*\mathcal{F}_Y, \pi_*\mathcal{F}_W)$ corresponding to the extension

$$0 \to \pi_*\mathcal{F}_W \to \pi_*\mathcal{F}_X \to \pi_*\mathcal{F}_Y \to 0$$

on E^2.

Proof. This follows from Theorem 4.6 and [Rav86, Thm 2.3.4]. □

Corollary 4.9. *Let M be an R-module and $x \in \pi_k M$ be of filtration 1 for $\mathrm{DSS}(\mathcal{F}_M)$. Denote by N the cone of the map $\Sigma^k R \xrightarrow{x} M$. Then the extension*

$$0 \to \pi_*\mathcal{F}_M \to \pi_*\mathcal{F}_N \to \pi_*\Sigma^{k+1}\mathcal{O}^{top} \to 0$$

is classified by the reduction $\overline{x} \in \mathrm{Ext}^1_{\pi_*\mathcal{O}^{top}}(\pi_*\Sigma^{k+1}\mathcal{O}^{top}, \pi_*\mathcal{F}_M) \cong H^1(\mathcal{X};\pi_{k+1}\mathcal{F}_M)$.

Proof. The map $\Sigma^k R \to M$ sends $1 \in \pi_k\Sigma^k R$ to $x \in \pi_k M$. The map of descent spectral sequences $DSS(\mathcal{O}^{top}) \to DSS(\mathcal{F}_M)$ from Theorem 4.8 sends $\overline{1} \in H^0(\mathcal{X};\pi_*\Sigma^k\mathcal{O}^{top})$ to the class $y \in \mathrm{Ext}^1_{\pi_*\mathcal{O}^{top}}(\pi_*\Sigma^k\mathcal{O}^{top}, \pi_*\mathcal{F}_M) \cong H^1(\mathcal{X};\pi_k\mathcal{F}_M)$ classifying the extension above. As this map of spectral sequences converges to the map $\Sigma^k R \to M$, we have $\overline{x} = y$. □

4.3. **Certain TMF-modules.** Throughout, $\mathcal{O}^{top}$ refers to the Goerss–Hopkins–Miller sheaf of E_∞-ring spectra on $\mathcal{M} = \mathcal{M}_{ell}$ and $\mathcal{O}$ to the structure sheaf on $\mathcal{M}$.

The step from algebra to topology is rather easy for $TMF_1(n)$ and $TMF(n)$.

Proposition 4.10. *Let $\mathcal{F}$ and $\mathcal{G}$ be locally free $\mathcal{O}^{top}$-module of finite rank on $\mathcal{M}[\frac{1}{n}]$ for $n \geq 2$ and assume that $\pi_*\mathcal{G}$ is concentrated in even degrees. Let $h\colon \mathcal{X} \to \mathcal{M}[\frac{1}{n}]$ for $\mathcal{X} = \mathcal{M}_1(n)$ or $\mathcal{M}(n)$ be the projection. Then every morphism $g_{alg}\colon h_*h^*\pi_0\mathcal{G} \to \pi_0\mathcal{F}$ can be realized by a map*

$$g\colon h_*h^*\mathcal{G} \to \mathcal{F}$$

with $\pi_0 g = g_{alg}$ and this realization is unique in the homotopy category of $\mathcal{O}^{top}$-modules. The analogous statement is true for morphisms $\pi_0\mathcal{F} \to h_*h^*\mathcal{G}$.

Proof. As h is finite etale, we have by Section 3.3

$$\mathcal{H}om_{\mathcal{O}}(h_*h^*\pi_0\mathcal{G}, \pi_k\mathcal{F}) \cong h_*\mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(h^*\pi_0\mathcal{G}, \pi_k\mathcal{F}).$$

By Lemma 3.5 the higher cohomology groups of $h_*\mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(h^*\pi_0\mathcal{G}, \pi_k\mathcal{F})$ vanish. Since $h_*h^*\mathcal{G}$ is locally free, we have

$$\pi_k\mathcal{H}om_{\mathcal{O}^{top}}(h_*h^*\mathcal{G}, \mathcal{F}) \cong \mathcal{H}om_{\pi_0\mathcal{O}^{top}}(h_*h^*\pi_0\mathcal{G}, \pi_k\mathcal{F})$$

(see Lemma 4.2 for the definition of the $\mathcal{H}om$-sheaf). Hence, the descent spectral sequence for

$$\mathcal{H}om_{\mathcal{O}^{top}}(h_*h^*\mathcal{G}, \mathcal{F})$$

is concentrated in the 0-line. Therefore, there is a (up to homotopy) a unique map

$$g\colon h_*h^*\mathcal{G} \to \mathcal{F}$$

realizing the algebraic map g_{alg}.

The proof for a map $\pi_0\mathcal{F} \to h_*h^*\mathcal{G}$ is the same using that

$$\mathcal{H}om_{\mathcal{O}}(\pi_k\mathcal{F}.h_*h^*\pi_0\mathcal{G}) \cong h_*\mathcal{H}om_{\mathcal{O}}(h^*\pi_k\mathcal{F}.h^*\pi_0\mathcal{G})$$

□

We want to combine this with the following lemma:

Lemma 4.11 ([Mei12], Lemma 3.5.4)**.** *For $p\colon \mathcal{M}(2) \to \mathcal{M}[\frac{1}{2}]$ the projection,*

$$p_*p^*\mathcal{O} \cong f_*f^*\mathcal{O} \oplus f_*f^*\mathcal{O} \otimes \omega^2.$$

This lemma is stated in the reference only 3-locally, but the proof gives actually the result just stated.

Corollary 4.12. *We have $TMF(2) \simeq TMF_0(2) \oplus \Sigma^4 TMF_0(2)$ as TMF-modules.*

The following is proven in [Mei16]:

Proposition 4.13. *We have $TMF(3)_{(2)} \simeq \bigoplus_{i=0}^5 \Sigma^{2i} TMF_1(3)_{(2)}$.*

We remark though that the existence of a splitting of $TMF(3)$ into shifts of $TMF_1(3)$ follows also rather directly from $TMF(3)$ being a $TMF_1(3)$-module and the fact that all vector bundles on $\mathcal{M}_1(3)_{(2)}$ are sums of $\omega^{\otimes i}$ by [MR09, Section 3] and [Mei15, Thm 3.9] (which applies in particular to the pushforward of $\mathcal{O}_{\mathcal{M}(3)_{(2)}}$ along $\mathcal{M}(3)_{(2)} \to \mathcal{M}_1(3)_{(2)}$).

Proposition 4.14. *Consider an etale surjective morphism $h\colon \mathcal{X} \to \mathcal{M}_{ell}[\frac{1}{n}]$. Then $\mathcal{O}^{top}(\mathcal{X})$ is a faithful $TMF[\frac{1}{n}]$-module. This applies in particular to $TMF_1(n)$ and $TMF(n)$.*

Proof. Let M be a nonzero $TMF[\frac{1}{n}]$-module. Then $\mathcal{O}^{top}(\mathcal{X}) \wedge_{TMF[\frac{1}{n}]} M \simeq \mathcal{F}_M(\mathcal{X})$. As M is nonzero and $\Gamma\mathcal{F}_M \simeq M$ by Section 4.1, $\mathcal{F}_M$ is nonzero and thus $\mathcal{F}_M(\mathcal{X})$ as well as $\mathcal{X} \to \mathcal{M}_{ell}$ is an etale cover. □

To compute the homotopy groups of cofibers of maps of TMF-modules, we need the following well-known lemma.

Lemma 4.15. *Let $y \in \pi_m R$ be an A_∞-ring spectrum and $x \in \pi_n R$, $y \in \pi_l R$ and $z \in \pi_k R$ be elements with $xy = 0$ and $yz = 0$. Denote by Cx the cofiber of the R-linear mapped*

$$\Sigma^n R \xrightarrow{x} R.$$

Let furthermore $\widetilde{y} \in \pi_* Cx$ be an element with $\beta(\widetilde{y}) = y$ and $w \in \pi_* R$ be an element such that the projection of w is mapped to $\widetilde{y}z$ under β. Then $w \in \langle x, y, z \rangle$.

Proof. This is clear by the following diagram:

$$\begin{array}{ccccccc}
\Sigma^{k+l+n}R & \xrightarrow{z} & \Sigma^{k+l}R & & & & \\
\vdots & & \downarrow{=} & & & & \\
\vdots & & \Sigma^{k+l}R & \xrightarrow{y} & \Sigma^k R & & \\
\downarrow & & \downarrow{\widetilde{y}} & & \downarrow{=} & & \\
\Sigma^{-1}R & \longrightarrow & Cx & \longrightarrow & \Sigma^k R & \xrightarrow{x} & R
\end{array}$$

□

We will invert implicitly the prime 2 for the rest of this section so that TMF means $TMF[\frac{1}{2}]$ etc. Define TMF_α to be the cofiber of

$$\alpha \colon \Sigma^3 TMF \to TMF.$$

The homotopy groups of TMF_α are easily computed using the long exact sequence of homotopy groups. For example, as $\alpha^2 = 0$, the class $\alpha \in \pi_7 \Sigma^4 TMF$ lifts to $\widetilde{\alpha} \in \pi_7 TMF_\alpha$. The action of $\pi_* TMF$ on $\pi_* TMF_\alpha$ follows from Lemma 4.15 and the result can be found in Section 7.4.

Observe that $D_{TMF} TMF_\alpha \simeq \Sigma^{-4} TMF_\alpha$. We denote the map

$$\Sigma^{-4} TMF_\alpha \simeq D_{TMF} TMF_\alpha \to D_{TMF} \Sigma^7 TMF \simeq \Sigma^{-7} TMF,$$

dual to $\widetilde{\alpha}$, by ${}^t\widetilde{\alpha}$.

Lemma 4.16. *The compositions*

$$\Sigma^{10} TMF \xrightarrow{\widetilde{\alpha}} \Sigma^3 TMF_\alpha \xrightarrow{{}^t\widetilde{\alpha}} TMF$$

and

$$\Sigma^{10} TMF_\alpha \xrightarrow{{}^t\widetilde{\alpha}} \Sigma^7 TMF \xrightarrow{\widetilde{\alpha}} TMF_\alpha$$

both equal (multiplication by) β.

Proof. We want to show that $\widetilde{\alpha} \circ \widetilde{\alpha}^t = m_\beta$, where m_β denotes multiplication by β. Since $\alpha\widetilde{\alpha} = \beta$ in $\pi_* TMF_\alpha$, we have the following commutative diagram:

$$\begin{array}{ccccc}
TMF & & \overset{\beta}{\searrow} & & \\
\downarrow & & & & \\
TMF_\alpha & \xrightarrow{{}^t\widetilde{\alpha}} & \Sigma^{-3} TMF & \xrightarrow{\widetilde{\alpha}} & \Sigma^{-10} TMF_\alpha
\end{array}$$

By mapping (over TMF) into TMF_α, the triangle

$$\Sigma^3 TMF \to TMF \to TMF_\alpha \to \Sigma^4 TMF$$

induces a triangle

$$\Sigma^{-4} TMF_\alpha \to \mathrm{Hom}_{TMF}(TMF_\alpha, TMF_\alpha) \to TMF_\alpha.$$

The diagram above shows that $\widetilde{\alpha} \circ {}^t\widetilde{\alpha} \in \pi_{10} \mathrm{Hom}_{TMF}(TMF_\alpha, TMF_\alpha)$ maps to β and so does multiplication by β. Therefore the difference $\widetilde{\alpha} \circ \widetilde{\alpha}^t - m_\beta$ comes from $\pi_{14} TMF_\alpha$. But $\pi_{14} TMF_\alpha = 0$ since $\pi_{14} TMF = 0$ and $\beta \in \pi_{10} TMF$ has non-trivial multiplication by α. Therefore $\widetilde{\alpha} \circ {}^t\widetilde{\alpha}$ equals multiplication by β.

Thus, we see that the composition

$$\Sigma^{10} TMF \xrightarrow{\widetilde{\alpha}} \Sigma^3 TMF_\alpha \xrightarrow{{}^t\widetilde{\alpha}} TMF \xrightarrow{\widetilde{\alpha}} \Sigma^{-7} TMF_\alpha$$

represents $\beta\widetilde{\alpha} \in \pi_{17} TMF_\alpha$. Since only $\beta \in \pi_{10} TMF$ is sent by $\widetilde{\alpha}\colon \Sigma^7 TMF \to TMF_\alpha$ to $\beta\widetilde{\alpha} \in \pi_{17} TMF_\alpha$, we see that ${}^t\widetilde{\alpha} \circ \widetilde{\alpha} = \beta$. □

Note that the modules TMF, TMF_α, $TMF_0(2)$ and their sums and shifts realize all standard vector bundles in the sense of Section 3.2 by TMF-modules. In particular, they are relatively free in the sense that their associated (graded) quasi-coherent sheaf on $\mathcal{M}$ is a vector bundle. Examples as in Section 7.4 show that the realization of a standard vector bundle by a TMF-module is far from unique. The next two sections will study how much we can say about a relatively free TMF-module when we just know that its associated vector bundle is standard.

5. From relatively free TMF-Modules to hook modules

In this section, we will define and investigate the concept of a relatively free $TMF_{(A)}$-module, where A is a set of primes. This will be mainly the content of the first subsection, where we will also show that every finite $TMF_{(A)}$-module has a length-2-resolution by relatively free modules. In the second subsection, we will discuss how one can build relatively free modules by killing torsion elements and generators. In the third subsection, we will discuss some low rank concrete examples. In the last (more technical) subsection, we investigate the relationship of killing torsion elements or generators with the class of standard vector bundles. The last two subsections work only at the prime 3.

5.1. **Relatively free TMF-modules.**

Definition 5.1. Let A be a set of primes. Then a $TMF_{(A)}$-module M is called *relatively free* if $\mathcal{F}_M$ is a locally free $\mathcal{O}^{top}_{(A)}$-module of finite rank. This rank is called the *rank* of M.

Here and in the following, an R-module is free if it is a sum of suspensions of R.

Proposition 5.2. *Let M be a $TMF_{(l)}$-module.*

1. *If $l \neq 2$, then M is relatively free if and only if $M \wedge_{TMF_{(l)}} TMF_0(2)_{(l)}$ is a free $TMF_0(2)_{(l)}$-module of finite rank. This happens if and only if $M \wedge_{TMF_{(l)}} TMF(2)_{(l)}$ is a free $TMF(2)_{(l)}$-module of finite rank.*
2. *If $l \neq 3$, then M is relatively free if and only if $M \wedge_{TMF_{(l)}} TMF_1(3)_{(l)}$ is a free $TMF_1(3)_{(l)}$-module of finite rank.*
3. *If $l > 3$, then M is relatively free if and only if its a free $TMF_{(l)}$-module of finite rank.*

Proof. Set $\mathcal{X} = \mathcal{M}(2)_{(l)}$, $\mathcal{M}_0(2)_{(l)}$, $\mathcal{M}_1(3)_{(l)}$ or $\mathcal{M}_{(l)}$ for l as above. These are weighted affine lines in the sense of [Mei15, Def 3.8]; for $\mathcal{M}(2)$ and $\mathcal{M}_0(2)$ this is discussed in Section 3.1 and for $\mathcal{M}_1(3)$ this follows from [MR09]. Thus, every vector bundle on $\mathcal{X}$ is a direct sum of tensor powers of ω by [Mei15, Theorem 3.9].

Denote the map $\mathcal{X} \to \mathcal{M}_{(l)}$ by h. This is an etale cover. Thus, $\mathcal{F}_M$ is locally free of finite rank if and only if $h^*\mathcal{F}_M$ is locally free of finite rank. As $h^*\mathcal{F}_M$ is locally free if and only if $\pi_*h^*\mathcal{F}$ is locally free, we see that this happens if and only if $\pi_*h^*\mathcal{F}$ is isomorphic to $\bigoplus_{j\in J} \omega^{\otimes n_j}$. As the corresponding descent spectral sequence is concentrated in the 0-line, every such isomorphism can be realized by an equivalence

$$\bigoplus_{j\in J} \Sigma^{2n_j}\mathcal{O}^{top} \to h^*\mathcal{F}.$$

Such an equivalence exists if and only if $\Gamma(h^*\mathcal{F}_M)$ is a free $\mathcal{O}^{top}(\mathcal{X})$-module.

We have

$$\mathcal{O}^{top}(\mathcal{X}) \wedge_{TMF_{(l)}} M \simeq h_*h^*\mathcal{F}_M$$

by the projection formula. (This topological version follows directly from the algebraic one because equivalences of quasi-coherent $\mathcal{O}^{top}$-modules are detected on π_*.) Thus,

$$\Gamma(h^*\mathcal{F}_M) \simeq M \wedge_{TMF_{(l)}} \mathcal{O}^{top}(\mathcal{X})$$

as was to be shown. □

Lemma 5.3. *Every relatively free $TMF_{(A)}$-module is finite.*

Proof. Every locally free $\mathcal{O}^{top}$-module of finite rank is compact in $\mathrm{QCoh}(\mathcal{M}_{(A)}, \mathcal{O}^{top})$ and

$$\mathrm{QCoh}(\mathcal{M}_{(A)}, \mathcal{O}^{top}) \simeq TMF_{(A)}\text{-mod}.$$ □

There are different ways to build examples of relatively free $TMF_{(A)}$-modules. We will see more ways of constructing them in the next subsection, but we will already give one:

Proposition 5.4. *Let $h\colon \mathcal{X} \to \mathcal{M}_{(A)}$ be finite etale. Then $\mathcal{O}^{top}(\mathcal{X})$ is a relatively free $TMF_{(A)}$-module.*

Proof. The sheaf $h_*\pi_*\mathcal{O}_{\mathcal{X}} \cong \pi_*h_*\mathcal{O}_{\mathcal{X}}$ is locally free and hence also $h_*\mathcal{O}_{\mathcal{X}}$ itself. The result follows as $\Gamma(h_*\mathcal{O}_{\mathcal{X}}) \simeq \mathcal{O}^{top}(\mathcal{X})$ and thus $\mathcal{F}_{\mathcal{O}^{top}(\mathcal{X})} \simeq h_*\mathcal{O}^{top}$. □

The understanding of relatively free modules is key to the understanding of all $TMF_{(A)}$-modules.

Lemma 5.5. *For every finite $TMF_{(A)}$-module M, there exists a relatively free module P with a morphism $P \to M$ that induces a surjection $\pi_*\mathcal{F}_P \to \pi_*\mathcal{F}_M$.*

Proof. First we assume that there is a prime l not in A. We know by [MM15, Theorem 1.5] that $TMF_{(A)} \to TMF(l)_{(A)}$ is a $GL_2(\mathbb{Z}/l)$-Galois extension. By Galois descent, the functor

$$TMF_{(A)}\text{-mod} \xrightarrow{- \wedge_{TMF_{(A)}} TMF(l)_{(A)}} \widetilde{TMF(l)_{(A)}[GL_2}(\mathbb{Z}/l)]\text{-mod}$$

is an equivalence of ∞-categories. Here, we denote for R a ring spectrum with G-action by $\widetilde{R}[G]$ the *twisted group ring spectrum*; it is build in a way such that $\widetilde{R}[G]$-mod are equivalent to R-modules with semilinear G-action. See [Mei12, Chapter 6] for details.

In particular, there is a surjective $\pi_* \widetilde{TMF(l)_{(A)}[GL_2(\mathbb{Z}/l)]}$-linear morphism

$$\bigoplus_{j\in J} \Sigma^{n_j} \pi_* \widetilde{TMF(l)_{(A)}[GL_2(\mathbb{Z}/l)]} \to \pi_* M \wedge_{TMF_{(A)}} TMF(l)_{(A)}.$$

As the source is free, this can be realized by a $\widetilde{TMF(l)_{(A)}[GL_2(\mathbb{Z}/l)]}$-linear map

$$F\colon \bigoplus_{j\in J} \Sigma^{n_j} \widetilde{TMF(l)_{(A)}[GL_2(\mathbb{Z}/l)]} \to M \wedge_{TMF_{(A)} TMF(l)_{(A)}} .$$

By Galois descent, this is induced by a morphism

$$f\colon \bigoplus_{j\in J} \Sigma^{n_j} TMF(l)_{(A)} \to M.$$

As $\mathcal{F}_M(\mathcal{M}(l)_{(A)}) \simeq M \wedge_{TMF_{(A)}} TMF(l)_{(A)}$, we see that the morphism

$$\mathcal{F}_{TMF(l)_{(A)}}(\mathcal{M}(l)_{(A)}) \to \mathcal{F}_M(\mathcal{M}(l)_{(A)})$$

induced by f agrees with F and is hence a surjection on π_*; thus $\pi_*\mathcal{F}_{TMF(l)_{(A)}} \to \pi_*\mathcal{F}_M$ is a surjection as well. Furthermore, $TMF(l)_{(A)}$ is relatively free by Proposition 5.4.

The case where A is the set of all primes is slightly more difficult. There exist TMF-modules $T(2)$ and $T(3)$ with $T(2)[\frac{1}{2}] \simeq TMF(2)$ and $T(3)[\frac{1}{3}] \simeq TMF(3)$. Indeed, there is a 3-cell complex C_3 and an 8-cell complex C_8 such that

$$TMF_0(2) \simeq TMF[\frac{1}{2}] \wedge C_3$$

and

$$TMF_1(3) \simeq TMF[\frac{1}{3}] \wedge C_8.$$

By Proposition 4.13, $TMF(3)_{(2)}$ is a free $TMF_1(3)_{(2)}$-module. Likewise, by Corollary 5.21, $TMF(2) \simeq TMF_0(2) \oplus \Sigma^4 TMF_0(2)$. Thus,

$$TMF(2) \simeq TMF[\frac{1}{2}] \wedge C_6$$

and

$$TMF(3)_{(2)} \simeq TMF[\frac{1}{3}]_{(2)} \wedge C_{24}$$

where $C_6 = C_3 \vee \Sigma^4 C_3$ and C_{24} is a wedge of four suspensions of C_8. We define $T(2)$ as $TMF \wedge C_6$ and $T(3)$ as $TMF \wedge C_{24}$.

For a module M, choose morphisms $f_2\colon \bigoplus_{i\in I} \Sigma^{n_i} TMF(2) \to M[\frac{1}{2}]$ and $f_3\colon \bigoplus_{j\in J} \Sigma^{n_j} TMF(3) \to M[\frac{1}{3}]$ inducing surjections to $\pi_*\mathcal{F}_{M[\frac{1}{2}]}$ and $\pi_*\mathcal{F}_{M[\frac{1}{3}]}$. As the source is finite, we have

$$\left[\bigoplus_{i\in I} \Sigma^{n_i} TMF(2), M[\frac{1}{2}]\right]^{TMF[\frac{1}{2}]} \cong [\bigoplus_{i\in I} \Sigma^{n_i} T(2), M]^{TMF}[\frac{1}{2}].$$

Thus, we can replace f_2 by $2^n f_2$ and see that f_2 is induced by a TMF-linear map

$$g_2\colon \bigoplus_{i\in I} \Sigma^{n_i} T(2) \to M.$$

Likewise for f_3, we get a map $g_3\colon \bigoplus_{j\in J} \Sigma^{n_j} T(3) \to M$. The resulting map

$$g_2 + g_3\colon \bigoplus_{i\in I} \Sigma^{n_i} T(2) \oplus \bigoplus_{j\in J} \Sigma^{n_j} T(3) \to M$$

is the map we are looking for. Indeed, a map of abelian groups is a surjection if and only if it is a surjection after inverting 2 and after localizing at 2. □

Proposition 5.6. *For every finite $TMF_{(A)}$-module M, there is a length-2-resolution by relatively free modules. More precisely, there are relatively free $TMF_{(A)}$-modules P_0, P_1 and P_2 together with cofiber sequences*

$$P_2 \to P_1 \to \mathrm{hofib}(f)$$

$$\mathrm{hofib}(f) \to P_0 \xrightarrow{f} M$$

Proof. Every quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{M}_{(A)}$ has locally projective dimension at most 2. For example, this can be seen by observing that $\mathcal{M}(3)_{(A)}$ and $\mathcal{M}(4)_{(3)}$ are regular affine schemes with Krull dimension 2 (see introduction of [DR73]).

For a finite module $TMF_{(A)}$-module M, we can choose by the last lemma a map $P_0 \xrightarrow{f} M$ that induces a surjection $\pi_*\mathcal{F}_{P_0} \to \pi_*\mathcal{F}_M$. Thus, the kernel $\pi_*\mathcal{F}_{\mathrm{hofib}(f)}$ has locally projective dimension at most 1. Choose a map $P_1 \to \mathrm{hofib}(f)$ with fiber P_2, inducing a surjection $\pi_*\mathcal{F}_{P_1} \to \pi_*\mathcal{F}_{\mathrm{hofib}(f)}$. Its kernel $\pi_*\mathcal{F}_{P_2}$ has locally projective dimension 0, i.e. is a vector bundle. Thus, $\mathcal{F}_{P_2}$ is locally free and P_2 is relatively free. □

This implies in particular that the algebraic K-theory $K_0(TMF_{(A)})$ is generated by the relatively free modules.

5.2. Killing torsion and generators: Building up and tearing down.

In this subsection, we will always implicitly localize at a set of primes A.

Given a relatively free TMF-module M, there are at least two ways to build a new relatively TMF-module M' from it:

Proposition 5.7. *Let M be a relatively free TMF-module, $\Sigma^k TMF \xrightarrow{\phi} M$ be a map and denote by M' its cofiber. Then:*

(a) *If M' is relatively free, then $\mathrm{rk}\, M' = \mathrm{rk}\, M + 1$ or $\mathrm{rk}\, M' = \mathrm{rk}\, M - 1$.*
(b) *The module M' is relatively free and $\mathrm{rk}\, M' = \mathrm{rk}\, M + 1$ if and only if*

$$\phi_*\colon \pi_*\Sigma^k \mathcal{O}^{top} \to \pi_*\mathcal{F}_M$$

is zero, if and only if $[\phi] \in \pi_k M$ is torsion. In this case, we say that M' is obtained from M by killing a torsion class.
(c) *If M' relatively free and $\mathrm{rk}\, M' = \mathrm{rk}\, M - 1$, then*

$$\pi_*\mathcal{F}_{\Sigma^{-1}M'} \to \pi_*\Sigma^k\mathcal{O}^{top}$$

is zero. In this case, we say that M' is obtained from M by killing a generator.

Proof. Let $\mathrm{Spec}\, A \to \mathcal{M}_{ell}$ be an etale map with $\mathrm{Spec}\, A$ connected, ω trivialized on $\mathrm{Spec}\, A$ and $\mathcal{F}_M$ free on $\mathrm{Spec}\, A$. Note that this map has dense image as the underlying topological space of $\mathcal{M}_{ell}$ is $\mathrm{Spec}\, \mathbb{A}^1$ and hence irreducible.

If M' is relatively free, then also assume that $\mathcal{F}_{M'}$ is free on $\mathrm{Spec}\, A$. Without loss of generality, assume that k is odd. We have a long exact sequence

$$0 \to M_0 \to M'_0 \to A \xrightarrow{\phi_1} M_1 \to M'_1 \to 0,$$

where

$$\phi_1 = \phi_*(\operatorname{Spec} A)_1,$$
$$M_0 = (\pi_0\mathcal{F}_M)(\operatorname{Spec} A),$$
$$M_1 = (\pi_1\mathcal{F}_M)(\operatorname{Spec} A)$$

and likewise for M'. We can split this into two short exact sequences

$$0 \to M_0 \to M_0' \to \ker(\phi_1) \to 0$$

and

$$0 \to \operatorname{im}(\phi_1) \to M_1 \to M_1' \to 0.$$

The ring A is an integral domain because it is regular. Let $\mathcal{Q}$ be its quotient field. If $\ker(\phi_1) \otimes_A \mathcal{Q} \neq 0$ and M' is relatively free, then $\operatorname{rk} M_0' = \operatorname{rk} M_0 + 1$ and $\operatorname{rk} M_1 = M_1'$. If $\operatorname{im}(\phi_1) \otimes_A \mathcal{Q} \neq 0$ and M' is relatively free, then $\operatorname{rk} M_1' = \operatorname{rk} M_1 - 1$ and $\operatorname{rk} M_0' = \operatorname{rk} M_0$. This proves (a).

Because $\ker(\phi_1)$ and $\operatorname{im}(\phi_1)$ are A-torsionfree, we have $\ker(\phi_1) = 0$ if and only if $\ker(\phi_1) \otimes_A \mathcal{Q} = 0$ and likewise for $\operatorname{im}(\phi_1)$. Thus, $\ker(\phi_1)$ is zero or A and consequently $\operatorname{im}(\phi_1)$ is isomorphic to A or zero, respectively. If M' is relatively free and $\ker(\phi_1) = 0$, then $\pi_*\mathcal{F}_{\Sigma^{-1}M'} \to \pi_*\Sigma^k\mathcal{O}^{top}$ is zero. This proves (c).

Clearly, M' is relatively free and $\operatorname{rk} M' = \operatorname{rk} M + 1$ if and only if $\ker(\phi_1) = A$. This happens if and only if $\phi_*\colon \pi_*\Sigma^k\mathcal{O}^{top} \to \pi_*\mathcal{F}_M$ is zero. We can show the the part about $[\phi]$ being torsion after inverting l for an arbitrary number $l > 1$. Then ϕ_* is zero iff the induced map

$$\pi_*\Sigma^k TMF(l) \to \pi_*\mathcal{F}_M(\mathcal{M}(l))$$

is zero. This happens iff the image of $[\phi]$ under the map $\pi_*M \to \pi_*M(l)$ for $M(l) = M \wedge_{TMF} TMF(l)$ is zero. The groups $\pi_*TMF(l)$ are torsionfree (as $\mathcal{M}(l)$ is flat over $\mathbb{Z}[\frac{1}{l}]$) and $M(l)$ is a projective $TMF(l)$-module (analogously to Proposition 5.2). Thus, every torsion class $[\phi]$ maps to zero. On the other hand, $\pi_*M/\text{torsion}$ injects into $\pi_*M(l)^{GL_2(\mathbb{Z}/l)}$ by the descent spectral sequence (because $M \simeq M(l)^{hGL_2(\mathbb{Z}/l)}$); thus every element in the kernel of $\pi_*M \to \pi_*M(l)$ must be torsion. This proves (b). ☐

Let us specialize to the case $2 \notin A$ for a moment. The module $M(2) = M \wedge_{TMF} TMF(2)$ carries an S_3-action induced by the S_3-action on $TMF(2)$. If M is relatively free, $M(2)$ is a free $TMF(2)$-module. We denote by $E(M)$ the set of generators $x \in \pi_*(M(2))$ of direct $TMF(2)_*$-summands which are invariant under the S_3-action. Let (by abuse of notation) denote $c\colon M \to M(2)$ the map induced by the algebra map $c\colon TMF \to TMF(2)$. We say that M *has an invariant generator* if $E(M) \cap \operatorname{im}(c_*) \neq \varnothing$. We can kill a generator from M if and only if it has an invariant generator.

Definition 5.8. A relatively free module X can be *built up* if there is a sequence $X_0 = 0, X_1, \ldots, X_n \cong X$ (for n the rank of X) with cofiber sequences $\Sigma^? TMF \to X_i \to X_{i+1}$. Dually, X can be *torn down* if there is a sequence of modules $X^0 = 0, X^1, \ldots, X^n = X$ with cofiber sequences $\Sigma^? TMF \to X^{i+1} \to X^i$.

Proposition 5.9. *Every module that can be torn down can be built up and vice versa. Such modules are called* standard modules*.*

Proof. Let $X^0, \ldots, X^n = X$ be a tearing down sequence. Then define X_i as the fiber of $X^n \to X^{n-i}$. By the octahedral axiom the left column of the following diagram is distinguished:

$$\begin{array}{ccccc} X_{i-1} & \longrightarrow & X^n & \longrightarrow & X^{n-i+1} \\ \downarrow & & \downarrow = & & \downarrow \\ X_i & \longrightarrow & X^n & \longrightarrow & X^{n-i} \\ \downarrow & & \downarrow & & \downarrow = \\ \Sigma^? TMF & \longrightarrow & X^{n-i+1} & \longrightarrow & X^{n-i} \end{array}$$

Clearly, $X_n = X$ and $X_0 = 0$, so X can be built up. The dual follows by the dual proof or Spanier–Whitehead duality. □

It is easy to see that every standard module is algebraically standard in the following sense:

Definition 5.10. A relatively free TMF-module M is *algebraically standard* if the vector bundles $\pi_0\mathcal{F}_M$ and $\pi_1\mathcal{F}_M$ are standard in the sense of Definition 3.9, i.e. these vector bundles can be built up iteratively by extensions with line bundles.

The first partial converse concerns low rank modules. We will prove the following two results in Section 5.4 and at the end of Section 6:

Proposition 5.11. *If every algebraically standard TMF-module of rank $\leq n$ has an invariant generator, every algebraically standard TMF-module of rank $\leq n$ is standard.*

Theorem 5.12. *Every algebraically standard TMF-module of rank ≤ 3 has an invariant generator and thus all these are standard.*

In general, we can build relatively free TMF-modules by more complicated procedures, e.g. by coning off 5 times a torsion element from TMF, then killing two generators, then coning off some torsion elements again etc. We will prove two statements that drastically limit this possibilities. In the following, we will always work 3-locally, i.e. with $A = \{3\}$. The second partial converse states that we can tore down every algebraically standard module in the weak sense that we can kill first two generators, then cone off a torsion element, then kill two generators etc. More precisely:

Definition 5.13. We define the notion of a finite TMF-module being *hook-standard* inductively: First, $\Sigma^k TMF$ is hook-standard for all k. Furthermore, a TMF-module M is hook-standard if there are cofiber sequences

$$\Sigma^{|a|}TMF \xrightarrow{a} M \to X$$
$$\Sigma^{|x_1|}TMF \xrightarrow{x_1} X \to X'$$
$$\Sigma^{|x_2|}TMF \xrightarrow{x_2} X' \to X''$$

with X'' hook-standard, where a corresponds to a torsion element and $c_*(x_1) \in E(X)$ and $c_*(x_2) \in E(X')$.

Every standard module is hook-standard: If $a = 0$, $X = \Sigma^{|a|+1}TMF \oplus M$ and we can choose $x_1 : \Sigma^{|a|+1}TMF \to X$ to be the inclusion of the direct summand.

Our main theorem, to be proven in Section 6, is:

Theorem 5.14 (The hook theorem)**.** *Every algebraically standard TMF-module is hook-standard.*

Note that in principle it is possible to classify all hook-standard TMF-modules up to a certain finite rank: For rank 1, we have just suspensions of TMF. Now suppose, we have classified all hook-standard modules up to rank $(n-1)$. Given a hook standard module Z of this rank, we can choose a torsion element in $\pi_* D_{TMF} Z$, cone it off to get a module Z' of rank n. Here, we choose again a torsion element, cone it off and get a module Z''. Here, we choose a $z \in \pi_* Z''$ with $c(z) \in E(Z'')$ and get a module $D_{TMF} M$ after coning it off whose dual M is hook-standard. All hook-standard modules of rank n are built in this way.

5.3. **Low-rank examples and the realification.** We will implicitly localize at a set of primes not containing 2 everywhere in this section. Denote again by $\mathcal{O}^{top}$ the Goerss–Hopkins–Miller sheaf of E_∞-ring spectra on $\mathcal{M} = \mathcal{M}_{ell}$. We want to topologify the realification map r of Section 3.3 (which we denote by r_{alg} in the following) to a map $r\colon p_* p^* \mathcal{O}^{top} \to \mathcal{O}^{top}$. Since

$$p_* p^* \mathcal{O}^{top} \cong f_* f^* \mathcal{O}^{top} \oplus \Sigma^4 f_* f^* \mathcal{O}^{top},$$

Proposition 4.10 gives us a unique map

$$r\colon p_* p^* \mathcal{O}^{top} \to \mathcal{O}^{top}$$

realizing the algebraic map r_{alg}.

Remark 5.15. Probably, the realification map $TMF(2) \to TMF \simeq TMF(2)^{hS_3}$ coincides with the transfer map, which can be defined using a form of Shapiro's lemma. Since this identification is not needed for our purposes, we abstain from a discussion.

Recall that we denote the unit map $TMF \to TMF(2)$ by c.

Lemma 5.16. *We have $rc = 6$ and $cr = \Sigma_{g \in S_3} g$.*

Proof. These identities hold at the level of vector bundles by Lemma 3.13. We know that realizations of sheaf map $\pi_* p_* p^* \mathcal{O}^{top} \to \pi_* p_* p^* \mathcal{O}^{top}$ are unique, hence the second equation. The descent spectral sequence (DSS) for $\mathcal{H}om_{\mathcal{O}^{top}}(\mathcal{O}^{top}, \mathcal{O}^{top})$ equals the DSS computing TMF. There are no permanent cycles in this spectral sequence in the 0-column above the 0-line. Indeed, the 0-column consists of the groups $H^{2k}(\mathcal{M}; \omega^{\otimes k})$. The even degree cohomology is generated by $\beta \in H^2(\mathcal{M}; \omega^{\otimes 6})$ and $\Delta \in H^0(\mathcal{M}; \omega^{\otimes 12})$. Thus, in the zero line we have non-trivial classes in filtration $2k$ iff k is divisible by 12. But because there is a vanishing line above filtration 8 (as follows from [Bau08]), these classes cannot represent non-trivial classes in $\pi_0 TMF$. Thus, we get the first equation. □

We will need again and again the following observation:

Lemma 5.17. *Let M be relatively free TMF-module and $x \in \operatorname{im}(r_*\colon \pi_* M(2) \to \pi_* M)$. Then $\alpha x = \beta x = 0$.*

Proof. Let $y \in \pi_* M(2)$ such that $r_*(y) = x$. Since $M(2)$ is a free $TMF(2)$-module, $\pi_* M(2)$ is torsion-free and hence $\alpha y = \beta y = 0$. Since r is a TMF-module map, the result follows. □

Recall that we have a map $\sigma\colon M_0(2) \to \Sigma^4 M_\alpha$ given as the cofiber of $c\colon M \to M_0(2)$, where $M_0(2) = M \wedge_{TMF} TMF_0(2)$.[1] Note that $E(M)$ is completely in the $M_0(2)$-summand of $M(2)$ since the map $\mathcal{M}(2) \to \mathcal{M}$ factors over $\mathcal{M}_0(2)$. Here, $E(M)$ is the set of invariant generators, as in Section 5.2. We can apply the realification to study σ:

[1] We abuse here the letter c since the usual map $c\colon M \to M(2)$ factors over $M \to M_0(2)$.

Lemma 5.18. *Every S_3-invariant element $x \in \pi_* M_0(2) \subset \pi_* M(2)$ is mapped by σ to a 3-torsion element in $\Sigma^4 M_\alpha$.*

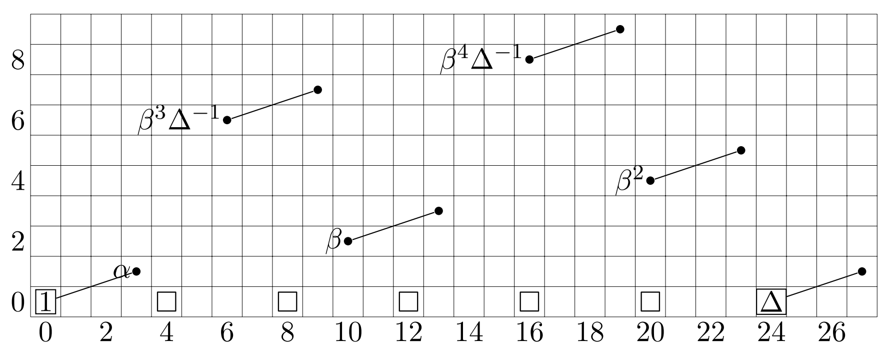
Proof. We have $cr(x) = \Sigma_{g \in S_3} gx = 6x$. Since 2 is invertible, this implies that $3x$ is in the image of c and, hence, $3\sigma(x) = \sigma(3x) = 0$. □

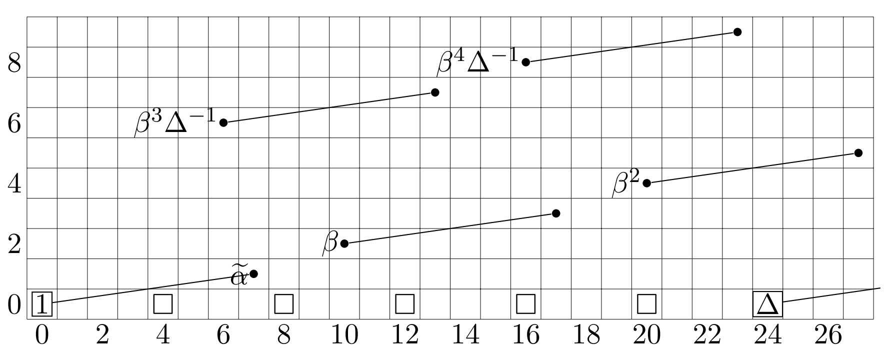
To identify the fiber of r, it will be convenient to discuss first some low-rank cases. Additionally, this will serve as an illustration of the general theory.

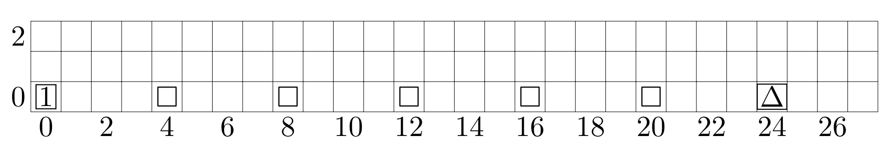
Lemma 5.19. *Let M be a algebraically standard TMF-module. We have an action of $\beta \in H^2(\mathcal{M}; \omega^6)$ on the DSS of M by Theorem 4.7, which commutes with the differentials since β is a permanent cycle in the DSS for TMF. Then β acts injectively on the E^2-term of the DSS for M beginning with the first line. In addition:*

- *If $\pi_* \mathcal{F}_M$ is concentrated in even degrees, β acts injectively on odd degrees (i.e. columns) on the E^r-term of the DSS beginning with the $(r-1)$-st line.*[2]
- *If the first line consists of permanent cycles, β acts injectively on the whole E^r-term of the DSS beginning with the $(r-1)$-st line.*

Proof. We know that $\pi_* \mathcal{F}_M$ decomposes into a direct sum of shifts of vector bundles of the form $\pi_* \mathcal{O}^{top}$, $E_\alpha \otimes_{\mathcal{O}} \pi_* \mathcal{O}^{top}$ and $E_{\alpha,\widetilde{\alpha}} \otimes_{\mathcal{O}} \pi_* \mathcal{O}^{top}$. The cohomology of these looks as follows (where the pattern continues to the left, right and top):







[2] To act proactively against possible confusion: That $\pi_* \mathcal{F}_M$ is concentrated in even degrees means that $\pi_k \mathcal{F}_M = 0$ for k odd, where π_k denotes the sheafified homotopy group. An element in the E^2-term $H^q(\mathcal{M}; \pi_p \mathcal{F}_M)$ of the DSS is in odd degree if $p - q$ is odd.

This follows from the results of Section 3.2 and Lemma 3.5. The injectivity of multiplication by β on E^2 beginning with the first line is now immediate. Now suppose, we have shown that β operates injectively on E^{r-1} beginning with the $(r-2)$-th line (on elements of odd degree). Now suppose $\beta\overline{a} = \beta\overline{b}$ for some $\overline{a} \neq \overline{b} \in E^r$ (in odd degrees) in line s and $s \geq r-1$. Then there are $a, b \in E^{r-1}$ reducing to $\overline{a}, \overline{b}$. Hence, there is an $x \in E^{r-1}$ with $d_{r-1}x = \beta(a-b) \neq 0$ and x is in line k with $k \geq 1$ (and of even degree). We want to show that there is a $y \in E^{r-1}$ such that $\beta y = x$: Let $x' \in E^2$ represent x. Then x' is divisible by β. Indeed, if $\pi_*\mathcal{F}_M$ is concentrated in even degrees, x' must be in every standard summand of $\pi_*\mathcal{F}_M$ of the form $\pm\Delta\beta^{k/2}$ or 0. The same holds if the first line of the DSS consists of permanent cycles since then all $\alpha\beta^l\Delta^i$ and $\widetilde{\alpha}\beta^l\Delta^i$ are permanent cycles as well and x' can be no permanent cycle. So, let $y' \in E^2$ such that $\beta y' = x'$. Suppose $d_l(y') \neq 0$ for some $l < r-1$. Then $d_l(x') = \beta d_l(y') \neq 0$ since β acts injectively beginning with $(l-1)$-st line on E^l. So, $d_l(y) = 0$ for $l < r-1$ and $x = \beta y$ for y denotes the reduction of y' to E^{r-1}. We have that $\beta d_{r-1}(y) = \beta(a-b) \in E^{r-1}$ for $d_{r-1}(y)$ and $(a-b)$ in the s-th line. Hence $d_{r-1}(y) = a-b$ and $\overline{a} = \overline{b}$. □

Proposition 5.20. *If M is relatively free of rank $n = 1$, we have $M \cong \Sigma^? TMF$. If M is relatively free of rank 2 and $\pi_0\mathcal{F}_M = E_\alpha$, then $M \cong \Sigma^{24i}TMF_\alpha$ for some $i \in \mathbb{Z}$.*

Proof. If the rank of M is 1, then $M \in \mathrm{Pic}(TMF)$. By [MS16], we have thus $M \cong \Sigma^? TMF$. An argument for this along the lines of the argument we give for $n = 2$ can be found in [Mei12, Proposition 8.3.6].

Assume that $\pi_0\mathcal{F}_M = E_\alpha$. Denote the images of Δ^i under the maps $\Gamma_*(\mathcal{O}) \to \Gamma_*(E_\alpha)$ and

$$\Gamma_*(\mathcal{O}) \to \Gamma_*(E_\alpha) = \Gamma_*(\pi_0\mathcal{F}_M) \to \Gamma_*(\pi_0\mathcal{F}_{M(2)})$$

also by Δ^i. We identify $M(2)$ with $TMF(2) \oplus TMF(2)$ and assume that no element of $E(M)$ is in $\mathrm{im}(c_*)$. By this contradiction assumption and Lemma 5.18, the $\Delta^i \in E(M)$ have to be mapped by σ to non-trivial torsion elements y_i in even degree in the exact sequence

$$\pi_*M \xrightarrow{c} \pi_*M(2) \xrightarrow{\sigma} \pi_{*-4}M_\alpha \oplus \pi_{*-4}M_0(2).$$

We can consider the y_i as lying in $\pi_{*-4}M_\alpha$ since $M_0(2)$ is a free $TMF_0(2)$-module and thus $\pi_*M_0(2)$ is torsionfree. We know that Δ^i in the DSS for $\mathcal{F}_M$ supports a non-zero d_{p_i}-differential: If it was a permanent cycle, the corresponding element in π_*M would map to $\Delta^i \in \pi_*M(2)$. Hence, $d_{p_i}(\beta^k\Delta^i) = \beta^k d_{p_i}(\Delta^i) \neq 0$ in E^{p_i} by Lemma 5.19. Thus, π_*M has no torsion elements in even degrees.

Now look at the exact sequence

$$\pi_{24i-4}M \to \pi_{24i-4}M_\alpha \to \pi_{24i-8}M$$

induced by the triangle $TMF \to TMF_\alpha \to \Sigma^4 TMF$. Since π_*M has no torsion elements in even degree, y_i is mapped to 0. For the same reason, it can come only from a non-torsion element in $\pi_{24i-4}M$. But

$$\pi_0\mathcal{F}_{M_\alpha} \cong E_\alpha \otimes E_\alpha$$

by Lemma 4.1 and the injection $E_\alpha \to E_\alpha \otimes E_\alpha$ induces an injection on graded global sections. Thus every non-torsion element in π_*M maps to a non-torsion element in π_*M_α (since it is in the 0-line of the DSS). This is a contradiction and one of the Δ^i must be a permanent cycle. Thus, we get a map $x\colon \Sigma^{24i}TMF \to M$ such that $c(x)\colon \Sigma^{24i}TMF(2) \to M(2)$ splits off a direct summand. Let Y be the fiber of x. Then $Y(2)$ has rank 1, therefore

Y is equivalent to some $\Sigma^k TMF$. Thus, we have a cofiber sequence

$$\Sigma^k TMF \xrightarrow{y} \Sigma^{24i} TMF \xrightarrow{x} M.$$

We know that y is of filtration (at least) 1 in the DSS for TMF since $\Sigma^{24i}TMF \to M$ induces an injective map $\pi_*\mathcal{F}_{\Sigma^{24i}TMF} \to \pi_*\mathcal{F}_M$. Thus, it equals $\pm\alpha\Delta^{3j}$ by Corollary 4.9 since else $\pi_*\mathcal{F}_M$ would split into two line bundles. Therefore, $M \cong \Sigma^{24i}TMF_\alpha$. □

The next case is that $\pi_0\mathcal{F}_M = f_*f^*\mathcal{O}$. We will treat a more general case:

Proposition 5.21. *Let M be a relatively free TMF-module and $\pi_0\mathcal{F}_M \cong f_*f^*\mathcal{O} \oplus Z_0$ for some vector bundle Z_0. Then there is a cofiber sequence*

$$TMF_0(2) \xrightarrow{\overline{y}} M \to Z \to \Sigma TMF_0(2)$$

such that $\pi_0\mathcal{F}_Z = Z_0$. This cofiber sequence splits.

Proof. This follows directly from Proposition 4.10. □

This implies, in particular, that we can always assume for the proof of Theorem 5.14 that $\pi_0(\mathcal{F}_M)$ contains no summand of the form $f_*f^*\mathcal{O}$ since we could compose the map $TMF_0(2) \to M$ with the unit map $TMF \to TMF_0(2)$ and get an invariant generator.

Now, we want to identify the fiber of r and begin by identifying the fiber of

$$r_{alg}\colon\ p_*p^*\mathcal{O} \to \mathcal{O}.$$

Recall that QCoh($\mathcal{M}$) is equivalent to the category of evenly graded $TMF(2)_{2*}$-modules with semilinear S_3-action. We have that $p_*p^*\mathcal{O}(\mathcal{M}(2)) \cong \bigoplus_{S_3} \mathcal{O}(\mathcal{M}(2))$ with diagonal S_3-action. By (the proof of) Lemma 3.13, r_{alg} maps on $\mathcal{M}(2)$ an element $(a_g)_{g\in S_3}$ to $\sum_{g\in S_3} a_g \in \mathcal{O}(\mathcal{M}(2))$. We can identify $f_*f^*\mathcal{O}(\mathcal{M}(2))$ with $\bigoplus_{i=1}^3 \mathcal{O}(\mathcal{M}(2))$ with the permutation action. Sending $(a_g)_{g\in S_3}$ to $(\sum_{g\colon\ g(1)=i} a_g)_{i=1}^3$ defines the projection to the direct summand $p_*p^*\mathcal{O} \to f_*f^*\mathcal{O}$ and the complement is isomorphic to $f_*f^*\mathcal{O} \otimes \omega^2$ (by Lemma 4.11). Thus, r_{alg} factors thus as $p_*p^*\mathcal{O} \to f_*f^*\mathcal{O} \to \mathcal{O}$, where the second map is the summing map on $\mathcal{M}(2)$. As recalled in Section 3.2, the latter map has kernel $E_\alpha \otimes \omega^4$. Thus $\ker(r_{alg}) \cong f_*^*f^*\mathcal{O} \otimes \omega^2 \oplus E_\alpha \otimes \omega^4$.

Let X be the fiber of $\Gamma(r)\colon\ TMF(2) \to TMF$.[3] Then $\pi_*\mathcal{F}_X \cong \omega^{2+*} \otimes f_*f^*\mathcal{O} \oplus \omega^{4+*} \otimes E_\alpha$. By the last proposition, we can decompose X as $\Sigma^4 TMF_0(2) \oplus Y$ with $\pi_*\mathcal{F}_Y \cong \omega^{4+*} \otimes E_\alpha$. Hence, by Proposition 5.20 $Y \cong \Sigma^{-8+24i}TMF_\alpha$ for some $i \in \mathbb{Z}$. Since there is no non-zero map $\Sigma^{-8+24i}TMF_\alpha \to \Sigma^5 TMF_0(2)$ (the groups $\pi_*TMF_0(2)$ vanish in odd degrees), we have $X \cong \Sigma^{-8+24i}TMF_\alpha \vee \Sigma^4 TMF_0(2)$. The fiber $\Sigma^{-1}TMF \to X$ of $X \to TMF(2)$ can only be of the form $\widetilde{\alpha} = (\widetilde{\alpha}, 0)$ by inspection of the homotopy groups of TMF_α (see Section 7.4). Thus, $i = 0$ and we have a triangle

$$\Sigma^{-1}TMF \xrightarrow{\widetilde{\alpha}} \Sigma^{-8}TMF_\alpha \vee \Sigma^{2?}TMF_0(2) \xrightarrow{d} TMF(2) \xrightarrow{r} TMF,$$

which, in turn, induces a triangle

$$\Sigma^{-1}M \xrightarrow{\widetilde{\alpha}} \Sigma^{-8}M_\alpha \vee \Sigma^{2?}M_0(2) \xrightarrow{d} M(2) \xrightarrow{r} M. \tag{5.22}$$

[3] This map and the induced map $M(2) \to M$ for a TMF-module M will often also be denoted by r.

5.4. **Tearing down algebraically standard modules.** The aim of this subsection is to show Proposition 5.11. Everything in this section will be implicitly localized at a set of primes A with $2 \notin A$. The basic idea is to have as induction hypothesis that every (algebraically standard) TMF-module of rank smaller than n is standard and then use an invariant generator to reduce from rank n to rank $n-1$. This works in an easy way without the hypothesis of being algebraically standard. But if we want to have an algebraically standard module again, we have to deal with the difficulty that the cokernel of a map of standard vector bundles is a priori not standard.

Lemma 5.23. *The element* $1 \in TMF(2)_* \cong \mathbb{Z}_{(A)}[x_2, y_2, \Delta^{-1}]$ *is not in the ideal* $(3, x_2+y_2)$.

Proof. Assume that $1 \in (3, x_2+y_2)$. This implies that 1 is divisible by x_2+y_2 in $TMF(2)_*/3$; hence $x_2 + y_2$ is a unit in this ring. This, in turn, implies that $(x_2 + y_2) \cdot z = \Delta^k$ for some $z \in \mathbb{F}_3[x_2, y_2]$. We know that $\mathbb{F}_3[x_2, y_2]$ is factorial and, hence, $x_2 + y_2$ is a prime element (since it is irreducible). Since $\Delta^k = 16^k x_2^{2k} y_2^{2k} (x_2 - y_2)^{2k}$, the element $x_2 + y_2$ has to divide x_2, y_2 or $x_2 - y_2$ in $\mathbb{F}_3[x_2, y_2]$, which is clearly impossible. □

Lemma 5.24. *Let M be a relatively free TMF-module and $y \in \pi_k M$ with $c(y) \in E(M)$. Assume that $\pi_*\mathcal{F}_M$ has a decompositions into shifts of $\pi_*\mathcal{O}^{top}$ and $\pi_*\mathcal{O}^{top} \otimes E_\alpha$. Then y is not in* $\mathrm{im}(r)$ *and the reduction* $\overline{y} \in \Gamma_*(\pi_*\mathcal{F}_M)$ *is not in* $\mathrm{im}(r_{alg})$.

Proof. For ease of notation, we assume $k = 0$. It is easy to see that the two statements to be proven are equivalent. Assume for contradiction that $\overline{y} \in \mathrm{im}(r_{alg})$. The module $\pi_*\mathcal{F}_M(\mathcal{M}(2)) = \pi_*M(2)$ is a free $TMF(2)_*$-module. We want to show that we can choose a basis such that $c(y)$ corresponds to an element $(a_1, \dots, a_n)$ with $a_i \in (3, x_2 + y_2) \subset TMF(2)_*$. This is enough since $1 \notin (3, x_2+y_2)$ by the last lemma and this is a contradiction to the assumption that $c(y) \in E(M)$.

The vector bundle $\pi_0\mathcal{F}_M$ decomposes into a sum $\bigoplus_i \omega^{n_i} \oplus \bigoplus_j E_\alpha \otimes \omega^{m_j}$. Thus, we can show the claim just for one of the standard summands. First assume $\pi_0\mathcal{F}_M \cong \omega^j$. Since $\overline{y} \in \mathrm{im}(r_{alg})$, we know that $\overline{y}$ lies in the ideal $(3, c_4, c_6)$ (see Scholium 3.17). As shown in Section 3.1, $c_{alg}(c_4)$ and $c_{alg}(c_6)$ are divisible by $(x_2 + y_2)$ after reducing mod 3. For $\pi_0\mathcal{F}_M \cong \omega^j \otimes E_\alpha$, we proceed as follows: In the proof of Proposition 3.15, it was shown that $\mathrm{im}(r_{alg})$ coincides with the image of the map $\Gamma(f_*f^*\mathcal{O} \otimes \omega^{2+j}) \to \Gamma(E_\alpha \otimes \omega^j)$. We know that $\Gamma_*(f_*f^*\mathcal{O}) \cong \mathbb{Z}_{(A)}[b_2, b_4, \Delta^{-1}]$, where b_2 maps to $-4(x_2 + y_2)$ and b_4 to $2x_2y_2$ in $\Gamma_*(p_*p^*\mathcal{O})$ (see also Section 3.1); thus, $\Gamma_*(f_*f^*\mathcal{O})$ is exactly the ring of invariant elements in $\Gamma_*(p_*p^*\mathcal{O})$ for a subgroup $C_2 \subset S_3$. The image of $\Gamma_*(f_*f^*\mathcal{O})$ in $\Gamma_*(p_*p^*f_*f^*\mathcal{O}) \cong \bigoplus_{i=1}^3 TMF(2)_*$ consists of (a, ta, t^2a) for $a \in \Gamma_*(f_*f^*\mathcal{O})$ and $t \in S_3$ an element of order 3.

Denote by I the composite functor

$$\mathbb{Z}_{(A)}[S_3]\text{-mod} \simeq \mathrm{QCoh}(\mathrm{Spec}\,\mathbb{Z}_{(A)}//S_3) \xrightarrow{i^*} \mathrm{QCoh}(\mathcal{M}),$$

where $i\colon \mathcal{M} \to \mathrm{Spec}\,\mathbb{Z}_{(A)}//S_3$ classifies the S_3-torsor $\mathcal{M}(2) \to \mathcal{M}$. In [Mei15, Section 4.3], it was shown that $E_\alpha \otimes \omega^{-2} \cong I\mathbb{Z}_{(A)}[\zeta_3]$ and that the map $f_*f^*\mathcal{O} \to E_\alpha \otimes \omega^{-2}$ is induced by the quotient map $\mathbb{Z}_{(A)}[C_3] \to \mathbb{Z}_{(A)}[\zeta_3]$ (given by quotienting out the diagonal). Thus, giving $\mathbb{Z}_{(A)}[\zeta_3]$ the basis $(1, \zeta_3)$, the element $(a, ta, t^2a) \in \Gamma_*(p_*p^*f_*f^*\mathcal{O})$ is sent to $(a - t^2a, a - ta) \in \Gamma_*(p_*p^*E_\alpha)$. We can assume that a is a monomial of the form $b_2^k b_4^l$ (since Δ is invariant). This is sent to

$$\left((x_2+y_2)^k x_2^l y_2^l - (y_2 - 2x_2)^k (y_2 - x_2)^l (-x_2)^l, (x_2+y_2)^k x_2^l y_2^l - (x_2 - 2y_2)^k (-y_2)^l (x_2 - y_2)^l\right)$$

by the formulas in Section 3.1. Modulo three, $y_2 - 2x_2$ equals $x_2 + y_2$, so both entries are in the ideal $(3, x_2 + y_2)$, which was to be proven. □

Proposition 5.25. *Let M be a relatively free TMF-module such that there is a $y \in \pi_k M$ with $c(y) \in E(M)$. Assume that $\pi_* \mathcal{F}_M$ has a decompositions into shifts of $\pi_* \mathcal{O}^{top}$ and $\pi_* \mathcal{O}^{top} \otimes E_\alpha$. Then there exists a $z \in \operatorname{im}(r)$ such that the cofiber of $\Sigma^k TMF \xrightarrow{y+z} M$ is algebraically standard of rank $\operatorname{rk} M - 1$.*

Proof. By the last lemma, $\overline{y} \notin \operatorname{im}(r_{alg})$. This implies that its projection to one of the standard summands $\mathcal{E}$ (isomorphic to ω^j or $E_\alpha \otimes \omega^j$) is not in $\operatorname{im}(r_{alg})$. Since every element in $\operatorname{im}(r_{alg})$ is a permanent cycle, we can by Scholium 3.17 find an element $z \in \operatorname{im}(r)$ such that for $y' = y + z$ the projection of the reduction $\overline{y'} \in \Gamma(\pi_0 \mathcal{F}_M)$ to $\mathcal{E}$ equals $\pm \Delta^{j/12}$. We have still $c(y') = E(M)$ since an element in a free module generates a direct summand if it projects to a unit in one of the summands. Thus, we get a diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \pi_0\mathcal{F}_M - \mathcal{E} & \longrightarrow & \pi_0\mathcal{F}_M - \mathcal{E} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O} & \xrightarrow{\overline{y'}} & \pi_0\mathcal{F}_M & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \\
 & & \downarrow{=} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Here, the map $\pi_0 \mathcal{F}_M \to \mathcal{E}$ is the projection. By the exactness of the lower two rows and the columns, the identification of the upper row follows by the Snake lemma. We have that $\mathcal{L} = 0$ if $\mathcal{E} \cong \mathcal{O}$ and $\mathcal{L} \cong \omega^{-2}$ for $\mathcal{E} \cong E_\alpha$. In both cases, $\mathcal{G}$ is standard since $\pi_0 \mathcal{F}_M - \mathcal{E}$ is.

If M' is the cofiber of $\Sigma^k TMF \xrightarrow{y'} M$, then $\mathcal{G} = \pi_0 \mathcal{F}_{M'}$. Thus, M' is algebraically standard since $\pi_1 \mathcal{F}_{M'} \cong \pi_1 \mathcal{F}_M$. □

Corollary 5.26 (Proposition 5.11)**.** *If every algebraically standard module M of rank $\leq n$ has an invariant generator, every algebraically standard TMF-module of rank $\leq n$ can be torn down and is thus standard.*

Proof. We will prove this by induction over the rank k of M. We will consider two cases.

First assume that $\pi_* \mathcal{F}_M$ has a summand of the form $f_* f^* \pi_* \mathcal{O}^{top}$ with complement a standard vector bundle. Then we can write $M = TMF_0(2) \oplus M'$ with M' algebraically standard as in Proposition 5.21. In particular, we can use $1 \in \pi_0 TMF_0(2)$ to get a map $TMF \to M$ whose cofiber is $\Sigma^4 TMF_\alpha \oplus M'$. By induction, M' is a standard module and thus also $\Sigma^4 TMF_\alpha \oplus M'$.

If $\pi_* \mathcal{F}_M$ has no summand of the form $f_* f^* \pi_* \mathcal{O}^{top}$, then we can apply the last proposition to get a cofiber sequence

$$\Sigma^? TMF \to M \to M'$$

where M' is a standard module by induction. □

6. Proof of the main theorem

The main goal of this section is to prove Theorem 5.14. We will always implicitly localize at a set A of primes not containing 2.

Let us first recall some notation. We denote the projections $\mathcal{M}_0(2) \to \mathcal{M}$ and $\mathcal{M}(2) \to \mathcal{M}$ by f and p. We set $TMF_0(2) = \mathcal{O}^{top}(\mathcal{M}_0(2))$ and $TMF(2) = \mathcal{O}^{top}(\mathcal{M}(2))$. For a TMF-module M, we write $M_\alpha = M \wedge_{TMF} TMF_\alpha$, $M_0(2) = M \wedge_{TMF} TMF_0(2)$ and $M(2) = M \wedge_{TMF} TMF(2)$. By $E(M)$ we denote the set of invariant generators as in Section 5.2. As in Section 4.1, we can associate to M the quasi-coherent $\mathcal{O}^{top}$-module $\mathcal{F}_M$. Its descent spectral sequence converges to M and will be denoted by $DSS(M)$; the corresponding filtration on π_*M will be denoted by $F_\bullet M$. Also of importance will be the maps $c\colon M \to M(2)$ and $r\colon M(2) \to M$ induced by the corresponding maps $c\colon TMF \to TMF(2)$ and $r\colon TMF(2) \to TMF$ discussed in Section 5.3.

Let us now sketch the strategy of the proof. An important observation (in Proposition 5.21) is that if we have a summand of the form $f_*f^*\mathcal{O}$ in $\pi_*\mathcal{F}_M$, M decomposes as $TMF_0(2) \oplus M'$. So our strategy is to enlarge M by coning off torsion elements in first filtration to get such summands to kill, which we will do in Section 6.5. For this, we need a sufficient supply of elements in first filtration for a module M without an invariant generator. The crucial results are Corollary 6.7 and Proposition 6.14. At the end, we will either get an invariant generator or a "hook". This all relies very much on the algebraic preliminaries from Section 3.

In the whole proof, the following triangle is very important:

$$M \xrightarrow{c} M(2) \xrightarrow{\sigma(2)} \Sigma^4 M_\alpha \vee \Sigma^4 M_0(2) \xrightarrow{t\widetilde{\alpha}} \Sigma M \tag{6.1}$$

It is induced by the triangle

$$TMF \xrightarrow{c} TMF(2) \xrightarrow{\sigma(2)} \Sigma^4 TMF_\alpha \vee \Sigma^4 TMF_0(2) \xrightarrow{t\widetilde{\alpha}} \Sigma TMF.$$

This in turn you get from the more well known triangle

$$TMF \to TMF_0(2) \xrightarrow{\sigma} \Sigma^4 TMF_\alpha \to \Sigma TMF$$

since $TMF(2) \cong TMF_0(2) \vee \Sigma^4 TMF_0(2)$ by Corollary 4.12. This triangle follows either from [Beh06, Lemma 2 in Section 2.4] or can be deduced from the short exact sequences in Section 3.2 and Proposition 5.20.

Throughout the proof we will make the following assumption:

Assumption 6.2. *We assume that M is an algebraically standard module without an invariant generator, i.e. that no $x \in E(M)$ is in the image of $c\colon \pi_*M \to \pi_*M(2)$.*

6.1. **Permanent cycles in first filtration.** It is our goal in this subsection to show that all elements in the 1-line of the DSS for M are permanent cycles. We start with the following proposition.

Proposition 6.3. *The restricted projection map* $\operatorname{Tors}\pi_*M \to \pi_*M/\operatorname{im}(r_*)$ *is a surjection.*

Proof. Look at the following diagram

$$\begin{array}{ccccc}
\pi_*\Gamma(p^*\mathcal{F}_M) \cong \pi_*M(2) & \xrightarrow{r_*} & \pi_*\Gamma(\mathcal{F}_M) \cong \pi_*M & \xrightarrow{c_*} & \pi_*\Gamma(p^*\mathcal{F}_M) \\
\downarrow l & & \downarrow \kappa & & \downarrow l \\
\Gamma(p^*\pi_*\mathcal{F}_M) & \xrightarrow{r_{alg}} & \Gamma(\pi_*\mathcal{F}_M) & \xrightarrow{c_{alg}} & \Gamma(p^*\pi_*\mathcal{F}_M)
\end{array}$$

Here κ and l denote the edge morphisms in the descent spectral sequences for $\mathcal{F}_M$ and $p^*\mathcal{F}_M$ respectively. Note that l is an isomorphism. Let $y \in \pi_*\Gamma(\mathcal{F}_M) = \pi_*M$. Then

$\kappa(y) \in \operatorname{im}(r_{alg})$, because else there is an element $a \in \Gamma(p^*\pi_*\mathcal{F}_M)$ such that $c_{alg}(\kappa(y)+r_{alg}(a))$ is in $l(E(M))$ by Proposition 3.15. This implies that

$$c_*(y + r_*(l^{-1}a)) \in E(M),$$

which is a contradiction to our assumption. Therefore, we can write $\kappa(y) = r_*l(a) = \kappa r_*(a)$ for some $a \in \pi_*\Gamma(p^*\mathcal{F}_M)$. So we see that $\kappa(y - r_*(a)) = 0$. Therefore, $c_*(y - r_*(a)) = 0$ and by the exact sequence induced by the triangle 6.1, we have that $y - r_*(a)$ is torsion, which implies the statement. □

Corollary 6.4. *Let $x \in E(M) \subset \pi_*M_0(2)$. Then $\sigma(x) = \beta^k g$, $k \geq 1$, where $g \in F_0\pi_*M_\alpha$. Here, $F_\bullet$ denotes the filtration associated to the DSS.*

Proof. Let $x \in E(M)$. By Lemma 5.18 and the contradiction assumption, $\sigma(x)$ is a non-zero 3-torsion element in $\pi_*\Sigma^4 M_\alpha$. Thus, $d(\sigma(x)) = 0$ and $\sigma x = \widetilde{\alpha} u_x$ for some $u_x \in \pi_*M$ (for d see the end of Section 5.3). The element u_x is only well-defined up to the image of r – therefore we can assume by the last proposition that u_x is torsion. Hence $u_x = {}^t\widetilde{\alpha} y_x$ for some $y_x \in \pi_*\Sigma^4 M_\alpha$ by (6.1) since $c(u_x) = 0$. By Lemma 4.16, we get that $\sigma(x) = \beta y_x$ for some $y_x \in \pi_*M_\alpha$. By the same argument, every torsion element in M_α is divisible by β and so we can repeat the process if y_x is not already in F_0. □

Recall now that on the level of vector bundles, $\sigma\colon M_0(2) \to \Sigma^4 M$ induces the map

$$\sigma_{alg}\colon \Gamma(f_*f^*\mathcal{O} \otimes \pi_*\mathcal{F}_M) \to \Gamma(E_\alpha \otimes \omega^{-2} \otimes \pi_*\mathcal{F}_M)$$

called σ in Section 3.

Corollary 6.5. *The 0-line of the DSS for M_α consists of permanent cycles.*

Proof. We will use a rank argument: Let $X \subset \Gamma(\pi_*\mathcal{F}_{M_\alpha})$ be the subgroup of permanent cycles. Then $\operatorname{im}(\sigma_{alg}) \subset X$ since the descent spectral sequence for $M_0(2)$ collapses on E^2. Define a filtration on X by setting

$$B_k = \{\overline{x} \in X : \beta^{k+1}x = 0 \text{ for some } x \in F_0\pi_*M_\alpha \text{ reducing to } \overline{x}\}.$$

Since β operates trivially on $M_0(2)$, we have

$$\operatorname{im}(\sigma_{alg}) \subset B_0.$$

Hence X/B_0 is a subquotient of $\operatorname{coker}(\sigma_{alg})$. The latter is an $\mathbb{F}_3[\Delta^{\pm 3}]$-vector space of rank $3n$ for n the number of irreducible direct summands of $\pi_*\mathcal{F}_M$ – this is proven in the proof of Proposition 3.15 and at the end of Section 3.3. So, if $X \neq \Gamma_*(\pi_*\mathcal{F}_{M_\alpha})$, then X/B_0 is an $\mathbb{F}_3[\Delta^{\pm 3}]$-vector space of rank smaller than $3n$.

Choose an isomorphism $M_0(2) \cong \bigoplus \Sigma^? TMF_0(2)$. We have $3n$ invariant generators of the form Δ^j for $j \in \{0, 1, 2\}$ in these direct summands of $\pi_*M_0(2)$. Denote the $\mathbb{Z}_{(A)}$ of these by G. Define a function $N\colon G \to \mathbb{N}$ by setting $N(g)$ for $g \in G$ to be the maximal k such that

$$\sigma(g) = \beta^k v$$

for some $v \in \pi_*M_\alpha$. Note that $\sigma(g) \neq 0$ as $g \in E(M)$ and thus cannot be in $\operatorname{im}(c_*)$. Let $I = \{1, \ldots, 3n\}$. Then we define a $\mathbb{Z}_{(A)}$-basis $(g_i)_{i\in I}$ of G inductively as follows. We take g_1 to be a primitive vector in G with maximal value under N. If we have already defined $g_1, \ldots, g_k$, we set g_{k+1} to be an element of G such that $(g_1, \ldots, g_{k+1})$ is part of a $\mathbb{Z}_{(A)}$-basis of G and such that it is among these one with maximal value under N. We set $n_i = N(g_i)$ so that

$$\sigma(g_i) = \beta^{n_i} v_i$$

It is easy to see that $N(\Sigma_{j\in J}a_jg_j)$ is less or equal $\min_{j\in J} n_j$ if $J \subset I$ and $a_j \in \mathbb{Z}_{(A)}^{\times}$ for $j \in J$.

We have $\overline{v_i} \in B_{n_i}$ since $\beta\sigma(g_i) = \sigma(\beta g_i) = 0$. Suppose, there exists a $v_i' \in \pi_*M_\alpha$ with the same reduction $\overline{v_i'} = \overline{v_i}$ in the zero-line, but $\beta^{n_i}v_i' = 0$. Then there is an $x \in \pi_*M_\alpha$ of higher filtration such that $v_i' = v_i - x$. Since x is torsion, it is by the (proof of the) last corollary of the form $\beta^l v$ for some $v \in F_0\pi_*M_\alpha$. Thus,

$$\beta^{l+n_i}v = \beta^{n_i}x = \beta^{n_i}v_i = \sigma(g_i)$$

in contradiction to the maximality of n_i. Thus, $\overline{v_i} \notin B_{n_i-1}$.

Assume now that X/B_0 is an $\mathbb{F}_3[\Delta^{\pm 3}]$-vector space of rank smaller than $3n$. Since $\oplus_{i\geq 1}B_i/B_{i-1} \cong X/B_0$, there is a $k \in \mathbb{N}$ and $J \subset I$ such that $v_j \in B_k \setminus B_{k-1}$ for $j \in J$ and the $(\overline{v_j})_{j\in J}$ are linear dependent over $\mathbb{F}_3$ in B_k/B_{k-1}. That is, there exist $a_j \in \{1, -1\}$ such that $\Sigma_j a_jv_j \in B_{k-1}$. As above, this implies

$$\beta^{l+k}v = \beta^k\Sigma_j a_jv_j = \sigma(\Sigma_j a_jg_j)$$

for some $v \in \pi_*M_\alpha$ and $l > 0$, i.e. $N(\Sigma_j a_jg_j) > \min_{j\in J} n_j = k$, which cannot happen. □

Notation 6.6. We recollect the notation from the last proof for the rest of the section: We have an index set I of cardinality $3n$, indexing elements $g_i \in \pi_*M_0(2) \subset \pi_*M(2)$ spanning $E(M)$ in the sense that every element in $E(M)$ is of the form $\sum a_ig_i$ for $a_i \in \mathbb{Z}_{(A)}$. We have numbers n_i and elements $v_i \in F_0\pi_*M_\alpha$ such that $\sigma(g_i) = \beta^{n_i}v_i$. The v_i reduce by the last proof to a basis $\{\overline{v_i}\}$ of $\operatorname{coker}\sigma_{alg}$. Note that the v_i are (thus, since $\operatorname{im}(r) = \operatorname{im}(\sigma)$ by the proof of Proposition 3.15) not in $\operatorname{im}(r_*)$ and can be modified by elements in $\operatorname{im}(r_*)$ so that the $\overline{v_i}$ are in the span of the elements of the form Δ^j in $H^0(\mathcal{M}; \pi_*M_\alpha)$ by Proposition 3.15 and the fact that $\beta \cdot \operatorname{im}(r) = 0$.

Corollary 6.7. *The* 1*-line of the DSS of* M *consists of permanent cycles.*

Proof. The map ${}^t\widetilde{\alpha}$ in the triangle (6.1) in the introduction induces as in Theorem 4.8 a morphism of descent spectral sequences, which is exactly ${}^t\widetilde{\alpha}$ on E^2. This implies that the whole first line of the descent spectral sequence in M consists of permanent cycles (which, of course, cannot be boundaries) since

$${}^t\widetilde{\alpha}\colon\, \Gamma(\pi_*\mathcal{F}_{M_\alpha}) \cong \Gamma(\pi_*\mathcal{F}_M \otimes E_\alpha) \to H^1(\mathcal{M}; \pi_{*+4}\mathcal{F}_M)$$

is surjective. Indeed, this follows by the short exact sequence

$$0 \to \pi_*\mathcal{F}_M \to \pi_*\mathcal{F}_M \otimes f_*f^*\mathcal{O} \to \pi_*\mathcal{F}_M \otimes E_\alpha \otimes \omega^{-2},$$

of whom ${}^t\widetilde{\alpha}$ is the boundary map and the fact that $H^1_*(\mathcal{M}; \pi_*\mathcal{F}_M \otimes f_*f^*\mathcal{O}) = 0$. □

6.2. **Bounding β-divisibility.** In the rest of this subsection, we want first to investigate how many times an element might be divided by β. Using the notation n_i from Notation 6.6, we will more precisely show that $n_i \leq 3$ for all i. This will be important for Proposition 6.14.

Before we begin with this, we have to compute a Toda bracket.

Lemma 6.8. *The Toda bracket* $\langle\widetilde{\alpha}, \beta^4, 3\rangle$ *(where we view* $\widetilde{\alpha}$ *again as a map* $\Sigma^7 TMF \to TMF_\alpha$*) contains* $\pm\{3\Delta^2\}$*.*

Proof. We first want to check that the Toda bracket is actually defined. Since $\beta^2\alpha = 0$ in π_*TMF, we see that $\beta^2\widetilde{\alpha} \in \pi_{27}TMF_\alpha$ is mapped to zero in the exact sequence

$$\pi_*TMF \to \pi_*TMF_\alpha \to \pi_{*-4}TMF$$

and is thus the image of an element $a \in \pi_{27}TMF$. The only non-zero elements in this degree are $\pm\{\alpha\Delta\}$.[4] These are annihilated by β^2 and thus $\beta^4\widetilde{\alpha} = 0$ and the Toda bracket is defined.

The element $\beta^4\widetilde{\alpha}$ in the E^2-term of the DSS of TMF_α is a permanent cycle (since $\widetilde{\alpha}$ is in $DSS(TMF_\alpha)$ and β^4 is one in $DSS(TMF)$) and can only be hit by a d_9-differential from $\pm\Delta^2$: Column 48 in lines below 9 consists only of line zero elements and by Scholium 3.17 and the fact that $\mathrm{im}(r)$ consists of permanent cycles, the existence of a non-trivial differential implies a non-trivial differential from $\pm\Delta^2$. Using Theorem 4.6, we could use that Massey products converge to Toda brackets and get the result.

Alternatively, one can use the definition of the Toda bracket and sees that it suffices to prove that the lift of $\beta^4 \in \pi_{40}TMF$ in the exact sequence

$$\pi_{48}TMF_\alpha \to \pi_{48}TMF_0(2) \to \pi_{40}TMF$$

is $\pm\Delta^2 \in \pi_{48}TMF_0(2)$. Indeed, these span the non-trivial elements in $\pi_{48}TMF_0(2)$ which are mapped trivially into the zero line of the DSS of TMF modulo the image of $\pi_{48}TMF_\alpha$ (as can be seen, for example, by an $\mathrm{im}(r)$-argument). □

We need a general observation. If Z is an R-module, we write $DZ = D_RZ = \mathrm{Hom}_R(Z, R)$ for the R-linear Spanier–Whitehead dual. If $z \in \pi_kZ$, we write ${}^tz$ for the dual map $DZ \to \Sigma^{-k}R$.

Lemma 6.9. *Let Z and M be R-modules and $a \in \pi_kZ$ and $z \in \pi_l(M \wedge_R DZ)$. Then the diagram*

$$\begin{array}{ccccc}
R & \xrightarrow{a} & \Sigma^{-k}Z & \xrightarrow{\mathrm{id}_Z \wedge_R z} & \Sigma^{-k-l}Z \wedge_R M \wedge_R DZ \\
\downarrow z & & & & \downarrow \cong \\
\Sigma^{-l}M \wedge_R DZ & \xrightarrow{\mathrm{id}_M \wedge_R {}^ta} & \Sigma^{-k-l}M & \xleftarrow{\mathrm{id}_M \wedge \mathrm{ev}} & \Sigma^{-k-l}M \wedge_R Z \wedge_R DZ
\end{array}$$

commutes. We will denote the composition $(\mathrm{id}_M \wedge \mathrm{ev}) \circ (\mathrm{id}_Z \wedge z)\colon\ \Sigma^{-k}Z \to \Sigma^{-k-l}M$ by ${}_tz$.

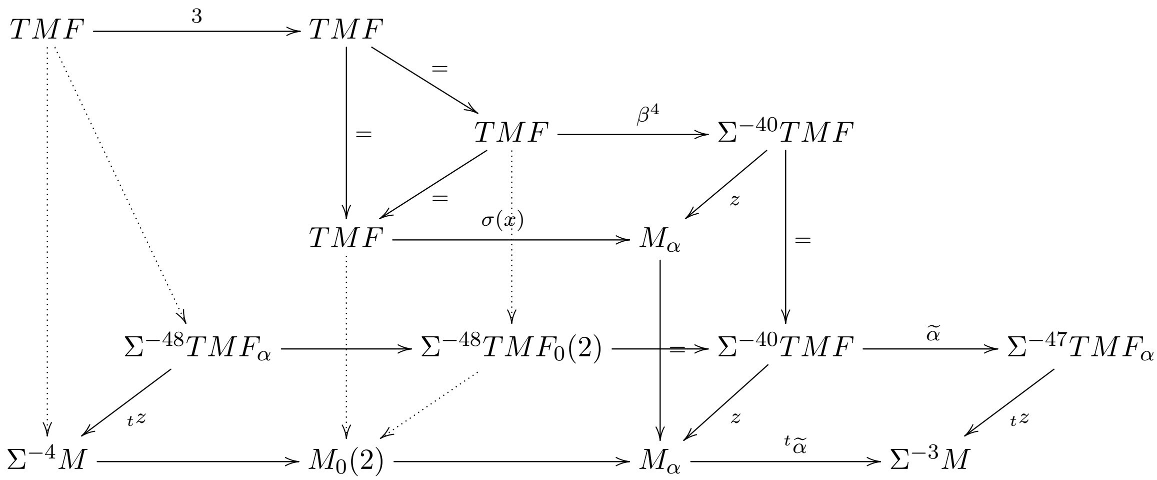
Proof. The only thing to observe is that ${}^ta$ is given as the composition

$$DZ \cong DZ \wedge_R R \xrightarrow{\mathrm{id} \wedge_R a} DZ \wedge_R \Sigma^{-k}Z \xrightarrow{\mathrm{ev}} \Sigma^{-k}R.$$ □

Lemma 6.10. *All n_i are smaller than 4.*

[4] One can check that $\beta\widetilde{\alpha}$ is non-zero and therefore a is non-zero as well. But this is not needed for our argument.

Proof. Assume we have an $x \in E(M)$ such that $\sigma(x) = \beta^4 z$, which is automatically $\neq 0$ since else x would be in the image of c. Look at the following diagram:



Here we use the isomorphism

$$DTMF_\alpha = \mathrm{Hom}_{TMF}(TMF_\alpha, TMF) \cong \Sigma^{-4} TMF_\alpha,$$

under which ${}_t z$ corresponds to z as in Lemma 6.9 (with $k = 7$, $l = -44$ and $Z = TMF_\alpha$). The Toda bracket $\langle \widetilde{\alpha}, \beta^4, 3\rangle$ contains $\{3\Delta^2\}$. Therefore, we get that $\langle {}^t\widetilde{\alpha}, \sigma(x), 3\rangle$ contains $b = {}_t z(\{3\Delta^2\})$. We have $c(b) = 3x'$ in $\pi_* M_0(2)$ by the definition of the Toda bracket with $\sigma(x) = \sigma(x')$. The element x' is invariant (since $3x'$ is), but is not in the image of c (since $\sigma(x') \neq 0$). Hence, the corresponding element

$$\overline{x'} \in H^0(\mathcal{M}; \pi_* \mathcal{F}_M) \cong \pi_*(M(2))^{S_3}$$

cannot be a permanent cycle in $DSS(M)$ and hence is not in the image of r. By Proposition 3.15, we can find a $y \in \pi_* M(2)$ with $c_{alg}(r_{alg}(\overline{y}) + \overline{x'})$ in $E(M)$. Set

$$x'' = x' + cr(y) \in \pi_* M(2).$$

This is clearly an invariant generator. We have that $c(b + 3r(y)) = 3x''$. Furthermore, for $w := {}_t z(1) + r(\Delta^{-2} y) \in \pi_* M$, the following holds:

$$\begin{aligned} 3\Delta^2 c(w) &= c(\{3\Delta^2\} w) \\ &= c(b) + cr(\{3\Delta^2\}\Delta^{-2} y) \\ &= 3x' + cr(3y) \\ &= 3x''. \end{aligned}$$

Hence, $c(w) = \Delta^{-2} x''$, which is an invariant generator. This is a contradiction to our global contradiction hypothesis. □

6.3. **Understanding the torsion if there are no E_α-summands.** Our next aim is to understand the torsion in $\pi_* M$ and $\pi_* M_\alpha$ more precisely if $\pi_* \mathcal{F}_M$ decomposes into a sum of shifts of $\pi_* \mathcal{O}^{top}$. This will be important for Proposition 6.14.

Lemma 6.11. *Let $I' \subset I$ be nonempty and $a_i = \pm 1$ for $i \in I'$. Then*

(1) $\sum_{i \in I'} a_i b^{k_i} v_i \notin \mathrm{im}(\sigma)$ *if $k_i < n_i$ for all $i \in I'$,*
(2) $\sum_{i \in I'} a_i b^{k_i} v_i \neq 0$ *in $\pi_* M_\alpha$ if $k_i \leq i$ for all $i \in I'$,*

Proof. Note first that (1) follows from (2) for each I'. Indeed, if $\sum_{i\in I'} a_i b^{k_i} v_i \in \mathrm{im}(\sigma)$, then $\sum_{i\in I'} a_i b^{k_i+1} v_i = 0$as it is always true that $\beta\sigma(x) = \sigma(\beta x) = 0$.

We will prove (2) and hence (1) by induction on $|I'|$. For $|I'| = 1$, assume that $\beta^{k_i} v_i = 0$. Then also $\beta^{n_i} v_i = \sigma(g_i) = 0$, which cannot happen (because else $g_i \in \mathrm{im}(c_*)$). Assume now that (2) has been proven for all indexing sets of size smaller than $|I'|$ and that $\sum_{i\in I'} a_i b^{k_i} v_i = 0$. We rewrite this as

$$\sum_{i\in I'_1} a_i b^{k_i} v_i = \sum_{i\in I'_2} (-a_i) b^{n_i} v_i$$

where $I'_2 \subset I'$ is the subset of all i such that $k_i = n_i$ and I'_1 is its complement. By multiplying with powers of β, we can assume that $I'_2 \neq \varnothing$ and thus $|I'_1| < |I'|$. As the right hand side is in $\mathrm{im}(\sigma)$, the induction hypothesis for (1) implies that $I'_1 = \varnothing$. This implies that

$$\sigma(\sum_{i\in I'} a_i g_i) = \sum_{i\in I'} (-a_i) b^{n_i} v_i = 0.$$

Thus, $\sum_{i\in I'} a_i g_i \in \mathrm{im}(c_*)$, which cannot be. □

Proposition 6.12. *Assume that $\pi_*\mathcal{F}_M$ decomposes into a sum of shifts of $\pi_*\mathcal{O}^{top}$. Then the torsion of π_*M is an $\mathbb{F}_3$-vector space with basis given by $\{\alpha\beta^k w_i\}$ with $k < n_i$ and $i \in I$. This means that there are elements $\overline{w_i} \in \Gamma_*(\pi_*\mathcal{F}_M)$ such that the $\{\alpha\beta^k w_i\}$ are detected by $\alpha\beta\overline{w_i}$ in the DSS for M. The torsion of π_*M_α is an $\mathbb{F}_3$-vector space with basis given by $\beta^k v_i$ with $k \leq n_i$ and $i \in I$.*

Warning 6.13. Similar to $\{\alpha\Delta\} \in \pi_{27}TMF$, the notation $\{\alpha\beta^k w_i\}$ does not entail that this element is divisible by α. But it is true that $\beta^k\{\alpha w_i\} = \{\alpha\beta^k w_i\}$.

Proof. We have that ${}^t\widetilde{\alpha}(\overline{v_i}) = \alpha\overline{w_i}$ for some elements $\overline{w_i}$ in the 0-line of the E^2-term of the descent spectral sequence of M. The $\overline{w_i}$ can be chosen to span the $\mathbb{Z}_{(A)}$-span of elements of the form Δ^i since these generate $H^0_*(\mathcal{M}; \pi_*\mathcal{F}_M)/\ker(\alpha)$ and ${}^t\widetilde{\alpha}$ is surjective onto $H^1(\mathcal{M}; \pi_*\mathcal{F}_M)$. All elements $\beta^k\alpha\overline{w_i}$ are permanent cycles since the β^k are permanent cycles in $DSS(TMF)$. The elements $\beta^k\alpha\overline{w_i}$ for $k \geq n_i$ must be boundaries since ${}^t\widetilde{\alpha}(\beta^k v_i) = {}^t\widetilde{\alpha}(\sigma(\beta^{k-n_i} g_i))$ is zero. Because M has no invariant generator, the $\overline{w_i}$ support non-trivial differentials $d_{p_i}\overline{w_i}$. Hence, also $d_{p_i}(\beta^k\overline{w_i}) \neq 0$ in E^{p_i} by Lemma 5.19 since the 1-line consists of permanent cycles. All in all, this implies that all torsion in π_*M is detected by the $\alpha\beta^k\ \overline{w_i}$ for $k < n_i$.

Note that this also implies that all elements of the form $\widetilde{\alpha}\beta^k\overline{v_i}$ cannot be permanent cycles in $DSS(M_\alpha)$ since $\beta^{k+1}\overline{w_i} = {}^t\widetilde{\alpha}(\widetilde{\alpha}\beta^k\overline{v_i})$ is not a permanent cycle. Therefore, the $\beta^k v_i$ span all the torsion in π_*M_α. By the last lemma, the $\beta^k v_i$ for $k \leq n_i$ form thus a basis for the torsion in π_*M_α. We want to show that they are actually detected by the corresponding elements in the DSS.

By the last lemma the $\mathbb{F}_3$-span of the $\beta^k v_i$ for $k < n_i$ gets mapped injectively into the torsion of π_*M by ${}^t\widetilde{\alpha}$. By rank comparison we see that no $\alpha\beta^k\overline{w_i}$ is a boundary. We set $\{\alpha\beta^k w_i\} := {}^t\widetilde{\alpha}(\beta^k v_i)$, which is detected by $\alpha\beta^k\overline{w_i}$ and is therefore in strict filtration $2k+1$. □

6.4. **Multiplication by α.** Assume again that $\pi_*\mathcal{F}_M$ decomposes into a sum of shifts of $\pi_*\mathcal{O}^{top}$. We use the notation of the last subsection concerning the v_i, $\{\alpha\beta^k w_i\}$ and n_i. Furthermore, we denote by $F_n = F_n\pi_*M$ the filtration coming from the descent spectral sequence and by $S_n\pi_*M$ the stratum $F_n\pi_*M - F_{n+1}\pi_*M$. The main result of this subsection is now the following:

Proposition 6.14. *There exists always an element x in $S_1\pi_*M$ such that $\alpha x = 0$.*

Proof. The proof will be by contradiction, so we assume that $\alpha x \neq 0$ for all $x \in S_1\pi_*M$. By Lemma 6.10, $n_i \leq 3$ for all i. The proof has now two parts. First we exclude the case that some $n_i \leq 2$. Finally, we lead the case that all n_i equal 3 to a contradiction.

We get a short exact sequence associated to

$$\Sigma^3 M \xrightarrow{\alpha} M \xrightarrow{i} M_\alpha \xrightarrow{p} \Sigma^4 M$$

of the form

$$0 \to \pi_*M/(\operatorname{im}\alpha) \to \pi_*M_\alpha \to \ker(\alpha) \to 0.$$

We will show that this restricts to a short exact sequence

$$0 \to \{\{\alpha\beta^k w_i\}\}_{\mathbb{F}_3}^{k=0,\dots n_i-1}/(\operatorname{im}\alpha) \to \{\beta^k v_i\}_{\mathbb{F}_3}^{k=0,\dots n_i} \to (\{\{\alpha\beta^k w_i\}\}_{\mathbb{F}_3}^{k=0,\dots n_i-1})_{\alpha=0} \to 0.$$

Here $()_{\alpha=0}$ denotes the elements where multiplication by α is zero. In addition note that the $\mathbb{F}_3$-spans run over all $i \in I$.

The first map restricts since all torsion in π_*M_α is spanned by the $\beta^k v_i$ as shown in the last subsection; it is automatically injective. The elements v_i map to torsion because the $\overline{v_i}$ get mapped to 0 in the spectral sequence since they are in the span of the elements of the form Δ^j and therefore come from M. Hence, the second map restricts.

Suppose an element $z \in \{\beta^k v_i\}_{\mathbb{F}_3}^{k=0,\dots,n_i}$ maps to zero via p; then it is in the image of i. Since

$$\{\{\alpha\beta^k w_i\}\}_{\mathbb{F}_3}^{k=0,\dots n_i-1} \subset \pi_*M \to \pi_*M/\operatorname{im}(r_*)$$

is a surjection by Proposition 6.3, we can write $z = i(x+y)$, where $x \in \{\{\alpha\beta^k w_i\}\}_{\mathbb{F}_3}^{k=0,\dots n_i-1}$ and $y \in \operatorname{im}(r_*)$. Since by Corollary 6.5 the whole 0-line of the DSS of the fiber of $r\colon M(2) \to M$ consists of permanent cycles, $\operatorname{im}(r_*)$ is completely detected by $\operatorname{im}(r_{alg})$ in the 0-line. Since $\beta^k\overline{v_i} \notin i_*(\operatorname{im}(r_{alg}))$, it follows $y = 0$ and we have exactness in the middle term.

If $p_*(x)$ is torsion for some $x \in \pi_*M_\alpha$, then either x is torsion or the reduction $\overline{x} \in \Gamma_*(\pi_*\mathcal{F}_{M_\alpha})$ maps to zero in $\Gamma_*(\pi_*\mathcal{F}_{\Sigma^4 M})$. We know that the $\overline{v_i}$ and $\overline{i_*(\operatorname{im}(r_*))} = i_*(\operatorname{im}(r_{alg}))$ span

$$\ker(\Gamma_*(\pi_*\mathcal{F}_{M_\alpha}) \to \Gamma_*(\pi_*\mathcal{F}_{\Sigma^4 M})) \cong \operatorname{im}(i_*)$$

by Scholium 3.17. Since $p_*i_*(\operatorname{im}(r_*)) = 0$ in π_*M and all torsion in π_*M_α is spanned by the $\beta^k v_i$, we have $p_*(x) = p_*(x')$ for some x' in the span of the $\beta^k v_i$. This proves exactness of the above short exact sequence.

Define $l := \dim_{\mathbb{F}_3[\Delta^{\pm 3}]}(\operatorname{im}(\alpha))$. Since $\operatorname{im}(\alpha) = \operatorname{im}(\alpha|_{\operatorname{tors}\pi_*M})$ (since $\operatorname{tors}\pi_*M$ surjects to $\pi_*M/\operatorname{im}(r_*)$), we see that

$$\Sigma_i(n_i+1) = 2\Sigma_i n_i - 2l.$$

This is equivalent to

$$2l + 3n = \Sigma_i n_i$$

since $|I| = 3n$ for n the rank of M. We know that all $n_i \leq 3$. Assume that $n_i < 3$ for one i. Then we see that $l < |I|$. Since there are $|I|$ elements $\{\alpha w_i\}$, we have $\alpha \sum_{j\in J} a_j\{\alpha w_j\} = 0$ for suitable $a_j \in \{1,2\}$ and non-empty $J \subset I$, which would imply the proposition.

Now, we are in the situation that all $n_i = 3$ and $l = |I|$. Furthermore, we still assume that α acts non-trivially on all non-zero elements of strict filtration 1. Thus, $\operatorname{im}(\alpha) = \alpha \cdot S_1\pi_*M$ for rank reasons. Suppose that $\alpha x \neq 0$ for some x of filtration greater than 1. Then $\alpha x = \alpha y$ for a $y \in S_1\pi_*M$. Thus, $\alpha(y-x) = 0$, which is not possible since $y - x \in S_1\pi_*M$. Thus, α acts trivially on all elements of higher filtration.

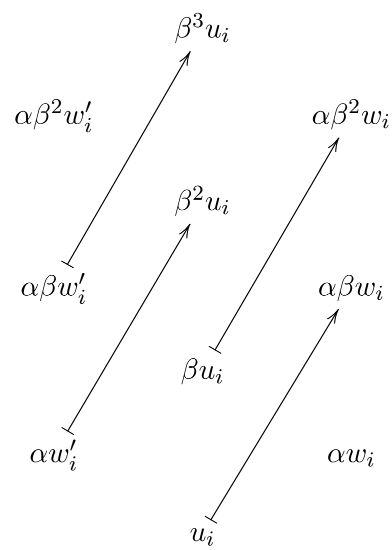
This implies $\beta\alpha x = \alpha\beta x = 0$ for $x \in \pi_* M$. Thus, multiplication by α has image in strict filtration 5 by our assumption that all n_i equal 3. More precisely, for rank reasons, it determines an isomorphism

$$F_1\pi_*M/F_2\pi_*M \to F_5\pi_*M/F_6\pi_*M.$$

Since $\alpha\{\alpha\beta w_i\} = \alpha\beta\{\alpha w_i\} = 0$, we must have

$$\{\alpha\beta w_i\} = p_*(\beta^k u_i)$$

with u_i of strict filtration 0 in $\pi_* M_\alpha$. Because p_* preserves filtration, $k \leq 1$. If $k = 1$, then $\beta p_*(u_i) = \{\alpha\beta w_i\}$, hence $p_*(u_i) = \{\alpha w_i\}$ and thus $\alpha\{\alpha w_i\} = 0$, which is a contradiction to our assumption. Therefore, $\{\alpha\beta w_i\} = p_*(u_i)$. We see that $p_*(\beta^2 u_i) = 0$. For similar reasons as above, $\beta^2 u_i = i_*(\{\alpha w_i'\})$ for some $\{\alpha w_i'\}$ in strict filtration 1; indeed, if $\beta^2 u_i$ is the image of an element of higher filtration, βu_i is in $\mathrm{im}(i_*)$, but $p_*(\beta u_i) = \beta\{\alpha\beta w_i\} \neq 0$. Thus we get the following picture of a part of the exact sequence induced by $M \to M_\alpha \to \Sigma^4 M$:



Note furthermore that we can write $\{\alpha\beta^2 w_i\} = \alpha\{\alpha w_i''\}$.

By a juggling formula (see e.g [Mei12, Lem 4.6.2]),

$$\langle \alpha, \{\alpha\beta^2 w_i\}, \beta^2\rangle$$

contains $\pm\{\alpha\Delta\}\{\alpha w_i''\}$ (since $\langle \alpha, \alpha, \beta^2\rangle$ contains $\{\alpha\Delta\}$). Furthermore, we know by Lemma 4.15 and the picture above that $\beta\{\alpha w_i'\} \in \langle \alpha, \{\alpha\beta^2 w_i\}, \beta^2\rangle$. The indeterminacy is

$$\beta^2\pi_{*-20}M + \alpha\pi_{*-3}M \subset F_5\pi_*M.$$

Hence $\beta\{\alpha w_i'\} = \pm\{\alpha\Delta\}\{\alpha w_i''\}$ in $F_3/F_4 \cong F_3/F_5$.

Suppose that the $\Sigma_{i\in I} a_i\{\alpha w_i'\} = 0$. Taking i_*, it follows $\beta^2\Sigma a_i u_i = \Sigma a_i \beta^2 u_i = 0$. The kernel of multiplication by β^2 on strict filtration 0 in $\pi_* M_\alpha$ is contained in $\mathrm{im}(r_*)$. Thus $\Sigma a_i u_i \in \mathrm{im}(r_*)$ and $\Sigma a_i\{\alpha w_i\} = p_*(\Sigma a_i u_i) \in \mathrm{im}(r_*)$. Since $\mathrm{im}(r_*)$ contains no torsion (as noted above), this implies $\Sigma a_i\{\alpha w_i\} = 0$ and hence $a_i = 0$ for all $i \in I$. Thus, the $\{\alpha w_i'\}$ are linearly independent and thus also the $\beta\{\alpha w_i'\}$.

Hence, multiplication by $\{\alpha\Delta\}$ defines a surjective map

$$F_1/F_2 = F_1/F_3 \to F_3/F_4 = F_3/F_5$$

and thus, by a dimension count, an isomorphism. But this isomorphism commutes with multiplication by β. Therefore, since multiplication by β is an isomorphism between F_1/F_2 and the F_3/F_4 and the F_3/F_4 and the F_5/F_6, multiplication by $\{\alpha\Delta\}$ is also an isomorphism between F_3/F_4 and F_5/F_6. This is obviously a contradiction since the square of $\{\alpha\Delta\}$ is zero as $\pi_{54}TMF = 0$. $\square$

6.5. **Enlargement and Shrinking.** Assume again that M has no invariant generator. Then $\pi_*\mathcal{F}_M$ has no $f_*f^*\mathcal{O}$-summand. Our strategy in this subsection is to enlarge our module M by coning off elements of first filtration to produce $f_*f^*\mathcal{O}$-summands, which can then be killed. This works in an easy way if we have an E_α-summand in $\pi_*\mathcal{F}_M$. If we have no E_α-summand, we get in general only a hook and no invariant generator.

So, suppose first that $\pi_0\mathcal{F}_M$ has an E_α-summand.[5] Furthermore assume that M has no invariant generator. Then we know that every element in the first line of the descent spectral sequence survives by Corollary 6.7, especially $\widetilde{\alpha}_{(0)}$ in the direct summand $H^1_*(\mathcal{M}; E_\alpha)$ of $H^1_*(\mathcal{M}; \pi_*\mathcal{F}_M)$. Take the map $\Sigma^7TMF \to M$ representing this $\widetilde{\alpha}_{(0)}$. We get a cofiber sequence

$$\Sigma^7TMF \xrightarrow{\widetilde{\alpha}_{(0)}} M \to X \to \Sigma^8TMF.$$

This corresponds to a short exact sequence

$$0 \to \pi_*\mathcal{F}_M \to \pi_*\mathcal{F}_X \to \pi_*\Sigma^8\mathcal{O}^{top} \to 0,$$

which corresponds again to the Ext-class $\widetilde{\alpha}_{(0)} \in \operatorname{Ext}^1(\omega^{-4}, \pi_0\mathcal{F}_M)$ by Corollary 4.9. That this is short exact can be seen as follows: The DSS of $\Sigma^{-7}M$ is equivalent to the DSS for $\mathcal{H}om(\Sigma^7\mathcal{O}^{top}, \mathcal{F}_M)$ and thus the map $\widetilde{\alpha}_{(0)}$ has filtration 1. Thus, it is send by the edge homomorphism

$$\begin{array}{c} [\Sigma^7TMF, M] \cong \pi_0\Gamma(\mathcal{H}om(\Sigma^7\mathcal{O}^{top}, \mathcal{F}_M)) \\ \downarrow \\ \operatorname{Hom}(\pi_*\Sigma^7\mathcal{O}^{top}, \pi_*\mathcal{F}_M) \cong \Gamma_0(\mathcal{H}om(\pi_*\Sigma^7\mathcal{O}^{top}, \pi_*\mathcal{F}_M)) \end{array}$$

to 0.

Thus $\pi_*\mathcal{F}_X$ contains a summand of the form $f_*f^*\mathcal{O}$ by the Extension 3.8 in Section 3.2. As in Proposition 5.21, we get a split map $\overline{y}\colon TMF_0(2) \to X$, which kills the $f_*f^*\mathcal{O}$-summand in $\pi_*\mathcal{F}_X$. Denote its cofiber by Y and the composition $M \to X \cong TMF_0(2) \oplus Y \xrightarrow{\mathrm{pr}_2} Y$ by g. Then g induces a surjective map $\pi_*\mathcal{F}_M \to \pi_*\mathcal{F}_Y$ with kernel $E_\alpha \otimes \pi_*\mathcal{O}^{top}$. Thus $\pi_*\mathcal{F}_{\mathrm{fib}(g)} \cong E_\alpha \otimes \pi_*\mathcal{O}^{top}$ and $Y \cong \Sigma^{24l}TMF_\alpha$ by Proposition 5.20. The element $1 \in \pi_{24l}TMF_\alpha$ maps to a $z \in \pi_{24l}M$ with $c(z) \in E(M)$. Thus, an M with an E_α-summand has always an invariant generator.

We can therefore assume that $\pi_*\mathcal{F}_M$ is a direct sum of shifts of $\pi_*\mathcal{O}^{top}$ and we assume again that M has no invariant generator. We want to play the same game as above. Choose a non-zero element $\alpha_{(0)} \in \pi_*M$ in filtration 1 such that $\alpha\alpha_{(0)} = 0$. The reduction $\overline{\alpha_{(0)}} \in H^1(\mathcal{M}; \pi_*\mathcal{F}_M)$ is of the form $\alpha \cdot 1_{(0)}$ for some $1_{(0)} \in \Gamma(\pi_*\mathcal{F}_\mathcal{M})$ and by a shift, we assume that $1_{(0)} \in \Gamma(\pi_0\mathcal{F}_\mathcal{M})$. Since $\alpha \cdot \operatorname{im}(r_{alg}) = 0$, we can by Proposition 3.15 furthermore assume

[5] If some other $\pi_n\mathcal{F}_M$ has a summand of the form E_α, we can deal with this the same way by shifting.

that the corresponding map $\pi_*\mathcal{O}^{top} \to \pi_*\mathcal{F}_\mathcal{M}$ is the inclusion of a direct summand and we call it the 0-*summand*. We get a cofiber sequence

$$\Sigma^3 TMF \xrightarrow{\alpha_{(0)}} M \to X \to \Sigma^4 TMF.$$

The (induced) 0-summand of X is of the form E_α and in first line of $DSS(X)$ we have elements $\Delta^i\widetilde{\alpha}$. Suppose one of these survives the descent spectral sequence. Then we have a map $\Sigma^k TMF \to X$ whose cofiber is a TMF-module Z of the form $TMF_0(2) \oplus Y$ as above. The fiber of the map

$$M \to X \to Z \cong TMF_0(2) \oplus Y \xrightarrow{\mathrm{pr}_2} Y$$

has rank 1 and is therefore isomorphic to $\Sigma^l TMF$ for some $l \in \mathbb{Z}$ by Proposition 5.20. The image z of $1 \in \pi_l\Sigma^l TMF$ in $\pi_l M$ satisfies $c(z) \in E(M)$. Thus, we can assume that none of the $\Delta^i\widetilde{\alpha}$ is a permanent cycle. Suppose that y is another element in the first line of the DSS of X projecting to the 0-summand as $\Delta^i\widetilde{\alpha}$. Then y can also be no permanent cycle since every element in the first line projecting to 0 in the 0-summand is in the image of $DSS(M) \to DSS(X)$ and therefore a permanent cycle.

Since $\alpha\alpha_{(0)} = 0$, there is an element $x \in \pi_7 X$ which is sent to $\alpha \in \pi_7\Sigma^4 TMF$. Since $\widetilde{\alpha} \in E^2(DSS(X))$ does not survive, x must live in filtration 0. The 0-summand has no elements in this degree and filtration. Therefore the projection of $\overline{x}$ to the 0-summand is zero. By Proposition 3.15, x can even be chosen such that $c(x) \in E(X)$ since outside the 0-summand $\mathrm{im}(r_*)$ maps to 0 in $\pi_*\Sigma^4 TMF$. Since X is algebraically standard, we can argue as in Proposition 5.25 that we can modify x by $\mathrm{im}(r_*)$ even in a way such that the cokernel of $\pi_*\Sigma^7\mathcal{O}^{top} \to \pi_*\mathcal{F}_X$ is standard.

Consider the cofiber sequence

$$\Sigma^7 TMF \xrightarrow{x} X \to X'.$$

Then $\pi_*\mathcal{F}_{X'}$ contains still a summand of the form E_α and is algebraically standard of $TMF(2)$-rank n. Therefore, we can apply the results of the beginning of the subsection and see that X' has an invariant generator, more precisely a $y \in \pi_{|y|}X'$ such that $\mathrm{Cofiber}(\Sigma^{|y|}TMF \to X')$ has rank one less than M. This provides a "hook" for M and the main theorem follows inductively:

Theorem 6.15. *Every algebraically standard module is hook-standard.*

We still have to show that every algebraically standard TMF-module M of rank ≤ 3 is standard. By Section 5.4, it is enough to show that every such module has an invariant generator. So, suppose that M has no invariant generator. Thus, we get a cofiber sequence

$$\Sigma^3 TMF \xrightarrow{\alpha_{(0)}} M \to X \to \Sigma^4 TMF.$$

and an $x \in \pi_7 X$ as above (reinstancing these shifting conventions). Furthermore, $\pi_*\mathcal{F}_M$ is a sum of shifts of $\pi_*\mathcal{O}^{top}$. We fix an element $1_{(0)} \in \Gamma_*(\mathcal{F}_M)$ such that $\alpha 1_{(0)}$ detects $\alpha_{(0)}$. Suppose that $d_5^M(1_{(0)}) = \alpha\beta^2\Delta^{-1}1_{(0)}$. Then

$$\begin{aligned} d_5^M(\Delta^2 1_{(0)}) &= d_5^{TMF}(\Delta^2)1_{(0)} + \Delta^2 \cdot d_5^M(1_{(0)}) \\ &= -\alpha\beta^2\Delta 1_{(0)} + \alpha\beta^2\Delta 1_{(0)} \\ &= 0 \end{aligned}$$

If $d_5^M(1_{(0)}) = -\alpha\beta^2\Delta^{-1}1_{(0)}$, we can do the same argumentation with Δ instead of Δ^2. Thus, we cannot have for all $i \in \{0,1,2\}$ that $d_5^M(\Delta^i 1_{(0)}) = \pm\alpha\beta^2\Delta^{i-1}1_{(0)}$.

Next, we want to show that a non-zero differential in $DSS(M)$ can only be of length 3, 5 or 7. With notation as in Proposition 6.12, we see that the $\alpha\beta^{n_i}\overline{w_i}$ must be boundaries, but $\alpha\beta^m\overline{w_i}$ with $m < n_i$ are not boundaries. If we have a differential of the form $d_i(\beta^k\overline{w}) = \alpha\beta^{n_i}\overline{w_i}$, then this implies $d_i(\overline{w}) = \alpha\beta^{n_i-k}\overline{w_i}$ by Lemma 5.19. Thus, $k = 0$. As everything in $\operatorname{im}(r_{alg})$ are permanent cycles and $\Gamma_*(\pi_*\mathcal{F}_M)/\operatorname{im}(r_{alg})$ has $\mathbb{F}_3[\Delta^{\pm 3}]$-rank $3n = |I|$, we see that all differentials with source in the 0-line are linear combinations of the differentials $d_i(\overline{w}) = \alpha\beta^{n_i}\overline{w_i}$. As $n_i = 1, 2$ or 3, these differentials have length $3, 5$ or 7. All differentials with source an element of the form $\beta^k\overline{w}$ have the same length.

The elements $\Delta^i 1_{(0)}$ must support non-zero differentials since otherwise M would have an invariant generator. Thus, $H^{2k+1}(\mathcal{M};\omega^k \otimes \pi_*\mathcal{F}_M)$ consists not only of $\pm\alpha\beta^2 1_{(0)}$ for $1 \leq k \leq 3$. Checking dimension, this yields that $\pi_*\mathcal{F}_M$ has an (additional) summand of the form $\pi_*\Sigma^k\mathcal{O}^{top}$ for $k = 14, 0$ or 10 (for $k = 0$ this means that we have two summands of the form $\pi_*\mathcal{O}^{top}$).

The element x reduces to an $\overline{x} \in \Gamma(\pi_7\mathcal{F}_X)$ not in $\operatorname{im}(r_{alg})$. Since $\Gamma(\pi_*\mathcal{F}_M) \to \Gamma(\pi_*\mathcal{F}_X)$ is an isomorphism in odd degrees, $\overline{x}$ is the image of an element $\overline{x}'$ in $\Gamma(\pi_*\mathcal{F}_M)$ not in $\operatorname{im}(r_{alg})$. Thus, $\pi_*\mathcal{F}_M$ has a summand of the form $\pi_*\Sigma^7\mathcal{O}^{top}$. Arguing for $\overline{x}'$ as for $1_{(0)}$ above, we get that $\pi_*\mathcal{F}_M$ has an (additional) summand of the form $\pi_*\Sigma^k\mathcal{O}^{top}$ for $k = 7, 17$ or 21. Thus, $\pi_*\mathcal{F}_M$ has rank at least 4 and it follows that every algebraically standard module of rank ≤ 3 has an invariant generator and is thus standard.

7. Examples and applications

This section is about examples. First, we will explain how equivariant TMF provides examples for relatively free TMF-modules. Then we will compute $TMF \wedge \mathbb{CP}^n$ and $TMF \wedge \mathbb{HP}^n$ when 2 is inverted. Still inverting 2, we will then construct an infinite family of (possibly indecomposable) standard TMF-modules and then present some low-rank examples.

7.1. Equivariant examples.

We will give in this section a short exposition of equivariant TMF as in [Lur09b] and will explain how it provides examples of relatively free TMF-modules.

For an abelian compact Lie group G, denote by $\widehat{G} = \operatorname{Hom}(G, \mathbb{C}^\times)$ the character group. Let $\mathcal{E}^{top}$ the universal derived (oriented) elliptic curve over the derived stack $(\mathcal{M}_{ell}, \mathcal{O}^{top})$. Note that the underlying classical Deligne–Mumford stack of $\mathcal{E}^{top}$ is just the classical universal elliptic curve over $\mathcal{M}_{ell}$ [!]. This is a (derived) group stack over $\mathcal{M}_{ell}$. Define the derived stack $(\mathcal{M}_G, \mathcal{O}_G^{top})$ to be the mapping stack $\operatorname{Hom}(\widehat{G}, \mathcal{E}^{top})$, whose underlying stack is $\operatorname{Hom}(\widehat{G}, \mathcal{E})$. This means that $\mathcal{M}_G(S)$ consists of an elliptic curve E/S together with a homomorphism $\widehat{G} \to E(S)$. For $G = S^1$, the stack $\mathcal{M}_G$ agrees with $\mathcal{E}$ and for $G = \mathbb{Z}/n$ it is equivalent to the n-torsion points $\mathcal{E}[n]$.

Lemma 7.1. *Let $h\colon \mathcal{M}_G \to \mathcal{M}_{ell}$ denote the projection map. Then $R^ih_*\mathcal{O}_{\mathcal{M}_G}$ is a vector bundle for $i = 0, 1$ and zero for $i > 1$.*

Proof. If $G = G_1 \oplus G_2$, then $\mathcal{M}_G \simeq \mathcal{M}_{G_1} \times_{\mathcal{M}_{ell}} \mathcal{M}_{G_2}$. Therefore, we can assume G to be cyclic of order n or $G \cong \mathbb{Z}$.

Assume the former and consider the diagram

$$\begin{array}{ccc} \mathcal{E}[n] & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow{\scriptstyle [n]} \\ \mathcal{M}_{ell} & \longrightarrow & \mathcal{E} \end{array}$$

This is a pullback diagram. By [KM85, Theorem 2.3.1], the map $[n]\colon \mathcal{E} \to \mathcal{E}$ is finite flat and hence also $\mathcal{E}[n] \to \mathcal{M}_{ell}$ is finite flat. Thus, $R^i h_*$ vanishes for $i > 0$ and $h_*\mathcal{O}_{\mathcal{M}_G}$ is a vector bundle.

Now assume that $G = \mathbb{Z}$. We have to show that $R^1 h_*\mathcal{O}_{\mathcal{E}}$ and $h_*\mathcal{O}_{\mathcal{E}}$ are vector bundles for the projection map $\mathcal{E} \to \mathcal{M}_{ell}$. We can show the corresponding statement for an elliptic curve $h\colon E \to S$ instead. By cohomology and base change, it is enough to show that $H^i(E_s; \mathcal{O}_{E_s})$ has constant dimension for s geometric points of S. Since E_s is an elliptic curve, the dimension is always 1 for $i = 0, 1$. □

Lurie defines the fixed points of G-equivariant TMF as $TMF^G = \Gamma(\mathcal{O}_G^{top}; \mathcal{M}_G)$.

Proposition 7.2. *The TMF-module TMF^G is relatively free.*

Proof. Let $h\colon \mathcal{M}_G \to \mathcal{M}_{ell}$ be the projection. By a descent spectral sequence [!],

$$\pi_i h_*\mathcal{O}_G^{top} \cong \begin{cases} h_*\mathcal{O}_{\mathcal{M}_G} & \text{for } i = 0 \\ R^1 h_*\mathcal{O}_{\mathcal{M}_G} & \text{for } i = -1. \end{cases}$$

As $\mathcal{O}_{\mathcal{M}_G}$ is even periodic, it follows that $h_*\mathcal{O}^{top}$ is a locally free $\mathcal{O}^{top}$-module on $\mathcal{M}_{ell}$ with global sections TMF^G. Thus, TMF^G is relatively free. □

7.2. **The TMF-module $TMF \wedge \mathbb{CP}^n$.** We will again localize everything in this section implicitly at a set of primes A not containing 2. The goal of this subsection will be to compute $TMF \wedge \mathbb{CP}^n$ as a TMF-module. Note that the homotopy groups of $tmf \wedge \mathbb{CP}^\infty$ have been computed as a $\pi_* tmf$-module before by [Bau03], even at the prime 2, by rather different techniques.

We will start in a slightly more general setting: If X is a CW-complex with cells only in even dimensions, $TMF \wedge X$ is relatively free and $\pi_0\mathcal{F}_{TMF} \wedge X$ is a vector bundle on $\mathcal{M}_{ell}$. How can one determine this vector bundle for well-known spaces like $X \cong \mathbb{CP}^n$? In other words, we are asking for description of the elliptic homology homology of X that is natural in the elliptic curve. The strategy is as follows: MU_*X has the structure of a MU_*MU-comodule (with explicit formulas), corresponding to a quasi-coherent sheaf on $\mathcal{M}_{FG}$. For an elliptic curve E over a ring R with automorphism group G, the formal group $\hat{E}$ gives rise to a morphism $\operatorname{Spec} R//G \to \mathcal{M}_{FG}$, factoring over $\mathcal{M}$, and we can pull the quasi-coherent sheaf back along this map to do concrete calculations.

The following is essentially routine though the details are somewhat lengthy:

Proposition 7.3 ([Mei12], Proposition 9.2.1)**.** *Let K be an evenly graded (MU_*, MU_*MU)-comodule with coaction map ψ and let $\mathcal{F}_K$ be the quasi-coherent sheaf on $\mathcal{M}_{FG}$ associated to it. Let E be an elliptic curve over a ring R and F the map $\operatorname{Spec} R \to \mathcal{M}_{FG}$ classifying its formal group. Then the action of an automorphism s of E on $\Gamma(F^*\mathcal{F}_K)$ can be described as follows:*

Choose a formal coordinate z on E (if possible) and let $f\colon MU_* \to R$ be the corresponding (ungraded) ring homomorphism classifying the associated formal group law. We write

- $\psi(x) = \sum x_i \otimes P_i$ *with* $P_i \in MU_*[b_1, b_2, \dots]$ *for* $x \in K$,

- $s(z) = z + a_1 z^2 + a_2 z^3 + \cdots \in R[\![z]\!]$

Then

$$\Gamma(F^*\mathcal{F}_K) \cong K \otimes_{MU_*} R$$

and

$$s \cdot (x \otimes 1) = \sum x_i \otimes f_*(P_i)(a_1, a_2, \dots),$$

where $f_*\colon MU_*[b_1,\dots] \to R[b_1,\dots]$ *denotes the morphism induced by* f.

At the prime 3, we consider the elliptic curve E with the equation $y^2 = x^3 - x$ over $\mathbb{F}_3$. We choose the automorphism s, mapping $y \mapsto y$ and $x \mapsto x+1$; it has order 3. The coordinate transformation $z = -\frac{x}{y}, w = -\frac{1}{y}$ sends the neutral element $(0,\infty)$ to $(0,0)$. In this coordinates, s has the form $z \mapsto z + w, w \mapsto w$. Note that $x = \frac{z}{w}$ and $y = -\frac{1}{w}$. The equation $y^2 = x^3 - x$ becomes transformed to

$$\begin{aligned} \frac{1}{w^2} &= \frac{z^3}{w^3} - \frac{z}{w} \\ \Leftrightarrow w &= z^3 - zw^2 \end{aligned}$$

We get

$$\begin{aligned} w &= z^3 - zw^2 = z^3 - z(z^3 - zw^2)^2 = z^3 - z^7 - z^5w^2 - z^3w^4 \\ &= \cdots = z^3 - z^7 + z^{11} - z^{15} + z^{19} \cdots \end{aligned}$$

This gives a formal expression for w in terms of z (note that the simple pattern does not continue). This implies that s is given in formal coordinates by

$$z \mapsto z + w = z + z^3 - z^7 + z^{11} - z^{15} + z^{19} \cdots .$$

To apply Proposition 7.3 to $X = \mathbb{CP}^n$, we have to recall its (MU_*, MU_*MU)-comodule structure. The Atiyah–Hirzebruch spectral sequence for $\mathbb{CP}^n$ collapses and so we have $\widetilde{MU}_*(\mathbb{CP}^n) \cong MU_*\{\beta_i\}_{i=1,\dots,n}$.

Theorem 7.4 ([Ada74], Proof of II.11.3)**.** *The coaction map*

$$\psi\colon \widetilde{MU}_*(\mathbb{CP}^n) \to MU_*MU \otimes_{MU_*} \widetilde{MU}_*(\mathbb{CP}^n)$$

is given by

$$\psi(\beta_i) = \sum_{0 \le j \le i} (\sum_{0 \le k} b_k)^j_{i-j} \otimes \beta_j.$$

Here, the lower index $i-j$ *denotes the degree of the term (where* $|b_k| = k$*) and* $b_0 = 1$.

We can easily deduce from this also the comodule structure for $\mathbb{HP}^n$. The map

$$p\colon \mathbb{CP}^{2n+1} \cong S^{2n+3}/U(1) \to S^{2n+3}/Sp(1) \cong \mathbb{HP}^n$$

is surjective on (MU_*-)homology. Set $\gamma_i = p_*\beta_{2i}$. We get the comodule structure for $\mathbb{HP}^n$ by replacing β_{2i} by γ_i and ignoring odd degree classes. We will compute the structure of $TMF \wedge \mathbb{HP}^n$.

Proposition 7.5. *The* TMF*-module* $TMF \wedge HP^n$ *decompose into shifts of* TMF*,* TMF_α *and* $TMF_0(2)$ *in the following way:*

$$TMF \wedge \mathbb{HP}^n \simeq \bigoplus_{i=1}^{\lfloor \frac{n}{3} \rfloor} \Sigma^{4i} TMF_0(2) \oplus Z$$

for

$$Z = \begin{cases} 0 & \textit{if } n \equiv 0 \mod 3 \\ \Sigma^{4n} TMF & \textit{if } n \equiv 1 \mod 3 \\ \Sigma^{4n} TMF_\alpha & \textit{if } n \equiv 2 \mod 3 \end{cases}.$$

Proof. Set $\mathcal{F}_n = \pi_0 \mathcal{F}_{M_{\mathbb{HP}^n}}$. By the skeletal filtration, $\mathcal{F}_n$ is a standard vector bundle and we want find out which one. Our first claim is that

$$\mathcal{F}_{3k}/\mathcal{F}_{3k-3} \cong f_* f^* \mathcal{O} \otimes \omega^{\otimes 2k}.$$

In the notation of Proposition 7.3, we have $a_2 = 1, a_6 = -1, \ldots$ by of the action of s on z. Thus, Proposition 7.3 implies in

$$(\widetilde{MU}_*(\mathbb{HP}^{3k}) \otimes_{MU_*} \mathbb{F}_3)/(\widetilde{MU}_*(\mathbb{HP}^{3k-3}) \otimes_{MU_*} \mathbb{F}_3) \cong \mathbb{F}_3\{\beta_{6k-4}, \beta_{6k-2}, \beta_{6k}\}$$

the following equations:

$$\begin{aligned} s \cdot \beta_{6k} &= \beta_{6k} + \binom{6k-2}{1}\beta_{6k-2} + \binom{6k-4}{2}\beta_{6k-4} = \beta_{6k} + \beta_{6k-2} + \beta_{6k-4} \\ s \cdot \beta_{6k-2} &= \beta_{6k-2} + \binom{6k-4}{1}\beta_{6k-4} = \beta_{6k-2} - \beta_{6k-4} \\ s \cdot \beta_{6k-4} &= \beta_{6k-4} \end{aligned}$$

This subquotient representation corresponds therefore to the matrix $\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Changing the basis to $(\beta_{6k-4}, -\beta_{6k-2}, \beta_{6k-2} + \beta_{6k})$, we get the matrix $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = J_3$. By Proposition 3.11, the only standard vector bundles on $\mathcal{M}_{(3)}$ inducing J_3 are $f_* f^* \mathcal{O} \otimes \omega^{\otimes i}$. Thus, $\mathcal{F}_{3k}/\mathcal{F}_{3k-3} \cong f_* f^* \mathcal{O} \otimes \omega^{\otimes i}$ for some i. As $\mathbb{HP}^{3k}$ decomposes into spheres rationally, we know that $\mathcal{F}_{3k}/\mathcal{F}_{3k-3}$ is rationally isomorphic to $\omega^{-6k} \oplus \omega^{-6k-2} \oplus \omega^{-6k-4}$, which agrees rationally with $f_* f^* \mathcal{O} \otimes \omega^{\otimes 2k}$ because $f_* f^* \mathcal{O} \otimes \omega^{\otimes i}$ is 4-periodic [Ref?]; as this is even rationally the lowest periodicity [Ref!!!], the claim follows.

By Proposition 5.21, this implies the proposition if n is divisible by 3; furthermore, it implies that $TMF \wedge \mathbb{HP}^{3k}$ splits off $TMF \wedge \mathbb{HP}^{3k+i}$ for $i = 1, 2$. As $\mathbb{HP}^{3k+1}/\mathbb{HP}^{3k} \simeq S^{12k+4}$ and $\mathbb{HP}^{3k+2}/\mathbb{HP}^{3k} \simeq \Sigma^{12k+4} \operatorname{Cone}(\nu)$ (and ν becomes α in $TMF_{(3)}$), the proposition follows. □

The following lemma is well-known:

Lemma 7.6. *Define $\mathbb{CP}^\infty_n$ as the Thom spectrum of the nL for L the dual of the canonical line bundle on $\mathbb{CP}^\infty$. Then after inverting 2, the 2-skeleton $\mathbb{CP}^2_{-1}$ of $\mathbb{CP}^\infty_{-1}$ is (stably) equivalent to $\Sigma^{-2} \operatorname{Cone}(\alpha_1) \vee S^0$.*

Proof. By the Thom isomorphism, we see that $\mathbb{CP}^2_{-1}$ has cells in dimensions -2, 0 and 2. As $\pi_1 \mathbb{S}[\frac{1}{2}] = 0$, we have a cofiber sequence

$$S^1 \to S^{-2} \vee S^0 \to \mathbb{CP}^2_{-1}.$$

We will show that this cofiber sequence does not split. This is enough as this implies that the map $S^1 \to S^{-2}$ is $\pm\alpha_1$.

The inclusion $\mathbb{CP}^\infty_{-1} \to \mathbb{CP}^\infty_0 = \mathbb{CP}^\infty_+$ induces an isomorphism on cohomology beginning with degree 2. We write $H^*(\mathbb{CP}^\infty; \mathbb{F}_3) = \mathbb{F}_3[x]$ and denote the Thom class of $-L$ by u. Furthermore, we denote by $\mathcal{P}$ the total 3-primary Steenrod power; e.g. $\mathcal{P}(x) = x(1+x^2)$. We have

$$\mathcal{P}(x)xu\mathcal{P}(ux^2) = \mathcal{P}(u)\mathcal{P}(x^2).$$

This gives

$$x^2(1+x^2)u = x^2(1+x^2)^2\mathcal{P}(u)$$

and thus

$$\mathcal{P}(u) = \frac{1}{1+x^2}u = (1 - x^2 + x^4 - \cdots)u.$$

In particular, we see that $\mathcal{P}^1(u) = x^2u$, which shows that S^2 is attached to S^{-2}. □

Proposition 7.7. *We have*

$$TMF \wedge \mathbb{CP}^n_{-1} \simeq TMF \wedge \mathbb{HP}^n \oplus \Sigma^{-4} TMF \wedge \mathbb{HP}^{n-2}$$

and

$$TMF \wedge \mathbb{CP}^n \simeq \Sigma^2 TMF_\alpha \oplus TMF \wedge \mathbb{HP}^n \oplus \Sigma^8 TMF \wedge \mathbb{HP}^{n-2}.$$

Proof. Let τ be complex conjugation on $\mathbb{CP}^n_{-1}$ or on $\mathbb{CP}^n$ and let e be the idempotent $\frac{1+\tau}{2}$. The map $\mathbb{CP}^{2n+1} \to \mathbb{HP}^n$ sends complex conjugation to the identity map (up to homotopy). Thus, we get induced maps $e\mathbb{CP}^{2n+i}_{-j} \to \mathbb{HP}^n$ (for $i, j = 0, 1$) that are equivalences as they are isomorphisms on homology. Thus, $\mathbb{HP}^n$ splits off $\mathbb{CP}^{2n+i}_{-j}$. Let us write X_n for the complement of $\mathbb{HP}^{[\![\frac{n}{2}]\!]}$ in $\mathbb{CP}^n_{-1}$.

We can determine $TMF \wedge X_n$ in a similar as $TMF \wedge \mathbb{HP}^n$ in Proposition 7.5. First, we analyze the 6-skeleton. By the last lemma, we know that the 2-skeleton of X_n is $\Sigma^{-2}\operatorname{Cone}(\alpha_1)$. Thus, $TMF \wedge X_2 \simeq \Sigma^{-2} TMF_\alpha$. Furthermore,

$$\begin{aligned} s \cdot \beta_3 &= \beta_3 + \binom{1}{1}\beta_1 = \beta_3 + \beta_1 \\ s \cdot \beta_1 &= \beta_1 \end{aligned}$$

in $\mathbb{F}_3\{\beta_1, \dots\} \cong \widetilde{MU}_*\mathbb{CP}^\infty \otimes_{MU_*} \mathbb{F}_3$. It is easy to deduce that s acts on the pullback of $\pi_0\mathcal{F}_{X_6}$ to $\operatorname{Spec}\mathbb{F}_3$ via $\begin{pmatrix} 1 & 1 & ? \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. By a change of basis, this is equivalent to an action via the Jordan block J_3. Thus, $TMF \wedge X_2$ splits off $TMF \wedge \mathbb{CP}^n_{-1}$ and is equivalent to $\Sigma^{-2}TMF_0(2)$. For the higher degrees, we need as input that

$$\begin{aligned} s \cdot \beta_{6k-3} &= \beta_{6k-3} + \binom{6k-5}{1}\beta_{6k-5} + \binom{6k-7}{2}\beta_{6k-7} = \beta_{6k-3} + \beta_{6k-5} + \beta_{6k-7} \\ s \cdot \beta_{6k-5} &= \beta_{6k-5} + \binom{6k-7}{1}\beta_{6k-7} = \beta_{6k-5} - \beta_{6k-7} \\ s \cdot \beta_{6k-7} &= \beta_{6k-7} \end{aligned}$$

modulo parts of degree lower than $6k-7$. Then the arguments runs as in Proposition 7.5.

We obtain the calculation for $TMF \wedge \mathbb{CP}^n$ by the fact that the cofiber of the standard map $\Sigma^{-2}TMF \to \Sigma^{-2}TMF_0(2)$ is $\Sigma^2 TMF_\alpha$ [Ref!]. □

Corollary 7.8. *A homotopy commutative and homotopy associative $TMF_{(3)}$-algebra R is complex orientable iff $\alpha \cdot 1 = 0$ in π_*R.*

Proof. Recall that a complex orientation is a class in $R^2(\mathbb{CP}^\infty)$ restricting to the standard generator $1 \in R^2(\mathbb{CP}^1) \cong R^2(S^2) \cong \pi_0 R$. The above discussion shows that the map $TMF_{(3)} \wedge \mathbb{CP}^1 \to TMF_{(3)} \wedge \mathbb{CP}^\infty$ factors as

$$TMF_{(3)} \wedge \mathbb{CP}^1 \to Z \to TMF_{(3)} \wedge \mathbb{CP}^3 \to TMF_{(3)} \wedge \mathbb{CP}^\infty$$

for a $TMF_{(3)}$-module $Z \cong \Sigma^? TMF_\alpha$ such that $Z \to TMF_{(3)} \wedge \mathbb{CP}^\infty$ is the inclusion of a direct summand; thus, we have also a factorization

$$R^2(\mathbb{CP}^\infty) \to R^2(\mathbb{CP}^3) \to [Z, \Sigma^2 R]_{TMF_{(3)}} \to R^2(\mathbb{CP}^1).$$

Hence, it is enough to show that $1 \in R^2(\mathbb{CP}^1)$ has a lift to $R^2(\mathbb{CP}^3)$. Since $\eta = 0$ at the prime 3, we have $\mathbb{CP}^2 \cong \mathbb{CP}^1 \vee S^4$ at 3. Thus, we have a cofiber sequence

$$S^5 \to S^2 \vee S^4 \to \mathbb{CP}^3.$$

The map $S^5 \to \mathbb{CP}^1 \vee S^4$ is non-zero stably at 3, since the Steenrod power operation $\mathcal{P}^1$ is non-zero on $\mathbb{CP}^3$, thus the map is stably equivalent to $(\pm\alpha_1, 0)$ (where we identify $\mathbb{CP}^1$ with S^2 again). Thus, 1 lifts to $\mathbb{CP}^3$ exactly iff $\alpha \cdot 1 = 0$ in $\pi_* R$. □

7.3. **An infinite family of modules.** In this subsection, we will again localize at a set of primes A not containing 2 and write TMF for $TMF_{(A)}$.

It is easy to see that for every given rank, there are up to equivalence only finitely many standard TMF-modules. In contrast, we want to construct an infinite family of standard TMF-modules (of growing rank) that do not decompose into other standard modules. Roughly the example of an infinite family is the following: Consider

$$C(\beta^3, \beta^4, \beta^3, \ldots, \beta^4, \beta^3) \text{ and } C(\beta^3, \beta^4, \beta^3, \ldots, \beta^4),$$

i.e. the result of iteratively coning off (lifts of) powers of β. These TMF-modules exist since $\langle \beta^3, \beta^4, \beta^3, \ldots, \beta^4, \beta^3 \rangle$ and $\langle \beta^3, \beta^4, \beta^3, \ldots, \beta^3, \beta^4 \rangle$ lie in $\pi_k TMF$ with $k = 70$ or $k = 29$ mod 72 and these groups are zero. If one of these modules splits into two standard modules, it would have two invariant generators (in the sense of Section 5.2). The second generator would have to lift from a torsion element somewhere – which is not possible for degree reasons.

More precisely define $X_1 = TMF$ and $x_1 \in \pi_{30} TMF$ to be β^3. Now assume that X_k has been defined and also $x_k \in \pi_{30} X_k$ if k is odd or $x_k \in \pi_{71} X_k$ if k is even. Furthermore, we assume inductively that $\pi_{70} X_k = 0$ and $\pi_{29} X_k = 0$. Define $X_{k+1} = \mathrm{Cone}(\Sigma^{|x_k|} TMF \to X_k)$. First consider the case that k is odd. Then we have an exact sequence

$$\pi_{71} X_k \to \pi_{71} X_{k+1} \to \pi_{71} \Sigma^{31} TMF \to \pi_{70} X_k.$$

This implies that there is a lift of $\beta^4 \in \pi_{71} \Sigma^{31} TMF$ to $\pi_{71} X_{k+1}$, which we define to be x_{k+1} (any choice is possible). Furthermore, we see that $\pi_{70} X_{k+1} = 0$ since $\pi_{70} X_k = 0$ and $\pi_{39} TMF = 0$. The same way, we see that $\pi_{29} X_{k+1} = 0$ since $\pi_{29} X_k = 0$ and $\pi_{70} TMF = 0$.

Now consider the case that k is even. Then we have an exact sequence

$$\pi_{30} X_k \to \pi_{30} X_{k+1} \to \pi_{30} \Sigma^{72} TMF \to \pi_{29} X_k.$$

This implies that there is a lift of $\beta^3 \in \pi_{30} \Sigma^{72} TMF$ to $\pi_{30} X_{k+1}$, which we define to be x_{k+1} (again, any choice is possible). Furthermore, we see that $\pi_{70} X_{k+1} = 0$ since $\pi_{70} X_k = 0$ and $\pi_{70} TMF = 0$. The same way, we see that $\pi_{29} X_{k+1} = 0$ since $\pi_{29} X_k = 0$ and $\pi_{29} TMF = 0$.

Before we go on, we want to define an invariant of TMF-modules. For a TMF-module M, consider $\pi_* M / \mathrm{im}(r_*)$. This is an $\mathbb{F}_3[\Delta^{\pm 3}]$-vector space since $rc = 6$. Set now

$$d(M) := \dim_{\mathbb{F}_3[\Delta^{\pm 3}]} (F_0 \pi_* M / (\mathrm{im}(r_*) + F_1 \pi_* M)),$$

where $F_\bullet$ denotes the filtration of the descent spectral sequence.

Lemma 7.9. *If $\pi_0\mathcal{F}_M$ and $\pi_1\mathcal{F}_M$ consist of a direct sum of $\omega^{\otimes i}$ and $E_\alpha \otimes \omega^{\otimes i}$, M has an invariant generator if and only if $d(M) > 0$. Furthermore, d sends direct sums to sums.*

Proof. The first part follows from Lemma 5.24 and the proof of Proposition 5.25. The second part is clear. □

Proposition 7.10. *The TMF-modules X_k are not decomposable in the homotopy category of TMF-modules into TMF-standard modules. If an X_k decomposes, it decomposes into two algebraically standard modules of which exactly one has an invariant generator.*

Proof. Let $X_k \cong A \oplus B$ for some k with A and B non-zero. We want to show first that $\pi_*\mathcal{F}_A$ and $\pi_*\mathcal{F}_B$ are sums of shifts of $\pi_*\mathcal{O}^{top}$: We know that $\pi_*\mathcal{F}_{X_k}$ decomposes into an even part $\bigoplus \pi_*\mathcal{O}^{top}$ and an odd part $\bigoplus \pi_*\Sigma^{31}\mathcal{O}^{top}$ (using Corollary 4.9), which can be treated separately. It is enough to show that every direct summand $\mathcal{E}$ of $\bigoplus \mathcal{O} = \bigoplus \pi_0\mathcal{O}^{top}$ is again a direct sum of the form $\bigoplus \mathcal{O}$. We know that $\Gamma(\mathcal{E})$ is a projective $\Gamma(\mathcal{O})$-module. Thus, $\Gamma(\mathcal{E})$ is a free $\mathbb{Z}_{(A)}[j]$-module by Seshadri's Theorem, a special case of Serre's conjecture (see [Lam06], II.6.1). Choose a basis $(a_1, \ldots, a_n)$ of $\Gamma(\mathcal{E})$ as a $\mathbb{Z}_{(A)}[j]$-module and consider the associated morphism $f\colon \bigoplus_{i=1}^n \mathcal{O} \to \mathcal{E}$. For a complement $\mathcal{G}$ of $\mathcal{E}$ in $\pi_0\mathcal{F}_{X_k} \cong \bigoplus \mathcal{O}$, we can do the same and get a morphism $g\colon \bigoplus \mathcal{O} \to \mathcal{G}$. The morphism

$$f \oplus g\colon \bigoplus \mathcal{O} \to \mathcal{E} \oplus \mathcal{G} \xrightarrow{\cong} \bigoplus \mathcal{O}$$

is an isomorphism on Γ, hence of the vector bundles. Therefore, also f is an isomorphism and $\mathcal{E}$ is free.

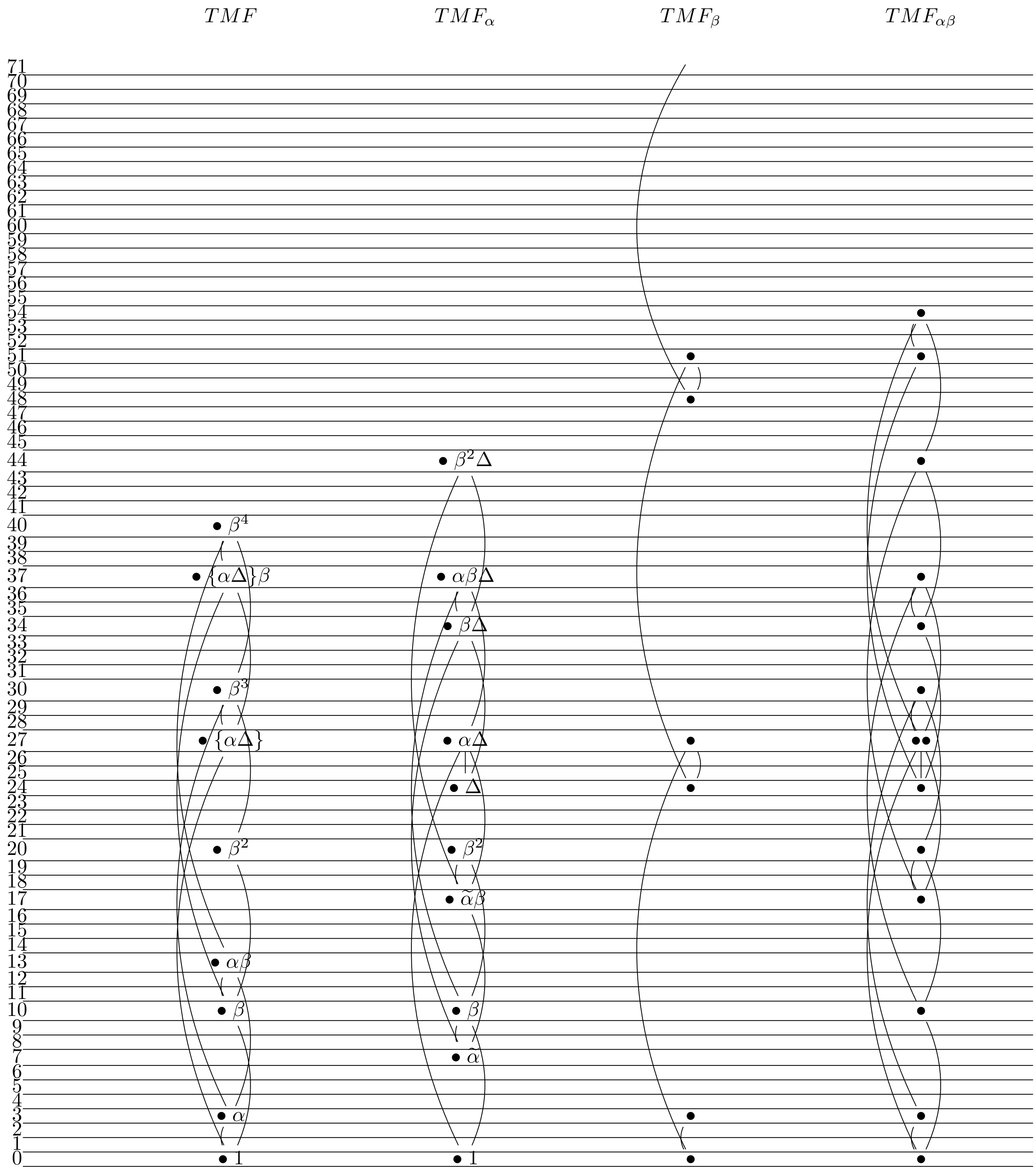
We want to prove by induction that $d(X_k) = 1$. This is obviously true for $k = 1$. The E^2-term of the DSS shows that X_k can have "generators" (that is, elements in $F_0\pi_*X_k/F_1\pi_*X_k$ which are not in $\operatorname{im}(r_*)$) only in dimensions $0, 24, 48, 31, 55$ and 7. It is easy to check that neither TMF nor $\Sigma^{31}TMF$ have any torsion there. So, given a generator x in π_*X_{k+1}, it has to map to some element y of (strict) filtration 0 in π_*TMF or $\pi_*\Sigma^{31}TMF$. Now note that $X_{k+1}(2)$ splits into $X_k(2)$ and (a suspension of) $TMF(2)$ and therefore every element in $\operatorname{im}(r_*)$ in TMF has a lift to an element in X_{k+1} (which lies also in $\operatorname{im}(r_*)$). The $\mathbb{Z}_{(A)}[\Delta^{\pm 3}]$-module $F_0\pi_*TMF/F_1\pi_*TMF$ is generated by $\operatorname{im}(r_*)$ and 1 by Scholium 3.17. Therefore, we can subtract from x an element z in $\operatorname{im}(r_*)$ and it maps (up to a unit) to 1 or 0 in TMF_*. But 1 cannot lift. Therefore, $x = i_*(x') + z$, where $x' \in \pi_*X_k$ is of strict filtration 0 and $i_*\colon \pi_*X_k \to \pi_*X_{k+1}$ is the map given by the construction of X_{k+1}. Since x is not in $\operatorname{im}(r_*)$, x' cannot be in $\operatorname{im}(r_*)$. Hence, x' is a generator and generators are by induction unique in π_*X_k up to the image of r. Therefore, generators in π_*X_{k+1} are unique up to multiplication by units and addition of $(\operatorname{im}(r_*) + F_1\pi_*X_{k+1})$ and $d(X_{k+1}) = 1$ follows.

Thus, $d(A) = 0$ and $d(B) = 1$ or $d(A) = 1$ and $d(B) = 0$. The result follows now from the last lemma. □

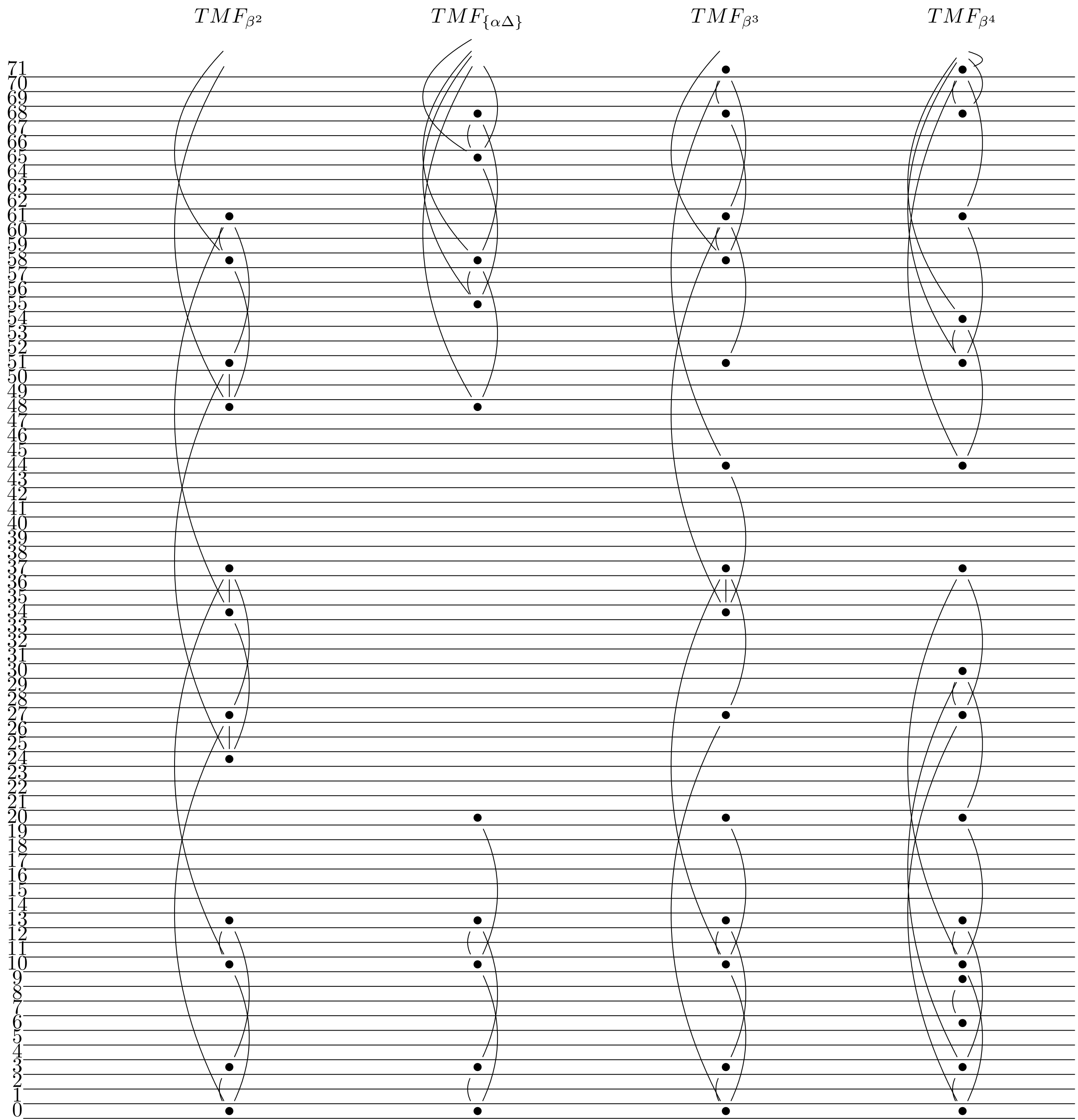
7.4. **Low-Rank Examples.** We want to present some examples of $TMF_{(3)}$-modules. Since we are mostly interested in torsion, we depict just $\pi_*M/\operatorname{im}(r_*)$ in the pictures, where every $\bullet$ stands for one $\mathbb{F}_3$. Thus, the same pictures can also be interpreted as pictures of $TMF_{(A)}$-modules for $2 \notin A$ as $\operatorname{coker}(r_*)$ does not see the difference. The (bend) vertical lines allude to non-zero multiplication by α, β or $\{\alpha\Delta\}$, depending on their length.

The computations of the homotopy groups of these low rank examples are straightforward from the knowledge of $\pi_*TMF_{(3)}$ – the latter can be obtained from $\pi_*tmf_{(3)}$ (as computed in [Bau08]) by inverting Δ. The action of α, β and $\{\alpha\Delta\}$ is computed via triple Toda bracekts using 4.15. Note that TMF_x denotes the cone of the map $\Sigma^{|x|}TMF_{(3)} \to TMF_{(3)}$ corresponding to an element $x \in \pi_* TMF_{(3)}$.

We have only named the elements in the cases of $TMF_{(3)}$ and TMF_α as these are the most important cases for us. The element $\widetilde{\alpha} \in \pi_7 TMF_\alpha$ lifts $\alpha \in \pi_3 TMF$. The element $\Delta \in \pi_{24} TMF_\alpha$ lifts $\beta^2 \in \pi_{20} TMF$; we call it Δ because it is in zeroth filtration in the DSS and can be chosen to reduce to Δ. Indeed, by the Toda bracket $\{3\Delta\} \in \langle \alpha, \beta^2, 3\rangle$, it follows that thrice this element is the image of $\{3\Delta\} \in \pi_{24} TMF$. Recall here the convention that we write $\{3\Delta\}$ instead of 3Δ for the element in $\pi_{24} TMF$ reducing to 3Δ as Δ itself is not a permanent cycle.



TMF
TMF_α
TMF_β
$TMF_{\alpha\beta}$
β^4
$\{\alpha\Delta\}\beta$
β^3
$\{\alpha\Delta\}$
β^2
$\alpha\beta$
β
α
1
$\beta^2\Delta$
$\alpha\beta\Delta$
$\beta\Delta$
$\alpha\Delta$
Δ
β^2
$\widetilde{\alpha}\beta$
β
$\widetilde{\alpha}$
1



TMF_{β^2}
$TMF_{\{\alpha\Delta\}}$
TMF_{β^3}
TMF_{β^4}
71 70 69 68 67 66 65 64 63 62 61 60 59 58 57 56 55 54 53 52 51 50 49 48 47 46 45 44 43 42 41 40 39 38 37 36 35 34 33 32 31 30 29 28 27 26 25 24 23 22 21 20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0

References

[Ada74] J. F. Adams. *Stable homotopy and generalised homology*. University of Chicago Press, Chicago, Ill.-London, 1974. Chicago Lectures in Mathematics. 46

[Ati66] M. F. Atiyah. K-theory and reality. *Quart. J. Math. Oxford Ser. (2)*, 17:367–386, 1966. 5

[Aut] The Stacks Project Authors. *Stacks Project*. `http://math.columbia.edu/algebraic_geometry/stacks-git`. 12

[Bau03] Tilman Bauer. Elliptic cohomology and projective spaces – a computation. `https://people.kth.se/~tilmanb/cpinfty.pdf`, 2003. 45

[Bau08] Tilman Bauer. Computation of the homotopy of the spectrum `tmf`. In *Groups, homotopy and configuration spaces*, volume 13 of *Geom. Topol. Monogr.*, pages 11–40. Geom. Topol. Publ., Coventry, 2008. 10, 16, 28, 50

[Beh06] Mark Behrens. A modular description of the $K(2)$-local sphere at the prime 3. *Topology*, 45(2):343–402, 2006. 8, 9, 34

[Bou90] A. K. Bousfield. A classification of K-local spectra. *J. Pure Appl. Algebra*, 66(2):121–163, 1990. 3, 4, 7

[DFHH14] Christopher L. Douglas, John Francis, André G. Henriques, and Michael A. Hill, editors. *Topological modular forms*, volume 201 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2014. 15, 17

[DR73] P. Deligne and M. Rapoport. Les schémas de modules de courbes elliptiques. In *Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, pages 143–316. Lecture Notes in Math., Vol. 349. Springer, Berlin, 1973. 9, 25

[FO10] William Fulton and Martin Olsson. The Picard group of $\mathcal{M}_{1,1}$. *Algebra Number Theory*, 4(1):87–104, 2010. 11

[HM17] Michael Hill and Lennart Meier. The C_2–spectrum $Tmf_1(3)$ and its invertible modules. *Algebraic & Geometric Topology*, 17(4):1953–2011, 2017. 10

[KM85] Nicholas M. Katz and Barry Mazur. *Arithmetic moduli of elliptic curves*, volume 108 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1985. 45

[Lam06] T. Y. Lam. *Serre's problem on projective modules*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2006. 50

[Lur09a] J. Lurie. Derived Algebraic Geometry VIII: Quasi-Coherent Sheaves and Tannaka Duality Theorems. 2009. 16

[Lur09b] J. Lurie. A survey of elliptic cohomology. In *Algebraic topology*, volume 4 of *Abel Symp.*, pages 219–277. Springer, Berlin, 2009. 44

[Lur09c] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009. 16

[Lur17] Jacob Lurie. Spectral algebraic geometry. `http://www.math.harvard.edu/~lurie/papers/SAG-rootfile.pdf`, 2017. 15

[Mat13] Akhil Mathew. The homology of tmf. *arXiv preprint arXiv:1305.6100v4*, 2013. 5, 6, 18

[Mei12] Lennart Meier. United elliptic homology (thesis). `http://people.virginia.edu/~flm5z/Thesis-FinalVersionb.pdf`, 2012. 20, 23, 30, 41, 45

[Mei15] Lennart Meier. Vector bundles on the moduli stack of elliptic curves. *J. Algebra*, 428:425–456, 2015. 10, 11, 12, 15, 20, 23, 32

[Mei16] Lennart Meier. (Topological) modular forms with level structure. *arXiv preprint arXiv:1609.09264*, 2016. 20

[ML63] Saunders Mac Lane. *Homology*. Die Grundlehren der mathematischen Wissenschaften, Bd. 114. Academic Press Inc., Publishers, New York, 1963. 13

[MM15] Akhil Mathew and Lennart Meier. Affineness and chromatic homotopy theory. *J. Topol.*, 8(2):476–528, 2015. 2, 3, 16, 17, 18, 23

[MR09] Mark Mahowald and Charles Rezk. Topological modular forms of level 3. *Pure Appl. Math. Q.*, 5(2, Special Issue: In honor of Friedrich Hirzebruch. Part 1):853–872, 2009. 20, 23

[MS16] Akhil Mathew and Vesna Stojanoska. The Picard group of topological modular forms via descent theory. *Geom. Topol.*, 20(6):3133–3217, 2016. 30

[Rav86] Douglas C. Ravenel. *Complex cobordism and stable homotopy groups of spheres*, volume 121 of *Pure and Applied Mathematics*. Academic Press Inc., Orlando, FL, 1986. 19

[Rog08] John Rognes. Galois extensions of structured ring spectra. Stably dualizable groups. *Mem. Amer. Math. Soc.*, 192(898):viii+137, 2008. 5, 6

[Sil09] Joseph H. Silverman. *The arithmetic of elliptic curves*, volume 106 of *Graduate Texts in Mathematics*. Springer, Dordrecht, second edition, 2009. 10

[Sto12] Vesna Stojanoska. Duality for topological modular forms. *Doc. Math.*, 17:271–311, 2012. 9

Mathematisches Institut, Endenicher Allee 60, 53115 Bonn, Germany
E-mail address: `lmeier@math.uni-bonn.de`