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Dicebat Bernardus Carnotensis nos esse quasi nanos, gigantium humeris insidentes, ut possimus plura eis et remotiora videre, non utique proprii visus acumine, aut eminentia corporis, sed quia in altum subvenimur et extollimur magnitudine gigantea.
(John of Salisbury)

If I have seen further it is by standing on ye sholders of Giants.
(Isaac Newton)

Dedicated to these Giants and my love.

Abstract

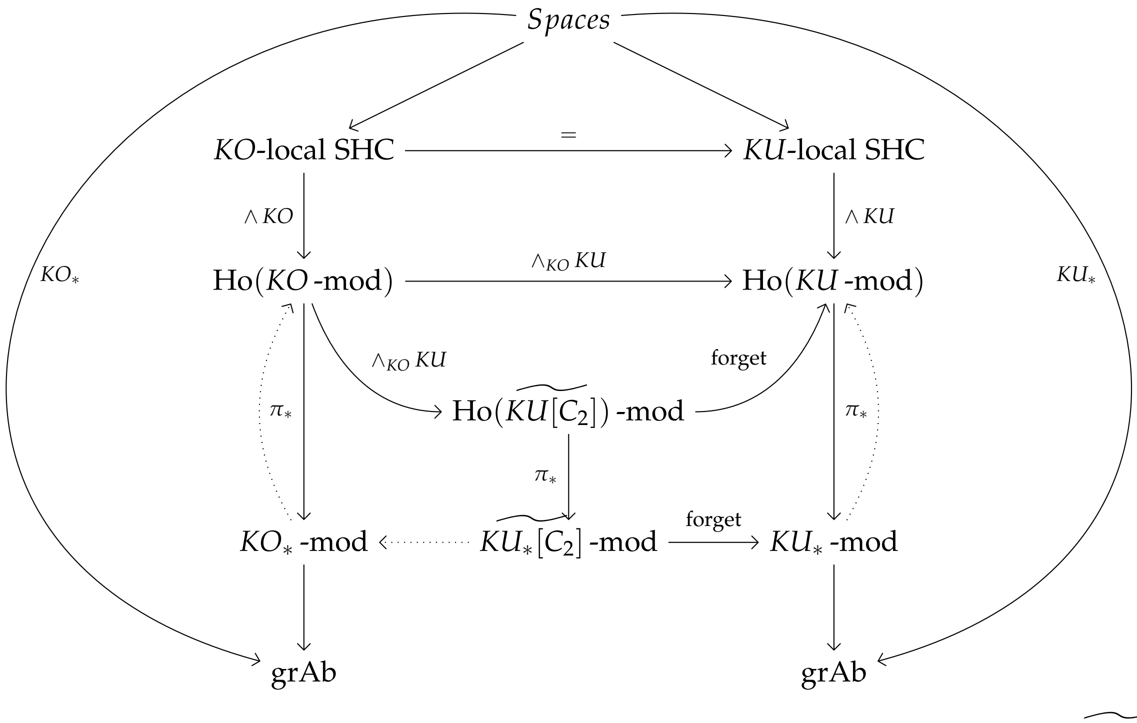
We study the categories of $KO-$ and TMF-modules. Inspired by work of Bousfield, we consider TMF-modules M at the prime 3 such that $M \wedge_{TMF} TMF(2)$ is a free $TMF(2)$-module. We show that a large class of these can be iteratively built from TMF by coning off torsion elements and killing generators. This is based on a detailed study of vector bundles on the moduli stack of elliptic curves. Furthermore, we consider examples of TMF-modules and also the relationship between the category of TMF-modules and the category of quasi-coherent sheaves on the derived moduli stack of elliptic curves.

Contents

Chapter 1

Introduction

Homology theories belong to the core techniques of algebraic topology. In the usual definition, a homology theory takes values in graded abelian groups. Yet it is well known that one often has extra structure. For example, ordinary homology with real coefficients takes values in graded $\mathbb{R}$-vector spaces and $\mathbb{F}_p$-homology in (graded) comodules over the dual Steenrod algebra. In addition, a homology theory factors through various homotopy categories. We present the example of real and complex K-theory in the form of the following commutative diagram:



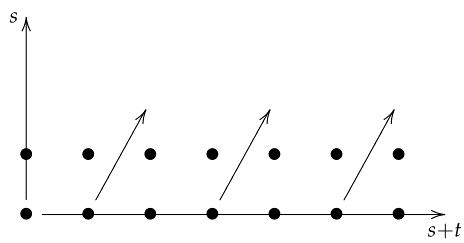
Here, SHC is an abbreviation for the stable homotopy category. Furthermore, $\widetilde{KU[C_2]}$ stands for category of KU-modules with a C_2-action which is semilinear with respect to complex conjugation; similarly, $\widetilde{KU_*[C_2]}$ stands for the category of KU_*-modules with semilinear C_2-action.

The diagram suggests that while the KO- or KU-local stable homotopy category may capture nearly all of the information real or complex K-theory tells us about a space, the homotopy category of KO- or KU-modules might be a useful approximation. Indeed, Bousfield used the theory of *KO*-modules in an essential way in his study of the *KO*-local category in [Bou90]. We want to review Bousfield's results on *KO*-modules from our perspective.

The dotted arrows in the diagram above indicate spectral sequences one can use for computation, which we want to describe in greater detail: Let R be a ring spectrum and M and N be module spectra. Then there is the universal coefficient spectral sequence (UCSS)

$$E^2 = \mathrm{Ext}^s_{R_*}(\pi_* M, \pi_* N[t]) \Rightarrow [M, N]^{s+t}_R$$

converging to the graded morphisms in the homotopy category of R-modules (for $\pi_* N[t] = \pi_{*-t} N$). The edge homomorphism sends an element in $[M, N]^k$ to the induced map in $\mathrm{Hom}_{R_*}(\pi_* M[k], \pi_* N)$. For example, we might consider the case of $R = KU$. We know that every graded module over $KU_* \cong \mathbb{Z}[u^{\pm 1}]$ has projective dimension at most 1. Therefore, the spectral sequence is concentrated in the first two rows and all differentials must vanish.



If we have two KU-modules M and N with an isomorphism $\overline{f}\colon \pi_* M \to \pi_* N$, then this isomorphism is realized as a map $f\colon M \to N$, which is then an isomorphism (in the homotopy category) of KU-modules. Therefore, the functor π_* classifies KU-modules in the sense that it detects isomorphisms. We can apply the same arguments to KO localized at an odd prime p. Both for $R = KU$ and $R = KO_{(p)}$ it follows by results of Franke and Patchkoria ([Pat11], 5.2.1) that the homotopy category of R-modules is equivalent to the derived category of graded R_*-modules. Thus, we get a good understanding of the homotopy category of KU-modules and $KO_{(p)}$-modules for an odd prime p.

Now it is known that KO_* has infinite homological dimension.[1] This means that the UCSS is potentially spread over the whole half-plane for these two ring spectra and we cannot use the approach above directly.

While the usual UCSS is based on resolutions by free modules, it is also possible to construct a modified UCSS based on *relatively free* KO-modules, i.e., (finite) KO-modules M such that $M \wedge_{KO} KU$ is a free KU-module. More precisely, based on ideas of [Bou90], Wolbert constructs in [Wol98] for $\mathcal{F}$ the collection of relatively free KO-modules a modified Ext-functor $\mathrm{Ext}_{\mathcal{F}}$ and a modified homotopy groups functor $\pi^{\mathcal{F}}_*$, which serve as input for a spectral sequence of the form

$$\mathrm{Ext}^s_{\mathcal{F}}(\pi^{\mathcal{F}}_*(M), \pi^{\mathcal{F}}_*(N)[t]) \Rightarrow [M, N]^{s+t}_{KO}$$

[1] We will sketch a proof for KO_* which is also valid for TMF_* and a much wider class of graded rings: Assume KO_* has finite global dimension. Let R denote the ungraded version of this ring. By [BH93, p.33], $\mathrm{Ext}^i_R(\mathbb{F}_2, M) \neq 0$ can only be true with bounded i for an R-module M. This is then also true after localizing R at the prime ideal $(2, \eta, \xi)$ unhomogeneously. But this localization is not a regular local ring. This is contradiction by [Eis95], Section 19, especially 19.12.

for KO-modules M and N. From the fact that KU_* has homological dimension 1 one can deduce by rather formal reasons that this spectral sequence is concentrated in the first two lines. Thus, $\pi_*^{\mathcal{F}}$ detects whether two KO-modules are isomorphic as above. The collection of relatively free KO-modules with all KO-module maps between them is called the *united K-theory* and the functor $\pi_*^{\mathcal{F}}$ is called the *united K-theory functor*.

Bousfield has a more explicit description of the functor $\pi_*^{\mathcal{F}}$ using the KO-modules KO, KU and KT, where the latter stands for K-theory with self-conjugation. It is easy to see that we can recover Bousfield's result (for finite modules) if we show that KO, KU and KT are (up to suspension) the only indecomposable relatively free KO-modules with respect to KU. Our approach is to introduce the following notion:

Definition 1.0.1. Let R be a ring spectrum. We define inductively the notion of a (topologically) *standard module*. First of all, all suspensions of R are standard modules. Then, if M is a standard module and $x \in \pi_* M$ is a torsion element, the cone of the map $\Sigma^{|x|} R \xrightarrow{x} M$ is a standard module. The collection of all standard modules is the collection of all R-modules which can obtained by this procedure in finitely many steps.

In Chapter 7 we will show then the following two propositions:

Proposition 1.0.2 (K-theory extension theorem)**.** *Every relatively free KO-module is a standard module.*

Proposition 1.0.3. *Every standard KO-module is a direct sum of suspensions of KO, KU and KT.*

This recovers then Bousfield's result (in the case of finite modules). Two of the three proofs we give for the K-theory extension theorem use a homotopy fixed point spectral sequence computing the homotopy groups of a KO-module M from $H^i(C_2; \pi_* M \wedge_{KO} KU)$. This can be also interpreted as an UCSS in the category $\widetilde{KU[C_2]}$-mod using that the functor $KO\text{-mod} \xrightarrow{\wedge_{KO} KU} \widetilde{KU[C_2]}\text{-mod}$ is an equivalence.

If one sees some story on KO, one often asks oneself: How about the spectrum of topological modular forms TMF? It will be the main aim of this thesis to investigate to what extent the above results translate to similar results in the more difficult world of TMF-modules.

For TMF localized at a prime p greater than 3, we have $(TMF_{(p)})_* \cong \mathbb{Z}_{(p)}[c_4, c_6, \Delta^{-1}]$, the ring of modular forms. This has homological dimension two.[2] Thus, we get in a similar way as above $Ho(TMF_{(p)}\text{-mod}) \simeq \mathcal{D}((TMF_{(p)})_*)$ by [Pat11], 1.1.3. Therefore, we want to concentrate on lower primes; more specifically, we will implicitly localize at 3 in

[2]The idea of proof is the following: Let M be a $\mathbb{Z}_{(p)}[c_4, c_6, \Delta^{-1}]$-module. Take an exact sequence

$$0 \to N \to P_1 \to P_0 \to M \to 0$$

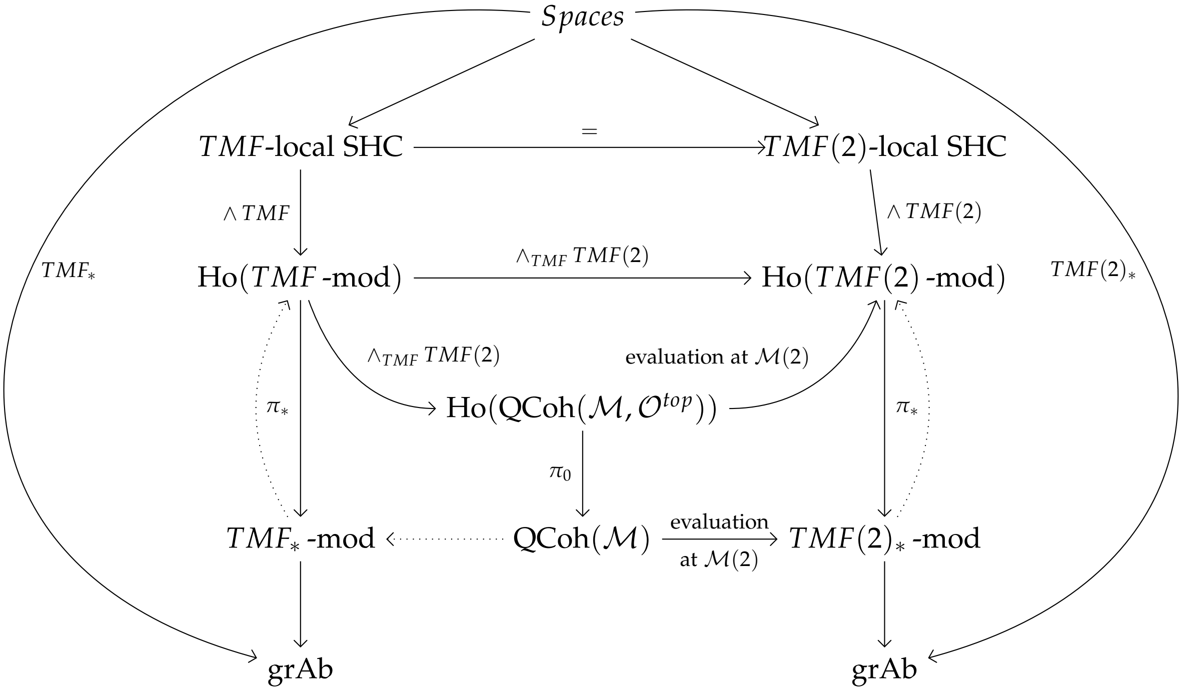
in the category of $\mathbb{Z}_{(p)}[c_4, c_6]$-modules such that P_0 and P_1 are projective. Since $\mathbb{Z}_{(p)}[c_4, c_6, c_4^{-1}]$ and $\mathbb{Z}_{(p)}[c_4, c_6, c_6^{-1}]$ have homological dimension ≤ 2, $N[c_4^{-1}]$ and $N[c_6^{-1}]$ are projective. If a module over $\mathbb{Z}_{(p)}[c_4, c_6]$ is projective if we invert c_4 and if we invert c_6, then it is also projective if we invert Δ because projectivity corresponds to being locally free on the spectrum and Δ can only be non-vanishing when c_4 or c_6 is (since 1728 is invertible). Thus,

$$0 \to N[\Delta^{-1}] \to P_1[\Delta^{-1}] \to P_0[\Delta^{-1}] \to M \to 0$$

is a projective resolution of M of length 2 in the category of $\mathbb{Z}_{(p)}[c_4, c_6, \Delta^{-1}]$-modules.

the following since at the prime 2 computations are much more difficult and most of our proofs do not work there.

The spectrum of topological modular forms TMF is constructed as the global sections of a certain sheaf of commutative ring spectra $\mathcal{O}^{top}$ on the moduli stack of elliptic curves $\mathcal{M}$. By evaluating $\mathcal{O}^{top}$ at the moduli stack of elliptic curves with level-2-structure $\mathcal{M}(2)$, we get a TMF-algebra called $TMF(2)$ with $TMF(2)_* \cong \mathbb{Z}_{(3)}[x_2, y_2, \Delta^{-1}]$. Thus, $TMF(2)_*$ has homological dimension 2 and can serve as an analogue of KU in the TMF-setting. As before, we have a diagram:



While we hope to apply our results at some point to the TMF-local stable homotopy category (or rather the $E(2)$-local one), this thesis will not contain any further discussion of the TMF-local stable homotopy category; we will concentrate on the category of TMF-modules.

As in the case of KO_*, one can show that TMF_* has infinite homological dimension. So, we want again to study *relatively free/projective* TMF-modules M in the sense that M is finite and $M \wedge_{TMF} TMF(2)$ is a free/projective $TMF(2)$-module. It is easy to see that the (derived) quasi-coherent sheaf on $(\mathcal{M}, \mathcal{O}^{top})$ associated to M is locally free if M is relatively free and thus the associated (classical) quasi-coherent sheaf on $\mathcal{M}$ is a vector bundle. Since we have an important spectral sequence, which has as input the cohomology of this vector bundle and converges to $\pi_* M$, the study of vector bundles on $\mathcal{M}$ becomes crucial.

Definition 1.0.4. We define inductively the notion of a *standard vector bundle* on $\mathcal{M}$. First of all, all line bundles are standard vector bundles. In addition, a vector bundle $\mathcal{E}$ is called *standard* if there is an injection $\mathcal{L} \hookrightarrow \mathcal{E}$ from a line bundle such that the cokernel is a standard vector bundle.

That every standard vector bundle is an iterated extension of line bundles will allow us to classify standard vector bundles; there are only finitely many indecomposable ones. We define a relatively free TMF-module M to be *algebraically standard* if $\pi_0 \mathcal{F}_M$ and $\pi_1 \mathcal{F}_M$ are standard vector bundles. It is unclear to the author whether every algebraically standard TMF-module is also standard, but we can define a slightly weaker notion:

Definition 1.0.5. We define the notion of a finite TMF-module being *hook-standard* inductively: First, $\Sigma^k TMF$ is hook-standard for all k. Furthermore, a TMF-module M is hook-standard if there are cofiber sequences

$$\Sigma^{|a|}TMF \xrightarrow{a} M \to X$$
$$\Sigma^{|x_1|}TMF \xrightarrow{x_1} X \to X'$$
$$\Sigma^{|x_2|}TMF \xrightarrow{x_2} X' \to X''$$

with X'' hook standard, where a corresponds to a torsion element and $c_*(x_1) \in E(X)$ and $c_*(x_2) \in E(X')$.

It is called 'hook-standard' since going up one rank and going down two ranks looks like a hook. Our main theorem is:

Theorem 1.0.6 (The hook theorem)**.** *Every algebraically standard TMF-module is a hook-standard module.*

For ranks ≤ 3, all algebraically standard modules are even standard. This allows, in principle, to classify all algebraically standard TMF-modules up to a certain rank, although computations quickly become complicated with growing rank.

If one looks for an analogy to the K-theory story, one might expect that there are only finitely many indecomposable standard modules. But the torsion of TMF is much more complicated (even at 3) and we can show the following:

Proposition 1.0.7. *There is an infinite sequence of standard modules (of arbitrary high rank) which do not decompose into standard modules of lesser rank.*

The infinitude of indecomposable relatively free TMF-modules makes it harder to use a modified UCSS in the case of TMF. Nevertheless, as indicated at the end of Section 4.3, for every finite TMF-module M, there is a kind of resolution of M into relatively projective modules. With other words, we have for every TMF-module a short resolution via relatively projective modules, which reduces the study of TMF-modules largely to the study of relatively free modules. The collection of all relatively free TMF-modules with all TMF-module maps between them deserves the name *united elliptic homology*.

As KO-mod $\simeq \widetilde{KU[C_2]}$-mod, the ∞-categories of quasi-coherent sheaves on the derived moduli stack of elliptic curves and the one of TMF-modules are equivalent as shown in Chapter 6. Unfortunately, the equivalence is only an abstract equivalence and we do not know if the global sections functor is an equivalence.

As a last point, we look at relatively free TMF-modules of the form $TMF \wedge X$ for a space X, both at the prime 2 and 3. For example, we show that $TMF \wedge \mathbb{CP}^\infty$ splits into summands of rank 2 and 3. It remains an open question whether we can find an infinite sequence of indecomposable relatively free TMF-modules of the form $TMF \wedge X_i$ for spaces X_i. In contrast, we show that $tmf \wedge BU(2)$ (for tmf being connective TMF) has an indecomposable summand (as tmf-module) of infinite rank.

All in all, many questions remain open and so the reader might view this thesis as a collection of preliminary studies on TMF-modules with an eye towards the study of the $E(2)$-local stable homotopy category.

After summarizing the results, we should hint at the structure of this thesis. The proofs of the K-theory extension theorem and the hook theorem rely crucially on algebraic results classifying integral representations of C_2 (for K-theory) and vector bundles on the moduli stack of elliptic curves (for TMF). Part I is purely algebraic and its main task is to prove the classification result for standard vector bundles (in Chapter 3) and also to provide in Chapter 2 foundations for the study of algebraic stacks in general and the moduli stack of elliptic curves in particular.

Part II is about the topological fruits of these algebraic enterprises. The Chapter 4 is mainly about some foundational material of abstract homotopy theory, module categories and derived algebraic geometry. The Section 4.3 gives more details on the treatment of relatively free modules in this introduction and the modified universal coefficient spectral sequence. The Chapter 5 introduces the main object of our study, the spectrum of topological modular forms TMF, and collects a few of its basic properties. As already mentioned, in Chapter 6 we will compare TMF-modules with quasi-coherent sheaf on the derived moduli stack of elliptic curves and will also study Galois extensions of TMF. The task of Chapter 7 will be to reprove Bousfield's results about the classification of relatively free KO-modules (in three ways). Chapter 8 is in some sense the core of this thesis and proves several properties of relatively free TMF-modules, especially the hook theorem, and it is probably the most technically complicated part of this thesis. In the last chapter, we will study some examples and construct, in particular, the infinite sequence of (indecomposable) TMF-modules. The appendix contains the details of some computer calculations and a list of notation.

Warning 1.0.8. Some time ago, I thought that I had a proof that every finite TMF-module is standard. I have stated this in several talks and offer my apologies since the proof was marred by two mistakes, which were discovered in March and June 2012.

Remark 1.0.9. Two words about referencing: One of our common sources is Jacob Lurie's DAG (Derived Algebraic Geometry). This can be (only) found on his homepage and we number the parts of DAG by Roman numbers (just as on his homepage). Another common source for us is the Stacks Project ([Aut]), an open source textbook on algebraic geometry. Since it is always changing, there is a system of tags that does not change. You can search for tags in the Stacks Project on the following web site: `http://www.math.columbia.edu/algebraic_geometry/stacks-git/query.php`.

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Part I

Stacks and Vector Bundles

Chapter 2

Moduli Stacks

The language of stacks is essential for this whole thesis. A friendly (and not too long) introduction may be found in [Góm01] and an in-depth treatments in [LMB00] and [Aut]. A treatment of Grothendieck topologies (and the more categorical aspects of stacks) is included in [Vis05]. The classical source on the moduli stack of elliptic curves is [DR73], where, for example, level structures and representability statements are discussed. An introduction to algebraic stacks in general and the stack of formal group in particular can be found in [Nau07]. We will review parts of the theory for the convenience of the reader and have to stress that, except for some minor points, this chapter contains no original research.

2.1 Stacks and Descent

Many moduli problems cannot be represented by schemes. One reason is that a functor representable by a scheme is a sheaf of sets, but many geometric objects (vector bundles, elliptic curves, ...) can be locally trivial without being globally trivial since we can use non-trivial automorphisms to glue them. The language of stacks is a way to study moduli problems with non-trivial automorphisms.

Let S be a base scheme and $\mathrm{Sch}\,/S$ be the categories of schemes over S (i.e., the overcategory of S). For our purposes, we have most of the time $S = \operatorname{Spec}\mathbb{Z}$ or $S = \operatorname{Spec}\mathbb{Z}_{(p)}$ for p a prime. Several Grothendieck topologies[1] can be chosen on the category of schemes and each of these restricts to a Grothendieck topology on $\mathrm{Sch}\,/S$. Three of the most important topologies are the Zariski, the étale and the fpqc topology, where the open covers consist of surjective morphisms which are

- disjoint unions of open immersions in the Zariski topology,
- étale in the étale topology, respectively,
- fpqc in the fpqc topology.

Here, a morphism $X \to Y$ is called *fpqc* if it is faithfully flat and has the property that every quasi-compact open subset (or, at least, every element of an affine open cover, see [Vis05, 2.33]) of Y is the image of a quasi-compact open subset of X. Recall that a morphism is

[1] For this notion, see [Vis05, Section 2.3]. Recall also that a *site* is just a category equipped with a Grothendieck topology.

called faithfully flat if it is flat and surjective. A morphism is called *étale* if it is flat and unramified. For the notion of a flat map, see [Har77], III.9, and for more information on étale and unramified morphisms, see [BLR90], 2.2.

An important property of the Zariski, étale and fpqc topology is that they are *subcanonical* in the following sense: For every scheme T over S, the presheaf $\operatorname{Hom}_S(-, T)$ on Sch/S is actually a sheaf with respect to these three topologies. Furthermore, every Zariski open cover is an étale open cover and every étale open cover is also an fpqc cover. Another important property of these three topologies is that if $U_i \to X$ for $i \in I$ are open covers, then also $\coprod_{i \in I} U \to X$ is an open cover. Thus it makes sense to define that $\{U_i \to X\}_{i \in I}$ is an open cover if $\coprod U_i \to X$ is an open cover.

A stack can be thought of as a sheaf of groupoids with respect to a choice of Grothendieck topology. Since taking pullback is usually only associative up to canonical isomorphism, one has to use 2-categorical language to make this precise. Therefore, one usually takes another route: The datum of a stack is a category $\mathcal{X}$ together with a functor $F\colon \mathcal{X} \to \mathcal{C}$ for a site $\mathcal{C}$. If F makes $\mathcal{X}$ into a category fibered in groupoids over $\mathcal{C}$ and $\mathcal{X}$ satisfies descent with respect to the Grothendieck topology on $\mathcal{C}$, $\mathcal{X}$ is called a *stack over* $\mathcal{C}$. If $\mathcal{C} = \operatorname{Sch}/S$ with some topology, then we speak of a *stack over* S. For the precise definitions of these terms, see [Góm01], Section 2. The (2-)category of stacks is the full (2-)subcategory of stacks of the category of categories over Sch/S. One sometimes denotes the fiber of F over a scheme T by $\mathcal{X}(T)$.

Definition 2.1.1. Let

$$\begin{array}{ccc} & & \mathcal{C} \\ & & \downarrow{\scriptstyle F} \\ \mathcal{D} & \xrightarrow{G} & \mathcal{E} \end{array}$$

be a diagram of categories fibered over a common category $\mathcal{G}$. The *fiber product* $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$ is defined as follows: An object in $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$ consists of a triple (c, d, f), where $c \in \operatorname{Ob} \mathcal{C}$, $d \in \operatorname{Ob} \mathcal{D}$ and $f\colon F(c) \to G(d)$ is an isomorphism. A morphism from (c, d, f) to (c', d', f') consists of two morphisms $g_{\mathcal{C}}\colon c \to c'$ and $g_{\mathcal{D}}\colon d \to d'$ such that $f' \circ F(g_{\mathcal{C}}) = G(g_{\mathcal{D}}) \circ f$. The fiber functor to $\mathcal{G}$ is defined as the composition $\mathcal{C} \times_{\mathcal{E}} \mathcal{D} \to \mathcal{C} \to \mathcal{G}$ and gives $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$ the structure of a fibered category again.

Example 2.1.2. For an arbitrary site $\mathcal{C}$, the groupoid of sheaves on it forms a stack. More precisely, define a category $\mathcal{X}$, where an object is a sheaf on $\mathcal{C}/U$ for some $U \in \mathcal{C}$. A morphism between $(\mathcal{F}, U)$ and $(\mathcal{G}, V)$ consists of a morphism $f\colon U \to V$ in $\mathcal{C}$ and an isomorphism $\mathcal{F} \to f^*\mathcal{G}$. The fiber functor is given by $(\mathcal{F}, U) \mapsto U$. It can be easily checked that this is a stack. Note that this is a general procedure producing out of a groupoid valued (2-)functor a category fibered in groupoids, the Grothendieck construction (see also [Góm01, bottom of p.8]).

Example 2.1.3. Let Sch/S be equipped with the fpqc topology. Consider the groupoid valued (pre)sheaf on Sch/S given by $U \mapsto \operatorname{QCoh}(U)$, the groupoid of quasi-coherent sheaves. Then, its Grothendieck construction forms a fpqc stack (see, for example, [Vis05, Section 4.2]). Spelled out, this means, in particular, the following: Let $f\colon Y \to X$ be fpqc. Then a quasi-coherent sheaf $\mathcal{F}$ on X is uniquely specified by the sheaf $f^*\mathcal{F}$ on Y together with an isomorphism $\operatorname{pr}_1^* f^*\mathcal{F} \to \operatorname{pr}_2^* f^*\mathcal{F}$ on $Y \times_X Y$ (satisfying a cocycle condition). One can use these results to show that the category of quasi-coherent sheaves on some U is equivalent to the category of those sheaves of $\mathcal{O}$-modules on Sch $/U$ in the fpqc-topology that have locally a presentation $\bigoplus_I \mathcal{O} \to \bigoplus_J \mathcal{O} \to \mathcal{F} \to 0$ (see [Aut, 03DX]).

Let $\mathcal{C}$ be a site with an action of a group G, i.e., we have an action of G on the category $\mathcal{C}$ preserving the notion of a cover. We define $G - \mathcal{C}$ to be the category consisting of the same objects as $\mathcal{C}$ and, as morphisms, pairs $(g, \phi)\colon x \to y$, consisting of a $g \in G$ and a morphism $\phi\colon g \cdot x \to y$ in $\mathcal{C}$. The composition is given by $(h, \Psi) \circ (g, \Phi) = (hg, \Psi \circ (h \cdot \Phi))$. A morphism in $G - \mathcal{C}$ is defined to be an open cover if the image under the map $\mathrm{Mor}(G - \mathcal{C}) \to \mathrm{Mor}(\mathcal{C})$, given by $(g, \phi) \mapsto \phi$, is an open cover. We have an (inclusion) functor $i\colon \mathcal{C} \to G - \mathcal{C}$. The datum of a sheaf $\mathcal{F}$ on $G - \mathcal{C}$ is equivalent to giving a sheaf $\mathcal{F}'$ on $\mathcal{C}$ together with isomorphisms $f_g\colon \mathcal{F}' \to g^*\mathcal{F}'$ such that $i^*\mathcal{F} = \mathcal{F}'$ and $f_{gh} = f_h \circ f_g$ (here $g^*\mathcal{F}'$ denotes the pullback of the presheaf $\mathcal{F}'$ along the functor $g\colon \mathcal{C} \to \mathcal{C}$). Sheaves on $G - \mathcal{C}$ are called *G-equivariant sheaves on $\mathcal{C}$*.

Definition 2.1.4. For G a finite group, an *(étale) G-torsor* over a scheme Y consists of an étale cover $X \to Y$ with a G-action of X over Y such that the morphism

$$\begin{array}{ccc} X \times G & \to & X \times_Y X \\ (x, g) & \mapsto & (x, gx) \end{array}$$

is an isomorphism. More generally, for G a group scheme, one considers fpqc covers $X \to Y$ instead of étale covers and gets the notion of a *G-torsor*. Note that every G-torsor for G a finite group is also an étale G-torsor since being étale is fpqc local on the target by [Aut, 02VN].[2]

For X an étale G-torsor over Y and G finite, we have a G-action on the site of open sets $\mathrm{Op}(X)$ of X and $\mathcal{O}_X$ gets the structure of a G-equivariant sheaf by the isomorphisms $\mathcal{O}(U) \to^{\cong} \mathcal{O}_X(gU) = (g^*\mathcal{O}_X)(U)$ induced by the map $g^{-1}\colon gU \to U$. An equivariant $\mathcal{O}$-module (i.e. a module on $(G - \mathrm{Op}(X), \mathcal{O}_X)$) is called *quasi-coherent* if its underlying sheaf is quasi-coherent and the category of equivariant quasi-coherent sheaves on X is denoted by $G - \mathrm{QCoh}(X)$. One can check that the category of descent data for quasi-coherent sheaves associated to the map $X \to Y$ is equivalent to $G - \mathrm{QCoh}(X)$ (see [BLR90, 6.2B] for a very similar situation).

Corollary 2.1.5. *For X an étale G-torsor over Y, we have an equivalence*

$$G - \mathrm{QCoh}(X) \simeq \mathrm{QCoh}(Y).$$

Definition 2.1.6. Given a scheme (or more generally, a stack) X with a G-action (G an algebraic group), define a stack $X//G$ as the fibered category, which associates to an $U \in \mathrm{Sch}/S$ the groupoid of G-torsors over U with a G-equivariant map into X. If X is a G-torsor over a scheme Y, then $X//G$ is isomorphic to Y (since $X \to Y$ is the final G-torsor with a map to X).

2.2 Algebraic Stacks

In this section, we will again fix a scheme S and view Sch $/S$ (equipped with some topology) as base site.

[2]Here, a property P of morphisms is called *fpqc-local on the target* if the following holds: Suppose $f\colon X \to Y$ is a morphism and $U \to Y$ an fpqc cover such that $f_U\colon U \times_Y X \to U$ has P, then f has P as well.

To a scheme T over S, we can associate a stack over S by taking $\mathcal{X} := \mathrm{Sch}\,/T$ and defining the fiber functor $\mathcal{X} \to \mathrm{Sch}\,/S$ by $F(Y \to T) = F(Y \to T \to S)$.[3] This embedding from Sch/S to stacks over S is fully faithful and a stack equivalent to an object in the image is called *representable*. For every stack $(\mathcal{X}, F\colon \mathcal{X} \to \mathrm{Sch}\,/S)$ over S, there is an equivalence of categories between morphisms between T and $\mathcal{X}$ over S and the fiber of F over T by the 2-categorical Yoneda lemma. Thus, we will often identify an object X of $\mathcal{X}$ with a map $F(X) \to \mathcal{X}$.

For $\mathcal{X}$ and $\mathcal{Y}$ stacks, a morphism $\mathcal{Y} \to \mathcal{X}$ is called *representable* if for every morphism $U \to \mathcal{X}$ over S (with $U \in \mathrm{Sch}\,/S$), the fiber product $U \times_{\mathcal{X}} \mathcal{Y}$ is representable. If the diagonal $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ of a stack is representable, every morphism from a scheme to $\mathcal{X}$ is representable (see [Góm01], 2.19).

Let P be a property of morphisms betweeen schemes which is local on the target and stable under arbitrary base changes (such as separated, affine, proper, quasi-compact, locally of finite type, flat, smooth, étale, surjective, ...). Then we say that a morphism $f\colon \mathcal{Y} \to \mathcal{X}$ *has* P if it is representable and for every morphism $U \to \mathcal{X}$ the pullback $U \times_{\mathcal{X}} \mathcal{Y} \to U$ has P.

Just as a scheme is not just a sheaf of sets on $\mathrm{Aff}\,/S$ (where Aff denotes the category of affine schemes, or, equivalently, the opposite category of commutative rings), but carries a kind of atlas by affine schemes, we have to impose similar conditions on stacks to really use the full power of algebraic geometry. In addition, one usually wants some compactness and separatedness since it is technically more convenient. There are different notions of algebraic stacks in the literature, which are good for different purposes. We present two of the most common ones:

Definition 2.2.1 (Deligne-Mumford stack)**.** Let (Sch/S) be equipped with the étale topology and let $\mathcal{X}$ be a stack over S. Then we call $\mathcal{X}$ a *Deligne-Mumford stack* if the following conditions hold:

1. The diagonal $\Delta\colon \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable, quasi-compact and separated.

2. There exists a scheme U (called *atlas*) with an étale surjective morphism $u\colon U \to \mathcal{X}$.

Definition 2.2.2 (Algebraic Stack)**.** Let (Sch/S) be equipped with the fpqc topology and let $\mathcal{X}$ be a stack over S. Then we call $\mathcal{X}$ an *algebraic stack* if the following conditions hold:

1. The diagonal $\Delta\colon \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable and affine.

2. There is an affine scheme U (called *atlas*) with an fpqc morphism $u\colon U \to \mathcal{X}$.

This notion corresponds to an algebraic stack in the sense of Goerss, Naumann, . . . and adapted to the needs of homotopy theorists. Algebraic geometers usually use the word "algebraic stack" for an Artin stack. We will not recall the general notion of an Artin stack (but see [LMB00, 4.1, 10.1] or [Góm01, 2.22]) since all our examples of Artin stacks fit in the following two special cases:

- Every Deligne–Mumford stack is an Artin stack.

- Every algebraic stack in our sense where u is locally of finite type is also an Artin stack (see [LMB00], 10.1).

[3] In the sheaf of groupoid picture, this corresponds to viewing the sheaf of sets represented by T as a sheaf of groupoids (via the usual embedding of the category of sets into the category of groupoids).

Remark 2.2.3. If $S = \operatorname{Spec} R$ is affine, then the atlas of an algebraic stack $U \to \mathcal{X}$ is an affine map. Indeed, for $V \to \mathcal{X}$ a map over S from an affine scheme V, the square

$$\begin{array}{ccc} U \times_{\mathcal{X}} V & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \Delta \\ U \times_S V & \longrightarrow & \mathcal{X} \times_S \mathcal{X} \end{array}$$

is cartesian (see [Góm01, 2.19]). Since the diagonal Δ is affine, so is $U \times_{\mathcal{X}} V \to U \times_S V$. Since $U \times_S V$ is affine, $U \times_{\mathcal{X}} V$ is as well. This implies that $U \to \mathcal{X}$ is affine.

Example 2.2.4. Separated schemes are examples of both Deligne–Mumford and algebraic stacks. Indeed, the diagonal in a separated scheme is a closed immersion, so in particular quasi-compact, separated and affine. An arbitrary cover by affine opens provides an atlas.

Let (A, Γ) be a Hopf algebroid, i.e., a cogroupoid object in rings such that Γ is a flat A-module. By taking Spec, we get a groupoid object $(\operatorname{Spec} A, \operatorname{Spec} \Gamma)$ in (affine) schemes, representing a groupoid valued functor on schemes (a "preasheaf of groupoids"). There is a procedure associating to a presheaf of groupoids a stack, called *stackification* (analogous to sheafification) ([LMB00, Lemme 3.2]). Stackification turns the presheaf of groupoids above into a stack $\mathcal{X}$ together with a faithfully flat map $u\colon \operatorname{Spec} A \to \mathcal{X}$, which makes $\mathcal{X}$ into an algebraic stack. On the other hand, given an algebraic stack $\mathcal{X}$ and an faithfully flat map $\operatorname{Spec} A \to \mathcal{X}$, we can form the stack $\mathcal{Y} := \operatorname{Spec} A \times_{\mathcal{X}} \operatorname{Spec} A$. Since $\operatorname{Spec} A \to \mathcal{X}$ is representable and affine, $\mathcal{Y}$ is an affine scheme of the form $\operatorname{Spec} \Gamma$ and one can write down the structure maps of a Hopf algebroid. As described in detail in [Nau07], Section 3, this defines an equivalence (of 2-categories) between Hopf algebroids and algebraic stacks with chosen atlas.

2.3 Quasi-Coherent Sheaves

In this section, we want to discuss the category of quasi-coherent sheaves associated to a stack and the cohomology of quasi-coherent sheaves. We start in the setting of an arbitrary ringed site $(\mathcal{C}, \mathcal{O})$, i.e., a site equipped with a sheaf of rings. Note that we can view $\mathcal{O}$ as a monoid in the category of abelian sheaves on $\mathcal{C}$ (i.e. sheaves of abelian groups).

Definition 2.3.1. An *$\mathcal{O}$-module* is an $\mathcal{O}$-module in the category of abelian sheaves on $\mathcal{C}$. We will denote the category of $\mathcal{O}$-modules by $\operatorname{Mod}(\mathcal{O})$ or $\mathcal{O}$-mod.

Definition 2.3.2. An $\mathcal{O}$-module $\mathcal{F}$ is called *quasi-coherent* (or *cartesian*) if for any morphism $f\colon U \to V$ in $\mathcal{C}$, the map $\mathcal{F}(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \to \mathcal{F}(U)$ is an isomorphism.[4] We call an $\mathcal{O}$-module $\mathcal{F}$ *coherent* if there is for every $U \in \mathcal{C}$ a cover $V \to U$ with a surjective map $\mathcal{O}^n|_V \to \mathcal{F}|_V$ for some $n \in \mathbb{N}$.[5]

Definition 2.3.3. An $\mathcal{O}$-module $\mathcal{F}$ is a *vector bundle* if for any $U \in \operatorname{Ob} \mathcal{C}$ there exists a cover $\{V \to U\}$ of U such that $\mathcal{F}|_{\mathcal{C}/V}$ is a free $\mathcal{O}$-module of finite rank. It is called a *line bundle* if it is a vector bundle of rank 1.

[4]This is probably non-standard terminology. Often an $\mathcal{O}$-module is rather called quasi-coherent if it has locally a presentation. But for our purposes, the given definition seems to be the most suitable one. In the case of algebraic stacks they agree anyhow, as shown later.

[5]This will give the right notion of coherent in noetherian situations, but it is not a good notion in a non-noetherian context. We will use it only in a noetherian context.

Example 2.3.4. A quasi-coherent sheaf on Sch/S is locally free in the Zariski topology iff it is locally free in the étale topology iff it is locally free in the fpqc topology. Indeed, faithfully flat maps of rings detect projective modules and the category of projective modules over a ring A is equivalent to the category of Zariski locally free quasi-coherent sheaves on $\operatorname{Spec} A$. See [Aut, 05B2].

To define quasi-coherent sheaves on a stack, we have to associate a site to a stack. So, let $(\mathcal{X}, F\colon \mathcal{X} \to \mathrm{Sch}/S)$ be a stack (or, more generally, a fibered category) for a topology τ on Sch/S. Then we put the following topology on $\mathcal{X}$: A morphism $f\colon X \to Y$ is a cover iff it is strongly cartesian[6] and $F(f)\colon F(X) \to F(Y)$ is a cover in τ. We denote this site by $\mathcal{X}_\tau$ or just by $\mathcal{X}$ if the topology is clear from the context.

Remark 2.3.5. We could (for any choice of τ finer than the Zariski topology) also restrict just to (disjoint unions of) affine schemes over S and get an equivalent category of sheaves since every scheme is covered by affine schemes.

Definition 2.3.6. Define a presheaf $\mathcal{O}_\mathcal{X}$ on $\mathcal{X}$ by $\mathcal{O}_\mathcal{X}(U) = \Gamma(\mathcal{O}_{F(U)})$. This is a sheaf of rings in the fpqc-topology by Example 2.1.3 and is called the *structure sheaf* of $\mathcal{X}$. Thus $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ gets the structure of a ringed site.

Definition 2.3.7. For a stack $\mathcal{X}$, a *quasi-coherent sheaf on $\mathcal{X}$* is a quasi-coherent sheaf on the associated ringed site $(\mathcal{X}, \mathcal{O}_\mathcal{X})$. We denote the category of quasi-coherent sheaves on $\mathcal{X}$ by $\mathrm{QCoh}(\mathcal{X})$.

We have the following equivalent characterization of quasi-coherent sheaves:

Proposition 2.3.8. *An $\mathcal{O}_\mathcal{X}$-module $\mathcal{F}$ (in the fpqc topology) is quasi-coherent iff it has a local presentation: Given $X \in \mathcal{X}$, there is a fpqc-cover $p\colon U \to X$ such that the associated sheaf $\mathcal{F}|_U$ on $\mathcal{X}/U$ admits an exact sequence*

$$\bigoplus_I \mathcal{O}_{\mathcal{X}/U} \to \bigoplus_J \mathcal{O}_{\mathcal{X}/U} \to \mathcal{F}|_U \to 0.$$

If $\mathcal{X}$ is a Deligne–Mumford stack, we get an equivalent category of quasi-coherent sheaves if we substitute the fpqc topology by the étale topology.

Proof. The first statement follows by [Aut, 57.11.3, 06WI] and [Aut, 57.11.5, 06WK]. Note that they use the fppf-topology instead of the fpqc-topology, but this is caused by their very strict set-theoretical policy – their results rely at the end only on fpqc-descent. Their Lemma 57.11.5 implies also that their definition of a quasi-coherent sheaf is equivalent to the definition of [LMB00]. The last statement of our proposition is [LMB00], 13.2.3. □

Example 2.3.9. The structure sheaf $\mathcal{O}_\mathcal{X}$ is quasi-coherent. Furthermore, every vector bundle is quasi-coherent. Both are even coherent.

Remark 2.3.10. An extension of two vector bundles on an algebraic or Deligne–Mumford stack in the category of quasi-coherent sheaves is a vector bundle again. Indeed, the extension splits locally since locally the stack is an affine scheme of the form $\operatorname{Spec} A$, the category of quasi-coherent sheaves on (the over-site) $\mathcal{X}/\operatorname{Spec} A$ is equivalent to the category of A-modules (using Example 2.1.3) and vector bundles correspond to projective modules.

[6] This means roughly that X is a kind of pullback of Y along $F(X) \to F(Y)$ - see [Vis05, Section 3.1] for a precise definition

Remark 2.3.11. Sometimes, it is convenient to evaluate a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$ not only on a scheme over $\mathcal{X}$, but also on a stack $\mathcal{Y}$ over $\mathcal{X}$. We define $\mathcal{F}(\mathcal{Y})$ as $\mathrm{Hom}_{\mathrm{Pre}(\mathcal{X})}(h_{\mathcal{Y}}, \mathcal{F})$, where $\mathrm{Pre}(\mathcal{X})$ denotes the category of presheaves on $\mathcal{X}$ and $h_{\mathcal{Y}}$ is the presheaf defined by $h_{\mathcal{Y}}(U) = \mathrm{Hom}_{\mathcal{X}}(\mathcal{Y}, F(U))$, where F is the fiber functor $\mathcal{X} \to \mathrm{Sch}/S$. In particular, the global sections functor $\Gamma(\mathcal{F}) = \mathcal{F}(\mathcal{X})$ is given as $\mathrm{Hom}_{\mathrm{Pre}(\mathcal{X})}(*, \mathcal{F})$ for $*$ being the final presheaf.

If $f\colon \mathcal{X} \to \mathcal{Y}$ is a (representable) morphism of stacks, there are adjoint functors

$$\mathrm{Mod}(\mathcal{O}_{\mathcal{Y}}) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{Mod}(\mathcal{O}_{\mathcal{X}}).$$

For $\mathcal{F} \in \mathcal{O}_{\mathcal{X}}$-mod, the $\mathcal{O}_{\mathcal{Y}}$-module $f_*\mathcal{F}$ is defined by $f_*\mathcal{F}(U) := \mathcal{F}(U \times_{\mathcal{Y}} \mathcal{X})$ for a map $U \to \mathcal{Y}$. We will not define f^* in general, but for f fpqc and $\mathcal{G} \in \mathcal{O}_{\mathcal{Y}}$-mod, it is defined by

$$f^*\mathcal{G}(U \to \mathcal{X}) := \mathcal{G}(U \to \mathcal{X} \to \mathcal{Y}).$$

For the behavior of these adjoint functors on quasi-coherent sheaf, we cite the following proposition:

Proposition 2.3.12. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of stacks. Then:*

1. *The functor $f^*\colon \mathrm{Mod}(\mathcal{O}_{\mathcal{Y}}) \to \mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ restricts to a functor $f^*\colon \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{X})$.*
2. *If f is quasi-compact and quasi-separated, then the functor $f^*\colon \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{X})$ has a right adjoint $f'_*\colon \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{X})$. For quasi-coherent sheaves where f_* of the underlying module sheaf is already quasi-coherent, f'_* coincides with f_* of the underlying module sheaf.*
3. *For $\mathcal{X}, \mathcal{Y}$ Artin stacks and f quasi-compact, f_* preserves quasi-coherence.*

Proof.
1. This is [Aut, 03DO(5)].
2. This is [Aut, 077A].
3. This is [LMB00, 13.2.6(iii)].

□

Lemma 2.3.13. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be an affine fpqc morphism and $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent $\mathcal{O}_{\mathcal{Y}}$-modules. Then*

$$f_*f^*(\mathcal{F}) \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{G} \cong f_*f^*(\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{G}).$$

Proof. The tensor product is defined as the sheafification of the (naive) tensor product $\otimes^{naive}$ of presheaves of $\mathcal{O}_{\mathcal{Y}}$-modules. We will first show an isomorphism on the level of (naive) tensor products of presheaves. Let $U \to \mathcal{Y}$ be a morphism. Then the natural morphism

$$\mathcal{O}_{\mathcal{Y}}(U \times_{\mathcal{Y}} \mathcal{X}) \otimes_{\mathcal{O}_{\mathcal{Y}}(U)} \mathcal{G}(U) \to \mathcal{G}(U \times_{\mathcal{Y}} \mathcal{X}).$$

is an isomorphism (by the definition of quasi-coherent sheaves).

This induces natural isomorphisms

$$\begin{aligned}
(f_*f^*(\mathcal{F}) \otimes^{naive}_{\mathcal{O}_{\mathcal{Y}}} \mathcal{G})(U) &= \mathcal{F}(U \times_{\mathcal{Y}} \mathcal{X}) \otimes_{\mathcal{O}_{\mathcal{Y}}(U)} \mathcal{G}(U) \\
&\cong \mathcal{F}(U \times_{\mathcal{Y}} \mathcal{X}) \otimes_{\mathcal{O}_{\mathcal{Y}}(U \times_{\mathcal{Y}} \mathcal{X})} \mathcal{O}_{\mathcal{Y}}(U \times_{\mathcal{Y}} \mathcal{X}) \otimes_{\mathcal{O}_{\mathcal{Y}}(U)} \mathcal{G}(U) \\
&\cong \mathcal{F}(U \times_{\mathcal{Y}} \mathcal{X}) \otimes_{\mathcal{O}_{\mathcal{Y}}(U \times_{\mathcal{Y}} \mathcal{X})} \mathcal{G}(U \times_{\mathcal{Y}} \mathcal{X}) \\
&= f_*f^*(\mathcal{F} \otimes^{naive}_{\mathcal{O}_{\mathcal{Y}}} \mathcal{G})(U).
\end{aligned}$$

Thus, we get an isomorphism $f_*f^*(\mathcal{F}) \otimes^{naive}_{\mathcal{O}_\mathcal{Y}} \mathcal{G} \to f_*f^*(\mathcal{F} \otimes^{naive}_{\mathcal{O}_\mathcal{Y}} \mathcal{G})$ (and also an isomorphism after sheafification). Hence, we have a diagram

$$\begin{array}{ccc}
f_*f^* \otimes^{naive}_{\mathcal{O}_\mathcal{Y}} \mathcal{G} & \longrightarrow & f_*f^*(\mathcal{F}) \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{G} \\
\downarrow \cong & & \downarrow \cong \\
f_*f^*(\mathcal{F} \otimes^{naive}_{\mathcal{O}_\mathcal{Y}} \mathcal{G}) & \longrightarrow & (f_*f^*(\mathcal{F} \otimes^{naive}_{\mathcal{O}_\mathcal{Y}} \mathcal{G}))^\dagger \\
\downarrow & \swarrow & \\
f_*f^*(\mathcal{F} \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{G}) & &
\end{array}$$

where $()^\dagger$ denotes sheafification. Since on affine schemes the naive and the sheafy tensor product agree, the arrow pointing downwards-left is an isomorphism and the lemma follows. □

A quasi-coherent sheaf $\mathcal{F}$ on a stack $\mathcal{X}$ is, in particular, a sheaf of abelian groups. As on any site, the category of abelian sheaves has enough injectives (see [Aut, 01DP]) and we define $H^i(\mathcal{X};\mathcal{F})$ to be the i-th right derived functor of the global sections functor

$$\mathcal{F} \mapsto \Gamma(\mathcal{F}) = \mathcal{F}(\mathcal{X})$$

(see Remark 2.3.11 for the definition) from abelian sheaves to abelian groups. By [Aut, 01DU], the category of $\mathcal{O}_\mathcal{X}$-modules has also enough injectives and the derived functor of the global sections in $\mathcal{O}_\mathcal{X}$ -mod agrees with the cohomology of the underlying abelian sheaves by [Aut, 03FD].

For two $\mathcal{O}_\mathcal{X}$-modules $\mathcal{F}$ and $\mathcal{G}$, we define a sheaf $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F},\mathcal{G})$ by

$$U \mapsto \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}/U}}(\mathcal{F}|_{\mathcal{X}/U}, \mathcal{G}|_{\mathcal{X}/U})$$

and with structure morphisms given by restriction. For $\mathcal{F}$ and $\mathcal{G}$ quasi-coherent, we get an isomorphism to the (pre-)sheaf $U \mapsto \mathrm{Hom}_{\mathcal{O}_\mathcal{X}(U)}(\mathcal{F}(U), \mathcal{G}(U))$ on all affine schemes by evaluating on U. We also fix the notation $\check{\mathcal{F}}$ for the dual $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{O}_\mathcal{X})$ of an $\mathcal{O}_\mathcal{X}$-module $\mathcal{F}$.

For $\mathcal{F}$ an $\mathcal{O}_\mathcal{X}$-module, we denote the value of the i-th right derived functor of

$$\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, --)$$

on an $\mathcal{O}_\mathcal{X}$-module $\mathcal{G}$ by $\mathcal{E}xt^i_{\mathcal{O}_\mathcal{X}}(\mathcal{F},\mathcal{G})$. This agrees with the sheafification of the presheaf $U \mapsto \mathrm{Ext}^i_{\mathcal{O}_{\mathcal{X}|U}}(\mathcal{F}|_U, \mathcal{G}_U)$.

For an algebraic stack $\mathcal{X}$ and an atlas $u\colon U \to \mathcal{X}$, the category of quasi-coherent sheaves on $\mathcal{X}$ is equivalent to the category of comodules over the associated Hopf algebroid (A,Γ) ([Nau07], 3.4). The global sections functor $\Gamma\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{Ab}$ corresponds to $\mathrm{Hom}_{(A,\Gamma)\text{-comod}}(A,-)$ and thus

$$(R^i_{\mathrm{QCoh}}\Gamma)(\mathcal{F}) \cong \mathrm{Ext}_{(A,\Gamma)\text{-comod}}(A, \Gamma(u^*\mathcal{F})).$$

So, the question becomes interesting if $(R^i_{\mathrm{QCoh}}\Gamma)(\mathcal{F}) \cong H^i(\mathcal{X};\mathcal{F})$.

This isomorphism seems not to be true for an arbitrary scheme, only for noetherian or quasi-compact and semi-separated ones (see [TT90, Appendix B] for a discussion). For our notion of algebraic stack, one can adapt the argument of [TT90, Appendix B] to see that we have indeed $R^i_{\mathrm{QCoh}}\Gamma(\mathcal{F}) \cong H^i(\mathcal{X};\mathcal{F})$. Note that for $\mathcal{E}$ a vector bundle and $\mathcal{F}$ a quasi-coherent sheaf, this implies $\mathrm{Ext}^n_{\mathrm{QCoh}}(\mathcal{E},\mathcal{F}) \cong \mathrm{Ext}^n(\mathcal{E},\mathcal{F})$ by the Grothendieck spectral sequence and the fact that all Ext-sheaves vanish. In particular, every extension between two vector bundles in the category of $\mathcal{O}$-modules is isomorphic to a vector bundle (and hence a vector bundle) by Remark 2.3.10 (since the Ext^1-groups agree).

The correspondence between quasi-coherent sheaves on algebraic stacks and comodules yields another instance of Galois descent: Let $\operatorname{Spec} R \to \mathcal{X}$ be a $\mathbb{G}_m$-torsor[7]. Then $\operatorname{Spec} R \times_{\mathcal{X}} \operatorname{Spec} R \simeq \operatorname{Spec} R[u^{\pm 1}]$. As the datum of an $(R, R[u^{\pm 1}])$-comodule is equivalent to a graded R-module, $\mathrm{QCoh}(\mathcal{X})$ is equivalent to the category of graded R-modules. The same argumentation works for other affine group schemes to give other instances Galois descent for stacks. For example, consider a finite group G. We can view G as the affine group scheme $\operatorname{Spec} AG$ where AG is the Hopf algebra defined as follows: As a ring, $AG = \mathrm{Map}(G,\mathbb{Z})$. Note that $AG \otimes AG \cong \mathrm{Map}(G \times G, \mathbb{Z})$. Thus, we can structure maps as follows: $(\Delta f)(g,h) = f(gh)$, $(\epsilon f)(g) = f(e)$ and $(af)(g) = f(g^{-1})$, where $f \in AG$, $g,h \in G$ and Δ, ϵ and a denote diagonal, counit and antipode. It is easy to see that $\operatorname{Spec} AG$ represents the functor $T \mapsto G^{\pi_0 T}$, where $\pi_0 T$ denotes the set of connected components of T.

If $\operatorname{Spec} R \to \mathcal{X}$ is a G-torsor, then the category of quasi-coherent sheaves on $\mathcal{X}$ is equivalent to $(R, R \otimes AG)$-comodules. As a ring, this Hopf algebroid is isomorphic to $\mathrm{Map}(G,R)$. The left unit $\eta_l \colon R \to \mathrm{Map}(G,R)$ sends r to $(\eta_l(r))(e) = r$, $(\eta_l(r))(g) = 0$, for $g \neq e$. The right unit $\eta_r \colon R \to \mathrm{Map}(G,R)$ sends r to $(\eta_r(r))(g) = g(r)$. The counit is the evaluation at e and the diagonal $\Delta\colon \mathrm{Map}(G,R) \to \mathrm{Map}(G \times G, R)$ is again given by $(\Delta f)(g,h) = f(g,h)$. Given a R-module M with twisted G-action, we associate to it the right comodule with structure map $M \to M \otimes_R \mathrm{Map}(G,R) \cong \mathrm{map}(G,M)$, $m \mapsto (g \mapsto g(m))$. It is easy to see that the category of comodules over this Hopf algebroid is equivalent to R-modules with twisted G-action.

If the Hopf algebroid (A,Γ) is graded, we can define graded cohomology groups of (A,Γ) by $H^q_k((A,\Gamma)) := \mathrm{Ext}^q_\Gamma(A, A[k])$. Graded comodules over (A,Γ) correspond to ungraded comodules over $(A,\Gamma[u^{\pm 1}])$. The comodule $A[1]$ corresponds to a line bundle ω on the algebraic stack $\mathcal{X}$ associated to $(A,\Gamma[u^{\pm 1}])$; more concretely, one has a descent datum consisting of the isomorphism $\Gamma[u^{\pm 1}] \cong \omega(\operatorname{Spec}\Gamma[u^{\pm 1}]) \to \omega(\operatorname{Spec}\Gamma[u^{\pm 1}]) \cong \Gamma[u^{\pm 1}]$ given by multiplication by u. The graded cohomology of the Hopf algebroid is isomorphic to $H^*(\mathcal{X};\omega^{\otimes *})$, which is also called the *graded cohomology of* $\mathcal{X}$. Tensor products $\mathcal{F} \otimes \omega^k$ are sometimes called *twists* of $\mathcal{F}$. We will often use the notation $\Gamma_k(\mathcal{F}) := \Gamma(\mathcal{F} \otimes \omega^k)$ and $H^i_k(\mathcal{X};\mathcal{F}) := H^i(\mathcal{X};\mathcal{F} \otimes \omega^k)$.

2.4 The Moduli Stack of Elliptic Curves

Elliptic curves over the complex numbers have a long history with its roots lying in the study of elliptic integrals. For our purposes, we have to consider elliptic curves not only over fields, but over general rings (or even general base schemes). The modern algebraic

[7] This is essentially defined as in the scheme case, but we will be more precise about this definition in Section 2.6.

geometry definition is the following.

Definition 2.4.1. An *elliptic curve* over a scheme S is a proper smooth morphism $p\colon E \to S$ together with a section $e\colon S \to E$ such that for every morphism $x\colon \operatorname{Spec} k \to S$ with k an algebraically closed field, the pullback x^*E is a connected curve of genus 1.

Note that this data induces on E the structure of an abelian group scheme over S (see [KM85], 2.1.2). Furthermore, we get a line bundle $\omega := p_*\Omega^1_{E/S} \cong e^*\Omega^1_{E/S}$ on S. If $S = \operatorname{Spec} R$ and ω is trivial (what is both true locally in the Zariski topology), we can choose elements $a_1, a_2, a_3, a_4, a_6 \in R$ such that E is the closure in $\mathbb{P}^2_R$ of the affine subscheme of $\mathbb{A}^2_R$ given by the equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \text{ (see [KM85], 2.2).}$$

If 2 is invertible, we can simplify to the form

$$y^2 = 4x^3 + b_2x^2 + 2b_4x + b_6$$

for $b_2, b_4, b_6 \in R$. If 2 and 3 are invertible, we can even simplify to

$$y^2 = x^2 - 27c_4x - 54c_6$$

for $c_4, c_6 \in R$. These are the forms of the equations that can be found in [Sil09], III.1, and the b_i are polynomials in the a_i and the c_i are polynomials in the b_i, both with integral coefficients. There is a polynomial Δ in the a_i such that $1728\Delta = c_4^3 - c_6^2$ and the equation defines an elliptic curve iff $\Delta \in R^*$.

The moduli stack of elliptic curves $\mathcal{M}$ classifies the functor from schemes to groupoids which sends a scheme S to the groupoid of elliptic curves over S with isomorphisms between them. As usual, we obtain it by a Grothendieck construction: An object of $\mathcal{M}$ is an elliptic curve E over some scheme S. A morphism from $(p\colon E \to S, e)$ to $(q\colon E' \to S', e')$ consists of a morphism $f\colon S \to S'$ and an isomorphism $F\colon E \to f^*E'$ over and under S. This defines a stack in the fpqc topology (see [DR73, III.2.1]).

We get a map $\operatorname{Spec} A = \operatorname{Spec} \mathbb{Z}[a_1, a_2, a_3, a_4, a_6, \Delta^{-1}] \to \mathcal{M}$ since a morphism $\operatorname{Spec} R \to \operatorname{Spec} A$ corresponds to elements $a_1, a_2, a_3, a_4, a_6 \in R$ with Δ invertible and we can associate to this an elliptic curve as above. By the definition of the fiber product of stacks, $\operatorname{Spec} A \times_{\mathcal{M}} \operatorname{Spec} A$ is the stack classifying automorphisms of elliptic curves with given coordinate presentation; this is equivalent to the scheme $\operatorname{Spec} A[r, s, t, u^{\pm 1}]$ (this is essentially contained in [Sil09, III.1]). This shows that $\operatorname{Spec} A \to \mathcal{M}$ is a flat affine map (since locally, every morphism to $\mathcal{M}$ factors over $\operatorname{Spec} A$);[8] thus $\mathcal{M}$ is algebraic and we get an associated Hopf algebroid (A, Γ). The associated graded Hopf algebroid agrees with the Weierstrass Hopf algebroid $(\mathbb{Z}[a_1, a_2, a_3, a_4, a_6], \mathbb{Z}[a_1, a_2, a_3, a_4, a_6][r, s, t])$ in [Bau08] after inverting Δ. Here, by the usual correspondence, an element $a \in A$ is homogeneous of degree k if the image of the coaction map in $A \otimes \Gamma$ is homogeneous in u of degree $-k$. As might be expected, $|a_i| = i$. Similarly, the b_i and c_i get natural degrees with $|b_i| = i$ and $|c_i| = i$.

[8] More precisely, the argument is as follows: If a morphism is flat, can be checked on some fpqc open cover. Choose such an fpqc cover $\operatorname{Spec} B \to \mathcal{M}$, factoring over $\operatorname{Spec} A \to \mathcal{M}$. Thus,

$$\operatorname{Spec} B \times_{\mathcal{M}} \operatorname{Spec} A \simeq \operatorname{Spec} B \times_{\operatorname{Spec} A} \operatorname{Spec} A \times_{\mathcal{M}} \operatorname{Spec} A \simeq \operatorname{Spec} B[r, s, t, u^{\pm 1}.$$

The map $\operatorname{Spec} B[r, s, t, u^{\pm 1} \to \operatorname{Spec} B$ is flat.

Denote by $\mathcal{M}_R$ the fiber product $\mathcal{M} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} R$ or, equivalently, the moduli stack of elliptic curves over $\operatorname{Spec} R$. This becomes particularly simple if $\frac{1}{2}, \frac{1}{3} \in R$. Then $\mathcal{M}_R \simeq \operatorname{Spec} R[c_4, c_6, \Delta^{-1}]//\mathbb{G}_m$ since in this case the automorphisms of the Weierstrass form are given just by the transformation $x \mapsto u^2 x$, $y \mapsto u^3 y$. Here $\mathbb{G}_m$ denotes again the group scheme $\operatorname{Spec} \mathbb{Z}[t^{\pm 1}]$. In particular, over a field K of characteristic not 2 or 3, the stack $\mathcal{M}_K$ embeds into the weighted projective stack $\mathcal{P}(4,6) = (\operatorname{Spec} K[c_4, c_6] - \{0\})//\mathbb{G}_m$.

For many purposes, it is nice to have a *compactified moduli stack*. If $\mathcal{M}$ itself was proper over $\operatorname{Spec} \mathbb{Z}$, the following would be true (by the valuative criterion for properness ([Góm01, 2.39]): Given a (discrete) valuation ring A with quotient field K and an elliptic curve E over K, there is a finite extension K' of K such that there is an elliptic curve E' over the integral closure A' of A in K' such that $E'_{K'} \cong E_{K'}$. This is only true if E has potentially good reduction. In general, it is only possible to define E' to be a group scheme with a nodal singularity. Roughly, generalized elliptic curves are elliptic curves with nodal singularities. Since this does not lie in our main line of study, we won't define here precisely what generalized elliptic curves and the compactified moduli stack of elliptic curves $\overline{\mathcal{M}}$ are. We only remark that we use the model $\mathfrak{M}_1$ of [DR73, IV.2.4] (see also [Sto11, Section 4]).

Theorem 2.4.2 ([DR73], III.2.6; [Con07], 3.1.7)**.** *$\mathcal{M}$ and $\overline{\mathcal{M}}$ are Deligne–Mumford stacks and $\overline{\mathcal{M}}$ is proper over* $\operatorname{Spec} \mathbb{Z}$.

2.5 Level Structures

There are several variations of moduli stacks of elliptic curve, based on the notion of a *level structure*. We will give the definition and a few simple properties and investigate then the moduli stacks of elliptic curves with level structure of niveau 2 and 4 in detail.

An elliptic curve E over S is, in particular, an abelian group scheme over S and we can consider (for given n) the finite sub group scheme $E[n]$ of n-torsion points.

Definition 2.5.1. Let E/S be an elliptic curve. A *level structure of niveau n* (or simply *level-n-structure*) is an isomorphism $S \times (\mathbb{Z}/n)^2 \to E[n]$.[9] The moduli stack of elliptic curves with level-n-structure is denoted by $\mathcal{M}(n)$.

One can also just choose a point of exact order n (i.e. fix an injection $S \times \mathbb{Z}/n \hookrightarrow E[n]$), without trivializing the whole n-torsion, which gives $\mathcal{M}_1(n)$. For $n = 2$, this is the same as choosing a sub group scheme of the torsion of order 2; therefore, $\mathcal{M}_1(2)$ is also often called $\mathcal{M}_0(2)$.

We have maps $\mathcal{M}(n) \to \mathcal{M}$ and $\mathcal{M}_1(n) \to \mathcal{M}$, which are étale and surjective (therefore étale covers) if we invert n. The surjectivity can be seen by the well-known fact that over an algebraically closed field of characteristic not dividing n, the n-torsion of an elliptic curve is isomorphic to $(\mathbb{Z}/n)^2$.

Let now R be a ring which contains $\frac{1}{2}$. Then every elliptic curve can be represented by an equation of the form $y^2 = 4x^3 + b_2 x^2 + 2b_4 x + b_6$ (in $\mathbb{P}^2_R$). A point of exact order 2 corresponds to a point with $y = 0$ (see also [Beh06], 1.3.2). Therefore, a level-structure

[9] This does not exist for every elliptic curve; for example, never if $S = \operatorname{Spec} \mathbb{F}_p$ and $p | n$.

of niveau 2 gives a splitting $4x^3 + b_2x^2 + 2b_4x + b_6 = 4(x - e_1)(x - e_2)(x - e_3)$. By a coordinate change $x \mapsto x + e_3$, we get an equivalent form $4(x - (e_1 - e_3))(x - (e_2 - e_3))x$. Set $x_2 := e_1 - e_3$ and $y_2 := e_2 - e_3$. One can see that (up to scaling) these two values are determined by the elliptic curve with level-2-structure uniquely – therefore, we get that $\mathcal{M}(2)[\frac{1}{2}] = \operatorname{Spec} \mathbb{Z}[\frac{1}{2}][x_2, y_2, \Delta^{-1}]//\mathbb{G}_m$, where Δ is the image of $\Delta \in H^0_*(\mathcal{M}; \mathcal{O})$ under the map $H^0_*(\mathcal{M}; \mathcal{O}) \to H^0_*(\mathcal{M}(2); \mathcal{O})$ (see [Sto11, Section 7] for more details). As usual, a $\mathbb{G}_m$-action corresponds to gradings and $|x_2| = |y_2| = |b_2| = 2$.

By [Beh06, Section 1.3.2], one gets a similarly $\mathcal{M}_0(2) \simeq \operatorname{Spec} \mathbb{Z}[\frac{1}{2}][b_2, b_4, \Delta^{-1}]//\mathbb{G}_m$.

As before, we can associate to every elliptic curve $p\colon E \to S$, we can associate the line bundle $p_*\Omega^1_{E/S} \cong s^*\Omega^1_{E/S}$, the direct image of the differentials. This yields a line bundle ω on $\mathcal{M}$. This line bundle generates the group of all line bundles and satisfies $\omega^{12} \cong \mathcal{O}_{\mathcal{M}}$. The isomorphism is given by the unit $\Delta \in H^0_{12}(\mathcal{M}; \mathcal{O})$, where here and in the following $H^i_j(\mathcal{M}; \mathcal{F})$ denotes $H^i(\mathcal{M}; \mathcal{F} \otimes \omega^j)$. We want to remark that, indeed, our line ω corresponds to $A[1]$ in the Weierstraß Hopf algebroid: There is a basis of ω over $\operatorname{Spec} A$ given by the *invariant differential* $\omega_0 = \frac{dx}{2y + a_1x + a_3}$. It is an easy computation that $f(r, s, t, u)^*\omega_0 = u\omega_0$, where $f(r, s, t, u)$ denotes the automorphism of the elliptic curve corresponding to $r, s, t, u \in A$ (see also [Rez02, Proposition 9.4]). Thus ω_0 equals $u^{-1}f(r, s, t, u)^*\omega_0$ and thus ω corresponds to $A[1]$. Similarly, ω corresponds on $\mathcal{M}(2) \simeq \operatorname{Spec} \mathbb{Z}[\frac{1}{2}][x_2, y_2, \Delta^{-1}]$ to the shift $\mathbb{Z}[\frac{1}{2}][x_2, y_2, \Delta^{-1}][1]$ and we have also the analogous statement for $\mathcal{M}_0(2)$. In particular, $H^0_*(\mathcal{M}(2); \mathcal{O}) \cong \mathbb{Z}[\frac{1}{2}][x_2, y_2, \Delta^{-1}]$ and $H^0_*(\mathcal{M}_0(2); \mathcal{O}) = \mathbb{Z}[\frac{1}{2}][b_2, b_4, \Delta^{-1}]$.

We localize now (implicitly) at 3. We get a map $H^0_*(\mathcal{M}; \mathcal{O}) \to H^0_*(\mathcal{M}(2); \mathcal{O})$ as above. The source is called *the ring of modular forms* and is multiplicatively generated by c_4,c_6 and $\Delta^{\pm 1}$ with the relation $1728\Delta = c_4^3 - c_6^2$. The target is, as indicated above, isomorphic to $\mathbb{Z}_{(3)}[x_2, y_2, \Delta^{-1}]$. What is the image of c_4 and c_6?

There are formulas (which can be found in [Sil09], III.1):

$$
\begin{aligned}
c_4 &= b_2^2 - 24b_4 \\
c_6 &= -b_2^3 + 36b_2b_4 - 216b_6
\end{aligned}
$$

These can be seen as equations of functions, which assign to an elliptic curve with chosen coordinates b_2, b_4 and b_6 the quantities c_4 and c_6. In our elliptic curve with level-2-structure, we have:

$$
\begin{aligned}
b_2 &= -4(x_2 + y_2) \\
b_4 &= 2x_2y_2 \\
b_6 &= 0
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
c_4 &= 16(x_2 + y_2)^2 - 48x_2y_2 = 16(x_2^2 + y_2^2 - x_2y_2) \\
c_6 &= 64(x_2 + y_2)^3 - 288x_2y_2(x_2 + y_2) = 64(x_2^3 + y_2^3) - 96(x_2^2y_2 + x_2y_2^2)
\end{aligned}
$$

Here, we denote the images of c_4 and c_6 in $H^0_*(\mathcal{M}(2); \mathcal{O})$ by the same name. If we reduce modulo 3, the formulas become much simpler and we have:

$$
\begin{aligned}
c_4 &= 16(x_2 + y_2)^2 \\
c_6 &= 64(x_2 + y_2)^3
\end{aligned}
$$

In general, we have the following formula:

$$\Delta = -27b_6^2 + (9b_2b_4 - \frac{1}{4}b_2^3)b_6 - 8b_4^3 + \frac{1}{4}b_2^2b_4^2 \text{ [Sil09, III.1]}$$

This gives in terms of x_2 and y_2:

$$\Delta = \frac{1}{4}b_4^2(b_2^2 - 32b_4) = x_2^2y_2^2(16(x_2+y_2)^2 - 64x_2y_2) = 16x_2^2y_2^2(x_2-y_2)^2$$

There is a group action S_3 on $\mathcal{M}(2)$ as a special case of the general action of $GL_n(\mathbb{Z}/n)$ on $\mathcal{M}(n)$, acting on the trivialization of the n-torsion. The $GL_2(\mathbb{Z}/2) \cong S_3$-action on $\mathcal{M}(2)$ permutes the e_1, e_2 and e_3. Therefore, we get formulas for the group action as follows:[10]

$$\begin{array}{llll}
(1\,2\,3): x_2 & \mapsto & x_2, & y_2 \mapsto y_2 \\
(2\,1\,3): x_2 & \mapsto & y_2, & y_2 \mapsto x_2 \\
(3\,2\,1): x_2 & \mapsto & -x_2, & y_2 \mapsto y_2 - x_2 \\
(1\,3\,2): x_2 & \mapsto & x_2 - y_2, & y_2 \mapsto -y_2 \\
(2\,3\,1): x_2 & \mapsto & y_2 - x_2, & y_2 \mapsto -x_2 \\
(3\,1\,2): x_2 & \mapsto & -y_2, & y_2 \mapsto x_2 - y_2
\end{array}$$

These formulas will be used in some way in Sections 8.2 and 8.4.

As a last point, we want to study $\mathcal{M}(4)$, based on [Shi73]. In [Shi73], the definition of a level structure of niveau n is slightly different: In general, we have for every elliptic curve E over a base scheme S, a pairing

$$e_n\colon\ E[n] \times E[n] \to \mu_n$$

where the latter is the sub group scheme $\mu_n \subset S \times \mathbb{G}_m$ of n-th roots of unity over S. The pairing e_n is called the *Weil pairing* and is alternating and bilinear in the sense that $e_n(P,P) = 0$ and $e_n(kP,Q) = e_n(P,kQ) = (e_n(P,Q))^k$ for $k \in \mathbb{Z}$ (see [KM85, Section 2.8]). A level structure of niveau n yields two points P and Q of order n as the images of $(1,0)$ and $(0,1)$ under the map $(\mathbb{Z}/n)^2 \to E[n]$. Thus, we can associate to a level structure a n-th root of unity $e_n(P,Q)$, giving a morphism $\mathcal{M}(n) \to \mu_n$, where $\mu_n \cong \mathbb{Z}[t]/(t^n-1)$. The image $e_n(P,Q)$ has to be primitive since e_n is a perfect pairing in the sense that $e_n(R,S) = 1$ for all $S \in E[n]$ implies $R = 0$ (see [KM85, 2.8.5.1]). A *Shioda level structure of niveau n*[11] is a usual level structure of niveau n such that $e_n(P,Q) = \zeta$ for ζ a chosen primitive n-th root of unity. The moduli problem of elliptic curves with Shioda level structure of niveau 4 (for $\zeta = i$) is classified by $\operatorname{Spec} A$ for $A = \mathbb{Z}[\frac{1}{2},i][\sigma,\sigma^{-1}(\sigma^4-1)^{-1}]$ (see [Shi73], Theorem 1 and the Remark after it). Since μ_4 over $\mathbb{Z}[\frac{1}{2},i]$ consists as a scheme just of 4 copies of $\operatorname{Spec}\mathbb{Z}[\frac{1}{2},i]$ (since $\mathbb{Z}[\frac{1}{2},i][t]/(t-1)(t+1)(t-i)(t+i)$ decomposes into

$$\mathbb{Z}[\frac{1}{2},i][t]/(t-1) \times \mathbb{Z}[\frac{1}{2},i][t]/(t+1) \times \mathbb{Z}[\frac{1}{2},i][t]/(t-i) \times \mathbb{Z}[\frac{1}{2},i][t]/(t+i)$$

by the Chinese remainder theorem, see [Lan02, II.2.2]), the stack $\mathcal{M}(4)$ is a disjoint union of the fibers over i and $-i$ of the map $\mathcal{M}(4) \to \mu_4 \times \operatorname{Spec}\mathbb{Z}[\frac{1}{2},i]$ above.

[10] For the notation for elements of S_3, see the list of notation B

[11] This is not standard terminology.

Consider the subgroup $G \subset GL_2(\mathbb{Z}/4)$ given by matrices of the form $\begin{pmatrix} 1+2a & 2b \\ 2c & 1+2d \end{pmatrix}$. This is a normal subgroup isomorphic to $(C_2)^4$ and equal to the kernel of the mod 2 reduction map $GL_2(\mathbb{Z}/4) \to GL_2(\mathbb{Z}/2)$. The group G operates on the set of all level-4-structures (P, Q) inducing the same level-2-structure $(2P, 2Q)$. This gives a G-action on $\mathcal{M}(4)$ over $\mathcal{M}(2)$. Concretely, the operation is given by

$$(P, Q) \mapsto (P + 2aP + 2bQ, Q + 2cP + 2dQ).$$

Thus, we have involutions $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ and $\tilde{d}$ acting on $\mathcal{M}(4)$ corresponding to

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \text{ respectively } \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The involution $\tilde{a}$ sends (P, Q) to $(-P, Q)$. Since $e_n(-P, Q) = e_n(P, Q)^{-1}$, the involution $\tilde{a}$ permutes the two components of $\mathcal{M}(4)$. Thus, $\mathcal{M}(4) \simeq \operatorname{Spec} A \coprod \operatorname{Spec} A$. The involution $\tilde{a}\tilde{d}$ (sending (P, Q) to $(-P, -Q)$) corresponds to the identity on $\operatorname{Spec} A \coprod \operatorname{Spec} A$ since the level structures are isomorphic (via $[-1]\colon E \to E$). The involutions $\tilde{b}$ and $\tilde{c}$ induce the involutions $\sigma \mapsto \frac{1}{\sigma}$ and $\sigma \mapsto -\frac{1}{\sigma}$ respectively on A by [Shi73], Proposition 2.

We have for $z_b = \sigma - \frac{1}{\sigma}$ that $\tilde{b}(z_b) = -z_b$ and $\tilde{c}(z_b) = z_b$. The element $z_b = \frac{1}{\sigma}(\sigma^2 - 1)$ is a unit in A. Hence, $z_b\cdot\colon A \to A$ is an isomorphism and has the effect on the G-action that the $\tilde{b}$-part is twisted by sign. Similarly, for $z_c = \sigma + \frac{1}{\sigma}$, we get that $z_c\cdot\colon A \to A$ twists (only) the $\tilde{c}$-action by sign. In summary, A is G-equivariantly isomorphic to A with either the $\tilde{b}$-action or the $\tilde{c}$-action or both twisted by sign.

2.6 Galois Coverings of Stacks

In this section, we will investigate the notion of a Galois covering of a stack in some detail.

Let $F\colon \mathcal{Y} \to \mathcal{X}$ be a morphism of categories fibered over the site of schemes over a base scheme S. For H a group (scheme), the easiest notion of an H-action of $\mathcal{Y}$ over $\mathcal{X}$ is an H-action on $\mathcal{Y}$ in the over-category over $\mathcal{X}$. This notion is too strict for some purposes (e.g., in the context of formal groups to be covered in Section 2.8). A possible definition is the following:

Definition 2.6.1. Let H be a group scheme over S; we denote the associated fibered category with the same letter. Denote by EH the category (fibered over Sch/S) which associates to each T over S the groupoid with the same objects as $H(T)$, but with exactly one morphism between each two objects.

An *H-action on $\mathcal{Y}$ over $\mathcal{X}$* is given by the following data: First, a 1-morphism $a\colon H \times \mathcal{Y} \to \mathcal{Y}$ satisfying the usual axioms of an action. This induces a 1-morphism $H \to \operatorname{Fun}(\mathcal{Y}, \mathcal{X})$ by the action on F, where $\operatorname{Fun}(\mathcal{Y}, \mathcal{X})(T) = \operatorname{Fun}(\mathcal{Y}(T), \mathcal{X}(T))$. The second datum is an extension of this morphism to a morphism $EH \to \operatorname{Fun}(\mathcal{Y}, \mathcal{X})$.

More concretely, for $Y \in \mathcal{Y}(T)$, $h \in H(T)$, the second datum gives an isomorphism $F(Y) \xrightarrow{\alpha_{Y,h}} F(hY)$ satisfying various compatibility conditions. We say that the action is *strictly over $\mathcal{X}$* if all $\alpha_{Y,h}$ are identity morphisms.

Example 2.6.2. The scheme $\operatorname{Spec} \mathbb{C}$ can be considered as the fibered category over $\operatorname{Sch}/\mathbb{R}$ sending an $\mathbb{R}$-scheme X to the set of points in $x \in X(\mathbb{C})$. Let $\mathcal{P}$ be the fibered category over $\operatorname{Sch}/\mathbb{R}$ sending an $\mathbb{R}$-scheme X to the groupoid of $\mathbb{C}$-schemes isomorphic to $X_{\mathbb{C}}$ over

$\mathbb{R}$ together with a chosen point in $Y(\mathbb{C})$. We have an morphism $F\colon \operatorname{Spec}\mathbb{C} \to \mathcal{P}$, with $F(X,x) = (X_{\mathbb{C}}, x)$.

The scheme $\operatorname{Spec}\mathbb{C}$ has a C_2-action by complex conjugation. For t, the generator of C_2, we have furthermore an isomorphism $F(X,x) \to F(t(X,x))$ given by the complex conjugation $X_{\mathbb{C}} \to X_{\mathbb{C}}$. This defines a C_2-action of $\operatorname{Spec}\mathbb{C}$ over $\mathcal{P}$.

Definition 2.6.3. For a group scheme H over S, a morphism $F\colon \mathcal{Y} \to \mathcal{X}$ of stacks is an *H-Galois covering* (or, equivalently, gives $\mathcal{Y}$ the structure of an *H-torsor over $\mathcal{X}$*) if F is an fpqc morphism and $\mathcal{Y}$ is equipped with an H-action such that the following morphism $\Psi_{\mathcal{Y}}\colon H \times \mathcal{Y} \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ is an equivalence of stacks: Let T be a scheme over S and $Y \in \mathcal{Y}(T)$ and $h \in H(T)$. Then we associate to (h, Y) the triple $(Y, hY, \alpha_{Y,h})$, where the latter is the isomorphism from $F(Y)$ to $F(hY)$ described above.[12] Note that if H is a finite group, then F is automatically étale since being étale is fpqc-local on the target by [Aut, 02VN].

Example 2.6.4. Let X be a scheme with a G-action. Recall that $X//G$ is the stack classifying G-torsors with equivariant maps to X. Given a morphism $Y \to X$, then the morphism $X \to X//G$ sends it to the trivial G-torsor $Y \times G$ together with the map $Y \times G \to X \times G \to X$ (the last map being the action of G). For every $g \in G$, we have a morphism $\alpha_g\colon = \mathrm{id}_Y \times (\cdot g)\colon Y \times G \to Y \times G$ of G-torsors, defining an action of G on X over $X//G$ in the sense above. To see that $X \to X//G$ is a G-torsor note that X is G-equivariantly equivalent over $X//G$ to $\tilde{X}$, the stack of *trivialized* G-torsors with an equivariant map to X. Since two trivializations of a G-torsor over Y differ by an element of $G(Y)$, the claim follows.

Construction 2.6.5. Let $F\colon \mathcal{Y} \to \mathcal{X}$ be an *H-Galois covering*. We have a natural transformation $\gamma^{\mathcal{Y}}\colon F\,\mathrm{pr}_1 \to F\,\mathrm{pr}_2$ of functors $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{X}$ given by the isomorphism in the definition in the fiber product. For $\mathcal{F}$ a quasi-coherent sheaf on $\mathcal{X}$, this defines an isomorphism $\gamma^{\mathcal{Y}}_*\colon \mathrm{pr}_1^* F^*\mathcal{F} \to \mathrm{pr}_2^* F^*\mathcal{F}$. We have the identities $\mathrm{pr}_1 \circ \Psi_{\mathcal{Y}} = p_1$ and $\mathrm{pr}_2 \circ \Psi_{\mathcal{Y}} = a$ as morphisms $\mathcal{Y} \times H \to \mathcal{Y}$, where $p_1$is the projection on the first factor and a denotes the action map. Thus, we get an induced map $\gamma^{\mathcal{Y}}_*\colon p_1^* F^*\mathcal{F} \to a^* F^*\mathcal{F}$. In particular, if $h \in H(\mathcal{Y})$, we get an isomorphism $F^*\mathcal{F}(\mathcal{Y}) \to h^* F^*\mathcal{F}(\mathcal{Y})$, defining a twisted group object in $\mathrm{QCoh}(\mathcal{Y})$. This agrees with the G-action used in the Hopf algebroid approach to Galois descent in Section 2.3 in the case of algebraic stacks and finite groups. If the action of H is trivial on $\mathcal{Y}$, then this defines actually a group object.

Proposition 2.6.6. *Let T be a scheme over S with a morphism $f\colon T \to \mathcal{Y}$ over S. Then $Ff\colon T \to \mathcal{X}$ defines an object $A \in \mathcal{X}(T)$.*

1. *Suppose that a finite group G acts on A in $\mathcal{X}(T)$. Then there is an associated map $\overline{f}$ and a natural transformation $\alpha\colon \overline{f}F^T \to Ff$ making the diagram*

$$\begin{array}{ccc} T & \xrightarrow{f} & \mathcal{Y} \\ \downarrow{\scriptstyle F^T} & & \downarrow{\scriptstyle F} \\ T//G & \xrightarrow{\overline{f}} & \mathcal{X} \end{array}$$

2-commutative. Here, the action on T by G is trivial.

[12] Recall for that purpose that the fiber product $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ is defined to be the category of triples (Y_1, Y_2, f), where Y_1 and Y_2 are objects in $\mathcal{Y}$ and $f\colon F(Y_1) \to F(Y_2)$ is an isomorphism. A morphism of such triples (Y_1, Y_2, f) and (Y_1', Y_2', f') consists of $\phi_1\colon Y_1 \to Y_1'$ and $\phi_2\colon Y_2 \to Y_2'$ such that $f' \circ F(\phi_1) = F(\phi_2) \circ f$.

2. *Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{X}$. Assume that $T = \operatorname{Spec} R$ and $\mathcal{Y} = \operatorname{Spec} A$ and denote $\mathcal{F}(\mathcal{Y})$ by M. A map $\varphi\colon G \to H(T)$ defines a map $\Phi\colon T \times G \cong \coprod_G T \to \mathcal{Y} \times H$ by $(t, g) \mapsto (f(t), (\varphi(g))(t)$.*

   Assume that Φ makes the diagram

$$\begin{array}{ccc} T \times G & \xrightarrow{\Phi} & \mathcal{Y} \times H \\ \downarrow{\scriptstyle \Psi_T} & & \downarrow{\scriptstyle \Psi_{\mathcal{Y}}} \\ T \times_{T//G} T & \xrightarrow{f \times f} & \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} \end{array}$$

   2-commutative via a natural transformation $\beta\colon (f \times f) \circ \Psi_T \to \Psi_{\mathcal{Y}} \circ \Phi$.[13] Then $T \to \mathcal{Y}$ is G-equivariant in the sense that

$$\begin{array}{ccc} T & \xrightarrow{(f, \varphi(g))} & \mathcal{Y} \times H \\ & {\scriptstyle f} \searrow & a^{\mathcal{Y}} \downarrow\downarrow p_1^T \\ & & \mathcal{Y} \end{array}$$

   is commutative for every $g \in G$. Furthermore, the following two G-actions on $M \otimes_A R$ agree:

   (a) *The transformation α induces an isomorphism $\alpha_*\colon \left((F^T)^* \overline{f}^* \mathcal{F}\right)(T) \to M \otimes_A R$. On $(F^T)^* \overline{f}^* \mathcal{F}(T)$, we have a G-action from the construction of Galois descent.*

   (b) *For $g \in G$, we get a morphism $(f, \varphi(g))\colon T \to Y \times H$. The transformation $\gamma^{\mathcal{Y}}$ induces an isomorphism*

$$f^* F^* \mathcal{F} = (f, \varphi(g))^* (p_1^{\mathcal{Y}})^* F^* \mathcal{F} \xrightarrow{(f,\varphi(g))^* (\gamma^{\mathcal{Y}})_*} (f, \varphi(g))^* (a^{\mathcal{Y}})^* F^* \mathcal{F} = f^* F^* \mathcal{F}.$$

   Evaluating on T, we get a morphism $M \otimes_A R \to M \otimes_A R$. Informally, it is the induced G-action on $M \otimes_A R$ from the H-action on M.

Proof. 1. Define a fibered category T/G as $T \times BG$, where we view T as a fibered category and BG as the category with one object with automorphisms equal to G. Then $T//G$ (for G acting trivially on T) is a stackification of T/G (see [BCE+12, Section 4.4]). The object A with the group of automorphisms G defines a morphism $T/G \to \mathcal{X}$; by the universal property of stackification, this gives a morphism $\overline{f}\colon T//G \to \mathcal{X}$ as desired (see [Aut, 0435] for this universal property).

2. The diagram

$$\begin{array}{ccc} T & \xrightarrow{(f, \varphi(g))} & \mathcal{Y} \times H \\ & {\scriptstyle f} \searrow & a^{\mathcal{Y}} \downarrow\downarrow p_1^T \\ & & \mathcal{Y} \end{array}$$

[13] The definition of $(f \times f)$ will be recalled in the proof of this proposition.

is 2-commutative via β, hence actually commutative since all occurring stacks are actually schemes. The other statement follows from the commutativity of the diagram

$$\begin{array}{ccccccc}
(F^T)^*\overline{f}^*\mathcal{F} & \xrightarrow{=} & (\mathrm{id}_T, g)^*(p_1^T)^*(F^T)^*\overline{f}^*\mathcal{F} & \xrightarrow{(\mathrm{id}_T,g)^*\gamma^T} & (\mathrm{id}_T, g)^*(a^T)^*(F^T)^*\overline{f}^*\mathcal{F} & \xrightarrow{=} & (F^T)^*\overline{f}^*\mathcal{F} \\
\downarrow{\scriptstyle \alpha} & & & & & & \downarrow{\scriptstyle \alpha} \\
f^*F^*\mathcal{F} & \xrightarrow{=} & (f, \varphi(g))^*(p_1^{\mathcal{Y}})F^*\mathcal{F} & \xrightarrow{(f,\varphi(g))^*(\gamma^{\mathcal{Y}})} & (f, \varphi(g))^*(a^{\mathcal{Y}})^*F^*\mathcal{F} & \xrightarrow{=} & f^*F^*\mathcal{F}
\end{array}$$

which we want to show now.

The following square of natural transformations of morphisms between $T \times_{T//G} T \to \mathcal{X}$ is commutative:

$$\begin{array}{ccc}
F\,\mathrm{pr}_1^{\mathcal{Y}}(f \times f) & \longrightarrow & F\,\mathrm{pr}_2(f \times f) \\
\downarrow & & \downarrow \\
\overline{f}F^T\,\mathrm{pr}_1^T & \longrightarrow & \overline{f}F^T\,\mathrm{pr}_2
\end{array}$$

The transformations are as follows: Let $t = (t_1, t_2, \delta\colon F^T(t_1) \to F^T(t_2))$ be a point in $(T \times_{T//G} T)(X)$ for some X. If we apply the square above to it, we get

$$\begin{array}{ccc}
F(f(t_1)) & \longmapsto & F(f(t_2)) \\
\downarrow & & \downarrow \\
\overline{f}F^T(t_1) & \longmapsto & \overline{f}F^T(t_2)
\end{array}$$

The morphisms down are given by α, the horizontal morphisms are induced from the isomorphisms in the definition of the fiber product, i.e., the lower one by δ. To that purpose recall that the isomorphism in $(f \times f)(t)$ is actually defined by the commutativity of this diagram.

Precomposing with Ψ_T, we get the lower square of the following commutative diagram:

$$\begin{array}{ccc}
Fp_1^{\mathcal{Y}}\Phi & \longrightarrow & Fa^{\mathcal{Y}}\Phi \\
\downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} \\
F\,\mathrm{pr}_1^{\mathcal{Y}}\Psi_{\mathcal{Y}}\Phi & \longrightarrow & F\,\mathrm{pr}_2^{\mathcal{Y}}\Psi_{\mathcal{Y}}\Phi \\
{\scriptstyle \beta}\uparrow & & {\scriptstyle \beta}\uparrow \\
F\,\mathrm{pr}_1^{\mathcal{Y}}(f \times f)\Psi^T & \longrightarrow & F\,\mathrm{pr}_2^{\mathcal{Y}}(f \times f)\Psi_T \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\
\overline{f}F^T\,\mathrm{pr}_1\Psi_T & \longrightarrow & \overline{f}F^T\,\mathrm{pr}_2\Psi_T
\end{array}$$

Precomposing the outer square with $(\mathrm{id}_T, g)\colon T \to T \times G$ gives

$$\begin{array}{ccc} Ff & \longrightarrow & Ff \\ \downarrow{=} & & \downarrow{=} \\ Fp_1^{\mathcal{Y}}(f, \varphi(g)) & \longrightarrow & Fa^{\mathcal{Y}}(f, \varphi(g)) \\ \downarrow & & \downarrow \\ \overline{f}F^T & \longrightarrow & \overline{f}F^T \end{array}$$

This is exactly what is needed for the commutativity of the diagram above.

□

Example 2.6.7. We want to show that $\mathcal{M}(n) \to \mathcal{M}$ is a $GL_2(\mathbb{Z}/n)$-Galois covering. There is an action of $GL_2(\mathbb{Z}/n)$ on $\mathcal{M}(n)$ via acting on the left of the isomorphism

$$(\mathbb{Z}/n)^2 \times T \to E[n]$$

for an elliptic curve E over T; this action is (strictly) over $\mathcal{M}$. We have a map $[n]\colon \mathcal{M} \to \operatorname{Spec} \mathbb{Z}//GL_2(\mathbb{Z}/n)$ by associating to each elliptic curve E over S the $GL_2(\mathbb{Z}/n)$-torsor associated to the finite abelian group scheme $E[n]$, the n-torsion. This is part of a 2-commutative diagram:

$$\begin{array}{ccc} \mathcal{M}(n) & \longrightarrow & \operatorname{Spec} \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathcal{M} & \longrightarrow & \operatorname{Spec} \mathbb{Z}//GL_2(\mathbb{Z}/n) \end{array}$$

Since $\operatorname{Spec} \mathbb{Z}$ is equivalent to the moduli stack of trivialized $GL_2(\mathbb{Z}/n)$-torsors, this is a (homotopy) pullback square of stacks. Since a pullback of a torsor is a torsor again, $\mathcal{M}(n)$ is a $GL_2(\mathbb{Z}/n)$-torsor over $\mathcal{M}$.

Let now T be a scheme over S, E an elliptic curve over T with a level structure

$$\alpha\colon (\mathbb{Z}/n)^2 \times T \to E[n]$$

and G a finite group acting on E. Then G acts also on $E[n]$ and we can send a $g \in G$ to $\varphi(g) = \alpha^{-1} g \alpha \in GL_2(\mathbb{Z}/n)(T)$. We want to check that this satisfies the condition on φ in the last proposition: The square in the proposition specializes to

$$\begin{array}{ccc} T \times G & \longrightarrow & \mathcal{M}(n) \times GL_2(\mathbb{Z}/n) \\ \downarrow & & \downarrow \\ T \times_{T//G} T & \longrightarrow & \mathcal{M}(n) \times_{\mathcal{M}} \mathcal{M}(n) \end{array}$$

and on a point $(t, g) \in (T \times G)(X)$ it looks as follows:

$$\begin{array}{ccccc} (t, g) & \longmapsto & & & ((E, \alpha), \alpha^{-1} g \alpha) \\ \downarrow & & & & \downarrow \\ (t, t, g) & \longmapsto & ((E, \alpha), (E, \alpha), g) & \overset{\beta}{\longmapsto} & ((E, \alpha), (E, g\alpha), \mathrm{id}_E) \end{array}$$

The isomorphism β is given by (id_E, g) and defines the natural transformation making the square 2-commutative.

2.7 The Cohomology of the Moduli Stack of Elliptic Curves

The aim of this section is to recall the cohomology of the the moduli stacks of elliptic curves when 2 is inverted from [Bau08]. We also want to sketch an alternative way to obtain it, based on the following lemma:

Lemma 2.7.1. *Let*

$$\pi\colon \mathcal{X} \to \mathcal{Y}$$

be a G-Galois covering of algebraic stacks, where G is a finite group and $\mathcal{Y}$ is flat over $\mathbb{Z}$. Then for every quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{Y}$, the adjunction unit defines an isomorphism $\mathcal{F} \to (\pi_*\pi^*\mathcal{F})^G$. Furthermore, there is a spectral sequence

$$H^p(G; H^q(\mathcal{X}; \pi^*\mathcal{F})) \Rightarrow H^{p+q}(\mathcal{Y}; \mathcal{F}).$$

Proof. The categories $G - \mathrm{QCoh}(\mathcal{X})$ and $\mathrm{QCoh}(\mathcal{Y})$ are equivalent via π^* by Galois descent. Locally, π looks like $\operatorname{Spec} \prod_G A \to \operatorname{Spec} A$ and an inverse of π^* is given by taking G-invariants as follows from the general formula for faithfully flat descent given on p. 134 of [BLR90]. Now suppose that $\mathcal{F}$ is a quasi-coherent sheaf on $\mathcal{Y}$. We can define another sheaf $\mathcal{G}$ on $\mathcal{Y}$ by $(\pi_*\pi^*\mathcal{F})^G$, i.e., $\mathcal{G}(U) = (\mathcal{F}(U \times_{\mathcal{Y}} \mathcal{X}))^G$ (this is a sheaf since taking invariants is left-exact). The usual adjunction morphism $\mathcal{F} \to \pi_*\pi^*\mathcal{F}$ factors over $\mathcal{G}$. Since π^* and taking G-invariants are locally inverses, locally, the morphism $\mathcal{F} \to \mathcal{G}$ of sheaves is an isomorphism; therefore, it is also globally an isomorphism. In particular, we have $H^0(\mathcal{X}; \pi^*\mathcal{F})^G \cong H^0(\mathcal{Y}; \mathcal{F})$. Thus, we have now a (2-)commutative diagram

$$\begin{array}{ccc} G - \mathcal{X}\text{-mod} & \xrightarrow{\Gamma_G} & \mathbb{Z}[G]\text{-mod} \\ \uparrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle ()^G} \\ \mathrm{QCoh}(\mathcal{Y}) & \xrightarrow{\Gamma} & \mathbb{Z}\text{-mod} \end{array}$$

The functor Γ_G is defined as global sections, remembering the G-action. The composition $\Gamma_G \circ \pi^*$ corresponds to h_* for $h\colon \mathcal{Y} \to \operatorname{Spec} \mathbb{Z}//G$. Thus, Γ_G preserves injectives since h_* has an exact left adjoint h^* because h is flat. Then one can first apply the Grothendieck spectral sequence

$$E_2^{pq} \cong H^p(G; R^q\Gamma_G(\pi^*\mathcal{F})) \Rightarrow H^{p+q}(\mathcal{Y}; \mathcal{F}).$$

Here we use that π^* is an equivalence. Since the forgetful functor $u\colon \mathbb{Z}[G]\text{-mod} \to \mathbb{Z}\text{-mod}$ is exact, we obtain $uR^q\Gamma_G = R^q(u\Gamma_G) = R^q\Gamma = H^q$. Thus, the claim follows. □

Recall that the 0th cohomology of $\mathcal{M}$ is classical (and originally due to Deligne and Tate): $H^0_*(\mathcal{M}; \mathcal{O}) \cong \mathbb{Z}[c_4, c_6, \Delta^{\pm 1}]$.

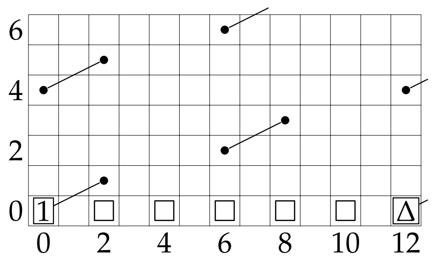
There are at least two routes to the computation of the higher cohomology of $\mathcal{M}$. The first is to use the results of [Bau08], where he computes the cohomology of the Weierstrass Hopf algebroid. Since the associated Hopf algebroid to $\mathcal{M}$ is the Weierstrass Hopf algebroid with Δ inverted, we have just to invert Δ in the cohomology to obtain the cohomology of $\mathcal{M}$.

The second way uses the lemma above. Since $\mathcal{M}(2)$ has a $\mathbb{G}_m$-torsor of the form $\operatorname{Spec}\Lambda$, quasi-coherent sheaves on $\mathcal{M}(2)$ are equivalent to graded Λ-modules. Thus

$$H^i(\mathcal{M}(2); \mathcal{F}) \cong \operatorname{Ext}^i_{\Lambda\text{-grmod}}(\Lambda, \mathcal{F}(\Lambda)) = 0$$

for every $\mathcal{F} \in \mathrm{QCoh}(\mathcal{M}(2))$ and $i > 0$. This implies that $H^i_*(\mathcal{M}[\frac{1}{2}]) \cong H^i(S_3; H^0_*(\mathcal{M}(2)))$ by the lemma above. The latter cohomology groups are computed in [Sto11].

Both ways yield as graded cohomology of the (uncompactified) moduli stack:



Here, our conventions are as follows: The position (p, q) corresponds to $H^q(\mathcal{M}; \omega^p)$. Bullets represent an $\mathbb{F}_3$ and boxes a $\mathbb{Z}[\frac{1}{2}][j]$ where $j = c_4^3/\Delta$. We choose a non-trivial class in $H^1(\mathcal{M}; \omega^2)$ and call it α; the lines denote α-multiplication. We choose a non-trivial class in $H^2(\mathcal{M}; \omega^6)$ and call it β; here we can even pin down the sign by choosing β such that it is in the Massey product $\langle \alpha, \alpha, \alpha \rangle$. All elements in higher cohomology are of the form $\pm \alpha^i \beta^j \Delta^k$ (for $i \in \{0, 1\}$, $j \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}$) and all this elements are non-zero. Note that Δ acts invertible, so the whole cohomology is 12-periodic.

In particular, $H^i_*(\mathcal{M}_{(p)}; \mathcal{O}) = 0$ for $p > 3$ and $i > 0$.

2.8 The Moduli Stack of Formal Groups

Completing an elliptic curve E at its identity section yields the formal group $\hat{E}$, a refinement of the Lie algebra of E. In this section, we will define precisely what a formal group is and show how to get a morphism from the moduli stack of elliptic curves to the moduli stack of formal groups. This will be essential for the definition of the spectrum of topological modular forms TMF.

Definition 2.8.1. Let S be a scheme. A *formal scheme over* S is a functor $(\mathrm{Sch}/S)^{op} \to \mathrm{Set}$, which is a (small) filtered colimit of functors representable by schemes over S. The cartesian product in the functor category restricts to a product on the category of formal schemes over S, denoted by $\times_S$. A commutative group object in formal schemes over S is called an *abstract formal group over* S.

Example 2.8.2. Let A be a ring with a chosen ideal I. Then $\mathrm{Spf}\, A := \mathrm{colim}_n \mathrm{Spec}(A(I^n))$ is a formal scheme. If we have a morphism $f\colon A \to B$ such that $f(I) \subset J$ for a chosen ideal $J \subset B$, then we get an induced map $\mathrm{Spf}\, B \to \mathrm{Spf}\, A$.

A *formal group law* over a ring R consists of a power series $F \in R[[X, Y]]$ satisfying the axioms of a commutative group in a formal way (see [Rav86], Appendix B, for a precise definition). The formal spectrum $\mathrm{Spf}\, R[[X]] := \mathrm{colim}\, \mathrm{Spec}\, R[x]/x^i$ is a formal scheme and F induces a morphism $\mathrm{Spf}\, R[[X]] \times_{\mathrm{Spec}\, R} \mathrm{Spf}\, R[[Y]] \cong \mathrm{Spf}\, R[[X, Y]] \to \mathrm{Spf}\, R[[X]]$ (by sending X to F), which defines an abstract formal group over R; here, the chosen ideal of $R[[X, Y]]$ is the augmentation ideal (X, Y). A (1-dimensional, commutative) *formal group* over a scheme S is an abstract formal group F which comes Zariski locally on S from a formal group law (i.e., we can cover S as $\bigcup U_i$ with $U_i \cong \mathrm{Spec}\, R_i$ such that $F|_{U_i}$ is isomorphic to an abstract formal group coming from a formal group law over R_i).

Definition 2.8.3. The *moduli stack of formal groups* $\mathcal{M}_{FG}$ is given by associating to each ring R the groupoid of formal groups over R.

The moduli stack of formal group laws FGL (without morphisms between them) is much simpler: It is isomorphic to $\operatorname{Spec} L$ for L (uncanonically) isomorphic to $\mathbb{Z}[x_1, x_2, \dots]$ and L carries an universal formal group law F^{univ}.[14] In concrete terms, this means the map $\operatorname{Hom}_{\mathrm{Rings}}(L, R) \to FGL(R)$ given by $\phi \mapsto \phi_*(F^{univ})$ is a bijection. The fiber product $\operatorname{Spec} L \times_{\mathcal{M}_{FG}} \operatorname{Spec} L$ is equivalent to $\operatorname{Spec} W$ for $W = L[u^{\pm 1}, b_1, b_2, \dots]$. As explained in [Nau07], this shows that $\mathcal{M}_{FG}$ is algebraic and (by a theorem of Quillen) $\mathrm{QCoh}(\mathcal{M}_{FG}) \simeq (MU_*, MU_*MU)$-comod (where the comodules are graded).

For future purposes, we want to be a bit more explicit: We set $H = \operatorname{Spec} \mathbb{Z}[u^{\pm 1}, b_1, b_2, \dots]$ and identify $H(\operatorname{Spec} R)$ with power series of the form $ux + b_1x^2 + b_2x^3 + \cdots$ with $b_1, b_2, \dots \in R$ and $u \in R^\times$ a unit. Composition of power series defines a natural group structure on $H(\operatorname{Spec} R)$ and thus the structure of a group scheme on H. The scheme $\operatorname{Spec} L \cong FGL$ gets the structure of an H-torsor over $\mathcal{M}_{FG}$ with H acting as follows: For $h \in H(\operatorname{Spec} R)$ and $F \in FGL(\operatorname{Spec} R)$, define a formal group law $h \cdot F$ over R by $hF(h^{-1}(x), h^{-1}(y))$. This defines an action of H on FGL. This can be extended to an action of H on $FGL \cong \operatorname{Spec} L$ over $\mathcal{M}_{FG}$ in the sense of Section 2.6: For $h \in H(\operatorname{Spec} R)$, $F \in FGL(\operatorname{Spec} R)$ the element h defines an isomorphism between the underlying formal groups of F and $h \cdot F$, which we take as our $\alpha_{F,h}$. That $\operatorname{Spec} L \times H \to \operatorname{Spec} L \times_{\mathcal{M}_{FG}} \operatorname{Spec} L$ is an equivalence boils down to the fact that an isomorphism between formal groups associated to formal group laws is given by a power series.

Let F be a formal group law over R and g be an automorphism of the associated formal group. Then we can write g as power series $\varphi(g) \in R[[x]]$ with

$$\varphi(g)^{-1}F((\varphi(g))(x), (\varphi(g))(y)) = F.$$

This defines a morphism from the automorphism group of the underlying formal group of F into $H(\operatorname{Spec} R)$. The check that this morphism fulfills the conditions on φ in Proposition 2.6.6 is analogous to the example of the moduli stack of elliptic curves with level structures.

To every elliptic curve E/S, we can associate a formal group as follows: Denote by $e\colon S \to E$ the unit section and by I the ideal sheaf on E corresponding to the reduced subscheme structure on $\operatorname{im}(e)$, i.e., $\operatorname{im}(e)$ equals the vanishing locus $V(I)$ of I. A map $f\colon X \to E$ factors over $V(I)$ iff the ideal sheaf f^*I is zero. It factors over $\hat{E} := \operatorname{colim} V(I^n)$ iff f^*I is locally nilpotent, hence, iff the morphism X^{red} factors over $V(I)$. Suppose now, we have points $a, b \in \hat{E}(X)$. Via the canonical map $\hat{E} \to E$, these induce maps $a', b'\colon X \to E$. We get a diagram

$$\begin{array}{ccccc} X^{red} & \longrightarrow & \operatorname{im}(e) \times_S \operatorname{im}(e) & \longrightarrow & \operatorname{im}(e) \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{a' \times b'} & E \times_S E & \longrightarrow & E \end{array}$$

The map $X^{red} \to E$ corresponds to a point $a \cdot b \in \hat{E}(X)$, inducing a group structure on $\hat{E}$. This defines indeed a formal group (as can be seen, e.g., in the Weierstrass form). For further information, see also [Rez02, 11.4].

[14] The ring L is called the *Lazard ring*.

Locally, the corresponding formal group law can be concretely calculated up to arbitrary precision using a Weierstraß form; either by hand, as in [Sil09], Chapter IV.1, or by Magma or similar programs.

Theorem 2.8.4. *The assignment $E \mapsto \hat{E}$ induces a morphism $\mathcal{M} \to \mathcal{M}_{FG}$, which is flat.*

The author is not aware of a published proof of the flatness statement, but, at least, this theorem is stated in Lecture 15 of [Lur10]. Furthermore, it can probably be deduced from [BL10, 8.1.6] and the Serre–Tate theorem (stating that elliptic curves have the same deformation theory as p-divisible groups, see [BL10, 7.2.1]).

Chapter 3

Vector Bundles

Our aim in this chapter is the study of vector bundles over the moduli stack of elliptic curves. Recall the following definition:

Definition 3.0.1. A *vector bundle* on a Deligne–Mumford $\mathcal{X}$ stack is an $\mathcal{O}_{\mathcal{X}}$-module that is locally free of finite rank in the étale topology.

As noted before, every vector bundle is a quasi-coherent (even coherent) sheaf since it has locally a presentation.

Recall the notation $\mathcal{M}_R$ for the moduli stack of elliptic curves over R. As a shorthand, denote by $\mathcal{M}_{(p)}$ the moduli stack of elliptic curves over $\mathbb{Z}_{(p)}$. Furthermore, we denote structure sheaves in general by $\mathcal{O}$ (with subscript if it is not clear from the context).

The *Picard group Pic* of a stack is the group of isomorphism classes of line bundles (with group structure given by the tensor product and the inverses by duals). The classification of line bundles on $\mathcal{M}_R$ is already known:

Theorem 3.0.2 ([FO10])**.** *Every line bundle over $\mathcal{M}_R$, for R a reduced ring, is a tensor power of ω and we have $\omega^{12} \cong \mathcal{O}$. Therefore, the Picard group* $\mathrm{Pic}(\mathcal{M}_R)$ *is isomorphic to* $\mathbb{Z}/12$.

We will prove that every vector bundle splits into line bundle on $\mathcal{M}_{\mathbb{Q}}$ using an argument by Angelo Vistoli. In general, the situation is more complicated and we will mainly restrict to the case $\mathcal{M}_{(3)}$. A particularly accessible class of vector bundles is the following:

Definition 3.0.3. We define the notion of a *standard vector bundle* for a prime p inductively: Every line bundle on $\mathcal{M}_{(p)}$ is called *standard*. Furthermore, a vector bundle $\mathcal{E}$ on $\mathcal{M}_{(p)}$ is called *standard* if there is an injection $\mathcal{L} \hookrightarrow \mathcal{E}$ from a line bundle on $\mathcal{M}_{(p)}$ such that the cokernel is a standard vector bundle.

Thus, standard vector bundles are those vector bundles which can be built as iterated extension of line bundles.

Lemma 3.0.4.

1. *Let $\mathcal{E}$ be a vector bundle with a surjective morphism $\mathcal{E} \to \mathcal{L}$ to a line bundle such that the kernel $\mathcal{F}$ is a standard vector bundle. Then $\mathcal{E}$ is a standard vector bundle.*

2. *Let $\mathcal{E}$ be a standard vector bundle. Then also $\check{\mathcal{E}}$ is a standard vector bundle.*

Proof. 1. Let $\mathcal{E}$ be of rank n. By induction, we assume that we have shown the first part of the lemma for all smaller ranks. By definition, we have an injection $\mathcal{L}' \hookrightarrow \mathcal{F}$ from a line bundle such that the cokernel $\mathcal{F}'$ is a standard vector bundle again. Consider the (snake lemma) diagramm

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{L}' & \xrightarrow{=} & \mathcal{L}' & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{=} & & \\
0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Here, $\mathcal{E}'$ is defined as the cokernel of $\mathcal{L}' \to \mathcal{F} \to \mathcal{E}$. It is a vector bundle since it is an extension of two vector bundles (see Remark 2.3.10). Furthermore, it is of rank $n-1$ and has a surjective morphism to $\mathcal{L}$ whose kernel $\mathcal{F}'$ is a standard vector bundle. By induction, $\mathcal{E}'$ is thus a standard vector bundle. This implies that also $\mathcal{E}$ is standard.

2. By induction, we assume that we know the statement for all standard vector bundles of smaller rank than $\mathcal{E}$. Consider a sequence

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{L}$ is a line bundle and $\mathcal{F}$ is standard. Dualizing gives

$$0 \to \check{\mathcal{F}} \to \check{\mathcal{E}} \to \check{\mathcal{L}} \to 0.$$

Note that the sequence is short exact because the Ext-sheaf $\mathcal{E}xt^i(\mathcal{F}, \mathcal{O})$ vanishes for $i > 0$ since $\mathcal{F}$ is a vector bundle. The morphism $\check{\mathcal{E}} \to \check{\mathcal{L}}$ is surjective and its kernel is standard by induction. Thus, we can use the first part of the lemma.

□

The main aim of this chapter is to show the following theorem:

Theorem 3.0.5. *Every standard vector bundle over $\mathcal{M}_{(3)}$ is isomorphic to a sum of copies of the vector bundles $\mathcal{O}$, E_α or $E_{\alpha,\widetilde{\alpha}}$ (and tensor products of line bundles with them). Here, the latter two are vector bundles of rank* 2 *and* 3*, respectively, to be introduced in Section 3.4.*

Conjecture 3.0.6. *Every vector bundle on $\mathcal{M}_{(3)}$ is standard.*

In addition, we prove that there are infinitely many indecomposable vector bundles on $\mathcal{M}_{(2)}$.

As a warm up, we will recall the classification of integral representations of the cyclic group C_2 or, what is equivalent, vector bundles over $\operatorname{Spec} \mathbb{Z}//C_2$ – this is easier but in some ways analogous to classification results on vector bundles on the moduli stack of elliptic curves. We must stress that the classification of integral C_2-representations is already known for a long time – if not since the beginning of time or the era of Archimedes, then at least since [Die40].

After that, we will classify vector bundles on $\mathcal{M}_{\mathbb{Q}}$ and show a few basic properties of the category of vector bundles on $\mathcal{M}_{(3)}$. Then we go on and study the vector bundles $\mathcal{O}$, E_α and $E_{\alpha,\widetilde{\alpha}}$ in detail. In the last section, we will prove the main theorem of this chapter.

3.1 Vector bundles over $\operatorname{Spec}\mathbb{Z}//C_2$

In this section, we will classify integral representations of the cyclic group with two elements, C_2. We remark that this category is both equivalent to the category of vector bundles over $\operatorname{Spec}\mathbb{Z}//C_2$ (by Galois descent) and to the category of modules over $\mathbb{Z}[C_2]$ that are free of finite rank as abelian groups.

Lemma 3.1.1. *Every $\mathbb{Q}[C_2]$-module is a direct sum of one-dimensional representations.*

Proof. Denote by t the generator of C_2. Then $e_1 = \frac{1+t}{2}$ and $e_2 = \frac{1-t}{2}$ are orthogonal idempotents in $\mathbb{Q}[C_2]$. Therefore, $\mathbb{Q}[C_2] \cong \mathbb{Q}e_1 \times \mathbb{Q}e_2$ and $\mathbb{Q}[C_2]\text{-mod} \simeq \mathbb{Q}\text{-mod} \times \mathbb{Q}\text{-mod}$. □

Lemma 3.1.2. *Every one-dimension C_2-representation over $\mathbb{Z}$ or $\mathbb{Q}$ is either the trivial or the sign representation. In particular, every C_2-representation over $\mathbb{Q}$ is of the form $M \otimes \mathbb{Q}$ for an integral C_2-representation M.*

Proof. The multiplicative groups $\mathbb{Q}^\times$ and $\mathbb{Z}^\times$ have only one non-trivial element of order 2, the element -1. □

Lemma 3.1.3. *Every integral C_2-representation M of dimension m sits in an extension*

$$0 \to L \to M \to N \to 0,$$

where L is a one-dimensional representation and N is an $(m-1)$-dimensional one.

Proof. By Lemma 3.1.1, we have an injection $L' \to L' \otimes \mathbb{Q} \to M \otimes \mathbb{Q}$ of $\mathbb{Z}[C_2]$-modules for some 1-dimensional integral C_2-representation L'. Multiply this map by a natural number to get an injection $L' \to M$ with cokernel C. Divide out the torsion of C to get a $\mathbb{Z}[C_2]$-module N, which is free as an abelian group. Denote the kernel of $M \to N$ by L, which is obviously also free as an abelian group. Since $L \otimes \mathbb{Q} \cong L' \otimes \mathbb{Q}$, we have that L is of rank 1. □

Example 3.1.4 (Examples of C_2-representations)**.** We have the two 1-dimensional representations $\mathbb{Z}$ and $\mathbb{Z}'$ (the sign representation) and the representation $\mathbb{Z}[C_2]$ of rank 2. We know that $\operatorname{Ext}^1_{\mathbb{Z}[C_2]}(\mathbb{Z},\mathbb{Z}) \cong \operatorname{Ext}^1_{\mathbb{Z}[C_2]}(\mathbb{Z}[C_2],\mathbb{Z}) = 0$ and $\operatorname{Ext}^1_{\mathbb{Z}[C_2]}(\mathbb{Z}',\mathbb{Z}) \cong \mathbb{F}_2$, where the non-trivial element corresponds to the extension

$$0 \to \mathbb{Z} \to \mathbb{Z}[C_2] \to \mathbb{Z}' \to 0.$$

Proposition 3.1.5. *Every integral representation of C_2 is a direct sum of (several copies of) the trivial representation, the sign representation and the free representation.*

Proof. For rank $n = 1$ this is true by Lemma 3.1.2. Assume by induction that the assertion of the proposition is true for representations of rank smaller than n, for some $n \in \mathbb{N}$. Now let M be a C_2-representation of rank $n+1$ and choose an extension

$$0 \to L \to M \to N \to 0$$

as above. We can assume that L is the trivial representation – else we could tensor the exact sequence with the sign representation. The extension above corresponds to a class x in $\mathrm{Ext}^1_{\mathbb{Z}[C_2]}(N, \mathbb{Z})$. By assumption $N \cong \mathbb{Z}^a \oplus (\mathbb{Z}')^b \oplus (\mathbb{Z}[C_2])^c$. We see that $\mathrm{Ext}^1_{\mathbb{Z}[C_2]}(N, \mathbb{Z}) \cong \mathbb{F}_2^b$. By a change of basis, we can assume that $x = (1, 0, \ldots, 0)$ or $x = 0$. So either $M \cong \mathbb{Z}^a \oplus \mathbb{Z}'^{b-1} \oplus \mathbb{Z}[C_2]^{c+1}$ or $M \cong \mathbb{Z}^{a+1} \oplus \mathbb{Z}'^b \oplus \mathbb{Z}[C_2]^c$. □

3.2 Vector bundles over $\mathcal{M}_\mathbb{Q}$

We will classify in this section vector bundles on $\mathcal{M}_\mathbb{Q}$. Everything in this section (except possibly mistakes) I have learned from Angelo Vistoli.

For $a_1, \ldots, a_n \in \mathbb{N}$ and a commutative ring R, the weighted projective stack $\mathcal{P}_R(a_1, \ldots, a_n)$ is the (stack) quotient of $\mathbb{A}^n_R - \{0\}$ by the multiplicative group $\mathbb{G}_m$ under the action which is the restriction of the map

$$\begin{array}{rcl} \phi\colon\ \mathbb{A}^1_R \times \mathbb{A}^n_R & \to & \mathbb{A}^n_R \\ R[t] \otimes R[t_1, \ldots, t_n] & \leftarrow & R[t_1, \ldots, t_n] \\ t^{a_i} \otimes_i & \leftarrow & t_i \end{array}$$

to $(\mathbb{G}_m \times \operatorname{Spec} R) \times (\mathbb{A}^n_R - \{0\})$. Here, $A^n_R - \{0\}$ denotes the complement of the zero point (corresponding to the ideal $(t_1, \ldots, t_n)$). On geometric points, the action corresponds to the map $(t_1, \ldots, t_n) \mapsto (t^{a_1} t_1, \ldots, t^{a_n} t_n)$. The restriction to $\{0\} \times \mathbb{A}^n_R = \operatorname{Spec}(R \otimes R[t_1, \ldots, t_n])$ (induced by $t \mapsto 0$) equals the projection to the 0-point (which is induced by $t_1, \ldots, t_n \mapsto 0$). In this section, the base ring R will always (implicitly) be $\mathbb{Q}$. As explained in 2.4, we have an open embedding $i\colon \mathcal{M}_\mathbb{Q} \hookrightarrow \mathcal{P}(4, 6)$ given by the Weierstraß form.

Now let ξ be a vector bundle on $\mathcal{M}_\mathbb{Q}$. The sheaf ξ is *reflexive* in the sense that the canonical map $\xi \to (\check{\xi})^\vee$ to the double-dual is an isomorphism. We want to extend ξ by a reflexive coherent sheaf on $\mathcal{P}(4, 6)$. Note first that $i_*\xi$ is quasi-coherent by [LMB00], 13.2.6, since i is quasi-compact. By [LMB00], 15.5, there is then a coherent sheaf $\mathcal{G}$ on $\mathcal{P}(4, 6)$ with $i^*\mathcal{G} = \xi$. Let $\mathcal{F}$ denote its double-dual. This is both reflexive ([Har80], 1.2 - which we can use also for stacks since both reflexivity and coherence are local conditions) and coherent and, in addition, we have $i^*\mathcal{F} = \xi$ since ξ is already reflexive.

Proposition 3.2.1. *Every reflexive sheaf $\mathcal{F}$ on $\mathcal{P}(m, n)$ is a direct sum of line bundles.*

Proof. By Galois descent, the sheaf $\mathcal{F}$ corresponds to a $\mathbb{G}_m$-equivariant sheaf on $\mathbb{A}^2 - \{0\}$, with respect to the action given by $t(x, y) = (t^m x, t^n y)$; we will denote this $\mathbb{G}_m$-equivariant sheaf by abuse of notation still by $\mathcal{F}$. This new sheaf $\mathcal{F}$ is reflexive since pullback by flat maps preserves reflexive sheaves (this is essentially [Ser00], p. 70, prop 12). Using the inclusion $(\mathbb{A}^2 - \{0\})//\mathbb{G}_m \hookrightarrow \mathbb{A}^2//\mathbb{G}_m$, we can, as above, extend $\mathcal{F}$ to a reflexive (hence locally free) $\mathbb{G}_m$-equivariant sheaf on $\mathbb{A}^2$, which we'll denote by abuse of notation again by $\mathcal{F}$. Since every reflexive sheaf on a regular 2-dimensional scheme is locally free ([Har80], 1.4), $\mathcal{F}$ is locally free.

Let F_0 be the fiber of $\mathcal{F}$ at the origin, i.e. the $\mathbb{Q}$-vector space corresponding to $i_0^*\mathcal{F}$, where $i_0\colon \operatorname{Spec}\mathbb{Q} \to \mathbb{A}^2$ is the inclusion of the origin. This fiber gets the structure of an (algebraic) representation of $\mathbb{G}_m$, which splits into a direct sum of 1-dimensional representations (as every $\mathbb{G}_m$-representation). Denote by $\mathcal{E}$ the $\mathbb{G}_m$-equivariant locally free sheaf $F_0 \otimes_\mathbb{Q} \mathcal{O}_{\mathbb{A}^2}$. The sheaf $\mathcal{E}$ is a direct sum of $\mathbb{G}_m$-equivariant invertible sheaves, so it is enough to show that $\mathcal{F}$ is isomorphic to $\mathcal{E}$. Let $\mathcal{H} = \mathcal{H}om(\mathcal{E}, \mathcal{F})$ be the sheaf of homomorphisms $\mathcal{E} \to \mathcal{F}$ on $\mathbb{A}^2$ and H_0 its fiber at the origin. The restriction homomorphism $H^0(\mathbb{A}^2; \mathcal{H}) \to H_0$ is surjective (say, since $\mathcal{E}$ and $\mathcal{F}$ are non-equivariantly trivial by Seshadri's Theorem, [Lam06, II.6.1]); since $\mathbb{G}_m$ is linearly reductive, it will stay surjective after taking $\mathbb{G}_m$ invariants (this can be seen in concrete terms using that $\mathbb{G}_m$-representations split in one-dimensional representations). This means that every $\mathbb{G}_m$-equivariant homomorphism $E_0 \to F_0$ will lift to a $\mathbb{G}_m$-equivariant homomorphism $\mathcal{E} \to \mathcal{F}$.

Now, consider a $\mathbb{G}_m$-equivariant homomorphism $f\colon \mathcal{E} \to \mathcal{F}$ that restricts to the identity $E_0 = F_0$ at the origin. Thus, it is also an isomorphism at the stalk at the origin. Indeed, let $A = \mathbb{Q}[X, Y]_{(X,Y)}$ be the local ring at the origin and E and F be the A-modules corresponding to $\mathcal{E}$ and $\mathcal{F}$. Then $f \otimes_A \mathbb{Q}\colon E \otimes_A \mathbb{Q} \to F \otimes_A \mathbb{Q}$ is the identity by assumption. Thus, $\operatorname{coker}(f) \otimes_A \mathbb{Q} = 0$ and also $\ker(f) \otimes_A \mathbb{Q} = 0$ since $\operatorname{Tor}_1^A(F, \mathbb{Q}) = 0$. Thus $\operatorname{coker}(f\colon E \to F) = \ker(f\colon E \to F) = 0$ by Nakayama's lemma.

Both $\operatorname{coker}(f)$ and $\ker(f)$ are generated by finitely many global sections. Thus, the set of points where their stalks are zero is open, containing the origin. Let C be the set of points at whose stalks $f\colon \mathcal{E} \to \mathcal{F}$ is not an isomorphism. It follows that this is a $\mathbb{G}_m$-invariant closed subset C of $\mathbb{A}^2$ not containing the origin. Considering the map

$$\phi\colon \mathbb{A}^1 \times \mathbb{A}^2 \to \mathbb{A}^2$$

as above, we see that $\mathbb{A}^1 \times C$ must have image in C since C is closed. But this image contains the origin if C is not empty; thus C must be empty and f an isomorphism between $\mathcal{E}$ and $\mathcal{F}$. This completes the proof. □

Corollary 3.2.2. *Every vector bundle on $\mathcal{M}_{\mathbb{Q}}$ is the direct sum of line bundles.*

Remark 3.2.3. Of course, the proof goes through also for any other field of characteristic $\neq 2, 3$ instead of $\mathbb{Q}$.

3.3 Kernels of Morphisms of Vector Bundles on $\mathcal{M}_{(3)}$

The aim of this section is to show two propositions, one about kernels of maps between vector bundles and one about global sections of vector bundles. We set, by abuse of notation, $\mathcal{M} = \mathcal{M}_{(3)}$.

We have an étale covering $q\colon \mathcal{M}(4) \to \mathcal{M}$ from the moduli stack of elliptic curves with level structure of niveau 4. The stack $\mathcal{M}(4)$ is representable by a scheme. Indeed, we can write it as $\operatorname{Spec} A \coprod \operatorname{Spec} A \simeq \operatorname{Spec} A \times A$, where $A = \mathbb{Z}_{(3)}[i][X, X^{-1}(X^4 - 1)^{-1}]$ (see the discussion at the end of Section 2.5).

Proposition 3.3.1. *Let $f\colon \mathcal{E} \to \mathcal{E}'$ be a morphism of vector bundles on $\mathcal{M}$. Then $\mathcal{L} := \ker(f)$ is a vector bundle again.*

Proof. A quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{M}$ is a vector bundle iff $\mathcal{F}(\mathcal{M}(4))$ is projective of finite rank. Since $\mathbb{Z}_{(3)}[i]$ is a principal ideal domain, A is of homological dimension 2. Thus a kernel of a map between projective modules is projective. Since A is noetherian, the kernel of $\mathcal{E}(\mathcal{M}(4)) \to \mathcal{E}'(\mathcal{M}(4))$ is also finitely generated. □

Proposition 3.3.2. *Let $\mathcal{E}$ be a vector bundle on $\mathcal{M}_{(3)}$. Then $\Gamma_*(\mathcal{E}) \neq 0$.*

Proof. Set $E = \mathcal{E}(\mathcal{M}(4))$. We know that $E_{\mathbb{Q}} = E \otimes_{\mathbb{Z}} \mathbb{Q}$ splits by the last section into a sum of $GL(2, \mathbb{Z}/4\mathbb{Z})$-equivariant projective modules of rank one[1] over $(A \times A)_{\mathbb{Q}} = (A \times A) \otimes_{\mathbb{Z}} \mathbb{Q}$ (using Galois descent). By the classification of line bundles on the moduli stack of elliptic curves, these are already defined over $(A \times A)$. Take now such an invertible module N so that we have an injection $(A \times A)_{\mathbb{Q}} \hookrightarrow (E \otimes N^{-1})_{\mathbb{Q}}$, which corresponds to an (invariant) element $s' \in (E \otimes N^{-1})_{\mathbb{Q}}$. Now take a d such that $3^d \cdot s' \in E \otimes N^{-1}$ and define $s := 3^d \cdot s'$. This is again invariant and therefore corresponds to an injection $\mathcal{O} \hookrightarrow \mathcal{E} \otimes \mathcal{L}^{-1}$ for $\mathcal{L}$ associated to N and, hence, an injection $\mathcal{L} \hookrightarrow \mathcal{E}$. For $\mathcal{L} \cong \omega^n$, this defines a non-trivial element in $\Gamma_{-n}(\mathcal{E})$. □

3.4 Examples of Vector Bundles on $\mathcal{M}_{(3)}$

In this section, we will give a detailed exposition of the vector bundles of low rank on the moduli stack of elliptic curves at $p = 3$. For this section, we set by abuse of notation $\mathcal{M} = \mathcal{M}_{(3)}$.

As already mentioned, line bundles on $\mathcal{M}$ are classified by the following result of Mumford and Fulton-Olsson:

Theorem 3.4.1 ([FO10])**.** *The Picard group of $\mathcal{M}$ is isomorphic to $\mathbb{Z}/12\mathbb{Z}$ and generated by the line bundle ω.*

By Section 2.7, we know that $\mathrm{Ext}^1(\omega^j, \omega^k) \cong \mathbb{Z}/3\mathbb{Z}$ (generated by an element α) for $k - j = 2$ and 0 else. Here and in the following $k - j = 2$ is understood as an equality in the Picard group,i.e., $k - j \equiv 2 (\mod 12)$. This implies that the only standard vector bundle over $\mathcal{M}$ of rank 2 that does not split into line bundles sits in an extension

$$0 \to \mathcal{O} \to E_\alpha \to \omega^{-2} \to 0 \tag{3.1}$$

or a twist of it (so that the vector bundle is isomorphic to $E_\alpha \otimes \omega^j$ for some j). Here it should be noted that we do not need to distinguish between an extension and its negative in the Ext-group since its middle terms are isomorphic.

We now want to compute some Ext-groups. We have an exact sequence

$$\mathrm{Hom}(\omega^j, \omega^{-2}) \xrightarrow{\delta_0}$$

$$\mathrm{Ext}^1(\omega^j, \mathcal{O}) \longrightarrow \mathrm{Ext}^1(\omega^j, E_\alpha) \longrightarrow \mathrm{Ext}^1(\omega^j, \omega^{-2}) \xrightarrow{\delta_1}$$

$$\mathrm{Ext}^2(\omega^j, \mathcal{O}) \longrightarrow \mathrm{Ext}^2(\omega^j, E_\alpha) \longrightarrow \mathrm{Ext}^2(\omega^j, \omega^{-2}) \xrightarrow{\delta_2}$$

$$\mathrm{Ext}^3(\omega^j, \mathcal{O}) \longrightarrow \cdots$$

[1] Recall that is equivalent to being an invertible module.

To handle this, we need the following lemma:

Lemma 3.4.2 ([ML63], II.9.1). *Let*

$$0 \to A \to B \to C \to 0$$

be an extension in an abelian category $\mathcal{A}$ (with enough injectives or projectives), corresponding to the Ext-class $x \in \mathrm{Ext}^1(C, A)$. The boundary map $\mathrm{Ext}^k(T, C) \to \mathrm{Ext}^{k+1}(T, A)$ of the long exact sequence for Ext-groups out of T equals right multiplication by x. Similarly, the boundary map $\mathrm{Ext}^k(A, T) \to \mathrm{Ext}^{k+1}(C, T)$ of the sequence for Ext-groups into T equals left multiplication by x.

The map δ_0 is therefore surjective, δ_1 is zero (since $\alpha^2 = 0$) and δ_2 is an isomorphism. Hence, we get isomorphisms $\mathrm{Ext}^1(\omega^j, E_\alpha) \cong \mathrm{Ext}^1(\omega^j, \omega^{-2})$ and $\mathrm{Ext}^2(\omega^j, E_\alpha) \cong \mathrm{Ext}^2(\omega^j, \mathcal{O})$. This results in the following Ext-groups

$$\mathrm{Ext}^1(\omega^j, E_\alpha) = \begin{cases} \mathbb{Z}/3\mathbb{Z} & \text{if } j = -4 \\ 0 & \text{else} \end{cases}$$

$$\mathrm{Ext}^2(\omega^j, E_\alpha) = \begin{cases} \mathbb{Z}/3\mathbb{Z} & \text{if } j = -6 \\ 0 & \text{else} \end{cases}$$

With the same arguments, we can show that multiplication with β defines isomorphisms $\mathrm{Ext}^i(\omega^j, E_\alpha) \cong \mathrm{Ext}^{i+2}(\omega^j, E_\alpha)$. We denote the generator of the Ext^1-group that maps to $\alpha \in \mathrm{Ext}^1(\omega^{-2}, \mathcal{O})$ by $\widetilde{\alpha}$.

By dualizing the extension (3.1) and tensoring with ω^{-2}, we get an extension

$$0 \to \mathcal{O} \to \check{E}_\alpha \otimes \omega^{-2} \to \omega^{-2} \to 0.$$

This is non-split (else the dual sequence would split as well), therefore $\check{E}_\alpha \cong E_\alpha \otimes \omega^2$. Now consider the following lemma:

Lemma 3.4.3. *Let $(\mathcal{X}, \mathcal{O})$ be a ringed site and $\mathcal{E}$ and $\mathcal{F}$ be vector bundles and $\mathcal{G}$ be a quasi-coherent sheaf. Then we have $\mathrm{Ext}^i(\mathcal{E}, \mathcal{F} \otimes \mathcal{G}) \cong \mathrm{Ext}^i(\mathcal{E} \otimes \check{\mathcal{F}}, \mathcal{G})$.*

Proof. Since vector bundles are strongly dualizable, we have a natural isomorphism

$$\mathcal{H}om_{\mathcal{O}}(\mathcal{E}, \mathcal{F} \otimes \mathcal{G}) \cong \mathcal{H}om_{\mathcal{O}}(\mathcal{E} \otimes \check{\mathcal{F}}, \mathcal{G}).$$

The same holds for all higher Ext-sheaves (they are all zero). Therefore,

$$\mathrm{Ext}^i(\mathcal{E}, \mathcal{F} \otimes \mathcal{G}) \cong H^i \mathcal{H}om_{\mathcal{O}}(\mathcal{E}, \mathcal{F} \otimes \mathcal{G}) \cong H^i \mathcal{H}om_{\mathcal{O}}(\mathcal{E} \otimes \check{\mathcal{F}}, \mathcal{G}) \cong \mathrm{Ext}^i(\mathcal{E} \otimes \check{\mathcal{F}}, \mathcal{G})$$

by the Grothendieck spectral sequence converging from the cohomology of the Ext-sheaves to the Ext-groups. □

In particular, we have

$$\mathrm{Ext}^i(E_\alpha \otimes \omega^j, \mathcal{O}) \cong \mathrm{Ext}^i(\omega^j, \check{E}_\alpha) \cong \mathrm{Ext}^i(\omega^{j-2}, E_\alpha).$$

This implies that the only non-vanishing Ext^1-class is in $j = -2$.

We can also conclude that

$$\mathrm{Ext}^i(E_\alpha \otimes \omega^j, E_\alpha) \cong \mathrm{Ext}^i(E_\alpha \otimes \omega^j \otimes E_\alpha, \omega^{-2}) \cong \mathrm{Ext}^i(\omega^{j-2}, E_\alpha \otimes E_\alpha),$$

which we will calculate later in this section.

A further, particularly important example of a vector bundle is the following: Let

$$f\colon \mathcal{M}_0(2) \to \mathcal{M}$$

be the usual projection map. Then $f_*f^*\omega^j = f_*f^*\mathcal{O} \otimes \omega^j$ (see Lemma 2.3.13) defines a family of rank 3 vector bundles on $\mathcal{M}$.

Lemma 3.4.4. *The cohomology groups $H^i(\mathcal{M}; f_*f^*\mathcal{F})$ vanish for $i > 0$ for every quasi-coherent sheaf $\mathcal{F}$.*

Proof. The map f is finite and, in particular, affine. Therefore, all higher direct images R^if_* vanish and, using a degenerate form of the Leray spectral sequence, we get

$$H^i(\mathcal{M}; f_*f^*\mathcal{F}) \cong H^i(\mathcal{M}_0(2); f^*\mathcal{F})$$

There is an affine $\mathbb{G}_m$-torsor over $\mathcal{M}_0(2)$ of the form $\operatorname{Spec} A$ for $A \cong \mathbb{Z}_{(3)}[b_2, b_4, \Delta^{-1}]$ (see [Beh06, Section 1.3.2]). Quasi-coherent sheaves on $\mathcal{M}_0(2)$ correspond to graded A-modules. Since $\operatorname{Hom}_{A\text{-grmod}}(A, -)$ is clearly exact, we get that

$$H^i(\mathcal{M}; f^*\mathcal{F}) \cong \operatorname{Ext}^i_{A\text{-grmod}}(A, \mathcal{F}(\operatorname{Spec} A)) = 0$$

for $i > 0$. □

In the next section, we will show that the existence of exact sequences

$$0 \to \mathcal{O} \to f_*f^*\mathcal{O} \to E_\alpha \otimes \omega^{-2} \to 0 \tag{3.2}$$

and

$$0 \to E_\alpha \otimes \omega^4 \to f_*f^*\mathcal{O} \to \mathcal{O} \to 0 \tag{3.3}$$

such that the map $\mathcal{O} \to f_*f^*\mathcal{O}$ is the adjunction map and the map $f_*f^*\mathcal{O} \to \mathcal{O}$ is its dual (under a chosen isomorphism $(f_*f^*\mathcal{O})^\vee \cong f_*f^*\mathcal{O}$). Since $f_*f^*\mathcal{O}$ is self-dual, we get $\operatorname{Ext}^i(f_*f^*\mathcal{O}, \omega^j) = 0$ for all $i > 0$ by Lemma 3.4.3 (with $\mathcal{E} = \mathcal{O}$ and $\mathcal{F} = f_*f^*\mathcal{O}$). Using this, we get inductively that $\operatorname{Ext}^i(f_*f^*\mathcal{O}, \mathcal{E}) \cong \operatorname{Ext}^i(\mathcal{E}, f_*f^*\mathcal{O}) = 0$ for all standard vector bundles $\mathcal{E}$ for $i > 0$.

The two extensions (3.2) and (3.3) are non-split (as can be seen by computing cohomology). Thus, the second extension corresponds to $\pm$ the Ext^1-class $\widetilde{\alpha}$ mentioned above. Therefore, we will often call $f_*f^*\mathcal{O}$ also by the name $E_{\alpha,\widetilde{\alpha}}$. Call the class in $\operatorname{Ext}^1(E_\alpha \otimes \omega^{-2}, \mathcal{O})$ corresponding to the first extension ${}^t\widetilde{\alpha}$. Since $\operatorname{Ext}^1(\omega^{-4}, \mathcal{O}) = 0$, this projects non-trivially to $\operatorname{Ext}^1(\omega^{-2}, \mathcal{O})$ and thus this projection equals $\pm\alpha$.

Note also that $f_*f^*\mathcal{O} \cong f_*f^*\mathcal{O} \otimes \omega^4$. Indeed, $b_4 \in \Gamma_4(f_*f^*\mathcal{O})$ defines a map $b_4\colon f_*f^*\mathcal{O} \to f_*f^*\mathcal{O} \otimes \omega^4$ (since $f_*f^*\mathcal{O}$ is a sheaf of algebras) and since b_4 is divisor of Δ and hence a unit (by the formulas in Section 2.5), this map is an isomorphism.

If we tensor the extension (3.2) with E_α, we get:

$$0 \to E_\alpha \to f_*f^*\mathcal{O} \otimes E_\alpha \to E_\alpha \otimes E_\alpha \otimes \omega^{-2} \to 0$$

The middle term splits into $f_*f^*\mathcal{O} \oplus f_*f^*\mathcal{O} \otimes \omega^{-2}$ (as can be seen by tensoring the extension (3.1) with $f_*f^*\mathcal{O}$) and therefore has vanishing higher graded cohomology (i.e., vanishing higher cohomology even after tensoring with an ω^j). Therefore,

$$\mathrm{Ext}^2(\omega^{j-4}, E_\alpha) \cong \mathrm{Ext}^1(\omega^{j-2}, E_\alpha \otimes E_\alpha) \cong \mathrm{Ext}^1(E_\alpha \otimes \omega^j, E_\alpha),$$

which is zero unless $j = -2$, when it is isomorphic to $\mathbb{Z}/3$. The extensions

$$0 \to E_\alpha \to E_\alpha \otimes E_\alpha \to E_\alpha \otimes \omega^{-2} \to 0$$

is non-split since the (graded) cohomology of $E_\alpha \otimes E_\alpha$ differs from that of $E_\alpha \oplus E_\alpha \otimes \omega^{-2}$ by the calculation above. It follows that it presents a generator of $\mathrm{Ext}^1(E_\alpha \otimes \omega^{-2}, E_\alpha)$.

Consider now the extension X corresponding to the element in $\mathrm{Ext}^1(E_\alpha \otimes \omega^{-2}, E_\alpha)$ coming from the generator in $\mathrm{Ext}^1(E_\alpha \otimes \omega^{-2}, \mathcal{O})$ via the map induced by $\mathcal{O} \to E_\alpha$. This extension sits in a diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{O} & \longrightarrow & E_{\alpha,\widetilde{\alpha}} & \longrightarrow & E_\alpha \otimes \omega^{-2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_\alpha & \longrightarrow & X & \longrightarrow & E_\alpha \otimes \omega^{-2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \omega^{-2} & \xrightarrow{\cong} & \omega^{-2} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

This implies $X \cong E_{\alpha,\widetilde{\alpha}} \oplus \omega^{-2}$ (because every extension of a standard vector bundle with $E_{\alpha,\widetilde{\alpha}}$ splits). By computing cohomology, this implies that the middle horizontal extension is non-split and, hence, $E_\alpha \otimes E_\alpha \cong X$.

3.5 Representation Theory and Vector Bundles Over $\mathcal{M}_{(3)}$ and $\mathcal{M}_{(2)}$

We first present a new viewpoint on vector bundles on $\mathcal{M}_{(3)}$ and then apply similar ideas to vector bundles on $\mathcal{M}_{(2)}$. This new viewpoint allows also to prove statement about vector bundles coming from level structure, which were used in the last section.

As before, we denote by $\mathcal{M}(2)$ the moduli stack of elliptic curves with level-2-structure at the prime 3. Recall that we have an S_3-Galois cover $\mathcal{M}(2) \to \mathcal{M}_{(3)}$ and that $\mathcal{M}(2) \cong \operatorname{Spec} \Lambda//\mathbb{G}_m$, where $\Lambda = \mathbb{Z}_{(3)}[x_2, y_2, \Delta^{-1}]$. Define a morphism $\Lambda \to \mathbb{F}_3$ by $x_2 \mapsto 1, y_2 \mapsto -1$ (with $\Delta = 1$). This corresponds to an elliptic curve $E : y^2 = x^3 - x$ over $\mathbb{F}_3$ with level structure given by ordering the points of exact order 2 as $(0,0)$, $(0,-1)$ and $(0,1)$. This elliptic curve has a subgroup C_3 of automorphisms generated by

$$\begin{aligned} y &\mapsto y \\ x &\mapsto x + 1. \end{aligned}$$

This induces a map $e\colon \operatorname{Spec}\mathbb{F}_3//C_3 \to \mathcal{M}_{(3)}$ by Proposition 2.6.6. In total, we get the following diagram of stacks:

$$\begin{array}{ccccc}
\operatorname{Spec}\mathbb{F}_3 & \longrightarrow & \operatorname{Spec}\Lambda & \longrightarrow & \operatorname{Spec}\mathbb{Z}_{(3)} \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow \\
\operatorname{Spec}\mathbb{F}_3 & \xrightarrow{\epsilon} & \mathcal{M}(2) \simeq \operatorname{Spec}\Lambda//\mathbb{G}_m & \longrightarrow & \operatorname{Spec}\mathbb{Z}_{(3)}//\mathbb{G}_m \\
\downarrow & & \downarrow{\scriptstyle p} & & \downarrow \\
\operatorname{Spec}\mathbb{F}_3//C_3 & \xrightarrow{e} & \mathcal{M}_{(3)} & \xrightarrow{i} & X = \operatorname{Spec}\mathbb{Z}_{(3)}//\mathbb{G}_m//S_3
\end{array}$$

The left hand side of the diagram was just explained. The upper right horizontal morphism is induced by the canonical morphism $\mathbb{Z}_{(3)} \to \Lambda$. We get the other two right horizontal morphisms by the facts that $\operatorname{Spec}\Lambda \to \mathcal{M}(2)$ is a $\mathbb{G}_m$-torsor and $\mathcal{M}(2) \to \mathcal{M}_{(3)}$ is an S_3-torsor (and $\operatorname{Spec}\mathbb{Z}_{(3)}//G$ is the moduli stack of G-torsors by definition).

We want to understand the composition $RI\colon \operatorname{QCoh}(X) \to \operatorname{QCoh}(\operatorname{Spec}\mathbb{F}_3//C_3)$ for $R = e^*$ and $I = i^*$. We have that $\operatorname{QCoh}(\operatorname{Spec}\mathbb{F}_3//C_3) \simeq \mathbb{F}_3[C_3]$-mod by Galois descent and $\operatorname{QCoh}(X)$ is equivalent to graded $\mathbb{Z}_{(3)}$-modules with S_3-action. Note also that $\operatorname{QCoh}(\mathcal{M}_{(3)}) \simeq \widetilde{\Lambda[S_3]}$-grmod. The functor I can be seen as associating to a graded abelian group M with S_3-action a module $M \otimes_{\mathbb{Z}_{(3)}} \Lambda \in \widetilde{\Lambda[S_3]}$-grmod with S_3-action on both factors. By Proposition 2.6.6 and the example thereafter, an $N \in \widetilde{\Lambda[S_3]}$-grmod is sent by R to $\operatorname{res}^{S_3}_{C_3} N \otimes_\Lambda R \in \mathbb{F}_3[C_3]$-mod (forgetting the grading)[2] since the group C_3 of automorphisms of E acts on $E[2]$ by cyclically permuting the 2-torsion points. In summary, $RI(M) = \operatorname{res}^{S_3}_{C_3} M \otimes_{\mathbb{Z}_{(3)}} \mathbb{F}_3$ (forgetting the grading).

The group C_3 has (exactly) three indecomposable representations J_1, J_2 and J_3 over $\mathbb{F}_3$ of dimensions 1, 2 and 3 respectively, given by mapping the generator of C_3 to the Jordan matrices (1), $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ respectively $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. We want to show that they are realizable by integral representations of S_3 and, hence, as vector bundles on $\mathcal{M}_{(3)}$. For the trivial representation $\mathbb{Z}_{(3)}$ of S_3, we have $RI\mathbb{Z}_{(3)} \cong J_1$. The group S_3 acts on $\mathbb{Z}_{(3)}[\zeta_3]$ by permutation of the roots of unity (here, ζ_3 is a primitive third root of unity). By choosing the basis $(\zeta_3^2 - \zeta_3, \zeta_3)$, we see that $RI\mathbb{Z}_{(3)}[\zeta_3] \cong J_2$. Let P be the rank 3 canonical permutation representation of S_3; this is as C_3-representation isomorphic to $\mathbb{Z}_{(3)}[C_3]$ (with generator $t \in C_3$). Thus, $RIP \cong \mathbb{F}_3[C_3]$. By choosing the basis $(1 + t + t^2, -t + t^2, t)$ of $\mathbb{F}_3[C_3]$, we see that $\mathbb{F}_3[C_3] \cong J_3$.

We have that S_3-equivariantly $I\mathbb{Z}_{(3)}[S_3](\mathcal{M}(2)) \cong \bigoplus_{S_3} \Lambda$; here we let S_3 act on S_3 from the left by $h \cdot g = gh^{-1}$; and on the right hand side S_3 acts simultaneously by permuting the factors (by the action just described) and on Λ. This convention is chosen for the following reason: Consider the map

$$S_3 \times \mathcal{M}(2) \to \mathcal{M}(2) \times_{\mathcal{M}} \mathcal{M}(2)$$

indicated by the formula $(g, m) \mapsto (m, gm)$. If S_3 acts just on the left factor in the right hand side, the map becomes equivariant if we act on $S_3 \times \mathcal{M}(2)$ via $h \cdot (g, m) = (gh^{-1}, hm)$. Thus, $I\mathbb{Z}_{(3)}[S_3] \cong p_* p^* \mathcal{O}$ for $p\colon \mathcal{M}(2) \to \mathcal{M}_{(3)}$ the projection as above. Similarly, we have

[2] Or rather taking the direct sum of all degrees, depending on the definition of graded objects.

that $IP \cong f_*f^*\mathcal{O}$ (since $\mathcal{M}(2) \times_{\mathcal{M}_{(3)}} \mathcal{M}_0(2) \simeq \coprod_{\{1,2,3\}} \mathcal{M}(2)$). As $\mathcal{M}(2) \times_{\mathcal{M}_{(3)}} \mathcal{M}_0(2) \to \mathcal{M}(2) \times_{\mathcal{M}_{(3)}} \mathcal{M}_{(3)}$ corresponds to the fold map $\coprod_{\{1,2,3\}} \mathcal{M}(2) \to \mathcal{M}(2)$, the functor I sends the diagonal map $\mathbb{Z}_{(3)} \to P$ to the adjunction unit $\mathcal{O} \to f_*f^*\mathcal{O}$.

We have two exact sequences

$$0 \to \mathbb{Z}_{(3)} \to P \to \mathbb{Z}_{(3)}[\zeta_3] \to 0$$

and

$$0 \to (1-\zeta_3)\mathbb{Z}_{(3)}[\zeta_3] \to P \to \mathbb{Z}_{(3)} \to 0$$

of $\mathbb{Z}_{(3)}[S_3]$-modules (sending t to ζ_3 respectively ζ_3 to t). Here, the map $\mathbb{Z}_{(3)} \to P$ is the diagonal and the map $P \to \mathbb{Z}_{(3)}$ is the summing map. Since i is flat, $I = i^*$ is exact and we get exact sequences

$$0 \to \mathcal{O} \to f_*f^*\mathcal{O} \to I\mathbb{Z}_{(3)}[\zeta_3] \to 0$$

and

$$0 \to I\left((1-\zeta_3)\mathbb{Z}_{(3)}[\zeta_3]\right) \to f_*f^*\mathcal{O} \to \mathcal{O} \to 0.$$

Lemma 3.5.1. *We have* $I\left((1-\zeta_3)\mathbb{Z}_{(3)}[\zeta_3]\right) \cong \omega^4 \otimes E_\alpha$*, with notation as in the last section.*

Proof. Since the higher cohomology of $f_*f^*\mathcal{O}$ vanishes, we have that

$$H^2_k(\mathcal{M}_{(3)}; I\left((1-\zeta_3)\mathbb{Z}_{(3)}[\zeta_3]\right)) = \begin{cases} \mathbb{F}_3 & \text{for } k = 2 \mod 12 \\ 0 & \text{else.} \end{cases}$$

Thus, $I\left((1-\zeta_3)\mathbb{Z}_{(3)}[\zeta_3]\right)$ is an indecomposable vector bundle of rank 2 (since every line bundle has non-trivial cohomology in every 12-th degree).

The sub S_3-representation $\mathbb{Z}_{(3)}\langle x_2, y_2\rangle \subset \Lambda$ (where $\mathbb{Z}_{(3)}\langle x_2, y_2\rangle$ denotes the free $\mathbb{Z}_{(3)}$-module of rank 2) is isomorphic to $(1-\zeta_3)\mathbb{Z}_{(3)}[\zeta_3]$ (see Section 2.5). This induces an S_3-equivariant map $\Lambda \otimes (1-\zeta_3)\mathbb{Z}_{(3)}[\zeta_3] \to \Lambda$, which is surjective (since x_2 is a unit) and, with respect to the grading of Λ, of degree 2. By Galois descent, this induces in turn a surjective map $I\left((1-\zeta_3)\mathbb{Z}_{(3)}[\zeta_3]\right) \to \omega^2$. By Proposition 3.3.1, its kernel is a vector bundle again; since $I\left((1-\zeta_3)\mathbb{Z}_{(3)}[\zeta_3]\right)$ does not decompose, this has to be ω^4. Thus, $(1-\zeta_3)\mathbb{Z}_{(3)}[\zeta_3] \cong \omega^4 \otimes E_\alpha$. □

Lemma 3.5.2. *We have* $I\mathbb{Z}_{(3)}[\zeta_3] \cong \omega^{-2} \otimes E_\alpha$ *and* $(f_*f^*\mathcal{O})\check{} \cong f_*f^*\mathcal{O}$.

Proof. Equip $\check{P} = \operatorname{Hom}(P, \mathbb{Z}_{(3)})$ with the action $(g \cdot f)(p) = f(g^{-1}(p))$. Then sending each basis vector of P to its dual vector defines an S_3-equivariant isomorphism $P \cong \check{P}$. With this identification, the dual of the diagonal is the summing map. Thus, $(\mathbb{Z}_{(3)}[\zeta_3])\check{} \cong (1-\zeta_3)\mathbb{Z}_{(3)}[\zeta_3]$ by dualizing the short exact sequences above. Since i is an fpqc map, $I = i^*$ sends duals to duals (since pulling back is just restricting). Hence, $I\mathbb{Z}_{(3)}[\zeta_3] \cong \omega^{-2} \otimes E_\alpha$ and $(f_*f^*\mathcal{O})\check{} \cong f_*f^*\mathcal{O}$. □

This implies the exact sequences stated in the last section. As a last point at the prime 3, we want to prove the following three lemmas:

Lemma 3.5.3. *Let* $\mathbb{Z}'_{(3)}$ *the one-dimensional* S_3 *representation with* $g \cdot x = \operatorname{sgn}(g)x$*. Then* $I\mathbb{Z}'_{(3)} \cong \omega^6$.

Proof. Consider the element $\sqrt{\Delta} = 4x_2y_2(x_2 - y_2) \in \Lambda$. Then $\sqrt{\Delta} \cdot \mathbb{Z}_{(3)}$ defines an S_3-subrepresentation of Λ isomorphic to $\mathbb{Z}'_{(3)}$ (as can be seen by the formulas in Section 2.5). Since $\sqrt{\Delta}$ is a unit, this defines an S_3-equivariant graded isomorphism $\Lambda \otimes \mathbb{Z}'_{(3)} \cong \Lambda[6]$ to the 6-fold shift (for $|x_2| = |y_2| = 2$). But $\Lambda[n]$ corresponds (under Galois descent) to ω^n (see Section 2.5). Thus, the result. □

Lemma 3.5.4. *For $p\colon \mathcal{M}(2) \to \mathcal{M}_{(3)}$ the projection, $p_*p^*\mathcal{O} \cong f_*f^*\mathcal{O} \oplus f_*f^*\mathcal{O} \otimes \omega^2$.*

Proof. We know that $p_*p^*\mathcal{O} \cong I\mathbb{Z}_{(3)}[S_3]$. Thus it suffices to show that $\mathbb{Z}_{(3)}[S_3] \cong P \oplus \left(P \otimes \mathbb{Z}'_{(3)}\right)$ since $f_*f^*\mathcal{O} \cong f_*f^*\mathcal{O} \otimes \omega^4$ (as shown in the last section) and $I\mathbb{Z}'_{(3)} \cong \omega^6$. Sending $(a_g)_{g\in S_3}$ to the triple $(\sum_{g\colon g(1)=i} a_g)_{i=1}^3$ defines an S_3-map $\mathbb{Z}_{(3)}[S_3] \to P$. This is split by the map $P \to \mathbb{Z}_{(3)}[S_3]$ sending $(a_i)_{i=1}^3$ to $(a_g)_{g\in S_3}$ with $a_g = \frac{a_i}{2}$ for $g(1) = i$. The kernel of the first map consists of all $(a_g)_{g\in S_3}$ such that for $s = (1\,3\,2) \in S_3$ we have $a_g = -a_{gs}$. Since $P \subset \mathbb{Z}_{(3)}[S_3]$ is defined by the conditions $a_g = a_{gs}$, we have that this kernel is isomorphic to $P \otimes \mathbb{Z}'_{(3)}$, as was to be shown. □

Lemma 3.5.5. *For $q\colon \mathcal{M}(4) \to \mathcal{M}_{(3)}$ the projection, $q_*q^*\mathcal{O}$ is a direct sum of 8 copies of $p_*p^*\mathcal{O}$.*

Proof. Recall from the end of Section 2.5 that $\mathcal{M}(4) \to \mathcal{M}(2)$ is a $G = (C_2)^4$-torsor generated by involutions $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ and $\tilde{d}$ and that $\mathcal{M}(4) \simeq \operatorname{Spec} A \coprod \operatorname{Spec} A$. As in Lemma 2.7.1, the G-fixed points of $q_*q^*\mathcal{O}$ are $p_*p^*\mathcal{O}$. We can compute these fixed points by taking iteratively the fixed points of the 4 involutions given above (since they all commute). Since 2 is invertible, every quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{M}[\frac{1}{2}]$ with a C_2-action splits into $\mathcal{F}^{C_2}$ and $(\mathcal{F}')^{C_2}$, where on $\mathcal{F}'$ the C_2-action is twisted by sign. Since $\tilde{a}$ just permutes the two components of $\mathcal{M}(4)$, we get that $(q_*q^*\mathcal{O})^{C_2}_{\tilde{a}} \cong q_*(q^*\mathcal{O}|_{\operatorname{Spec} A})$ (and the same for the action twisted by sign). The involution $\tilde{a}\tilde{d}$ is trivial, so taking fixed points with respect to $\tilde{a}\tilde{d}$ changes nothing. Taking fixed points with respect to $\tilde{b}$ and $\tilde{c}$ gives now $p_*p^*\mathcal{O}$; if we twist by signs, we get the same result since A is $C_2 \times C_2$-equivariantly isomorphic to itself with $\tilde{b}$ and $\tilde{c}$ possibly twisted by signs (see the end of Section 2.5). □

We will come now to the situation of $\mathcal{M}_{(2)}$, which is in some respects quite different; we will see that we have here infinitely many indecomposable vector bundles (of arbitrary high rank). Recall that we have a $GL_2(\mathbb{F}_3)$-Galois covering $\mathcal{M}(3) \to \mathcal{M}_{(2)}$ for $\mathcal{M}(3)$ the moduli stack of elliptic curves with level-3 structure at the prime 2. Set $G = GL_2(\mathbb{F}_3)$. We have $\mathcal{M}(3)_{(2)} \cong \operatorname{Spec} B$, where $B \cong \mathbb{Z}_{(2)}[\zeta_3][X, (X^3 - 1)^{-1}]$ (as stated in the introduction of [DR73]).

Consider the elliptic curve $E\colon y^2 + y = x^3$ over $\overline{\mathbb{F}}_2$ (which has, according to [Sil09], III.10.1, automorphism group S of order 24). By [KM85, 2.7.2], the morphism $S \to G$ (given by the operation of S on $E[3]$) is injective. Using elementary group theory, we get that G has a unique subgroup of order 24, namely $SL_2(\mathbb{F}_3)$; thus S embeds onto $SL_2(\mathbb{F}_3)$. The group $SL_2(\mathbb{F}_3)$ has as a 2-Sylow group the quaternion group Q, the multiplicative subgroup of the quaternions generated by i and j. This defines an action of Q on E. Since the finite group scheme $E[3]$ over $\overline{\mathbb{F}}_2$ is isomorphic to $(\mathbb{Z}/3)^2$, we can choose a level-3 structure on E. This gives (as for the prime 3 before) the following diagram

$$\begin{array}{ccccc} \operatorname{Spec}\overline{\mathbb{F}_2} & \longrightarrow & \mathcal{M}(3)_{(2)} \simeq \operatorname{Spec} B & \longrightarrow & \operatorname{Spec}\mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec}\overline{\mathbb{F}_2}//Q & \longrightarrow & \mathcal{M}_{(2)} & \longrightarrow & \operatorname{Spec}\mathbb{Z}//G \end{array}$$

Thus, we get functors $I\colon\ \mathbb{Z}[G]\text{-mod} \to \mathrm{QCoh}(\mathcal{M}_{(2)})$ and $R\colon\ \mathrm{QCoh}(\mathcal{M}_{(2)}) \to \overline{\mathbb{F}}_2[Q]\text{-mod}$ as above. Again, the functor RI is given by tensoring with $\overline{\mathbb{F}}_2$ and restricting to $Q \subset GL_2\mathbb{F}_3$ (using Proposition 2.6.6).

There is a family of $C_2 \times C_2$-representations over $\mathbb{Z}$ given as $M_n = \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_n \oplus \mathbb{Z}y_0 \oplus \cdots \oplus \mathbb{Z}y_n$ and

$$(g_1 + (-1)^i)x_i = y_{i-1}, \qquad (g_2 + (-1)^i)x_i = y_i$$
$$(g_1 - (-1)^i)y_i = (g_2 + (-1)^i)y_i = 0$$

where g_1 and g_2 generate $C_2 \times C_2$ (see [HR62], 6.2). The modules $\overline{M_n} = M_n \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_2$ (and, hence, also the M_n) are indecomposable (see [HR61], Proposition 5(ii) and its corollary). The same holds if we pull them back to representations of Q via the surjective morphism $\pi\colon\ Q \to C_2 \times C_2$ (given by dividing out $i^2 \in Q$); we denote these pullbacks the same way.

Let $Y_1, Y_2, \ldots$ be the collection of indecomposable vector bundles on $\mathcal{M}_{(3)}$. Decompose $I\operatorname{ind}_Q^G M_n$ as $\bigoplus_{i=1}^\infty a_i Y_i$ (with almost all $a_i = 0$). Thus, $RI\operatorname{ind}_Q^G M_n \cong \bigoplus_{i=1}^\infty a_i R(Y_i)$. Since

$$RI\operatorname{ind}_Q^G M_n \cong \operatorname{res}_Q^G \operatorname{ind}_Q^G \overline{M}_n \cong \bigoplus_{G/Q} \overline{M}_n,$$

we see that $\overline{M_n}$ is a direct summand of this module. Therefore, by the theorem of Krull–Remak–Schmidt[3], $\overline{M_n}$ has to be a summand of one of the RY_i. Since $\operatorname{rk} M_n = 2n+1$, the rank of RY_i (and hence of Y_i) must be at least $2n+1$. Therefore, $\mathcal{M}_{(2)}$ has indecomposable vector bundles of arbitrary high rank.

3.6 Classification of Standard Vector Bundles on $\mathcal{M}_{(3)}$

In this section, we want to classify all standard vector bundles on $\mathcal{M}_{(3)}$. We set again by abuse of notation $\mathcal{M} = \mathcal{M}_{(3)}$.

Theorem 3.6.1. *Every standard vector bundle on $\mathcal{M}$ is a direct sum of the form $\bigoplus_I \omega^{n_i} \oplus \bigoplus_J E_\alpha \otimes \omega^{n_j} \oplus \bigoplus_K E_{\alpha,\widetilde{\alpha}} \otimes \omega^{n_k}$.*

Proof. We will prove this theorem by induction on the rank of the vector bundle. The rank 1 case is the classification of line bundles.

So assume that we have proven the theorem for all standard vector bundles of rank smaller than n and that X is a standard vector bundle of rank n. By the induction hypothesis, we have a short exact sequence

$$0 \to \omega^k \to X \to Y \to 0, \tag{3.4}$$

[3] This states that every noetherian and artinian module has a (up to permutation and isomorphisms) unique decomposition in indecomposable modules.

where Y is of the form $\bigoplus_{I_Y} \omega^{n_i} \oplus \bigoplus_{J_Y} E_\alpha \otimes \omega^{n_j} \oplus \bigoplus_{K_Y} E_{\alpha,\widetilde{\alpha}} \otimes \omega^{n_k}$ and of rank $(n-1)$. We call the depicted summands of Y the *standard summands of* Y. We can assume that Y is chosen with I_Y of minimal cardinality among all choices of morphisms $X \to Y$ with Y a direct sum of twists of $\mathcal{O}$, E_α and $E_{\alpha,\widetilde{\alpha}}$ and with a line bundle as kernel. Furthermore, we assume (for notational simplicity) that $k = 0$.

We assume that X is not of the form which is demanded by the theorem we want to prove. Then the extension (3.4) is non-trivial. Since the Ext functor commutes with (finite) direct sums, there is at least one standard summand S of Y such that the map $\mathrm{Ext}^1(Y, \mathcal{O}) \to \mathrm{Ext}^1(S, \mathcal{O})$ (induced by the inclusion) sends the class $x \in \mathrm{Ext}^1(Y, \mathcal{O})$ corresponding to (3.4) to a non-trivial class. We will prove the theorem case by case:

1) $S = E_{\alpha,\widetilde{\alpha}} \otimes \omega^j$: this cannot happen since $\mathrm{Ext}^1(E_{\alpha,\widetilde{\alpha}} \otimes \omega^j, \mathcal{O}) = 0$.

2) $S = E_\alpha \otimes \omega^j$: Since the only non-split extension of $\mathcal{O}$ and an $E_\alpha \otimes \omega^j$ is $E_{\alpha,\widetilde{\alpha}}$ with $j = -2$, we get a diagram (with rows and columns exact) of the form:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O} & \to & E_{\alpha,\widetilde{\alpha}} & \to & E_\alpha \otimes \omega^{-2} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O} & \to & X & \to & Y & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Y - (E_\alpha \otimes \omega^{-2}) & \xrightarrow{=} & Y - (E_\alpha \otimes \omega^{-2}) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

The left vertical extension is trivial since $\mathrm{Ext}^1(Y - (E_\alpha \otimes \omega^{-2}), E_{\alpha,\widetilde{\alpha}}) = 0$ (note to that purpose that $Y - (E_\alpha \otimes \omega^{-2})$ is standard since it is a sum of standard summands). Therefore,

$$X \cong E_{\alpha,\widetilde{\alpha}} \oplus (Y - (E_\alpha \otimes \omega^{-2})).$$

3) $S = \omega^j$: Since the only non-split extension of $\mathcal{O}$ and an ω^j is E_α with $j = -2$, we get a diagram (with rows and columns exact) of the form:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O} & \to & E_\alpha & \to & \omega^{-2} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O} & \to & X & \to & Y & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Y - \omega^{-2} & \xrightarrow{=} & Y - \omega^{-2} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

If the left vertical extension in the diagram is non-split, there is a standard summand S' of $Y - \omega^{-2}$ such that the map $\mathrm{Ext}^1(Y - \omega^{-2}, E_\alpha) \to \mathrm{Ext}^1(S', E_\alpha)$ induced by the inclusion sends the element $y \in \mathrm{Ext}^1(Y - \omega^{-2}, E_\alpha)$ corresponding to the left vertical extension to a non-trivial class. If $S' \cong \omega^l$, then the argument is similar to the case before and we get $X \cong (Y - \omega^{-2} - \omega^{-4}) \oplus E_{\alpha,\tilde{\alpha}}$. The case $S' \cong E_{\alpha,\tilde{\alpha}} \otimes \omega^l$ can again not occur because of the vanishing of Ext. Therefore, we can assume that S' is isomorphic to a twist of E_α. The only non-trivial extensions of two vector bundles of type E_α are $E_{\alpha,\tilde{\alpha}} \oplus \omega^{-2}$ and its twists (as proven at the end of Section 3.4). So we can assume that we get a commutative diagram (with rows and columns exact) of the form:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_\alpha & \longrightarrow & E_{\alpha,\tilde{\alpha}} \oplus \omega^{-2} & \longrightarrow & E_\alpha \otimes \omega^{-2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_\alpha & \longrightarrow & X & \longrightarrow & Y - \omega^{-2} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Y - \omega^{-2} - (E_\alpha \otimes \omega^{-2}) & \xrightarrow{=} & Y - \omega^{-2} - (E_\alpha \otimes \omega^{-2}) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Pushing the left vertical extension forward along the projection map $E_{\alpha,\tilde{\alpha}} \oplus \omega^{-2} \to E_{\alpha,\tilde{\alpha}}$ produces the following diagram (with rows and columns exact):

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \omega^{-2} & \xrightarrow{=} & \omega^{-2} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & E_{\alpha,\tilde{\alpha}} \oplus \omega^{-2} & \longrightarrow & X & \longrightarrow & Y - \omega^{-2} - (E_\alpha \otimes \omega^{-2}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow {\scriptstyle =} & & \\
0 & \longrightarrow & E_{\alpha,\tilde{\alpha}} & \longrightarrow & Y' & \longrightarrow & Y - \omega^{-2} - (E_\alpha \otimes \omega^{-2}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

The lower horizontal extension splits so that $Y' \cong E_{\alpha,\tilde{\alpha}} \oplus (Y - \omega^{-2} - (E_\alpha \otimes \omega^{-2}))$. Thus, Y' would have been possible as a choice for Y, but $|I_{Y'}| = |I_Y| - 1$, which is a contradiction to the minimality of $|I_Y|$. □

Scholium 3.6.2. *In every non-trivial extension*

$$0 \to \omega^k \to E \to E' \to 0$$

and in every non-trivial extension

$$0 \to E' \to E \to \omega^k \to 0$$

of standard vector bundles the total dimension of the non-line bundle indecomposable summands of E is, at least, one bigger than that of E'.

Proof. The first statement follows from the proof of the theorem (note that the total dimension of non-line bundle components is bigger in Y' than in Y in the last step of the proof). The second follows by dualizing. □

Part II

KO, TMF and Their Categories of Modules

Chapter 4

Module Categories

4.1 Foundations of Homotopy Theory

Nowadays, there is a plethora of settings for abstract homotopy theory. The most traditional theory is Quillen's language of *model categories*. These are categories with the extra structure of chosen classes of weak equivalences, fibrations and cofibrations satisfying certain axioms. The choice of (co)fibrations gives a very tight structure, which is particularly well-adapted to handle derived functors.

Sometimes, it is more convenient to have a way of doing abstract homotopy theory in a less structured or tight way, leading to the philosophy of $(\infty, 1)$-categories. This philosophy has several incarnations and the most important for us is the theory of quasi-categories (which we will often just call *∞-categories*). Other popular choices are simplicial categories, relative categories and complete Segal spaces. We want to sketch also these theories and indicate their relationship, which we want to exploit to prove a certain statement about homotopy limits of quasi-categories. We will not care about set-theoretical issues since there are standard ways to deal with them (say, via choices of Grothendieck universes).

4.1.1 Simplicial Categories and Quasi-Categories

In homotopy theory it is crucial to have a good theory of mapping spaces between objects. The theory of simplicial categories is the most straightforward answer to this desideratum as a *simplicial category* is just defined to be a category $\mathcal{C}$ enriched in simplicial sets. Its homotopy category $Ho(\mathcal{C})$ has the same objects as $\mathcal{C}$ and $\mathrm{Ho}(\mathcal{C})(x, y) = \pi_0\mathcal{C}(x, y)$ as morphism sets for $x, y \in \mathrm{Ob}(\mathcal{C})$. A functor between simplicial categories is called a *Dwyer–Kan equivalence* if it induces an equivalence of homotopy categories and weak equivalences on the mapping spaces. One can equip the category sCat of simplicial categories with the *Bergner model structure*, where the weak equivalences are the Dwyer–Kan equivalences and a object is fibrant iff each mapping space is a Kan complex (see [Ber07] or [Lur09b, A.3.2.4 and A.3.2.24]).

Example 4.1.1. If $\mathcal{M}$ is a simplicial model category, the sub simplicial category $\mathcal{M}^\circ$ of bifibrant objects is fibrant in the Bergner model structure. The homotopy category $\mathrm{Ho}(\mathcal{M}^\circ)$ is equivalent to the homotopy category of the model category $\mathcal{M}$.

While there is a strictly defined composition of morphisms in simplicial categories, this will be no longer the case in the theories of quasi-categories and (complete) Segal spaces.

The theory of quasi-categories begins with the observation that the nerve $N\mathcal{C}$ of a category $\mathcal{C}$ has the property that every morphism $\Lambda^n_k \to N\mathcal{C}$ from an inner horn (i.e. $0 < k < n$) can be filled uniquely to a map $\Delta^n \to N\mathcal{C}$. For example, the existence of the composition

$$\begin{array}{ccc} X & & \\ {\scriptstyle f}\downarrow & \searrow^{g\circ f} & \\ Y & \xrightarrow{g} & Z \end{array}$$

is just a filling of the two-horn Λ^2_1 and the associativity of composition is forced by a filling of a three-horn ([Lur09b, 1.1.2.2]).

Definition 4.1.2. A *quasi-category* (or *∞-category*) is a simplicial set $\mathcal{C}$ such that every morphism $\Lambda^n_k \to \mathcal{C}$ from an inner horn (i.e. $0 < k < n$) can be (possibly non-uniquely) filled to a map $\Delta^n \to \mathcal{C}$.

Thus, the composition is not unique, but only unique up to contractible choice.[1] Interestingly, there is a model structure on sSet (the *Joyal model structure*) such that every object is cofibrant and the fibrant objects are exactly the ∞-categories. There is a Quillen equivalence

$$\mathrm{sSet} \underset{N}{\overset{\mathfrak{C}}{\rightleftarrows}} \mathrm{sCat}$$

between simplicial sets with the Joyal model structure and simplicial categories with the Bergner model structure (see, for example, [Lur09b, Section 2.2]). Here, N stands for the coherent nerve in the sense of Cordier and Porter (see [Lur09b, Section 1.1.5] for a definition). Since N is a right Quillen functor, the image of a fibrant simplicial category is an ∞-category. In particular, this holds for $N(\mathcal{M}^\circ)$ for a simplicial model category $\mathcal{M}$. Note that we have $\mathrm{Ho}(\mathcal{M}) \simeq \mathrm{Ho}(N(\mathcal{M}^\circ))$.

A *functor* between ∞-categories $\mathcal{C}$ and $\mathcal{D}$ is defined to be a map $\mathcal{C} \to \mathcal{D}$ of simplicial sets. We say that a functor $f\colon \mathcal{C} \to \mathcal{D}$ is a *(categorical) equivalence* if $\mathfrak{C}(f)\colon \mathfrak{C}(\mathcal{C}) \to \mathfrak{C}(\mathcal{D})$ is a Dwyer–Kan equivalence. The homotopy category $\mathrm{Ho}(\mathcal{C})$ of an ∞-category $\mathcal{C}$ is defined as $\mathrm{Ho}(\mathfrak{C}(\mathcal{C}))$. In particular, we see that $\mathrm{Ho}(N(\mathcal{M}^\circ)) \simeq \mathrm{Ho}(\mathcal{M})$. For a deeper, yet accessible introduction to ∞-categories see [Lur09b, Chapter 1].

4.1.2 Comparison to Other Approaches

The aim of this section is to compare the quasi-categorical approach to the theory of complete Segal spaces. In this thesis, this will only be used to transfer results by Julie Bergner about homotopy limits of complete Segal spaces to homotopy limits of quasi-categories, so it might be skipped in first reading.

Before explaining the theory of complete Segal spaces, we introduce the theory of relative categories. A *relative category* is a category $\mathcal{C}$ equipped with a chosen subcategory $\mathcal{W}$ (called the sub category of *weak equivalences*) which contains all objects of $\mathcal{C}$. Important examples are model categories with their weak equivalences.

[1] More precisely, $X \in \mathrm{sSet}$ is an ∞-category iff $\mathrm{Map}(\Delta^2, X) \to \mathrm{Map}(\Lambda^2_1, X)$ is an acyclic Kan fibration.

Given a relative category $\mathcal{C}$, we can construct a simplicial category $L^H\mathcal{C}$, the *hammock localization* (for a definition, see [DK80b, Section 3.1]). We get a diagram

$$\begin{array}{ccc} \mathit{ModCat} & \longrightarrow & \mathit{relCat} \\ \uparrow & & \downarrow{\scriptstyle L^H} \\ \mathit{sModCat} & \xrightarrow{()^\circ} & \mathit{sCat} \end{array}$$

This commutes up to a natural zig zag of Dwyer–Kan equivalences of simplicial categories by [DK80b, Proposition 4.8].

Definition 4.1.3. A *relative functor* between relative categories $(\mathcal{C}, \mathcal{W})$ and $(\mathcal{C}', \mathcal{W}')$ is a functor $F\colon \mathcal{C} \to \mathcal{C}'$ such that $F(\mathcal{W}) \subset \mathcal{W}'$. It is a *homotopy equivalence* if there is a relative functor $G\colon \mathcal{C}' \to \mathcal{C}$ such that FG and GF are naturally equivalent to the identity functors (i.e., there is a zig zag of natural transformations consisting of weak equivalences).

Lemma 4.1.4. *A homotopy equivalence $F\colon (\mathcal{C}, \mathcal{W}) \to (\mathcal{C}', \mathcal{W}')$ induces a Dwyer-Kan equivalence $L^H F\colon L^H(\mathcal{C}, \mathcal{W}) \to L^H(\mathcal{C}', \mathcal{W}')$.*

Proof. By [DK80b, Proposition 3.2], the homotopy category $\mathrm{Ho}(L^H\mathcal{C})$ is a localization of $\mathcal{C}$ at the class of weak equivalences. Thus, $\mathrm{Ho}(L^H F)\colon \mathrm{Ho}(L^H\mathcal{C}) \to \mathrm{Ho}(L^H\mathcal{C})$ is essentially surjective.

Therefore, it is enough to show the following: Suppose that I and J are relative endofunctors of $(\mathcal{C}, \mathcal{W})$ with a natural transformation $s\colon I \to J$ consisting of weak equivalences between them. Then $L^H\mathcal{C}(X, Y) \to L^H\mathcal{C}(IX, IY)$ is a weak equivalence iff $L^H\mathcal{C}(X, Y) \to L^H\mathcal{C}(JX, JY)$ is a weak equivalence.

By [DK80a, Proposition 3.5], we have a commutative diagram

$$\begin{array}{ccccc} & & L^H\mathcal{C}(IX, IY) & & \\ & \nearrow & & \searrow{\scriptstyle s_*} & \\ L^H\mathcal{C}(X, Y) & & & & L^H\mathcal{C}(IX, JY) \\ & \searrow & & \nearrow{\scriptstyle s^*} & \\ & & L^H\mathcal{C}(JX, JY) & & \end{array}$$

By [DK80a, Proposition 3.3], s_* and s^* are weak equivalences. Thus, the result. □

We equip the category of relative categories (with relative functors between them) with the model structure from [BK12b], which we call the *Barwick–Kan model structure*, and denote it by RelCat. With the weak equivalences of the Bergner respectively Barwick–Kan model structures, both the category of simplicial categories and the category of relative categories get the structures of relative categories.

Proposition 4.1.5 ([BK12a], Theorem 1.7)**.** *The Hammock localization is a homotopy equivalence between the relative categories of relative categories and simplicial categories.*

The theory of Segal spaces begins with the observation that for the nerve $N\mathcal{C}$ of a category $\mathcal{C}$, we have an isomorphism

$$(N\mathcal{C})_n \to (N\mathcal{C})_1 \times_{(N\mathcal{C})_0} \cdots \times_{(N\mathcal{C})_0} (N\mathcal{C})_1$$

whose inverse is given by composition.

Definition 4.1.6. A simplicial space[2] W is called a *Segal space* if it is Reedy fibrant and

[2] Here, 'space' stands for a 'simplicial set'.

the Segal map $W_n \to W_1 \times_{W_0} \cdots \times_{W_0} W_1$ is a weak equivalence of simplicial sets. A Segal space is said to be *complete* if the Rezk completion map is an equivalence (see [Rez01, §4-6] for details).

For two 0-simplices $x, y \in W_0$ in a (Reedy fibrant) simplicial space W, define the mapping space $\mathrm{map}_W(x, y)$ to be the fiber of the map $(d_1, d_0)\colon W_1 \to W_0 \times W_0$ over (x, y). The homotopy category $\mathrm{Ho}(W)$ has $W_{0,0}$ as objects and $\pi_0 \,\mathrm{map}\, W(x, y)$ as Hom-sets. We say that a map of (Reedy fibrant) simplicial spaces is a *Dwyer–Kan equivalence* if it induces an equivalence of homotopy categories and weak equivalences of mapping spaces.

The category of simplicial spaces can be equipped with a (simplicial) model structure ([Rez01, Theorem 7.2]) such that the fibrant objects are exactly the complete Segal spaces, the weak equivalences between Segal spaces are given by Dwyer–Kan equivalences and every object is cofibrant. This model structure is Quillen equivalent both to the Joyal and the Bergner model structure. For example, we have a Quillen equivalence:

$$\mathrm{sSet} \underset{i_1^*}{\overset{p_1^*}{\rightleftarrows}} \mathrm{ssSet}$$

Here, the two Quillen functors a induced by the projection $p_1\colon \Delta \times \Delta \to \Delta$ to the first coordinate and the map $i_1\colon \Delta \to \Delta \times \Delta$ sending $[n]$ to $([n], [0])$.

We can associate to every relative category a simplicial space as follows: Let $\mathcal{C}^{[n]}$ be category of chains of n composable morphisms in $\mathcal{C}$. A morphism between two chains is called a weak equivalence if it is a weak equivalence on every object. Then, we define a simplicial space $N(\mathcal{C}, \mathcal{W})$ by

$$N(\mathcal{C}, \mathcal{W})_n = N(\mathrm{we}(\mathcal{C}^{[n]})),$$

which is called the *classifying diagram*.

Theorem 4.1.7 ([BK12b], Theorem 6.1 and Key Lemma 5.4). *There is a Quillen equivalence*

$$\mathrm{ssSet} \underset{N_\xi}{\overset{K_\xi}{\rightleftarrows}} \mathrm{RelCat}$$

such that there is a natural transformation $N \to N_\xi$ which consists of Reedy equivalences (which are, in particular, equivalences in the Rezk model structure). In addition, $N_\xi(f)$ is a weak equivalence (fibration) iff f is a weak equivalence (fibration), for f a morphism in ssSet. *In particular, the right derived functor RN_ξ is weakly equivalent to N.*

Next, we want to discuss an amazing result by Toën.

Theorem 4.1.8 ([Toë05],Theorem 6.3). *For $\mathcal{C}$ a simplicial category, denote by $RAut(\mathcal{C})$ the simplicial monoid consisting of those components of the derived mapping space $\mathrm{Map}_{\mathrm{sCat}}(\mathcal{C}, \mathcal{C})$ consisting of Dwyer–Kan equivalences. Then there is a weak equivalence of simplicial monoids $RAut(L^H \mathrm{ssSet}) \simeq C_2$. An endomorphism F of L^H* ssSet *lies in the component of the identity iff there is weak equivalence between the diagrams*

$$F(\Delta^0_\delta) \rightrightarrows F(\Delta^1_\delta) \quad \text{and} \quad \Delta^0_\delta \rightrightarrows \Delta^1_\delta\,.$$

Here, Δ^0_δ and Δ^1_δ are Δ^0 and Δ^1 viewed as discrete simplicial spaces and the maps are (induced by) the inclusion of the end points.

Note that if

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

is a Quillen equivalence, the derived functors LF and RG define a homotopy equivalence of relative categories and hence Dwyer–Kan equivalences of the Hammock localizations $L^H\mathcal{C}$ and $L^H\mathcal{D}$ by Lemma 4.1.4.

An object $X \in \mathrm{ssSet}$ can be viewed as an object in $L^H\,\mathrm{ssSet}$. By [DK80b, 4.8], we have a zig zag of Dwyer–Kan equivalences

$$L^H\,\mathrm{ssSet} \longrightarrow \mathrm{diag}\,L^H\,\mathrm{ssSet} \longleftarrow \mathrm{ssSet}^\circ$$

which are all identity on objects. We choose a fibrant replacement functor $()^f$ in simplicial categories, which preserves objects. One possibility is to apply to apply $S_\bullet||$ to every morphism space (where $S_\bullet$ denotes the singular complex). This is functorial an we get a zig zag

$$(L^H\,\mathrm{ssSet})^f \longrightarrow (\mathrm{diag}\,L^H\,\mathrm{ssSet})^f \longleftarrow \mathrm{ssSet}^\circ$$

Applying nerves, we get a zig zag of equivalences

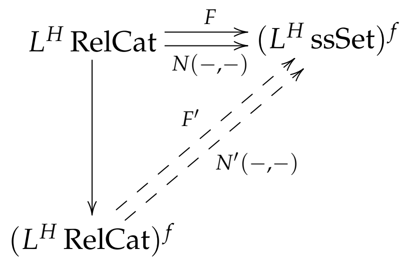
$$N((L^H\,\mathrm{ssSet})^f) \longrightarrow N((\mathrm{diag}\,L^H\,\mathrm{ssSet})^f) \underset{G}{\overset{G'}{\rightleftarrows}} N(\mathrm{ssSet}^\circ)$$

using the Ken Brown lemma ([Hov99, 1.1.12]). The dashed arrow is an inverse weak equivalence to G, which exists since both nerves are bifibrant. The map Q from the objects of ssSet to the 0-simplices of $N((\mathrm{diag}\,L^H\,\mathrm{ssSet})^f)$ is the identity. We denote the composition FQ by κ. Given $X \in \mathrm{ssSet}^\circ$, we have $QX = GX$. Since FG is equivalent to the identity functor, κX is naturally equivalent to X. Note also that κ preserves equivalences between objects (here, an equivalence is a morphism inducing an isomorphism in the homotopy category).

Corollary 4.1.9. *With this notation, there is a natural equivalence between $\kappa(p_1^*N(\mathcal{M}^\circ))$ and $\kappa(N(\mathcal{M},\mathcal{W}))$ in $N(\mathrm{ssSet}^\circ)$.*

Proof. By Proposition 4.1.5 and the fact that p_1^* and N are Quillen equivalences, $F = Lp_1^*RNL^H$ defines a homotopy equivalence between the relative categories RelCat and ssSet (with the Rezk model structure). By Theorem 4.1.7, $N(-,-)$ is also a homotopy equivalence from *RelCat* to ssSet. Thus, $N(-,-)K_\xi$ and FK_ξ are auto homotopy equivalences of ssSet (note that every object of ssSet is cofibrant). By Lemma 4.1.4, they define auto Dwyer–Kan equivalences of $L^H\,\mathrm{ssSet}$. By [BK12b, Proposition 7.3], $K_\xi([n]) \simeq K([n])$, where $K([n])$ is the relative category $[n]$ where weak equivalences are just identities. The Hammock localization of $K([n])$ is just the discrete category $[n]$ and $p_1^*N([n]) = p_1^*\Delta^n = \Delta^n_\delta$ in ssSet. Similarly, we get that $N([n], \mathrm{id}_{[n]}) = \Delta^n_\delta$. Both identifications are compatible with the structure maps in the category Δ. Thus, FK_ξ and $N(-,-)K_\xi$ lie in the same path components in the derived mapping space $\mathrm{Map}_{\mathrm{sCat}}(L^H\,\mathrm{ssSet}, L^H\,\mathrm{ssSet})$. Since K_ξ is a Dwyer–Kan equivalence, also F and $N(-,-)$ lie in the same path component of the derived mapping space $\mathrm{Map}_{\mathrm{sCat}}(L^H\,\mathrm{RelCat}, L^H\,\mathrm{ssSet})$. If we postcompose with the fibrant replacement

$L^H \,\mathrm{ssSet} \to (L^H \,\mathrm{ssSet})^f$, both F and $N(-,-)$ factor over the fibrant replacement:



We denote these maps $(L^H \,\mathrm{RelCat})^f \to (L^H \,\mathrm{ssSet})^f$ by F' and $N'(-,-)$. The induced maps NF' and $NN'(-,-)$ from $N((L^H \,\mathrm{RelCat})^f)$ to $N((L^H \,\mathrm{ssSet})^f)$ lie in the same path component of

$$\mathrm{Map}_{\mathrm{sSet}}(N((L^H \,\mathrm{RelCat})^f), N((L^H \,\mathrm{ssSet})^f))$$

(which is, at the same time, the derived mapping space since all objects in sSet are cofibrant and $N((L^H \,\mathrm{ssSet})^f)$ is fibrant). Thus, also NF and $NN(-,-)$ lie in the same path component of $\mathrm{Map}_{\mathrm{sSet}}(N(L^H \,\mathrm{RelCat}), N((L^H \,\mathrm{ssSet})^f)))$ and κNF and $\kappa NN(-,-)$ lie in the same path component of

$$\mathrm{Map}_{\mathrm{sSet}}(N(L^H \,\mathrm{RelCat}), N(\mathrm{ssSet}^\circ)),$$

i.e., there is a natural equivalence between $F(\mathcal{M}, \mathcal{W})$ and $N(\mathcal{M}, \mathcal{W})$ in $N \,\mathrm{ssSet}^\circ$.

The simplicial category $\mathcal{M}^\circ$ is a fibrant replacement of $L^H \mathcal{M}$ and all objects of sSet are cofibrant. Thus $F(\mathcal{M}, \mathcal{W}) = Lp_1^* RNL^H(\mathcal{M}) = p_1^* N(\mathcal{M}^\circ)$ and the result follows. □

Let *sModCat* be the category of simplicial model categories where morphisms are given by simplicial functors preserving fibrations, cofibrations and weak equivalences. Furthermore, we denote by holim the derived functor of the limit in a model category. The following corollary owes much to Chris Schommer-Pries.

Corollary 4.1.10. *Let $I \to sModCat, i \mapsto \mathcal{M}_i$ be a diagram of simplicial model categories. Then $p_1^* \operatorname{holim}_I N(\mathcal{M}_i^\circ)$ is weakly equivalent to $\operatorname{holim}_I N(\mathcal{M}_i, \mathcal{W}_i)$; here, the homotopy limits are built in the Joyal model structure on* sSet *and the Rezk model structure on* ssSet *respectively. In particular, for a simplicial model category $\mathcal{M}$, the nerve $N\mathcal{M}^\circ$ is weakly equivalent to $\operatorname{holim}_I N(\mathcal{M}_i^\circ)$ iff $N(\mathcal{M}, \mathcal{W})$ is weakly equivalent to $\operatorname{holim}_I N(\mathcal{M}_i, \mathcal{W}_i)$.*

Proof. We have two diagrams $\kappa(p_1^* N\mathcal{M}_i^\circ)$ and $\kappa(N(\mathcal{M}_i, \mathcal{W}_i))$ of the form $NI \to N(\mathrm{ssSet}^\circ)$, which are homotopic by the last corollary. These are naturally equivalent to $(p_1^* N\mathcal{M}_i^\circ)^f$ and $(N(\mathcal{M}_i, \mathcal{W}_i))^f$, where $()^f$ denotes fibrant replacement in the Rezk model structure. Thus, $\operatorname{holim}_{NI}(p_1^* N\mathcal{M}_i^\circ)^f \simeq \operatorname{holim}_{NI}(N(\mathcal{M}_i, \mathcal{W}_i))^f$, where the homotopy limit is taken in the ∞-categorical sense. By [Lur09b, Theorem 4.2.4.1], $\operatorname{holim}_I(p_1^* N\mathcal{M}_i^\circ) \simeq \operatorname{holim}_I(p_1^* N\mathcal{M}_i^\circ)^f$ (in the model categorical sense), where the diagram is in ssSet, is equivalent to $\operatorname{holim}_{NI}(p_1^* N\mathcal{M}_i^\circ)^f$ and $\operatorname{holim}_I N(\mathcal{M}_i, \mathcal{W}_i)$ to $\operatorname{holim}_{NI}(N(\mathcal{M}_i, \mathcal{W}_i))^f$ as well. Since p_1^* is (the derived functor of) a Quillen equivalence, $p_1^* \operatorname{holim}_I N\mathcal{M}_i^\circ$ is weakly equivalent to $\operatorname{holim}_I p_1^* N\mathcal{M}_i^\circ$ and the first statement follows.

For the second, note that $N\mathcal{M}^\circ \simeq \operatorname{holim}_I N(\mathcal{M}_i^\circ)$ iff

$$N(\mathcal{M}, \mathcal{W}) \simeq p_1^* N\mathcal{M}^\circ \simeq p_1^* \operatorname{holim}_I N(\mathcal{M}_i^\circ) \simeq \operatorname{holim}_I N(\mathcal{M}_i, \mathcal{W}_i).$$

□

4.2 Category of Modules over a Ring Spectrum

There are many different choices about the basic framework for ring spectra and their module categories. First, one has to decide whether to use model categories or ∞-categories (usually in the setting of Joyal's quasi-categories); second, there are many choices of model categories modelling spectra and one has to make a choice there. Since most of our main results are in the homotopy category of R-modules for a fixed ring spectrum R the choices barely matter – if they do at all, then in this and the next chapter.

For concreteness, we choose to work in the setting of symmetric spectra in simplicial sets, equipped with the stable (projective) model structure from [HSS00]. A (commutative) ring spectrum is for us always a strictly (commutative and) associative monoid in this category.

Given a ring spectrum R, one has an associated category of (left-)modules over it. As described in [SS00], it has an induced model structure with weak equivalences and fibrations the underlying ones. We denote its homotopy category by $\mathrm{Ho}(R\text{-mod})$. If we claim an isomorphism between two R-modules, it is always meant as an isomorphism in $\mathrm{Ho}(R\text{-mod})$. For two R-modules M and N, we denote their (derived) mapping spectrum by $\mathrm{Hom}_R(M,N)$ and set $[M,N]^n_R := \pi_{-n}\mathrm{Hom}_R(M,N)$. We have an isomorphism $[M,N]^n_R \cong \mathrm{Ho}(R\text{-mod})(M,\Sigma^n N)$. We will sometimes denote the mapping spectrum $\mathrm{Hom}_\mathbb{S}$ with respect to the sphere spectrum by the letter F to stress that it is not Hom over a background ring spectrum R.

Note also that the notions of a ring spectrum and a module spectrum have more explicit descriptions (as described in the beginning of the book project [Sch07]), which is equivalent to the more abstract one (see [Sch07, Theorem 3.8]).

As we will see, a priori, the commutative ring spectrum TMF is not constructed[3] as a symmetric spectrum in simplicial sets, but only in topological spaces. The results of [MMSS01, §19] give Quillen equivalences between these two categories of spectra and also of their categories of (commutative) monoids and corresponding module categories. Precise statements about equivalences to the ∞-category approach can be found in [Lur11], in particular in 4.1.4.6, 4.3.3.17, 4.4.4.9 and 6.3.2.18.

One of the most important tools in the study of module categories over ring spectra is the (generalized) universal coefficient spectral sequence.

Theorem 4.2.1 (Universal Coefficient Spectral Sequence, [EKMM97], IV.4.1[4]). *Let R be a ring spectrum and $M, N \in R\text{-mod}$. Then there is a spectral sequence*

$$E_2^{s,t} = \mathrm{Ext}^s_{R_*}(\pi_* M, \pi_* N[t]) \Rightarrow [M,N]^{s+t}_R.$$

Here $\pi_* N[t] = \pi_{*-t}N$. The edge homomorphism $[M,N]^n_R \to \mathrm{Hom}_{R_*}(\pi_* M, \pi_* N[n])$ is given by the induced map on homotopy groups.

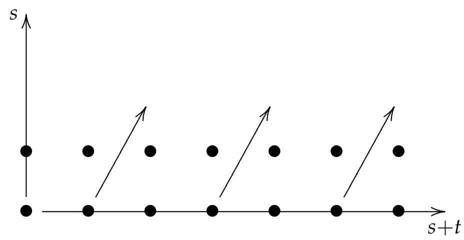
Example 4.2.2. By [Laz01, 11.8], there is an associative ring spectrum structure on Morava K-theory $K(n)$ for $p > 2$. Recall that $K(n)_* \cong \mathbb{F}_p[v_n^{\pm 1}]$ is a graded field. Therefore, all higher Ext-groups vanish over this ring and $\mathrm{Ho}(K(n)\text{-mod}) \simeq K(n)_*\text{-grmod}$.

---

[3] 'Constructed' is here used in a loose sense. Important steps in the "construction" are only existence proofs.

[4] While EKMM is set in S-modules, the proof of the universal coefficient spectral sequence is just happening in the homotopy category of R-modules and can be adapted also to symmetric spectra.

Example 4.2.3. As a slightly more interesting example, we might consider the case of $R = KU$; we already discussed this in the introduction, but will recapitulate it. We know that $KU_* \cong \mathbb{Z}[u^{\pm 1}]$ has homological dimension 1 in the sense that every graded module over KU_* has projective dimension at most 1. Therefore, the spectral sequence is concentrated in the first two rows and all differentials must vanish.



If we have two KU-modules M and N with an isomorphism $\overline{f}\colon \pi_*M \to \pi_*N$, then this isomorphism is realized by a map $f\colon M \to N$, which is an isomorphism (in the homotopy category) of KU-modules. Therefore, the functor π_* classifies KU-modules in the sense that it detects isomorphisms. We can apply the same arguments to KO localized at an odd prime p. For $R = KU$ or $R = KO_{(p)}$ it is even true by results of Franke and Patchkoria ([Pat11], 5.2.1) that the homotopy category of R-modules is equivalent to the derived category of R_*-modules. The same holds for TMF localized at a prime p greater than 3 since then $(TMF_{(p)})_* \cong \mathbb{Z}_{(p)}[c_4, c_6, \Delta^{-1}]$ has homological dimension two (as proved in the introduction):[5] so $\mathrm{Ho}(TMF_{(p)}\text{-mod}) \simeq \mathcal{D}((TMF_{(p)})_*)$ ([Pat11], 1.1.3). Yet another example is $TMF(2)$, which we get by evaluating the sheaf of commutative ring spectra $\mathcal{O}^{top}$ on the moduli stack of elliptic curves with level 2-structure $\mathcal{M}(2)$ at the prime 3. We have $TMF(2)_* \cong \mathbb{Z}_{(3)}[x_2, y_2, \Delta^{-1}]$ and thus we have also homological dimension 2 (by the same proof as for $TMF_{(p)}$) and get also the equivalence to the derived category.

At the end of this section, we want to collect a few definitions and simple lemmas. In these, R will always be a commutative ring spectrum.

Lemma 4.2.4. *If M is an R-module, then the map $[R, M]^k_R \to \pi_0(\Sigma^k M) \cong \pi_{-k}M$ (sending $[f]$ to $f_*(1)$ for $1 \in \pi_0 R$ the unit element) is an isomorphism. Furthermore, if $f\colon R \to S$ is a ring map and $x \in \pi_k M$ an element, then the element $(\mathrm{id}_M \wedge_R f)_*(x) \in \pi_k(M \wedge_R S)$ corresponds to the map*

$$x \wedge_R S\colon\ S \cong R \wedge_R S \to \Sigma^k M \wedge_R S.$$

Proof. The first part follows from the usual adjunction properties. The second part follows from the commutative diagram

$$\begin{array}{ccccc} \mathbb{S} & \xrightarrow{\eta_S} & S = R \wedge_R S & \xrightarrow{x \wedge_R S} & \Sigma^k M \wedge_R S \\ & \searrow^{\eta_R} & \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle x \wedge_R f} \\ & & R = R \wedge_R R & \longrightarrow & \Sigma^k M \wedge_R R. \end{array}$$

□

[5]For the definition of TMF, see the next chapter.

Lemma 4.2.5. *If π_*R is noetherian, then π_*M is a finitely generated π_*R-module for every finite R-module M.*

Proof. We use induction over the number of cells. The statement is obviously true for $M = \Sigma^k R$. Assume that $\pi_* M_0$ is a finitely generated π_*R-module and that we have a cofiber sequence

$$\Sigma^k R \xrightarrow{x} M_0 \to M.$$

We can split the corresponding long exact sequence of homotopy groups into short exact sequences like follows:

$$0 \to \pi_* M_0 / \operatorname{im}(x_*) \to \pi_* M \to \ker(x_*) \to 0$$

Both outer terms are finitely generated π_*R-modules since π_*R is noetherian. Thus, also the middle term is finitely generated. □

If M is an R-module, we write $DM = D_R M = \operatorname{Hom}_R(M, R)$ for the R-linear Spanier–Whitehead dual. If $z \in \pi_k M$, we write ${}^t z$ for the dual map $DM \to \Sigma^{-k} R$.

Lemma 4.2.6. *Let Z and M be R-modules and $a \in \pi_k Z$ and $z \in \pi_0(M \wedge_R DZ)$. Then the diagram*

$$\begin{array}{ccccc}
R & \xrightarrow{a} & \Sigma^{-k} Z & \xrightarrow{\mathrm{id}_Z \wedge_R z} & \Sigma^{-k} Z \wedge_R M \wedge_R DZ \\
\downarrow{\scriptstyle z} & & & & \downarrow{\scriptstyle \cong} \\
M \wedge_R DZ & \xrightarrow{\mathrm{id}_M \wedge_R {}^t a} & \Sigma^{-k} M & \xleftarrow{\mathrm{id}_M \wedge \mathrm{ev}} & \Sigma^{-k} M \wedge_R Z \wedge_R DZ
\end{array}$$

commutes. We will denote the composition $(\mathrm{id}_M \wedge \mathrm{ev}) \circ (\mathrm{id}_Z \wedge z)\colon \Sigma^{-k} Z \to \Sigma^{-k} M$ by ${}_t z$.

Proof. The only thing to observe is that ${}^t a$ is given as the composition

$$DZ \cong DZ \wedge_R R \xrightarrow{\mathrm{id} \wedge_R a} DZ \wedge_R Z \xrightarrow{\mathrm{ev}} R.$$

□

The following proposition will be important, especially for the next section:

Proposition 4.2.7 ([Rog08], Lemma 3.3.2)**.** *For R-modules X, Y and Z such that X or Z is finite, the canonical map*

$$\operatorname{Hom}_R(X, Y) \wedge_R Z \to \operatorname{Hom}_R(X, Y \wedge_R Z)$$

is an equivalence. Furthermore, the map $X \to D_R(D_R X)$ is an equivalence if X is finite.

Definition 4.2.8. An R-module M is called *free* if $M \cong \oplus_I R$ in $\operatorname{Ho}(R\text{-mod})$ for some set I. It is called *projective* if there is an R-module N such that $M \oplus N$ is free.

Lemma 4.2.9. *An R-module M is free (projective) iff $\pi_* M$ is a free (projective) π_*R-module.*

Proof. Let $\pi_* M$ be free as an π_*R-module with generators $(x_i)_{i \in I}$. The $x_i \in \pi_{k_i} M$ correspond to maps $f_i\colon \Sigma^{k_i} R \to M$ such that $(f_i)_*(1) = x_i$. Thus, the map $\Sigma f_i\colon \bigoplus_I \Sigma^{k_i} R \to M$ is an isomorphism on homotopy groups and thus an isomorphism in $\operatorname{Ho}(R\text{-mod})$.

Let now $\pi_* M$ be projective as an π_*R-module. Thus, there exists a free module N_0 over π_*R and another module P_0 over π_*R such that $P_0 \oplus \pi_* M \cong N_0$. We can find a free R-module N with $\pi_* N \cong N_0$ and realize the projection $N_0 \to \pi_* M$ by an R-module map $N \to M$. Denote its fiber by P. Clearly $\pi_* P \cong P_0$. Since $\pi_*(P \oplus M)$ is a free π_*R-module, $P \oplus M$ is free and M is projective.

The other implication is clear. □

4.3 Relatively Free Modules

As already explained in the introduction, one has often situations where R is a commutative ring spectrum and S an R-algebra such that $\pi_* R$ has infinite global dimension while $\pi_* S$ has finite global dimension. Then it makes sense to work in a relative setting:

Definition 4.3.1. Let R be a commutative ring spectrum and S be an R-algebra. A finite R-module M is called *relatively free* (with respect to S) if $M \wedge_R S$ is a free S-module. It is called *relatively projective* (with respect to S) if $M \wedge_R S$ is a projective S-module. We will leave out the "with respect to S" if it is clear from the context.

This idea can be used as input in a modified universal coefficient spectral sequence. Choose a collection $\mathcal{F}$ of finite R-modules and let $\mathcal{C}_\mathcal{F}$ denote the full (graded) subcategory of the homotopy category of R-modules spanned by $\mathcal{F}$. We denote by $\mathcal{C}_\mathcal{F}$-mod the category of graded additive functors from $\mathcal{C}_\mathcal{F}$ to graded abelian groups. We can define now a functor

$$\pi_*^\mathcal{F} \colon R\text{-mod} \to \mathcal{C}_\mathcal{F}\text{-mod}$$

by sending an $M \in R$-mod and an $F \in \mathcal{F}$ to $\pi_*(M \wedge_R F)$. We assume that $\pi_*^\mathcal{F}$ detects isomorphisms of R-modules. There is then a modified universal coefficient spectral sequence by Wolbert ([Wol98, Section 11]) of the following form:[6]

$$\mathrm{Ext}^s_{\mathcal{C}_\mathcal{F}\text{-mod}}(\pi_*^\mathcal{F}(M), \pi_*^\mathcal{F}(N)[t]) \Rightarrow [M, N]_R^{s+t}$$

The edge homomorphism is again defined to be the induced map on homotopy groups.

Let R be a commutative ring spectrum and S be an R-algebra with $\pi_* S$ of global dimension $\leq n$, $\pi_* R$ noetherian and $D_R S \cong S$. Let $\mathcal{F}$ be the collection of all finite R-modules M such that $\mathrm{Hom}_R(S, M) \simeq S \wedge_R M$ is a projective S-module. Note that $D_R M \in \mathcal{F}$ if $M \in \mathcal{F}$.

Lemma 4.3.2. *For $N \in \mathcal{F}$, the module $\pi_*^\mathcal{F} N$ is projective in $\mathcal{C}_\mathcal{F}$-mod.*

Proof. Define, for $N \in \mathcal{F}$, the functor $H_N \in \mathcal{C}_\mathcal{F}$-mod by $H_N(M) = [N, M]_R^*$ for $M \in \mathcal{F}$. We have $H_{D_R N} \cong \pi_*^\mathcal{F} N$ since $\mathrm{Hom}_R(D_R N, M) \simeq \mathrm{Hom}_R(D_R N, R) \wedge_R M \simeq N \wedge_R M$ by Proposition 4.2.7.

Let $F \to G$ be an epimorphism in $\mathcal{C}_\mathcal{F}$-mod and $f \colon H_{D_R N} \to G$ be a morphism (of degree 0). By the (enriched) Yoneda lemma, morphisms of degree 0 from $H_{D_R N}$ to $G \in \mathcal{C}_\mathcal{F}$-mod are in bijection with $G(D_R N)_0$; thus f corresponds to an element $f_0 \in G(D_R N)_0$. Since epimorphisms are surjective objectwise, we can lift f_0 to an element in $F(D_R N)_0$, giving the desired morphism $H_{D_R N} \to F$. □

We assume now that $\mathcal{F}$ has up to suspensions only finitely many indecomposable objects and that $S \in \mathcal{F}$. Then we have the following proposition:

Proposition 4.3.3. *For every finite R-module X, $\pi_*^\mathcal{F} X$ has projective dimension $\leq n$.*

Proof. For $N \in \mathcal{F}$, maps of degree k from H_N into $\pi_*^\mathcal{F} X$ are in bijection with $(\pi_k^\mathcal{F} X)(N) = \pi_k N \wedge_R X$ by the enriched Yoneda lemma. Since R_* is noetherian, $\pi_* N \wedge_R X$ is a finitely

[6] Wolbert only considers finite R-modules of the form $R \wedge X$, but this is an unnecessary restriction.

generated R_*-module by Lemma 4.2.5. Since $\mathcal{C}_\mathcal{F}$-mod is an R_*-linear category, we can thus choose finitely many maps $f_{N,i}\colon H_{\Sigma^{k_i}N} \to (\pi_*\mathcal{F}X)$ (of degree 0) such that

$$\bigoplus_i H_{\Sigma^{k_i}N}(N) \xrightarrow{\Sigma f_{N,i}} (\pi_*\mathcal{F}X)(N)$$

is surjective.

Now, we select finitely many $N_j \in \mathcal{F}$ such that every object in $\mathcal{F}$ is a suspension of one of the N_j and choose maps $f_{N_j,i}$ as above. The sum $\bigoplus_{i,j} H_{\Sigma^{k_i}N_j} \to \pi_*\mathcal{F}X$ is an epimorphisms since epimorphisms can be detected objectwise. Set $M := D_R\left(\bigoplus_{i,j}\Sigma^{k_i}N_j\right)$. Thus, we get a degree 0 morphism $f\colon H_{D_RM} \to \pi_*^{\mathcal{F}}X$, corresponding by Yoneda to an element $f(1) \in \pi_0 D_RM \wedge_R X \cong [M,X]_R$ (using Proposition 4.2.7). Define K to be the fiber of the corresponding map $M \to X$. Smashing with an $N \in \mathcal{F}$ gives a cofiber sequence

$$K \wedge_R N \to M \wedge_R N \to X \wedge_R N.$$

We want to show that the second map is surjective on homotopy groups. By definition, it agrees (up to sign) with the composition

$$\begin{array}{ccccc} M \wedge_R N & & & & X \wedge_R N \\ \downarrow\cong & & & & \uparrow \mathrm{ev}\wedge\mathrm{id} \\ (M\wedge_R N)\wedge_R R & \xrightarrow{\mathrm{id}\wedge f(1)} & (M\wedge_R N)\wedge_R(X\wedge_R D_RM) & \xrightarrow{\cong} & (M\wedge_R D_RM)\wedge_R(X\wedge_R N) \end{array}$$

This in turn agrees (up to sign) with the composition

$$\begin{array}{ccc} \mathrm{Hom}_R(D_RM,N) \cong \mathrm{Hom}_R(D_RM,N)\wedge_R R & \xrightarrow{\mathrm{id}\wedge f(1)} & \mathrm{Hom}_R(D_RM,N)\wedge_R D_RM \wedge_R X \\ \uparrow & & \downarrow \mathrm{ev}\wedge\mathrm{id}_X \\ M\wedge_R N \cong D_R(D_RM)\wedge_R N & & X\wedge_R N \end{array}$$

The first map in this composition is an equivalence (of R-modules). The composition of the latter two induces the morphism $f(S)\colon H_{D_RM}(N) \to (\pi_*^{\mathcal{F}}X)(N)$, which is surjective. Thus the morphism $M\wedge_R N \to X\wedge_R N$ is surjective on homotopy groups and we get a short exact sequence

$$0 \to \pi_*^{\mathcal{F}}K \to \pi_*^{\mathcal{F}}M \to \pi_*^{\mathcal{F}}X \to 0.$$

For $N = S$, this gives

$$0 \to \pi_*K\wedge_R S \to \pi_*M\wedge_R S \to \pi_*X\wedge_R S \to 0.$$

Since the middle term is projective as an π_*S-module, the homological dimension of $\pi_*K\wedge_R S$ as a π_*S-module is one less than that of $\pi_*\mathrm{Hom}_R(S,X)$ unless the latter is already 0. Since K is as the fiber of a morphism between finite modules also finite, we can repeat the same procedure (at most) n times and at the end get a finite K such that $\pi_*\mathrm{Hom}_R(S,K)$ is π_*S-projective, hence $K \in \mathcal{F}$ and $\pi_*^{\mathcal{F}}K$ is projective by the last lemma. Since also $\pi_*^{\mathcal{F}}M$ is projective in $\mathcal{C}_\mathcal{F}$-mod, this proves the proposition. □

Remark 4.3.4. We can restrict in the statement of the proposition to the indecomposable objects in $\mathcal{F}$ since the values on them determine every additive functor from $\mathcal{C}_\mathcal{F}$.

The following theorem was in originally shown (in a different form) by Bousfield in [Bou90] and we will show it again in Chapter 7.

Theorem 4.3.5. *Every relatively free KO-module is a sum of shifts of KO, KU and* $KT = KO \wedge \mathrm{Cone}(\eta^2)$.

Corollary 4.3.6. *For* $\mathcal{F} = \{KO, KU, KT\}$*, the functor* $\pi_*^{\mathcal{F}} X$ *classifies finite KO-modules.*

Proof. By the last theorem and the proposition above, for every finite KO-module X, the module $\pi_*^{\mathcal{F}}(X) \in \mathcal{C}_{\mathcal{F}}$-mod has projective dimension at most 1 (since projective implies free over $\pi_* KU$). Then we can use the modified universal coefficient spectral sequence to argue as in the case of KU. □

Even without assuming that there are only finitely many indecomposable relatively projective modules, one can often produce short resolutions by relatively projective modules. Let S again be a R-algebra such that $\pi_* S$ has global dimension $\leq n$. In addition, we assume that for every finite $M \in R$-mod, there is a map $N \to M$ from a relatively projective R-module such that $\pi_* S \wedge_R N \to \pi_* S \wedge_R M$ is surjective. Then we can produce for every finite $M \in R$-mod cofiber sequences of the form

$$M_1 \to N_0 \to M$$

$$M_2 \to N_1 \to M_1$$

$$\dots$$

$$M_k \to N_{k-1} \to M_{k-1}$$

such that $k \leq n$ and all N_i ($i \in \{0, \dots, k-1\}$) and M_k are relatively projective. These assumption are (for $n = 1$ respectively $n = 2$) true for $R = KO$, $S = KU$ and $R = TMF_{(3)}$, $S = TMF(2)$ (see Lemma 6.3.7 for the TMF-case).

4.4 Sheaves

Definition 4.4.1. Let $\mathcal{C}$ be a site and $\mathcal{D}$ be an ∞-category. Then a *sheaf on* $\mathcal{C}$ *with values in* $\mathcal{D}$ is a functor $\mathcal{F} \colon (\mathcal{C})^{op} \to \mathcal{D}$ such that we have descent for coverings in the following sense: For $U \to V$ a covering in $\mathcal{C}$, the map $\mathcal{F}(V) \to \operatorname{holim}_\Delta \mathcal{F}(U^{\times_V \bullet})$ is an equivalence. We denote the ∞-category of sheaves by $\mathrm{Shv}(\mathcal{C}; \mathcal{D})$. We say that a sheaf is *hypercomplete* if it satisfies descent with respect to all hypercovers; we will not define this since it is barely relevant for our purposes, but see [DHI04, Definition 4.3] and [Lur09b, Section 6.5].

For a sheaf $\mathcal{F}$ on a site $\mathcal{C}$ (with values in an ∞-category $\mathcal{D}$) and $\mathcal{G}$ another sheaf on $\mathcal{C}$, we define $\mathcal{F}(\mathcal{G})$ as $\mathrm{Fun}_{(\mathcal{C}^{op}, \mathcal{D})}(\mathcal{G}, \mathcal{F})$. Suppose now that $\mathcal{D} = \mathrm{Sp}$, the ∞-category of spectra. Via the functor Σ^∞_+, the enrichment of $\mathcal{C}$ in sets induces an enrichment in symmetric spectra Sp^Σ. The sheaf $\mathcal{F}$ comes from a functor $\mathcal{F}' \colon \mathcal{C}^{op} \to \mathrm{Sp}^\Sigma$ by Proposition 4.2.4.4 of [Lur09b] and Example 4.1.4.6 of [Lur11]; more precisely, we get $\mathcal{F}$ as the composition

$$N\mathcal{C}^{top} \xrightarrow{N\mathcal{F}'^\circ} N(\mathrm{Sp}^\Sigma)^\circ \simeq \mathrm{Sp}\,.$$

Now let U be in $\mathcal{C}$ and h_U be the presheaf $\mathcal{C}^{op} \to \mathrm{Sp}^\Sigma$ represented by U. Then enriched Yoneda implies $\mathcal{F}'(h_U) \cong \mathcal{F}'(U)$ in Sp^Σ and therefore $\mathcal{F}(h_U) \simeq \mathcal{F}(U)$ is Sp. Thus, we recover the usual evaluation of a sheaf.

We can associate to every sheaf of spectra $\mathcal{F}$ a presheaf of graded abelian groups $\pi_*^{pre}(\mathcal{F})$ by $(\pi_*^{pre}(\mathcal{F}))(U) = \pi_*(\mathcal{F}(U))$. We will denote the *sheafification* of this presheaf by $\pi_*\mathcal{F}$.

For a (commutative) monoid $\mathcal{O}$ in $\mathrm{Shv}(\mathcal{C};\mathrm{Sp})$ (Sp the ∞-category of spectra), we get in the usual way the notion of an $\mathcal{O}$-module. It turns out that the datum of a commutative monoid in $\mathrm{Shv}(\mathcal{C};\mathrm{Sp})$ is equivalent to a sheaf of commutative ring spectra (DAG VII.2.1.1).

Lemma 4.4.2. *Let $(\mathcal{X}, \mathcal{O}^{top})$ be a site equipped with a sheaf of ring spectra and let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}^{top}$-modules. Then the presheaf defined by $\mathcal{H}om(\mathcal{F},\mathcal{G})(U) := Hom_{\mathcal{O}^{top}|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ is already a sheaf.*

Proof. Probably, a more elementary proof (following, e.g., the lines of [KS06, p.430]) is possible, but we will base our proof on DAG VIII, Remark 2.1.11. This states that the construction $(U \in \mathcal{X}) \mapsto \mathcal{O}^{top}|_U$-mod is a sheaf on $\mathcal{X}$ with values in the ∞-category of ∞-categories. Analogously to [Lur09b, 1.2.13.8], the forgetful functor from ∞-categories under $\Delta^0 \coprod \Delta^0$ to ∞-categories preservers limits. Let I be the ∞-category Δ^1 together with the inclusion $\Delta^0 \coprod \Delta^0 \hookrightarrow \Delta^1$ of end points. Then, for an arbitrary ∞-category $\mathcal{C}$ together with a morphism $\Delta^0 \coprod \Delta^0 \xrightarrow{(X,Y)} \mathcal{C}$, the space of morphisms $I \to \mathcal{C}$ under $\Delta^0 \coprod \Delta^0$ is equivalent to the space of morphisms from X to Y in $\mathcal{C}$. Thus, $\mathrm{Hom}_{\mathcal{O}^{top}|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ defines a sheaf. □

Let $\mathcal{X}$ be a Grothendieck site with terminal object $*$. If $\mathcal{F}$ is a sheaf of spectra on $\mathcal{X}$, then there is a spectral sequence

$$H^q(\mathcal{X}; \pi_p(\mathcal{F})) \Rightarrow \pi_{p-q}\Gamma(\mathcal{F})$$

where $\Gamma(\mathcal{F}) := \mathcal{F}(*)$ and π_* denotes sheafified homotopy groups. This is called the *descent spectral sequence* and is denoted by $DSS(\mathcal{F})$. Details on construction and convergence can be found in [Dou07]. The DSS is natural with respect to maps of sheaves and its edge homomorphism

$$\pi_n\Gamma(\mathcal{F}) = \Gamma(\pi_n^{pre}(\mathcal{F})) \to \Gamma(\pi_n\mathcal{F})$$

is induced by the sheafification map. Thus, the DSS can be seen as a measure of the difference between $\pi_*^{pre}\mathcal{F}$ and $\pi_*\mathcal{F}$.

4.5 Quasi-Coherent Sheaves in Derived Algebraic Geometry

We will introduce here a bit of derived algebraic geometry, which will be used in the next chapters. Our main source is Jacob Lurie's Derived Algebraic Geometry (DAG), but we will use only a fraction of its generality. In particular, the following definitions are often just special cases of his definitions. In this section, a commutative ring spectrum will always denote a commutative monoid in the ∞-category of spectra.

Definition 4.5.1 (DAG VII, Remark 2.9)**.** Let A be a commutative ring spectrum. For $f \in \pi_0 A$, a localization $\phi\colon A \to A[\frac{1}{f}]$ of A at f is a map inducing isomorphisms

$$(\pi_n A) \otimes_{\pi_0 A} (\pi_0 A)[\frac{1}{f}] \to \pi_n(A[\frac{1}{f}]).$$

This localization always exists.

Construction 4.5.2. Let A be a commutative ring spectrum. Since $\pi_0 A$ is an ordinary commutative ring, we can associate to it the topological space $\operatorname{Spec} \pi_0 A$. For every $f \in \pi_0 A$, we define $\mathcal{O}(D(f)) := A[\frac{1}{f}]$ (for $D(f)$ the non-vanishing locus of f in $\operatorname{Spec} A$). This determines a sheaf of ring spectra $\mathcal{O}_A$ on $\operatorname{Spec} A$ (since the $D(f)$ are a basis of topology). The pair $(\operatorname{Spec} \pi_0 A, \mathcal{O}_A)$ is called $\operatorname{Spec} A$.

Definition 4.5.3. A *derived affine scheme* is a spectrally ringed space $(X, \mathcal{O}_X)$ such that there is a commutative ring spectrum A such that $X \cong \operatorname{Spec} \pi_0 A$ and there is an equivalence $\mathcal{O}_A \to \mathcal{O}_X$ in the ∞-category of sheaves of commutative ring spectra on X.

Definition 4.5.4. A *derived Deligne–Mumford* stack consists of a Grothendieck site $\mathcal{X}$ equipped with a sheaf of commutative ring spectra $\mathcal{O}$ such that

- the pair $(\mathcal{X}, \pi_0 \mathcal{O})$ is (the ringed site associated to) a Deligne–Mumford stack,
- the $\pi_0 \mathcal{O}$-modules $\pi_n \mathcal{O}$ are quasi-coherent (in the classical sense), and
- the sheaf $\Omega^\infty \mathcal{O}$ is hypercomplete.

Remark 4.5.5. This is a special case of the definition of a derived Deligne–Mumford stack in DAG VII by Theorem DAG VII.8.42.

Remark 4.5.6. Let $(\mathcal{X}, \mathcal{O})$ be a derived Deligne–Mumford stack and $X \in \mathcal{X}$ be an object projecting to $\operatorname{Spec} \Lambda$ (for Λ a commutative ring) in Sch. Then $(\operatorname{Spec} \Lambda, \mathcal{O}|_{\operatorname{Spec} \Lambda})$ is a derived affine scheme. Indeed, we have $\pi_0 \mathcal{O}|_{\operatorname{Spec} \Lambda} \cong \mathcal{O}_{\operatorname{Spec} \Lambda}$ and isomorphisms

$$\pi_n \mathcal{O}(\operatorname{Spec} \Lambda) \otimes_\Lambda \Lambda[\frac{1}{f}] \to \pi_n \mathcal{O}(D(f))$$

for $f \in \Lambda$ since the sheaves $\pi_n \mathcal{O}$ are quasi-coherent. Hence, $\mathcal{O}(D(f)) \simeq \mathcal{O}(\operatorname{Spec} \Lambda)[\frac{1}{f}]$.

Similarly to classical case, $\mathcal{O}$ corresponds to a commutative algebra object in (étale) sheaves of spectra on $\mathcal{X}$ and an $\mathcal{O}$-module is just a module over this algebra.

Remark 4.5.7. If $P\colon \mathcal{X} \to \mathrm{Sch}$ is the fiber functor associated to $(\mathcal{X}, \pi_0 \mathcal{X})$, then we define $\mathcal{X}_{\mathrm{Aff}}$ as the full subcategory on the preimages of all affine schemes under P. Since every scheme can be covered by affine schemes, the categories of sheaves on $\mathcal{X}$ and on $\mathcal{X}_{\mathrm{Aff}}$ are equivalent and so are the categories of $\mathcal{O}$-modules. Therefore, we can restrict our attention to the sub site $\mathcal{X}_{\mathrm{Aff}}$ if it is convenient.

Now we are ready to define quasi-coherent sheaves on derived Deligne–Mumford stacks $(\mathcal{X}, \mathcal{O})$:

Definition 4.5.8. An $\mathcal{O}$-module $\mathcal{F}$ is called *quasi-coherent* if for any (2-)commutative diagram (with U and V affine schemes)

$$\begin{array}{ccc} U & & \\ \downarrow{\scriptstyle f} & \searrow^{x} & \\ & & \mathcal{X} \\ & \nearrow_{y} & \\ V & & \end{array}$$

the associated morphism $\mathcal{F}(V) \wedge_{\mathcal{O}(V)} \mathcal{O}(U) \to \mathcal{F}(U)$ is an equivalence.

Proposition 4.5.9. *An $\mathcal{O}$-module $\mathcal{F}$ is quasi-coherent if and only if $\pi_n\mathcal{F}$ is quasi-coherent for every $n \in \mathbb{Z}$ and $\Omega^\infty\mathcal{F}$ is hypercomplete.*

Proof. This is proven in DAG VIII 2.3.12 and 2.3.21. □

Remark 4.5.10. We can evaluate a (quasi-coherent) sheaf on a derived Deligne–Mumford stack $\mathcal{X}$ not only on $U \in \mathcal{X}$ (corresponding to a morphism from the underlying scheme of U to $(\mathcal{X}, \pi_0\mathcal{O}_\mathcal{X})$), but also on a Deligne–Mumford stack $(\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ with a map to $(\mathcal{X}, \pi_0\mathcal{O}_\mathcal{X})$: It defines a sheaf $h_\mathcal{Y}$ on $\mathcal{X}$ by $U \mapsto \mathrm{Hom}_{(\mathcal{X},\pi_0\mathcal{O}_\mathcal{X})}(U, (\mathcal{Y}, \mathcal{O}_\mathcal{Y}))$ and we define $\mathcal{F}(\mathcal{Y})$ as $\mathcal{F}(h_\mathcal{Y}) = \mathrm{Hom}_{\mathrm{Fun}(\mathcal{X}^{op},\mathrm{Sp}))}(h_\mathcal{Y}, \mathcal{F})$.

As one might expect, on derived affine schemes, quasi-coherent sheaves are equivalent to modules:

Proposition 4.5.11 (DAG VIII, 2.3.11)**.** *Let $(X, \mathcal{O})$ be a derived affine scheme of the form* Spec *A. Then there are inverse equivalences*

$$\mathcal{O}\text{-mod} \underset{\widetilde{()}}{\overset{\Gamma}{\rightleftarrows}} A\text{-mod}.$$

Here Γ is given by taking global sections. For $M \in A$-mod *and* $\mathrm{Spec}\, A[\frac{1}{f}] \cong D(f) \subset \mathrm{Spec}\, A$*, the sheaf $\widetilde{M}$ is given by $\widetilde{M}(D(f)) \simeq M \wedge_A A[\frac{1}{f}]$.*[7]

For a map between derived Deligne–Mumford stacks $f\colon \mathcal{X} \to \mathcal{Y}$, there are adjoint functors

$$\mathrm{Mod}(\mathcal{O}_\mathcal{Y}) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{Mod}(\mathcal{O}_\mathcal{X}).$$

For $\mathcal{F} \in \mathcal{O}_\mathcal{X}$-mod, the $\mathcal{O}_\mathcal{Y}$-module $f_*\mathcal{F}$ is defined for a map $U \to \mathcal{Y}$ by $f_*\mathcal{F}(U) := \mathcal{F}(U \times_\mathcal{Y} \mathcal{X})$. We will not define f^* in general, but for f an étale morphism of the underlying classical Deligne–Mumford stacks and $\mathcal{G} \in \mathcal{O}_\mathcal{Y}$-mod, we have $f^*\mathcal{G}(U \to \mathcal{X}) \simeq \mathcal{G}(U \to \mathcal{X} \to \mathcal{Y})$. The functor f^* is symmetric monoidal (see Section 2.5 of DAG VIII).

The adjunction between ringed topoi and commutative ring spectra gives as a special case for every derived Deligne–Mumford stack $(\mathcal{X}, \mathcal{O}^{top})$ a morphism $f\colon (\mathcal{X}, \mathcal{O}^{top}) \to \mathrm{Spec}(\mathcal{O}^{top}(\mathcal{X}))$ (see DAG VII.8.4). This gives a functor

$$(\mathcal{O}^{top}(\mathcal{X}))\text{-mod} \simeq \mathrm{QCoh}(\mathrm{Spec}(\mathcal{O}^{top}(\mathcal{X}))) \xrightarrow{f^*} \mathrm{QCoh}(\mathcal{X}, \mathcal{O}^{top})$$

by Proposition 2.5.1 of DAG VIII. Denote the value of this functor on $M \in \mathcal{O}^{top}(\mathcal{X})$-mod by $\mathcal{F}_M$. The functor $\mathcal{F}$ is left adjoint to taking global sections and the unit $\mathcal{O}^{top} \to \mathcal{F}_{\mathcal{O}^{top}(\mathcal{X})}$ is (equivalent to) the identity. In particular, the map $\mathcal{O}^{top}(U) \to \mathcal{F}_{\mathcal{O}^{top}}(U)$ is an equivalence for every U. Recall that every left adjoint between stable ∞-categories preserves cofiber sequences, so in particular $\mathcal{F}$ does. This implies that the map $\mathcal{O}^{top}(U) \wedge_{\mathcal{O}^{top}(\mathcal{X})} M \to \mathcal{F}_M(U)$ is an equivalence for every finite $\mathcal{O}^{top}(\mathcal{X})$-module M.

Lemma 4.5.12. *Let $(\mathcal{X}, \mathcal{O}^{top})$ be again a derived Deligne–Mumford stack and M and N be finite $X = \mathcal{O}^{top}(\mathcal{X})$-modules such that $\mathcal{F}_M$ is locally free. Furthermore, set $\mathcal{O} = \pi_*\mathcal{O}^{top}$. Then*

$$\pi_*\mathcal{F}_{M \wedge_X N} \cong (\pi_*\mathcal{F}_M) \otimes_\mathcal{O} (\pi_*\mathcal{F}_N).$$

[7]This is the unwinded form of the Spec functor used in DAG.

Proof. For every $U \in \mathcal{X}$, we have a canonical map

$$\begin{aligned}
\pi_*(\mathcal{F}_M(U)) \otimes_{\mathcal{O}(U)} (\pi_*\mathcal{F}_N(U)) &\to \pi_*(\mathcal{F}_M(U) \wedge_{\mathcal{O}^{top}(U)} \mathcal{F}_N(U)) \\
&\cong \pi_*(M \wedge_X \mathcal{O}^{top}(U) \wedge_{\mathcal{O}^{top}(U)} N \wedge_X \mathcal{O}^{top}(U)) \\
&\cong \pi_*(M \wedge_X N \wedge_X \mathcal{O}^{top}(U)) \\
&= \pi_*(\mathcal{F}_{M \wedge_X N}).
\end{aligned}$$

Locally, this map is an isomorphism by the Künneth spectral sequence since $\mathcal{F}_M$ is locally free over $\mathcal{O}^{top}$. □

4.6 Toda Brackets

Since we have at a few places in this thesis the opportunity to use Toda brackets, we will dedicate this section to them. Let $\mathcal{T}$ be a triangulated category and $f\colon X \to Y$, $g\colon Y \to Z$ and $h\colon Z \to W$ maps in $\mathcal{T}$ with $gf = 0$ and $hg = 0$. We first recall a standard definition of the triple Toda bracket: Consider the diagram

$$\xymatrix{ X \ar[r]^f & Y \ar[r]^\iota \ar[dr]_g & Cf \ar[r] \ar@{.>}[d] & \Sigma X \ar@{.>}[d] \\ & & Z \ar[r]^h & W }$$

Since $gf = 0$, there is an extension $G\colon Cf \to Z$ of g. Since $hG\iota = hg = 0$, there is an extension $H\colon \Sigma X \to W$. Again, choices are involved. The set of all maps $H\colon \Sigma X \to W$ coming to existence in this way we denote by $\langle h, g, f\rangle \subset [\Sigma X, W]$. It is called the *Toda bracket* of f, g and h. It is easy to see that this is a coset with respect to $h_*[\Sigma X, Z] + (\Sigma g)^*[\Sigma Y, W]$ (this is called the *indeterminacy* of $\langle h, g, f\rangle$).

There are also other ways of describing Toda brackets. For example, look at the following diagram:

$$\xymatrix{ Y \ar[r] \ar[d]^= & Cf \ar[r] \ar@{.>}[d] & \Sigma X \ar[r]^{-\Sigma f} \ar@{.>}[d] & \Sigma Y \ar[d]^= & & \\ Y \ar[r]^g & Z \ar[r]^i \ar[d]^= & Cg \ar[r]^p \ar@{.>}[d] & \Sigma Y \ar[r]^{-\Sigma g} \ar@{.>}[d] & \Sigma Z \ar[d]^= & \\ & Z \ar[r]^h & W \ar[r]^j & Ch \ar[r]^q & \Sigma Z \ar[r]^{-\Sigma h} & \Sigma W }$$

We have a map $\gamma\colon \Sigma Y \to Ch$ extending $-\Sigma g$ (since $hg = 0$). Furthermore, there exists a map $\phi\colon \Sigma X \to W$ such that $-\gamma\Sigma f = j\phi$ (since $gf = 0$). We denote the set of all maps $\Sigma X \to W$ coming to existence in this way by $\langle h, g, f\rangle'$. Since this set has the same indeterminacy as $\langle h, g, f\rangle$, we just have to give one common element to prove the two sets to be equal. Recall that we have a map $G\colon Cf \to Z$ extending g. The two maps G and γ give us by the axioms of a triangulated category maps $\alpha\colon \Sigma X \to Cg$ and $\beta\colon Cg \to W$ completing the maps of triangles. The composition $\beta\alpha$ is now both in $\langle h, g, f\rangle$ and in $\langle h, g, f\rangle'$.

Most of the time, we will be interested in the case $\mathcal{T} = \mathrm{Ho}(R\text{-mod})$ for a (strictly) commutative ring spectrum R. Let $x, y, z \in \pi_*(R)$ with $xy = yz = 0$. We can interpret

x, y, z as self (R-module) maps of R, e.g., $x\colon \Sigma^{|x|}R \cong S^{|x|} \wedge R \to R \wedge R \to R$ as in Lemma 4.2.4.[8] So, this defines the Toda bracket $\langle x, y, z\rangle \subset \pi_{|x|+|y|+|z|-1}(R)$. One important feature of Toda brackets is that they control the homotopy groups of finite R-cell complexes. To be more precise: Let $x \in \pi_n R$ be an element in the coefficients and denote by Cx the cone of $\Sigma^n R \xrightarrow{x} R$. Then we have a long exact sequence

$$\cdots \to \pi_*\Sigma^n R \to \pi_* R \to \pi_* Cx \to \pi_{*-1}\Sigma^n R \to \pi_{*-1} R \to \cdots$$

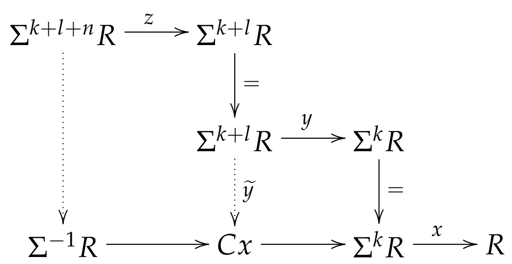
which splits into short exact sequences of the form

$$0 \to \pi_* R/x\pi_* R \xrightarrow{\alpha} \pi_* Cx \xrightarrow{\beta} \{\pi_{*-n}R\}_x \to 0$$

where $\{\pi_{*-n}R\}_x$ denotes all elements which are annihilated by x.

Lemma 4.6.1. *With notation as above, let $y \in \pi_m R$ and $z \in \pi_k R$ be elements in the coefficients of R with $xy = 0$ and $yz = 0$. Let $\widetilde{y} \in \pi_* Cx$ be an element with $\beta(\widetilde{y}) = y$. Let $w \in \pi_* R$ be an element such that the projection of w is mapped to $\widetilde{y}z$ under β. Then $w \in \langle x, y, z\rangle$.*

Proof. This is clear by the following diagram:



□

Lemma 4.6.2. *Let $a, b, c \in \pi_* R$ with $ab = bc = 0$. Furthermore, let M be a left R-module and $x \in \pi_* M$. Then*

$$\langle a, b, c\rangle \cdot x \subset \pm\langle a, xb, c\rangle.$$

More precisely, the relevant maps for the second Toda bracket are

$$c\colon\ \Sigma^{|c|+|b|+|a|+|x|}R \to \Sigma^{|b|+|a|+|x|}R,$$

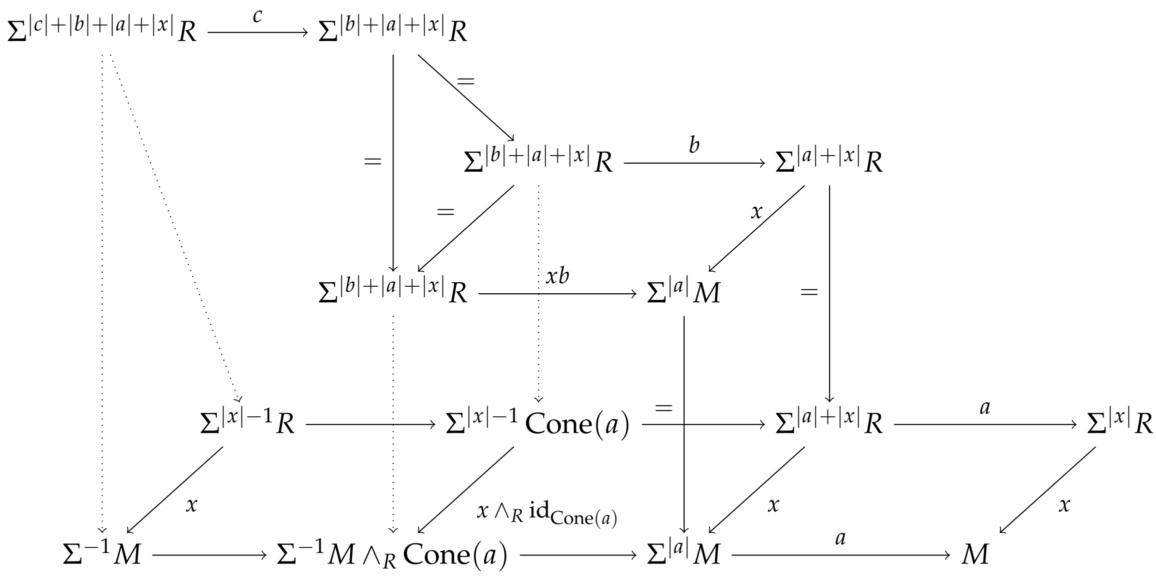
$$xb\colon\ \Sigma^{|b|+|a|+|x|}R \xrightarrow{b} \Sigma^{|a|+|x|}R \xrightarrow{x} \Sigma^{|a|}M \text{ and}$$

$$a\colon\ \Sigma^{|a|}M \cong S^{|a|} \wedge M \to R \wedge M \to M.$$[9]

Proof. We have the following diagram, which is (up to sign) commutative:

[8] Working with suspensions can bring delicate sign issues with it; since we do not add Toda brackets, the signs will not matter for our purposes and all statements should be interpreted in a ±-way in doubt.

[9] Since we permute an R with suspension variables in the definitions of these maps, it might be more sensible to introduce signs. But since we give only a ±-statement, we do not care.



For example, in the square in the lower right corner both compositions are

$$\Sigma^{|a|+|x|}R \cong S^{|a|} \wedge S^{|x|} \wedge R \xrightarrow{a \wedge x \wedge \mathrm{id}_R} R \wedge M \wedge R \cong R \wedge R \wedge M \to R \wedge M \to M,$$

where the last two arrows are the multiplication map of R and the left multiplication on M.

One fills first the two dotted arrows $\Sigma^{|b|+|a|+|x|}R \to \Sigma^{|x|-1}\,\mathrm{Cone}(a)$ and $\Sigma^{|c|+|b|+|a|+|x|}R \to \Sigma^{-1}R$ in the background. These determine the two dotted arrows in the foreground, making the diagram commute.

The diagram in the background defines an element in the Toda bracket $\langle a, b, c\rangle$ and the diagram in the foreground an element in the Toda bracket $\langle a, b \cdot x, c\rangle$. Thus the lemma. $\square$

Chapter 5

Topological Modular Forms

The aim of this chapter is to explain what TMF is and to describe its homotopy groups. Furthermore, we will study some important TMF-modules.

5.1 TMF and its Properties

As described in Section 2.8, we have a flat map $q\colon \mathcal{M} \to \mathcal{M}_{FG}$ from the moduli stack of elliptic curves to that of formal groups, associating to an elliptic curve its formal group. Let $C\colon \operatorname{Spec} R \to \mathcal{M}$ be a flat map associated to an elliptic curve C over R. Then the composite qC is flat and we can associate a Landweber exact homology theory $E(C)$ to it. This can be done as follows (see also [Goe09], section 3): Define a periodic version of MU as the homology theory $MUP_*(X) := \mathbb{Z}[x^{\pm 1}] \otimes MU_*(X)$, where $|x| = 2$. We have that $MUP_0 \cong L$, the Lazard ring, and $MUP_0 MUP \cong W$, the ring of isomorphisms of formal group laws. Thus $MUP_*(X)$ carries a (L, W)-comodule structure for every space X. As explained in Section 2.8, there is an equivalence of categories between (L, W)-comodules and quasi-coherent sheaves on $\mathcal{M}_{FG}$, associating to a comodule M a sheaf $\mathcal{G}_M$. For $X = S^0$, we have $\mathcal{G}_{MUP_{2n}(S^0)} = \omega^n$ for ω as at the end of Section 2.3. We define now the homology theory $E(C)$ by $E(C)_*(X) = \mathcal{G}_{MUP_*(X)}(\operatorname{Spec} R, C)$. If the formal group $\hat{C}$ has a chosen coordinate corresponding to a map $f\colon L \to R$, we get the more familiar formula $E(C) \cong MU_*(X) \otimes_L R$. All in all, we get a presheaf of homology theories on the category $\mathrm{Aff}^{flat}/\mathcal{M}$ of affine schemes with flat maps to the moduli stack of elliptic curves[1]

We would like to evaluate this presheaf of homology theories on the whole moduli stack $\mathcal{M}$. For this purpose, the following deep theorem is necessary:

Theorem 5.1.1 (Goerss–Hopkins–Miller)**.** *There is a lift*

$$\begin{array}{ccc} & & \textit{Commutative Ring Spectra} \\ & \overset{\mathcal{O}^{top}}{\nearrow} & \downarrow \\ (\mathrm{Aff}^{etale}/\mathcal{M})^{op} & \longrightarrow & \textit{Homology Theories} \end{array}$$

The presheaf $\mathcal{O}^{top}$ is actually a (hypercomplete) sheaf.[2]

[1] By [HS99], Cor. 2.15, there are no phantom maps between Landweber exact homology theories; so even the spectrum is well-defined up to unique isomorphism in the homotopy category.

[2] A complete proof can be found in [Beh11], which is based on [GH04] and [GH05]. These use commutative

Here, $\mathrm{Aff}^{etale}/\mathcal{M}$ denotes the site of affine schemes with étale maps to $\mathcal{M}$ and the étale topology. For the notion of a sheaf of commutative ring spectra, see Section 4.4. There is also explained how to extend $\mathcal{O}^{top}$ to the site of all stacks with an étale morphism to $\mathcal{M}$. Therefore, it makes sense to define $TMF := \mathcal{O}^{top}(\mathcal{M})$, the spectrum of **Topological Modular Forms**. Note that $\pi_{2n}\mathcal{O}^{top} = \omega^n$ since ω on $\mathcal{M}_{FG}$ pulls back to ω on $\mathcal{M}$.

There is a similar procedure using the compactified moduli stack of elliptic curves, extending $\mathcal{O}^{top}$ to the étale site over $\overline{\mathcal{M}}$. The global sections of this sheaf are denoted by Tmf and its connective cover by tmf.

Remark 5.1.2. Both $(\mathcal{M}, \mathcal{O}^{top})$ and $(\overline{\mathcal{M}}, \mathcal{O}^{top})$ are derived Deligne–Mumford stacks in the sense of Definition 4.5.4 since $\mathcal{M}$ and $\overline{\mathcal{M}}$ are (classical) Deligne–Mumford stacks and ω^n is quasi-coherent.

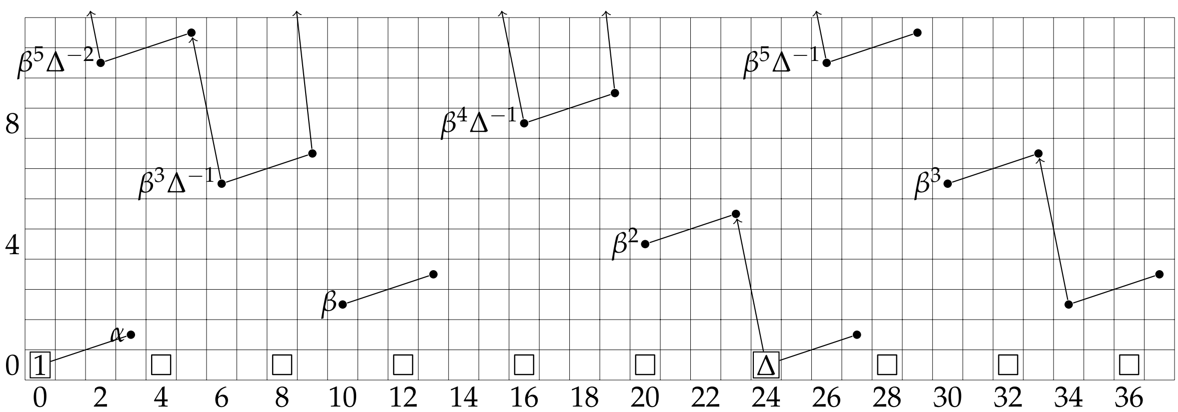
A computation of the homotopy groups of tmf can be found in [Bau08] or in the preprint [HM98]. Since our main concern is for TMF, we will give its homotopy groups for primes $p > 2$ here. Recall that we have for a sheaf of spectra $\mathcal{F}$ on a stack $\mathcal{X}$ the descent spectral sequence

$$H^q(\mathcal{X}; \pi_p\mathcal{F}) \Rightarrow \pi_{p-q}\mathcal{F}(\mathcal{X})$$

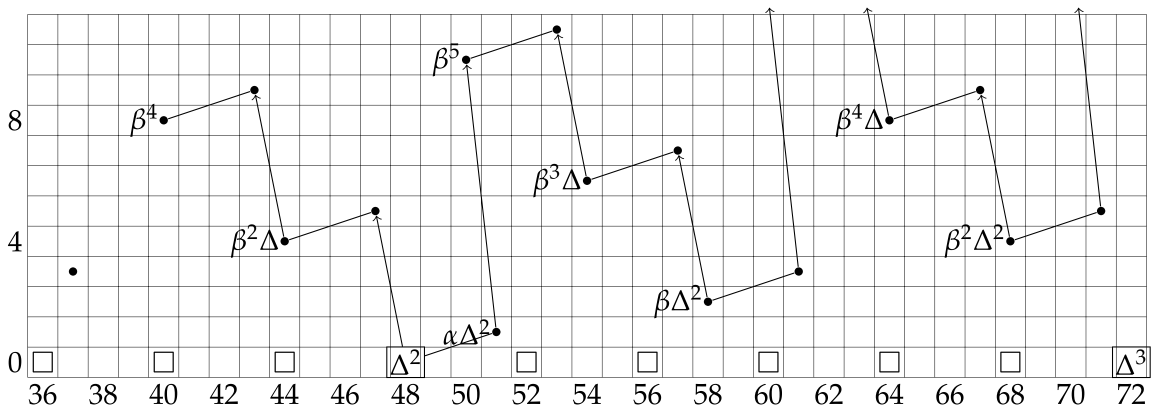
as in Sections 4.4 and 6.4. Here, π_* denotes the sheafified homotopy groups.

For $p > 3$, we have $H^q(\mathcal{M}_{(p)}, \omega^k) = 0$ for $q > 0$ (see Section 2.7). Therefore, the DSS collapses and we have $\pi_*TMF_{(p)} \cong H^0(\mathcal{M}; \omega^{2*}) \cong \mathbb{Z}_{(p)}[c_4, c_6, \Delta^{-1}]$.

For $p = 3$, the DSS is 72-periodic and looks as follows (as reference, see [Bau08], where have just to invert Δ):



monoids in symmetric spectra in topological spaces as their model for commutative ring spectra, but say that they could also have used S-modules or orthogonal spectra from the very beginning. A sketched proof can also be found in [Lur09a], which probably uses the concepts of [Lur11]. Since we will only really be interested in the homotopy categories of the module categories of these ring spectra, the choice of model does not really affect our results.



Here, we use the Adams convention for the grading of the spectral sequence, i.e., the (p,q)-spot corresponds to $H^q(\mathcal{M}_{(3)}; \omega^{\frac{p+q}{2}})$. The boxes stand for $\mathbb{Z}_{(3)}[j]$-summands (as in Section 2.7) and the dots for $\mathbb{F}_3$. The lines with positive slope indicate multiplication by α and the arrow of negative slopes are differentials.

All in all, this implies that the torsion elements in TMF are (up to Δ^3-periodicity) exactly the following:

$$\begin{aligned}
\alpha &\in \pi_3 TMF \\
\beta &\in \pi_{10} TMF \\
\alpha\beta &\in \pi_{13} TMF \\
\beta^2 &\in \pi_{20} TMF \\
\{\alpha\Delta\} &\in \pi_{27} TMF \\
\beta^3 &\in \pi_{30} TMF \\
\beta\{\alpha\Delta\} &\in \pi_{37} TMF \\
\beta^4 &\in \pi_{40} TMF
\end{aligned}$$

Here we use the same letters for the homotopy elements as for the cohomology elements. Multiplication is as in the spectral sequence except for $\alpha \cdot \{\alpha\Delta\} = \beta^3$ and $\alpha \cdot \beta\{\alpha\Delta\} = \beta^4$. Here, the name $\{\alpha\Delta\}$ is chosen since this element reduces to $\alpha\Delta$ in the spectral sequence, but is not divisible by α since Δ does not survive the spectral sequence.

The spectral sequence chart above is a Δ-periodic version of the one that can be found in [Bau08]. A spectral sequence chart computing the homotopy groups of $\pi_* tmf_{(3)}$ can be found in [Sto11], p. 22.

5.2 Extensions of TMF

We will work in this section only at the prime 3 and everything is implicitly localized at 3. The aim is to study certain (comparatively simple) TMF-modules, both as illustration and for the sake of the general theory in Chapters 6 and 8. In particular, we will investigate some TMF-modules coming from level structure, namely $TMF_0(2)$, $TMF(2)$ and $TMF(4)$, and show how they arise as finite TMF-modules.

A first example of an extension of TMF is the cofiber of $\Sigma^3 TMF \xrightarrow{\alpha} TMF$ (in the category of TMF-modules), which we denote by TMF_α. In other words, we have a cofiber sequence

$$\Sigma^3 TMF \to TMF \to TMF_\alpha \to \Sigma^4 TMF.$$

A table of the (torsion part of the) homotopy groups of TMF_α can found in Section 9.4. One particularly important element is $\widetilde{\alpha} \in \pi_7 TMF_\alpha$, which is obtained as the (unique) lift of $\alpha \in \pi_3 TMF$ along the map $TMF_\alpha \to \Sigma^4 TMF$. Since $\beta = \langle \alpha, \alpha, \alpha \rangle$ (see [Bau08]), we have by Lemma 4.6.1 the identity $\alpha\widetilde{\alpha} = \beta$, where we denote $\beta \in \pi_{10} TMF$ and its image in $\pi_{10} TMF_\alpha$ by the same letter.

Recall that we defined for a finite TMF-module M an $\mathcal{O}^{top}$-module sheaf $\mathcal{F}_M$ with $\mathcal{F}_M(U) \simeq \mathcal{O}^{top}(U) \wedge_{TMF} M$ (see the end of Section 4.5). Note that $\mathcal{F}_{TMF} \simeq \mathcal{O}^{top}$ and

$$\pi_k \mathcal{F}_M \cong \pi_0 \mathcal{F}_{M \wedge_{TMF} \Sigma^{-k} TMF} \cong \pi_0 \mathcal{F}_M \otimes_{\mathcal{O}} \pi_k \mathcal{O}^{top}$$

by Lemma 4.5.12.

We now want to determine $\pi_0 \mathcal{F}_{TMF_\alpha}$: There is a short exact sequence

$$0 \to \mathcal{O} = \pi_0 \mathcal{F}_{TMF} \to \pi_0 \mathcal{F}_{TMF_\alpha} \to \pi_0 \mathcal{F}_{\Sigma^4 TMF} = \omega^{-2} \to 0$$

since the (connecting) morphism has target $\pi_{-1} \mathcal{O}^{top} = 0$. Assume (for contradiction) that this extension splits. Then $\alpha \in H^1_2(\mathcal{M}; \mathcal{O})$ maps non-trivially to $\alpha' \in H^1_2(\mathcal{M}; \pi_0 \mathcal{F}_{TMF_\alpha})$. The element α' detects the image of α in $\pi_3 TMF_\alpha$ and cannot be hit by a differential in the DSS for TMF_α since it is in the first line. Therefore the image of α in $\pi_3 TMF_\alpha$ is non-zero; this is a contradiction since α is in the image of $\alpha\cdot$. Hence, the extension

$$0 \to \pi_* \mathcal{F}_{TMF} \to \pi_* \mathcal{F}_{TMF_\alpha} \to \pi_* \mathcal{F}_{\Sigma^4 TMF} \to 0$$

is non-split and we have $\pi_0 \mathcal{F}_{TMF_\alpha} \cong E_\alpha$.

The TMF-module TMF_α has as its dual $D_{TMF} TMF_\alpha \cong \Sigma^{-4} TMF_\alpha$, which can be seen by dualizing the defining cofiber sequence. Dualizing $\widetilde{\alpha}$, we get a map ${}^t\widetilde{\alpha}\colon \Sigma^{-4} TMF_\alpha \cong D_{TMF} TMF_\alpha \to \Sigma^{-7} TMF$. Precomposing with $\Sigma^{-4} TMF \to \Sigma^{-4} TMF_\alpha \cong D_{TMF} TMF_\alpha$ (which is dual to $TMF_\alpha \to \Sigma^4 TMF$), this agrees with $\Sigma^{-7}\alpha$ as it the dual of $\Sigma^4 \alpha$.

Lemma 5.2.1. *The compositions*

$$\Sigma^{10} TMF \xrightarrow{\widetilde{\alpha}} \Sigma^3 TMF_\alpha \xrightarrow{{}^t\widetilde{\alpha}} TMF$$

and

$$\Sigma^{10} TMF_\alpha \xrightarrow{{}^t\widetilde{\alpha}} \Sigma^7 TMF \xrightarrow{\widetilde{\alpha}} TMF_\alpha$$

both equal (multiplication by) β.

Proof. We want to show that $\widetilde{\alpha} \circ \widetilde{\alpha}^t = \cdot\beta$: Since $\alpha\widetilde{\alpha} = \beta$ in $\pi_* TMF_\alpha$, we have the following commutative diagram:

$$\begin{array}{ccccc}
TMF & & \xrightarrow{\quad\quad\beta\quad\quad} & & \Sigma^{-10} TMF_\alpha \\
\downarrow & & & & \| \\
TMF_\alpha & \xrightarrow{{}^t\widetilde{\alpha}} & \Sigma^{-3} TMF & \xrightarrow{\widetilde{\alpha}} & \Sigma^{-10} TMF_\alpha
\end{array}$$

By mapping (over TMF) into TMF_α, the triangle

$$\Sigma^3 TMF \to TMF \to TMF_\alpha \to \Sigma^4 TMF$$

induces a triangle

$$\Sigma^{-4} TMF_\alpha \to \mathrm{Hom}_{TMF}(TMF_\alpha, TMF_\alpha) \to TMF_\alpha.$$

The diagram above shows that $\widetilde{\alpha} \circ {}^t\widetilde{\alpha} \in \pi_{10} \mathrm{Hom}_{TMF}(TMF_\alpha, TMF_\alpha)$ maps to β and so does multiplication by β. Therefore the difference $\widetilde{\alpha} \circ \widetilde{\alpha}^t - (\cdot\beta)$ comes from $\pi_{14} TMF_\alpha$. But $\pi_{14} TMF_\alpha = 0$ since $\pi_{14} TMF = 0$ and $\beta \in \pi_{10} TMF$ has non-trivial multiplication by α. Therefore $\widetilde{\alpha} \circ {}^t\widetilde{\alpha}$ equals multiplication by β.

Thus, we see that the composition

$$\Sigma^{10} TMF \xrightarrow{\widetilde{\alpha}} \Sigma^3 TMF_\alpha \xrightarrow{{}^t\widetilde{\alpha}} TMF \xrightarrow{\widetilde{\alpha}} \Sigma^{-7} TMF_\alpha$$

represents $\beta\widetilde{\alpha} \in \pi_{17} TMF_\alpha$. Since only $\beta \in \pi_{10} TMF$ is sent by $\widetilde{\alpha}\colon \Sigma^7 TMF \to TMF_\alpha$ to $\beta\widetilde{\alpha} \in \pi_{17} TMF_\alpha$, we see that ${}^t\widetilde{\alpha} \circ \widetilde{\alpha} = \beta$. □

We define $TMF_0(2) := \mathcal{O}^{top}(\mathcal{M}_0(2))$ and $TMF(2) := \mathcal{O}^{top}(\mathcal{M}(2))$. Denote, as before, by $f\colon \mathcal{M}_0(2) \to \mathcal{M}$ the projection map. By definition, we have $TMF_0(2) = \Gamma(f_* f^* \mathcal{O}^{top})$. The sheaf $\pi_*(f_* f^* \mathcal{O}^{top}) \cong f_* f^* \pi_* \mathcal{O}^{top}$ has no higher cohomology by Lemma 3.4.4; therefore, the descent spectral sequence implies that $\pi_* TMF_0(2)$ is isomorphic to

$$\Gamma(f_* f^* \pi_* \mathcal{O}^{top}) \cong \Gamma_*(f_* f^* \mathcal{O}) \cong \mathbb{Z}_{(3)}[b_2, b_4, \Delta^{-1}].$$

The sheaf $f_* f^* \mathcal{O}^{top}$ is especially important because of the following lemma:

Lemma 5.2.2. *Let $\mathcal{F}$ be a locally free $\mathcal{O}^{top}$-module of finite rank. Then every morphism $g_{alg}\colon f_* f^* \mathcal{O} \to \pi_0 \mathcal{F}$ can be realized (uniquely in the homotopy category) by a map*

$$g\colon f_* f^* \mathcal{O}^{top} \to \mathcal{F}$$

with $\pi_0 g = g_{alg}$. The same holds if we replace $f_* f^* \mathcal{O}$ by a sum of twists of $f_* f^* \mathcal{O}$ by line bundles.

Proof. Since $f_* f^* \mathcal{O}$ is self-dual, we have that

$$\mathcal{H}om_{\mathcal{O}}(f_* f^* \mathcal{O}, \pi_k \mathcal{F}) \cong f_* f^* \mathcal{O} \otimes_{\mathcal{O}} \pi_k \mathcal{F}.$$

By Lemma 2.3.13, this is isomorphic to $f_* f^* \pi_k \mathcal{F}$ and by Lemma 3.4.4 the higher cohomology groups of $f_* f^* \pi_k \mathcal{F}$ vanish. Since $f_* f^* \mathcal{O}^{top}$ is locally free, we have

$$\pi_k \mathcal{H}om_{\mathcal{O}^{top}}(f_* f^* \mathcal{O}^{top}, \mathcal{F}) \cong \mathcal{H}om_{\pi_0 \mathcal{O}^{top}}(f_* f^* \mathcal{O}, \pi_k \mathcal{F})$$

(see Lemma 4.4.2 for the definition of the $\mathcal{H}om$-sheaf). Hence, the descent spectral sequence for

$$\mathcal{H}om_{\mathcal{O}^{top}}(f_* f^* \mathcal{O}^{top}, \mathcal{F})$$

is concentrated in the 0-line. Therefore, there is a (up to homotopy) a unique map

$$g\colon f_* f^* \mathcal{O}^{top} \to \mathcal{F}$$

realizing the algebraic map g_{alg}. The arguments for sums of twists is the same. □

Define another TMF-module $TMF_{\alpha,\widetilde{\alpha}}$ as the cofiber of the map $\widetilde{\alpha}\colon \Sigma^7 TMF \to TMF_\alpha$ (in the category of TMF-modules). Taking homotopy groups of the associated cofiber sequence gives a short exact sequence

$$0 \to E_\alpha \cong \pi_0\mathcal{F}_{TMF_\alpha} \to \pi_0\mathcal{F}_{TMF_{\alpha,\widetilde{\alpha}}} \to \pi_0\mathcal{F}_{\Sigma^8 TMF} \cong \omega^{-4} \to 0$$

since $\pi_1\mathcal{F}_{TMF_\alpha} = 0$. Suppose (for contradiction) that this extension splits. Then $\widetilde{\alpha} \in H^1_4(\mathcal{M}; E_\alpha)$ maps non-trivially to $\widetilde{\alpha}' \in H^1_4(\mathcal{M}; \pi_0\mathcal{F}_{TMF_{\alpha,\widetilde{\alpha}}})$. The element $\widetilde{\alpha}'$ detects the image of $\widetilde{\alpha}$ in $\pi_7 TMF_{\alpha,\widetilde{\alpha}}$ and cannot be hit by a differential in the DSS for $TMF_{\alpha,\widetilde{\alpha}}$ since it is in the first line. Therefore the image of $\widetilde{\alpha}$ in $\pi_7 TMF_{\alpha,\widetilde{\alpha}}$ is non-zero; a contradiction since $\widetilde{\alpha}$ is in the image of $\widetilde{\alpha}\cdot$. Therefore, the extension is non-split and $\pi_0\mathcal{F}_{TMF_{\alpha,\widetilde{\alpha}}} \cong E_{\alpha,\widetilde{\alpha}} \cong f_*f^*\mathcal{O}$.

Lemma 5.2.3 ([Beh06], 2.4, Lemma 2)**.** *We have $TMF_0(2) \simeq TMF_{\alpha,\widetilde{\alpha}}$.*

Proof. The sheaf $\mathcal{F}_{TMF_{\alpha,\widetilde{\alpha}}}$ is a locally free $\mathcal{O}^{top}$-module since the maps $\alpha\colon \Sigma^3\mathcal{O}^{top} \to \mathcal{O}^{top}$ and $\widetilde{\alpha}\colon \Sigma^7\mathcal{O}^{top} \to \mathcal{F}_{TMF_\alpha}$ induce locally zero (in the homotopy category). Indeed, for $U \to \mathcal{M}$ a morphism from an affine scheme, $\mathcal{O}^{top}(U)$ and $\mathcal{F}_{TMF_\alpha}(U)$ are torsion-free because U has no higher cohomology. Thus, $\alpha\colon \Sigma^3\mathcal{O}^{top}(U) \to \mathcal{O}^{top}(U)$ and $\widetilde{\alpha}\colon \Sigma^7\mathcal{O}^{top}(U) \to \mathcal{F}_{TMF_\alpha}(U)$ are zero in the homotopy category of $\mathcal{O}^{top}(U)$-modules since both maps are torsion.

By the last lemma, the isomorphism $f_*f^*\mathcal{O} \cong \pi_0\mathcal{F}_{TMF_{\alpha,\widetilde{\alpha}}}$ is realized by a map $f_*f^*\mathcal{O}^{top} \to \mathcal{F}_{TMF_{\alpha,\widetilde{\alpha}}}$, which is therefore an equivalence, thus also an equivalence on global sections. Hence, the result follows. □

By the lemma, we have a cofiber sequence

$$\Sigma^7 TMF \xrightarrow{\widetilde{\alpha}} TMF_\alpha \to TMF_0(2) \to \Sigma^8 TMF,$$

which dualizes to

$$\Sigma^{-5} TMF_\alpha \xrightarrow{t\widetilde{\alpha}} \Sigma^{-8} TMF \to D_{TMF} TMF_0(2) \to \Sigma^{-4} TMF_\alpha.$$

As above, one can show that the vector bundle associated to $D_{TMF}TMF_0(2)$ is a non-split extension of ω^4 and $\omega^2 \otimes E_\alpha$ and hence isomorphic to $f_*f^*\mathcal{O} \otimes \omega^4 \cong f_*f^*\mathcal{O}$ (by the results from Section 3.4). Using Lemma 5.2.2 again, we can show that $D_{TMF}TMF_0(2)$ is equivalent to $\Sigma^{-8}TMF_0(2) \simeq TMF_0(2)$. If we suspend 8 times, we get thus a cofiber sequence

$$TMF \to TMF_0(2) \to \Sigma^4 TMF_\alpha \to \Sigma TMF.$$

The map $f_*f^*\mathcal{O} \to \omega^{-4}$ induced by $TMF_{\alpha,\widetilde{\alpha}} \to \Sigma^8 TMF$ above is (up to isomorphism) the dual to the adjunction unit $\mathcal{O} \to f_*f^*\mathcal{O}$ tensored with ω^{-4} since there is up to isomorphism only two non-trivial extension of E_α and ω^{-4} (which are connected by a sign reversing isomorphism). Thus, the map $TMF \to TMF_0(2)$ in the cofiber sequence induces also the adjunction $\pi_0\mathcal{F}_{TMF} \cong \mathcal{O} \to f_*f^*\mathcal{O} \cong \pi_0\mathcal{F}_{TMF_0(2)}$. Since the set of (homotopy classes of) TMF-module maps $TMF \to TMF_0(2)$ agrees with the set of (homotopy classes of) $\mathcal{O}$-module maps $\mathcal{O} \to f_*f^*\mathcal{O}$, this shows that we have indeed the canonical map $TMF \to TMF_0(2)$ induced by $f\colon \mathcal{M}_0(2) \to \mathcal{M}$ in the cofiber sequence if $TMF_{\alpha,\widetilde{\alpha}}$ and $TMF_0(2)$ are suitably identified.

Lemma 5.2.4. *We have equivalences of TMF-module spectra*

$$TMF(2) \simeq TMF_0(2) \oplus \Sigma^4 TMF_0(2)$$

and

$$TMF(4) \simeq \bigoplus_{i=1}^{8} TMF(2).$$

Proof. The spectrum $TMF(2)$ has the structure of a $TMF_0(2)$-module (via the map $\mathcal{M}(2) \to \mathcal{M}_0(2)$) and 1 and $x_2 \in \pi_* TMF(2)$ form a basis to give $\pi_* TMF(2)$ the structure of a free $\pi_* TMF_0(2) \cong \mathbb{Z}_{(3)}[b_2, b_4, \Delta^{-1}]$-module (see the formulas of Section 2.5). Alternatively, we can use Lemma 5.2.2 again: For $p\colon \mathcal{M}(2) \to \mathcal{M}$ the usual projection, we have $p_* p^* \mathcal{O} \cong f_* f^* \mathcal{O} \oplus \omega^2 \otimes f_* f^* \mathcal{O}$ (by Lemma 3.5.4). By Lemma 5.2.2, we can realize this isomorphism by a map $f_* f^* \mathcal{O}^{top} \oplus \Sigma^{-4} f_* f^* \mathcal{O}^{top} \to p_* p^* \mathcal{O}^{top}$. Thus, we get an equivalence on global sections.

The last argument can also be applied to $TMF(4)$: Let $q\colon \mathcal{M}(4) \to \mathcal{M}$ be the usual projection. Then, by Lemma 3.5.5, we have that $q_* q^* \mathcal{O}$ is a sum of 8 copies of $p_* p^* \mathcal{O}$, hence also a sum of twists of $f_* f^* \mathcal{O}$. Thus, we can apply Lemma 5.2.2 as above and get that $TMF(4)$ is a sum of 8 copies of $TMF(2)$. □

Definition 5.2.5. For R a ring spectrum, an R-module M is called *faithful* if for every R-module N the condition $M \wedge_R N \cong 0$ implies that already $N \cong 0$.

Lemma 5.2.6. *The TMF-modules $TMF_0(2)$, $TMF(2)$ and $TMF(4)$ are faithful over TMF.*

Proof. In the light of the last lemma, it suffices to show the statement for $TMF_0(2)$. The element $\alpha \in \pi_3 TMF$ is the Hurewicz image of $\alpha_1 \in \pi_3 \mathbb{S}_{(3)}$. Thus, $TMF_\alpha \cong TMF \wedge C(\alpha_1)$. Since $\alpha_1^2 = 0$, there is a lift of α_1 to $\widetilde{\alpha_1} \in \pi_7 C(\alpha_1)$ and the Hurewicz image of this equals $\widetilde{\alpha} \in \pi_7 TMF_\alpha$. Thus, $TMF_0(2) \simeq TMF \wedge C(\alpha_1, \widetilde{\alpha_1})$. Clearly, the $\mathbb{Z}_{(p)}$-homology of $C(\alpha_1, \widetilde{\alpha_1})$ is non-trivial and torsionfree. Thus, $M \wedge_{TMF} TMF_0(2) \cong M \wedge C(\alpha_1, \widetilde{\alpha_1}) \cong 0$ implies $M = 0$ by [DHS88, Proposition 4.1] for every TMF-module M. □

Chapter 6

Galois Extensions and Descent

There are three main goals of this section. The first is to define Galois extensions of ring spectra and prove a version of Galois descent for them. The second is to give examples of Galois extensions of $TMF_{(p)}$ for a prime p. The third is to show an equivalence between the ∞-category of quasi-coherent sheaves on the derived moduli stack of elliptic curves $(\mathcal{M}, \mathcal{O}^{top})$ and the ∞-category of TMF-modules, at least for primes bigger than 2.

Besides this, we will give an introduction to homotopy fixed points and give an alternative account of the descent spectral sequence, using Galois descent.

6.1 Homotopy Fixed Points

In the theory of Galois descent, the notion of homotopy fixed points is extremely important. We will give the definition and a few properties in this section.

Definition 6.1.1. Let $\mathcal{C}$ be an ∞-category and $X\colon BG \to \mathcal{C}$ be a morphism for G a finite group. Then the homotopy fixed points X^{hG} are defined as the homotopy limit $\lim_{BG} X$. Similarly, if $\mathcal{M}$ is a model category and $X \in \mathcal{M}$ an object with a group action by a group G, then $X^{hG} := \operatorname{holim}_G X$.

There is also an alternative description of the homotopy fixed points: As described after Remark 4.13 in DAG XI, in the ∞-categorical context, X^{hG} is equivalent to the limit over the functor $X^\bullet\colon N(\Delta) \to \mathcal{C}$ given as X^{BG} (more precisely, X^n is given as $\prod_{BG_n} X$ with the structure maps induced by the structure maps of BG). A similar formula holds for fibrant objects in a simplicial model category via the Bousfield–Kan construction of a homotopy limit.

It will be particularly important to study homotopy fixed points in the ∞-category of ∞-categories. We begin with some preliminary definitions and constructions.

Definition 6.1.2. Let $\mathcal{C}$ be a category with an action of a group G. Then a twisted group object is a G-equivariant functor from the category EG to $\mathcal{C}$. Here, EG stands for the category with objects indexed by G and unique morphisms between them. More concretely, we are given an object $X \in \mathcal{C}$ together with morphisms $g \cdot X \to X$ for $g \in G$, satisfying some compatibility. We denote the category of twisted group objects by G-$\mathcal{C}$.

Let R be a symmetric ring spectrum with an action by a group G. Then the category R-mod gets a G-action as follows: Let $M = (M, a\colon R \wedge M \to M)$ be an R-module. Then we define $g \cdot M$ to equal M as a spectrum to have $R \wedge M \xrightarrow{g \wedge \mathrm{id}_M} R \wedge M \to M$ as structure map. Clearly, $g\cdot$ preserves fibrations and weak equivalences. As it is an equivalence inverse to g^{-1}, it also preserves cofibrations and is, in particular, a left Quillen functor.

It will turn out that the category of twisted group objects in R-mod is itself the category of modules over the *twisted group ring*, which we now want to define:

Construction 6.1.3. Let M be a monoid with unit e and multiplication μ and R be a symmetric ring spectrum with multiplication also denoted by *mu*. We denote by R_n the n-th space in R. Suppose, we have an action a of M on R via ring maps. The we define $\widetilde{R[M]}$ as a spectrum by $\widetilde{R[M]}_n := R_n \wedge M_+$ with S_n-action on the left factor. Thus $\widetilde{R[M]} = R \wedge M_+$. The different notation is chosen to emphasize the ring structure, which is given as follows:[1]

The unit map is given by the composition of the unit map of R with the morphism $R \to \widetilde{R[M]}$ given by $R_n \cong R_n \wedge \{e\}_+ \to R_n \wedge M_+$. The multiplication is given as the composition

$$\begin{array}{ccc}
(R_n \wedge M_+) \wedge (R_m \wedge M_+) & & R_{n+m} \wedge M_+ \\
\downarrow{\scriptstyle (R_n \wedge \Delta_M) \wedge (R_M \wedge M_+)} & & \uparrow{\scriptstyle m \wedge \mu} \\
(R_n \wedge M_+ \wedge M_+) \wedge (R_m \wedge M_+) & & (R_n \wedge R_m) \wedge (M \times M)_+ \\
\downarrow{\scriptstyle \cong} & & \uparrow{\scriptstyle \cong} \\
R_n \wedge M_+ \wedge (M_+ \wedge R_m) \wedge M_+ & \xrightarrow{R_n \wedge M_+ \wedge a \wedge M_+} & R_n \wedge M_+ \wedge R_m \wedge M_+
\end{array}$$

as indicated by the formula $(r_1, m_1), (r_2, m_2) \mapsto (r_1 \cdot m_1(r_2), m_1 m_2)$.

Lemma 6.1.4. *For G a group and R a symmetric ring spectrum with a G-action, the module category $\widetilde{R[G]}$-mod is equivalent (as a category) to the category of R-modules with twisted G-action.*

Proof. We will only sketch the proof:

Let N be a $\widetilde{R[G]}$-module. Thus, we have maps

$$R_k \wedge G^+ \wedge N_n \to N_{n+k}$$

satisfying the usual axioms. Using the unit map $S^0 \to R_0$, we get a map

$$\bigvee_G N_n \cong S^0 \wedge G^+ \wedge N_n \to R_0 \wedge G^+ \wedge N_n \to N_n.$$

This induces, for every $g \in G$, a map $g\colon N \to N \cong N$, which becomes an R-module map if we identify the target with $N \wedge_R R^g$. This defines a twisted G-action on N.

On the other hand, let P be a twisted group object in R-mod and let $e \in G$ be the unit. We define a map

$$\widetilde{R[G]}_k \wedge P(e)_n \cong \bigvee_G R_k \wedge P(e)_n \to P(e)_{n+k}$$

on the wedge summand correspond to $g \in G$ as the composition

$$R_k \wedge P(e)_n \to P(e)_{n+k} \xrightarrow{g} (P(e) \wedge_R R^g)_{n+k} \cong P(e)_{n+k}.$$

One can check that this defines a module structure over $\widetilde{R[G]}$. □

[1] Here, we use the explicit description of a ring spectrum given at the beginning of [Sch07].

In particular, this equips G-(R-mod) with a projective (simplicial) model structure.

Proposition 6.1.5. *Let R be a symmetric ring spectrum with an action by a group G. Then*

$$N(\widetilde{R[G]}\text{-mod}^\circ) \simeq (N(R\text{-mod}^\circ))^{hG}.$$

Proof. This result is essentially due to Julie Bergner and we will recall a special case of what she has proven in [Ber10, Theorem 4.1]: Let $\mathcal{M}$ be a model category with an action by G by left Quillen functors, then $N(G\text{-}\mathcal{M}, \mathcal{W}) \simeq (N(\mathcal{M}, \mathcal{W}))^{hG}$, where $N(-,-)$ denotes the classifying diagram functor as in Section 4.1.2.

In our case, $\mathcal{M} = R$-mod and $G\text{-}\mathcal{M} \simeq \widetilde{R[G]}$-mod by the lemma above. By Proposition 4.1.10, the result follows. □

We want to end this section with a spectral sequence:[2]

Theorem 6.1.6. *Given a spectrum X with an action by a discrete group G, we have a spectral sequence*

$$E_2^{pq} \cong H^q(G; \pi_{p+q}X) \Rightarrow \pi_p(X^{hG}).$$

This is called the homotopy fixed point spectral sequence (HFPSS). The edge morphism $\pi_* X^{hG} \to H^0(G; \pi_* X)$ is induced by the canonical morphism $X^{hG} \to X$.

If R is a ring spectrum with a multiplicative G-action, the HFPSS associated to R is multiplicative and agrees up to sign on the E^2-term with the multiplication induced by the product on $\pi_* R$. If M is an $\widetilde{R[G]}$-module, then the HFPSS gets the structure of a module spectral sequence over the HFPSS associated to R and the action on the E^2-term agrees up to sign with the action induced by $\pi_* M \otimes \pi_* R \to \pi_* M$.

Proof. The first part is standard. For the multiplicativity, see [Dug03, Theorem 6.1]. The statement about module structures is similar. □

6.2 Galois Descent

The aim of this section is to give some basics about Galois extensions of ring spectra and, in particular, to prove a version of Galois descent for them. We work again with symmetric spectra and all smash products (of spectra) and Hom-spectra are understood to be derived.

Definition 6.2.1 ([Rog08])**.** Let A be a commutative (symmetric) ring spectrum and B be a commutative A-algebra. Let G be a finite group acting on B via A-algebra maps from the left. Then B is a *G-Galois extension* of A if the maps $A \to B^{hG}$ and $B \wedge_A B \to F(G_+, B)$ are equivalences. Here, the latter map is indicated by the formula $(b_1 \wedge b_2, g) \mapsto g(b_1) \cdot b_2$.

Example 6.2.2 ([Rog08], 5.3.1)**.** Complex K-theory KU is a C_2-Galois extension of KO. On the other hand, connective ku is *not* a Galois extension of connective ko.

Conventions 6.2.3. Let B be a commutative A-algebra with G-action as above. Then we equip $B \wedge_A B$ with the B-module structure, which acts only on the right factor, and with the G-action, which acts only on the left factor. Furthermore, equip $F(G_+, B)$ with the B-module structure map adjoint to

$$F(G_+, B) \wedge B \wedge G_+ \cong (G_+ \wedge F(G_+, B)) \wedge B \xrightarrow{\text{ev} \wedge \text{id}} B \wedge B \xrightarrow{\mu} B$$

[2] Perhaps, every good section should end with a spectral sequence.

and with the G-action indicated by $g \cdot f = (h \mapsto f(hg))$. Then the map $B \wedge_A B \to F(G_+, B)$ is both a G- and a B-module map. It is also an equivalence of A-algebras for the algebra structure on $B \wedge_A B$ indicated by $(b_1 \wedge b_2) \cdot (b_1' \wedge b_2') = (b_1 b_1' \wedge b_2 b_2')$.

Proposition 6.2.4. *Let $A \to B$ be a faithful G-Galois extension. Then*

$$\begin{array}{rcl} G_+ \wedge B & \to & \mathrm{Hom}_A(B, B) \\ (g, b) & \mapsto & (b' \mapsto (g(b') \cdot b)) \end{array}$$

is an equivalence.

Proof. We first consider the case that B is equivalent to $F(G_+, A)$ as an A-algebra with G-action; so B might be thought of as a trivial G-Galois extension of A. While it is easy to show that both sides in the statement of the lemma are weakly equivalent, we have to consider the following diagram in order to show that the *map* is a weak equivalence:

$$\begin{array}{ccc} G_+ \wedge F(G_+, A) & \longrightarrow & \mathrm{Hom}_A(F(G_+, A), F(G_+, A)) \\ \uparrow & & \downarrow \\ G_+ \wedge G_+ \wedge A & & \mathrm{Hom}_A(G_+ \wedge A, F(G_+, A)) \\ \Big\downarrow & & \downarrow \\ & & \mathrm{Hom}_A(G_+ \wedge G_+ \wedge A, A) \\ & & \downarrow \\ G_+ \wedge G_+ \wedge A & \longrightarrow & F(G_+ \wedge G_+, A) \end{array}$$

The upper horizontal map is the one of the statement of the proposition. The upper two vertical maps are here given by the equivalence $G_+ \wedge A \to F(G_+, A)$ corresponding to the inclusion of the wedge into the product; the bottom map correspondingly for $G \times G$ instead of G. The other right vertical maps are given by the usual adjunctions, the other left vertical map corresponds to the isomorphism $(g_1, g_2, a) \mapsto (g_2 g_1, g_2, a)$. It is straightforward to see that the diagram commutes. By 2 out of 3 we get our result and we go back to the general case.

We consider the following equivalences

$$\begin{array}{rcl} \mathrm{Hom}_A(B, B) & \simeq & \mathrm{Hom}_B(B \wedge_A B, B) \simeq \mathrm{Hom}_B(F(G_+, B), B) \\ & \simeq & \mathrm{Hom}_B(G_+ \wedge B, B) \simeq F(G_+, B) \end{array}$$

of B-modules, where B acts only on the target. So we see that $\mathrm{Hom}_A(B, B)$ is a free B-module of rank $|G|$.

Next consider the composition

$$G_+ \wedge B \wedge_A B \to \mathrm{Hom}_A(B, B) \wedge_A B \to \mathrm{Hom}_A(B, B \wedge_A B) \simeq \mathrm{Hom}_B(B \wedge_A B, B \wedge_A B)$$

Here, the first map is the map of the statement of the proposition, smashed from the right with B. The other two are the obvious ones. The composition sends (g, b_1, b_2) to $(b_1', b_2') \mapsto (g(b_1')b_1, b_2' b_2)$, informally. By assumption, we know that $B \wedge_A B$ is G-equivalent as an A-algebra to $F(G_+, B)$, so we know that the composition is an equivalence by the case of a trivial Galois extension. Thus, the first map is a split injection on homotopy groups from a free $\pi_* B$-module of rank $|G|^2$ into a free $\pi_* B$-module of the same rank.

To see that this implies that the first map is an equivalence, one must prove: If M is a free R-module (for R a commutative ring) of rank n and $i\colon M \to M$ an inclusion of a direct summand, then i is an isomorphism. Otherwise, we get a projective cokernel P, which is non-zero. We can choose a maximal ideal $\mathfrak{m} \subset R$ such that $P_{\mathfrak{m}}$ is free of positive rank. Thus, $M_{\mathfrak{m}}$ would be isomorphic to a free module of rank bigger than n over $R_{\mathfrak{m}}$, which is a contradiction as can be seen by taking exterior powers.

Since B is faithful over A, it follows that

$$G_+ \wedge B \to \mathrm{Hom}_A(B, B)$$

is an equivalence. □

Lemma 6.2.5. *The map $\widetilde{B[G]} = G_+ \wedge B \to \mathrm{Hom}_A(B, B)$ in Proposition 6.2.4 is a map (and thus an equivalence) of ring spectra, where the ring structure on the right hand side is given as composition.*

Proof. We give an informal proof, which might easily be translated into a diagrammatic proof: Given an "element" $(b_1, g_1) \wedge (b_2, g_2) \in \widetilde{B[G]} \wedge \widetilde{B[G]}$, its product is given by $(b_1 \cdot g_1(b_2), g_1g_2) \in \widetilde{B[G]}$; this is mapped to

$$(b' \mapsto (g_1g_2)(b') \cdot b_1 \cdot g_1(b_2)) \in \mathrm{Hom}_A(B, B).$$

On the other hand, the composition of the images of (b_1, g_1) and (b_2, g_2) in $\mathrm{Hom}_A(B, B)$ is given as

$$b' \mapsto g_2(b') \cdot b_2 \mapsto g_1(g_2(b')) \cdot g_1(b_2) \cdot b_1,$$

which agrees with the value above by commutativity. □

Proposition 6.2.6. *Let B be a faithful G-Galois extension of A (with G finite) which is compact as an A-module (e.g. finite). Then the model category of A-modules is Quillen equivalent to the model category of $\widetilde{B[G]}$-modules via $\wedge_A B$.*

Proof. By [SS03], 3.1.1, it is enough to show that $\mathrm{Hom}_A(B, B)$ is equivalent to $\widetilde{B[G]}$ as an A-algebra since B is a compact generator of the category of A-modules. This is shown above. □

6.3 Galois Extensions of TMF

The aim of this section is to provide examples of Galois extensions of $TMF_{(p)}$ for a prime p. Recall that we defined for a finite $TMF_{(p)}$-module X a sheaf $\mathcal{F}_X$ of spectra on $\mathcal{M}_{(p)}$ with $\mathcal{F}_X(U) \simeq X \wedge_{TMF_{(p)}} \mathcal{O}^{top}(U)$.[3] The next proposition is our first goal:

Proposition 6.3.1. *Let $h\colon \mathcal{X} \to \mathcal{M}_{(p)}$ be a G-Galois covering such that $H^i(\mathcal{X}; \mathcal{O}_{\mathcal{X}}) = 0$ for $i > 0$. Assume in addition that $X := \mathcal{O}^{top}(\mathcal{X})$ is finite, $\pi_0\mathcal{F}_X$ is a vector bundle with*

$$H^i_*(\mathcal{M}_{(p)}; \pi_0\mathcal{F}_X) = 0$$

for $i > 0$ and $\pi_0\mathcal{E}$ for $\mathcal{E} := h_*h^*\mathcal{O}^{top} = \mathcal{O}^{top}(\mathcal{X} \times_{\mathcal{M}} -)$ is a standard vector bundle.[4] Then X is a G-Galois extension of $TMF_{(p)}$.

[3] This is equivalent to the construction at the end of Section 4.5.

[4] In practice, the third condition implies often the first two.

By abuse of notation, we set $\mathcal{M} := \mathcal{M}_{(p)}$ and $TMF := TMF_{(p)}$.

Lemma 6.3.2. *In the situation of the last proposition, the map $TMF \to X^{hG}$ induced by $h^*\colon TMF = \mathcal{O}^{top}(\mathcal{M}) \to \mathcal{O}^{top}(\mathcal{X}) = X$ is an equivalence.*

Proof. We have $\mathcal{X} \times_{\mathcal{M}} \mathcal{X} \simeq \mathcal{X} \times G$ and more generally $\mathcal{X}^{\times_{\mathcal{M}} n} \simeq \mathcal{X} \times G^{n-1} \simeq \coprod_{G^{n-1}} \mathcal{X}$. Thus, $\mathcal{O}^{top}(\mathcal{X}^{\times_{\mathcal{M}} n}) \simeq F(G_+^{n-1}, X)$. By projections and diagonal maps, $\mathcal{O}^{top}(\mathcal{X}^{\times_{\mathcal{M}} \bullet})$ gets the structure of a cosimplicial object, which is equivalent to the cotensor $X^{BG_\bullet}$. As mentioned in Section 6.1, $\operatorname{holim}_{N\Delta} X^{BG_\bullet} \simeq X^{hG}$. Thus also

$$TMF = \mathcal{O}^{top}(\mathcal{M}) \simeq \operatorname{holim}_{N\Delta} \mathcal{O}^{top}(\mathcal{X}^{\times_{\mathcal{M}} \bullet}) \simeq X^{hG}.$$

□

Lemma 6.3.3. *Let $\mathcal{Y}$ be a site and $\mathcal{F}$ be sheaf of spectra on $\mathcal{Y}$; let $\mathcal{Y}_0$ be the full subsite of all $U \in \mathcal{Y}$ with $H^i(\mathcal{Y}/U; \pi_*\mathcal{F}) = 0$ for all $i > 0$ (here $\pi_*\mathcal{F}$ denotes, as always, the sheafified homotopy groups). Then we have $(\pi_*(\mathcal{F}))(U) \cong \pi_*(\mathcal{F}(U))$ for every $U \in \mathcal{Y}_0$. In particular, the presheaf of homotopy groups of $\mathcal{F}$ is already a sheaf on $\mathcal{Y}_0$ and coincides there with $\pi_*\mathcal{F}$.*

Proof. We can assume U is terminal. The descent spectral sequence

$$H^i(\mathcal{Y}; \pi_*\mathcal{F}) \Rightarrow \pi_*(\mathcal{F}(U))$$

collapses and the edge homomorphism $\pi_*(\mathcal{F}(U)) \to (\pi_*(\mathcal{F}))(U)$ is an isomorphism. □

Lemma 6.3.4. *In the situation of the proposition, the map $f\colon \mathcal{F}_X \to \mathcal{E}$ of sheaves of ring spectra induced by restriction is an equivalence.*

Proof. We want to prove that $f_*\colon \pi_*\mathcal{F}_X \to \pi_*\mathcal{E}$ is an isomorphism. It is enough to show this on π_0 since we get all other homotopy groups by tensoring with powers of ω. Since $\pi_0\mathcal{F}_X$ and $\pi_0\mathcal{E}$ are vector bundles, the kernel $\mathcal{L} := \ker(f)$ is by Proposition 3.3.1 a vector bundle again. Since $\pi_*\mathcal{F}_X$ and $\pi_*\mathcal{E}$ have vanishing higher cohomology groups, they agree with the presheaves of homotopy groups of $\mathcal{F}_X$ and $\mathcal{E}$ on a subsite containing $\mathcal{X}$ by the last lemma. Thus, we have that $\Gamma_*(\pi_0\mathcal{F}_X) \to \Gamma_*(\pi_0\mathcal{E})$ is an isomorphism induced by the identity $X \to \mathcal{O}^{top}(\mathcal{X})$. Hence, $\Gamma_*(\mathcal{L}) = 0$ since Γ_* is left exact. This implies $\mathcal{L} = 0$ by Proposition 3.3.2.

Thus we get a short exact sequence

$$0 \to \pi_0\mathcal{F}_X \xrightarrow{f_*} \pi_0\mathcal{E} \to \mathcal{G} \to 0$$

where $\mathcal{G} = \operatorname{coker}(f_*)$. This induces a short exact sequence

$$0 \to \Gamma_*(\mathcal{F}_X) \to \Gamma_*(\mathcal{E}) \to \Gamma_*(\mathcal{G}) \to 0$$

since $\pi_0\mathcal{F}_X$ has vanishing graded cohomology. Therefore, $\operatorname{Hom}(\omega^{-*}, \mathcal{G}) = \Gamma_*(\mathcal{G}) = 0$. But this implies inductively that every morphism from a standard vector bundle to $\mathcal{G}$ vanishes. This shows that $\mathcal{G}$ itself is zero since $\pi_0\mathcal{E}$ is standard. Therefore, the map f_* is an isomorphism and hence f is an equivalence of $\mathcal{O}$-modules. □

Proof of Proposition: We only need to show that the map $X \wedge_{TMF} X \to F(G_+, X)$ given (informally) by $(x \wedge x', g) \mapsto g(x) \cdot x'$ is an equivalence. The map $\gamma\colon G \times \mathcal{X} \to \mathcal{X} \times_{\mathcal{M}} \mathcal{X}$ given by $(g, x) \mapsto (gx, x)$ is an equivalence, hence also the induced map $\mathcal{O}^{top}(\mathcal{X} \times_M \mathcal{X}) \to \mathcal{O}^{top}(G \times \mathcal{X}) \simeq \mathbb{F}(G_+, \mathcal{O}^{top}(\mathcal{X}))$. If we precompose this map with

$$f(\mathcal{X})\colon\ X \wedge_{TMF} X\mathcal{O}^{top}(\mathcal{X}) \wedge_{TMF} \mathcal{O}^{top}(\mathcal{X}) \to \mathcal{O}^{top}(\mathcal{X} \times_{\mathcal{M}} \mathcal{X})$$

of the last lemma, it coincides with γ. □

Example 6.3.5. We set $\mathcal{X} = \mathcal{M}(2)$ at $p = 3$. Since $\mathcal{M}(2)$ has an affine $\mathbb{G}_m$-torsor, its cohomology vanishes. By the results in Section 5.2, $TMF(2)$ is a finite $TMF_{(3)}$-module of the form $TMF_{\alpha,\widetilde{\alpha}} \vee \Sigma^4 TMF_{\alpha,\widetilde{\alpha}}$.

Thus, $\pi_*\mathcal{F}_{TMF(2)} \cong p_*p^*\pi_*\mathcal{O}^{top}$ (here, p denotes the projection $\mathcal{M}(2)_{(3)} \to \mathcal{M}$ and should not be confused with the prime we are working at). Furthermore, $p_*p^*\mathcal{O} \cong f_*f^*\mathcal{O} \oplus f_*f^*\mathcal{O} \otimes \omega^2$ is standard as proven in Section 3.5, in particular Lemma 3.5.4), and therefore the conditions of the proposition are fulfilled. We can conclude that $TMF(2)$ is an S_3-Galois extension of TMF at the prime 3. The same argument works for every $p > 2$.

We could also replace $p\colon \mathcal{M}(2) \to \mathcal{M}$ by $q\colon \mathcal{M}(4) \to \mathcal{M}$ since by the remarks at the end of Section 5.2, $TMF(4)$ is a sum of 8 copies of $TMF(2)$ (and, hence, $TMF(4)$ is finite and $\pi_*\mathcal{F}_{TMF(4)}$ has vanishing higher cohomology) and, by Lemma 3.5.5, we have also that $q_*q^*\mathcal{O}$ is a sum of 8 copies of $p_*p^*\mathcal{O}$, hence it is standard.

Remark 6.3.6. We suspect that at $p = 2$ and for $q\colon \mathcal{X} = \mathcal{M}(3) \to \mathcal{M}$, we also get that X is a finite $TMF_{(2)}$-module and $\mathcal{F}_X \simeq q_*q^*\mathcal{O}^{top}$. Indeed, we suspect that $X = TMF \wedge Y$, where Y is closely related to the complex C_γ of the proof of 5.4.5 in [Rog08] (which has the property that $eo_2 \wedge C_\gamma \simeq BP\langle 2\rangle$), perhaps Y consists just of six copies of C_γ. Evidence is provided by the paper [MR09] by Mahowald and Rezk, where they show that $\mathcal{M}_1(3) = \operatorname{Spec} \mathbb{Z}_{(2)}[a_1, a_3, \Delta^{-1}]//\mathbb{G}_m$ ([MR09, 3.2]) and that $TMF_0(3)$ and $TMF_1(3)$ are finite $TMF_{(2)}$-modules ([MR09, 7.2] and [MR09, 4.2]).

We want to give a little application of Galois descent, already used in the section about relatively free modules.

Lemma 6.3.7. *Let M be a finite TMF-module. Then there exists a map*

$$X\colon = \oplus_{j\in J}\Sigma^{n_j} TMF(2) \to M$$

which induces a surjection $\pi_*(X \wedge_{TMF} TMF(2)) \to \pi_*(M \wedge_{TMF} TMF(2))$.

Proof. TMF-modules are equivalent to $\widetilde{TMF(2)}[S_3]$-modules via the functor $\wedge_{TMF} TMF(2)$ by Proposition 6.2.6 and Example 6.3.5. Then $TMF(2)$ corresponds to $\widetilde{TMF(2)}[S_3]$ and we can simply realize the algebraic map. □

6.4 Intermezzo on the Descent Spectral Sequence

In this section, we will use the results of the last section to give an alternative account of the descent spectral sequence for sheaves $\mathcal{F}_M$ associated to finite $TMF_{(3)}$-modules M, which we will be our main case of interest. We use the notation $DSS(\mathcal{F}_M)$ or just $DSS(M)$ for the descent spectral sequence.

The following theorem seems to be known to experts:

Theorem 6.4.1. *The descent spectral sequence associated to a finite TMF-module agrees with its Adams–Novikov spectral sequence (based on MU).*

Since there is no published proof for this theorem (and I also haven't seen a unpublished one), I will present an approach circumventing this theorem.

By abuse of notation, we set again $TMF = TMF_{(3)}$. We will use the Adams spectral sequence in TMF-modules with respect to the TMF-algebra $TMF(2)$ as a model for the descent spectral sequence. To study this, let's begin with a few generalities on the Adams spectral sequence in R-modules (we follow here [BL01]).

Let R be a commutative ring spectrum and E be a homotopy commutative R-algebra. Assume that $E^R_* E = \pi_*(E \wedge_R E)$ is flat as an E_*-module. Then by [BL01], 1.1 and 2.1, $E^R_* E$ is a Hopf algebroid and we have for M an R-module an Adams spectral sequence with E^2-term

$$\mathrm{Ext}^{s,t}_{E^R_* E}(E_*, E^R_* M).$$

It converges (if the pages stabilize in every bidegree) to the completion $\pi_* \hat{L}^R_E(M)$. To define the latter, we consider the canonical R-module Adams resolution: Set $D_0 = M$ and let D_{s+1} be the (homotopy) fiber of $D_s \cong R \wedge_R D_s \to E \wedge_R D_s$. Furthermore let K_s be the cofiber of the map $D_s \to D_0 = M$. The maps $D_{s+1} \to D_s$ induce maps $K_{s+1} \to K_s$ and we set $\hat{L}^R_E(M) := \mathrm{holim}_s K_s$. By [Rog08, Lemmas 8.2.3 and 8.2.4], $\hat{L}^R_E(R) \simeq R$ if E is faithful and dualizable as an R-module. Since both the canonical Adams resolution and homotopy limits commute with smashing with a finite module, this implies $\hat{L}^R_E(M) \simeq M$.

For example, $TMF(2)$ is faithful and dualizable over TMF. Furthermore, by Example 6.3.5, $TMF(2) \wedge_{TMF} TMF(2) \simeq \widetilde{TMF(2)}[S_3]$, the twisted group ring. Therefore, in this case, $E^R_* E$-comodules correspond to graded $TMF(2)_*$-modules with twisted S_3-action, which is by Galois descent the same as quasi-coherent sheaves over the moduli stack of elliptic curves. Therefore,

$$\mathrm{Ext}^{s,t}_{E^R_* E}(E_*, E^R_* M) \cong \mathrm{Ext}^{s,t}_{\mathrm{QCoh}(\mathcal{M})}(\mathcal{O}, \pi_* \mathcal{F}_M)$$

and our $TMF(2)$-based Adams spectral sequence has the same E^2-term and the same convergence properties as the descent spectral sequence and we can use it as a substitute. Note also that (due to the maps $S^0 \to TMF$ and $MU \to TMF(2)$) we have a map of spectral sequence from the Adams–Novikov spectral sequence to our version of the descent spectral sequence as required in Tilman Bauer's paper [Bau08].

Theorem 6.4.2. *Let M be a finite TMF-module. Then $DSS(\mathcal{F}_M)$ possesses the structure of a module spectral sequence over $DSS(\mathcal{O}^{top})$ which induces the canonical module action of the E^2-terms.*

Proof. This is analogous to a special case of [Rav86], 2.3.3, in our case just replacing arguments in spectra by arguments in TMF-modules. □

Theorem 6.4.3. *Let*

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} \Sigma W$$

be a cofiber sequence of finite TMF-modules. Assume that the induced map $h\colon \pi_* \mathcal{F}_Y \to \pi_{*-1} \mathcal{F}_W$ is zero (π_* denotes here again the sheafified homotopy groups). Then we have a map of spectral *sequences* $DSS(\mathcal{F}_Y) \to DSS(\mathcal{F}_W)$ *(raising filtration by* 1*) which induces multiplication by the class in* $\mathrm{Ext}^1_{\pi_*\mathcal{O}^{top}}(\pi_*\mathcal{F}_Y, \pi_*\mathcal{F}_W)$ *corresponding to the extension*

$$0 \to \pi_*\mathcal{F}_W \to \pi_*\mathcal{F}_X \to \pi_*\mathcal{F}_Y \to 0$$

on E^2.

Proof. This is analogous to [Rav86], 2.3.4, in our case just replacing arguments in spectra by arguments in TMF-modules. □

Corollary 6.4.4. *Let* $x \in \pi_k M$ *be of DSS-filtration* 1*. Then the cone N of the map* $\Sigma^k TMF \xrightarrow{x} M$ *satisfies that the extension*

$$0 \to \pi_*\mathcal{F}_M \to \pi_*\mathcal{F}_N \to \pi_*\Sigma^k\mathcal{O}^{top} \to 0$$

is classified by the reduction $\overline{x} \in \mathrm{Ext}^1_{\pi_*\mathcal{O}^{top}}(\pi_*\Sigma^k\mathcal{O}^{top}, \pi_*\mathcal{F}_M) \cong H^1(\mathcal{M}; \pi_k\mathcal{F}_M)$.

Proof. The map $\Sigma^k TMF \to M$ sends $1 \in \pi_k\Sigma^k TMF$ to $x \in \pi_k M$. It sends also $\overline{1} \in H^0(\mathcal{M}; \pi_*\Sigma^k\mathcal{O}^{top})$ to $y \in \mathrm{Ext}^1_{\pi_*\mathcal{O}^{top}}(\pi_*\Sigma^k\mathcal{O}^{top}, \pi_*\mathcal{F}_M) \cong H^1(\mathcal{M}; \pi_k\mathcal{F}_M)$ classifying the extension above. Thus, by Theorem 6.4.3, $\overline{x} = y$. □

6.5 Galois Descent, the Second

In this section, we want to prove a version of Galois descent in derived algebraic geometry and will use it to investigate the relationship between quasi-coherent modules on the derived moduli stack of elliptic curves and TMF-modules. We will work again in the ∞-categorical setting.

Definition 6.5.1. Let $\mathcal{F}\colon \mathcal{C}^{op} \to \mathcal{D}$ be a contravariant functor from (the underlying category of) a site to an ∞-category with all limits. For a finite group G, we call a morphism $f\colon X \to Y$ in $\mathcal{C}$ a *G-torsor* if f is a cover and X is equipped with a G-action over Y such that $\coprod_G X \cong G \times X \to X \times_Y X$ (given, informally, by $(g,x) \mapsto (gx, x)$) is an isomorphism. We say that $\mathcal{F}$ *satisfies Galois descent* with respect to G if for any G-torsor $X \to Y$, we have that $\mathcal{F}(Y) \to \mathcal{F}(X)^{hG}$ is an equivalence.

Proposition 6.5.2. *If* $\mathcal{F}$ *is a sheaf on* $\mathcal{C}$ *(with values in an* ∞*-category* $\mathcal{D}$*), then it satisfies Galois descent with respect to every finite group G.*

Proof. We have for $X \to Y$ a G-torsor an equivalence

$$\mathcal{F}(Y) \longrightarrow \operatorname{holim}\left(\mathcal{F}(X) \rightrightarrows \mathcal{F}(X \times_Y X) \mathrel{\substack{\to\\[-2pt]\to\\[-2pt]\to}} \cdots\right)$$

since $X \to Y$ is in particular a cover. Since $X^{\times_Y n} \cong \coprod_{G^{n-1}} X$, we get an equivalence of the cosimplicial object above with a cosimplicial object $Z^\bullet$ in $\mathcal{D}$ satisfying $Z^n \simeq \prod_{G^{n-1}} \mathcal{F}(X)$. Just as in Lemma 6.3.2, we see that $Z^\bullet \simeq X^{BG_\bullet}$ and that we get therefore an equivalence

$$\mathcal{F}(Y) \to \mathcal{F}(X)^{hG}.$$

□

Proposition 6.5.3. *Let $(\mathcal{X}, \mathcal{O}^{top})$ be a derived Deligne–Mumford stack with fiber functor $P\colon \mathcal{X} \to \mathrm{Sch}$. Let $\mathcal{X}'$ be the site of all Deligne–Mumford stacks over $(\mathcal{X}, \pi_0\mathcal{O}^{top})$ and let*

$$P'\colon \mathcal{X}' \to \textit{Deligne–Mumford stacks}$$

denote the obvious extension of this fiber functor. Then the assignment

$$X \mapsto \mathrm{QCoh}(P'(X), \mathcal{O}^{top}|_X)$$

is a sheaf on $\mathcal{X}'$ and satisfies thus Galois descent (with respect to every group).

Proof. For every $X \in \mathcal{X}'$ we write h_X for sheaf represented by X in $\mathrm{Shv}_{\mathrm{sSet}}(\mathcal{X})$. The following is shown in the proof of Proposition 2.3.13 of DAG VIII: If $h_X \simeq \mathrm{colim}_i\, h_{X_i}$ in $\mathrm{Shv}_{\mathrm{sSet}}(\mathcal{X})$ for $X_i \in \mathcal{X}'$, then $\mathrm{QCoh}(P'(X), \mathcal{O}^{top}|_X) \simeq \lim_i \mathrm{QCoh}(P'(X_i), \mathcal{O}^{top}|_{X_i})$.

Since a covering $\{X_i\} \to X$ implies an equivalence $h_X \simeq \mathrm{colim}_i\, h_{X_i}$, the assignment $X \to \mathrm{QCoh}(P'(X), \mathcal{O}^{top}|_X)$ defines a sheaf on $\mathcal{X}'$. □

Theorem 6.5.4. *For $l > 2$ a prime, we have an equivalence*

$$\mathrm{QCoh}(\mathcal{M}_{(l)}, \mathcal{O}^{top}) \simeq TMF_{(l)}\text{-mod}$$

of ∞-categories.

Proof. Let $q\colon \mathcal{M}(4)_{(l)} \to \mathcal{M}_{(l)}$ be the $G = GL_2(\mathbb{Z}/4)$-Galois covering considered before. Recall that $\mathcal{M}(4)_{(l)}$ is affine (see Section 2.5). Thus,

$$\begin{aligned}
\mathrm{QCoh}(\mathcal{M}_{(l)}, \mathcal{O}^{top}) &\overset{6.5.3}{\simeq} (\mathrm{QCoh}(\mathcal{M}(4)_{(l)}, \mathcal{O}^{top}))^{hG} \\
&\overset{4.5.11}{\simeq} (TMF(4)_{(l)}\text{-mod})^{hG} \\
&\overset{6.1.5}{\simeq} \widetilde{TMF(4)_{(l)}}[G]\text{-mod} \\
&\overset{6.3.5,6.2.6}{\simeq} TMF_{(l)}\text{-mod}
\end{aligned}$$

□

Comment 6.5.5. The reader might have noticed that we proved the last theorem by going up and down via two different kinds of Galois descent. The latter was of an algebraic flavor as it was induced by a Galois covering in classical algebraic geometry. The former though was based on a deeper, topological notion as the map $TMF \to TMF(2)$ is no Galois extension on homotopy groups. In a similar vein, one might compare the discussion of étale morphism between ring spectra in Chapter 9 of [Rog08] with the definition of Lurie in [Lur11], Definitions 8.5.0.4 and 8.2.2.10. It goes without saying that this is not meant to be derogatory in any way with respect to Lurie's treatment.

Chapter 7

The Case of K-Theory

In this chapter, we want to classify (finite) KO-modules which are relatively free with respect to KU. After collecting some basic facts on K-theory in the first section, we prove (in three different ways) that all relatively free KO-modules are standard. In the third section, we classify all standard modules. We want to stress again that the results here are essentially due to [Bou90], but are proven here in a different way.

7.1 Basics on K-theory

We want to collect some basics on real and complex K-theory in this section. Most of these results can be found in [Bou90, Section 1].

Denote by $B_C \in \pi_2 KU$ the Bott periodicity element. Then $\pi_* KU \cong \mathbb{Z}[B_C^{\pm 1}]$.

The homotopy groups of KO are 8-periodic via the Bott periodicity element $B_R \in \pi_8 KO$. We have

$$\pi_i KO \cong \begin{cases} \mathbb{Z}/2 & \text{for } i \equiv 1,2 \mod 8 \\ \mathbb{Z} & \text{for } i \equiv 0,4 \mod 8 \\ 0 & \text{else.} \end{cases}$$

Degree 0 is generated by the unit 1 and we choose a generator ξ of $\pi_4 KO$. The Hurewicz image of $\eta \in \pi_1^{st} S^0$ in $\pi_1 KO$ is non-zero and, by abuse of notation, we denote it also by η. We have $\eta^2 \neq 0$, so η^2 generates $\pi_2 KO$.

We have (geometrically defined) maps $c\colon KO \to KU$ and $r\colon KU \to KO$, *complexification* and *realification*. The first is a morphism of ring spectra and gives KU the structure of a KO-module. Complex conjugation induces an involution τ on KU, which acts as a KO-algebra map. We have $cr = \mathrm{id} + \tau$ and $rc = 2$. Complex conjugation satisfies furthermore $\tau(B_C) = -B_C$ in $\pi_* KU$. If we view B_C as an equivalence $\Sigma^2 KU \to KU$, the map $B_C \tau B_C^{-1}$ sends $1 \in \pi_0 KU$ to -1 and $B_C \in \pi_2 KU$ to B_C. By Example 6.2.2 and Proposition 6.2.4, we have a basis consisting of 1 and τ of $[KU, KU]_{KO}$. Thus $B_C \tau B_C^{-1} = -\tau$.

Since η acts trivially on KU, we can extend c to a map $KO \wedge C\eta \to KU$, which can be chosen to be an equivalence.[1] More precisely, we get a triangle

$$\Sigma KO \xrightarrow{\eta} KO \xrightarrow{c} KU \xrightarrow{\pm r B_C^{-1}} \Sigma^2 KO.$$

[1] It is hard to find a complete proof for this statement in the literature. A short discussion can be found in [Rog08, p.23]. In [Ati66, Proposition 3.2], there is a proof showing that KU and $KO \wedge C\eta$ represent the same cohomology theories (on spaces).

of KO-modules. Indeed, $D_{KO}KU \simeq D_{KO}KO \wedge C\eta \simeq \Sigma^{-2}KU$. Thus,

$$[KU, \Sigma^2 KO]_{KO} \cong \pi_{-2} D_{KO} KU \cong \pi_0 KU \cong \mathbb{Z}.$$

The element $rB_C^{-1} \in [KU, \Sigma^2 KO]_{KO}$ is indivisible (by any natural number > 1) as $r(B_C^2) = \xi$ since $cr(B_C^2) = 2B_C^2$ and $c(\xi) = 2B_C^2$. Since the boundary map $KU \to \Sigma^2 KO$ is also indivisible (it has also ξ in its image since $\eta\xi = 0$), it has to be equal to $\pm rB_C^{-1}$.

For an $M \in KO$-mod, we set $M_{KU} := M \wedge_{KO} KU$. By abuse of notation, we denote the maps $M \to M_{KU}$ and $M_{KU} \to M$ induced by c and r also by c and r. By smashing the above triangle with M, we get a triangle

$$\Sigma M \xrightarrow{\eta} M \xrightarrow{c} M_{KU} \xrightarrow{\pm rB_C^{-1}} \Sigma^2 M,$$

which induces a long exact sequence

$$\cdots \to \pi_{*-1} M \xrightarrow{\eta} \pi_* M \xrightarrow{c_*} \pi_* M_{KU} \xrightarrow{\rho} \pi_{*-2} M \to \cdots \tag{7.1}$$

for $\rho = (rB_C^{-1})_*$. Observe that $B_C c_* \rho = \mathrm{id} - \tau$.

An important variant of K-theory is K-theory with self-conjugation KT. While it has also a geometric interpretation, for our purposes, we can define it as the KO-module $KO \wedge C(\eta^2)$. We have

$$\pi_i KT \cong \begin{cases} \mathbb{Z}/2 & \text{for } i \equiv 1 \mod 4 \\ \mathbb{Z} & \text{for } i \equiv 0, 3 \mod 4 \\ 0 & \text{else.} \end{cases}$$

7.2 The KO-Extension Theorem

The aim of this section is to prove the following proposition:

Proposition 7.2.1. *Let M be a nonzero finite KO-module such that M_{KU} is KU-free. Then there is a map $f\colon \Sigma^j KO \to M$ such that the map*

$$(f \wedge_{KO} KU)_*\colon \pi_* \Sigma^j KU \to \pi_* M_{KU}$$

is split injective (equivalently as map of abelian groups in every degree or as map of $\pi_* KU$-modules).

Remark 7.2.2. Since maps between free modules are determined by their effect on homotopy groups, in $f \wedge_{KO} KU\colon \Sigma^j KU \to M_{KU}$ splits for M_{KU} free iff $(f \wedge_{KO} KU)_*\colon \pi_* \Sigma^j KU \to \pi_* M_{KU}$ splits.

Corollary 7.2.3. *Every relatively free (finite) KO-module M is a standard module.*[2]

Proof. For a relatively free M, the dual $D_{KO}M$ is also relatively free since

$$\mathrm{Hom}_{KO}(M, KO) \wedge_{KO} KU \simeq \mathrm{Hom}_{KU}(M_{KU}, KU)$$

by Proposition 4.2.7. Thus, using the proposition, we can choose an $f\colon \Sigma^j KO \to D_{KO}M$ splitting off a direct summand after smashing with KU and call the Spanier–Whitehead

[2]Recall from the induction that a standard module is a KO-module which arises by iteratively coning off torsion elements from a suspension of KO.

dual of the cofiber N. Note that this is relatively free of one rank less than M (since the map $f_{KU}\colon \Sigma^j KU \to D_{KO}M \wedge_{KO} KU$ splits).

After dualizing f, we get a cofiber sequence $\Sigma^{-j-1}KO \xrightarrow{g} N \to M \to \Sigma^{-j}KO$. As the dual of a split map $M_{KU} \to \Sigma^{-j}KU$ has a section. Thus, $g_{KU}\colon \Sigma^{-j-1}KU \to N_{KU}$ is zero. Therefore, the corresponding element $x = g(1) \in \pi_{-j-1}N$ satisfies $c_*(x) = 0$. Hence, x is in the image of η and therefore torsion.

All in all, we get that we can obtain M from a relatively free module of smaller rank by coning off a torsion element. Now, we can assume inductively that every relatively free module of smaller rank than M (e.g., N) is standard and get that M is standard. Note that we use as an induction start that $M_{KU} = 0$ implies $M = 0$. Indeed, $M_{KU} = 0$ implies that $\eta\colon \Sigma M \to M$ is an isomorphism of KO-modules, but $\eta^3 = 0$. □

Remark 7.2.4. We will give three proofs of this proposition. The first two proofs use the homotopy fixed point spectral sequence as their main tool, the third a Toda bracket argument. Toda bracket arguments will come up again in Section 8.6 and descent spectral sequence arguments are central to the whole proof of Theorem 8.1.5. It might be helpful for the reader to keep the easier analogues from this section in mind. The third argument was actually the first proof of this proposition I came up with and has motivated the earlier parts of my attempts to prove Theorem 8.1.5, especially the search for divisibility by large powers of β.

Proof. We start with a few observations which are important for all three proofs.

- It is enough to find an indivisible element $e \in \pi_* M_{KU}$ which is in the image of c_* (since every indivisible element in a free abelian group generates a direct summand). Here e is called indivisible if $k \cdot x = e$ for $k \in \mathbb{Z}$ implies $k = \pm 1$.

- Every torsion element in $\pi_* M$ is in the image of η and thus 2-torsion. Thus, for k odd, $k \cdot x$ is in $\operatorname{im}(c_*)$ iff $x \in \operatorname{im}(c_*)$. Therefore, it suffices to find an element in $\operatorname{im}(c_*)$ which is not divisible by 2 in $\pi_* M_{KU}$.

- Suppose $\operatorname{im}\rho$ is torsionfree in every degree. Then $\rho(kx) = k\rho(x) = 0$ implies that $\rho(x) = 0$. Therefore, $kx \in \operatorname{im} c_*$ implies $x \in c_*$. Therefore, either $c_* = 0$ or $\operatorname{im} c_*$ contains an indivisible element. But if $c_* = 0$, the whole module $\pi_* M$ is contained in the image of η and is therefore completely torsion. This implies $\rho = 0$ and every element in $\pi_* M_{KU}$ is in $\operatorname{im}(c_*)$, hence $M_{KU} = 0$ (implying $M = 0$), which is a contradiction. Hence, $\operatorname{im}(\rho)$ has 2-torsion.

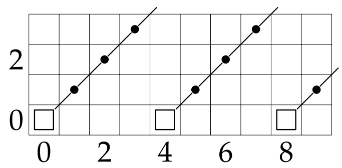
First proof: Since $KU^{hC_2} \simeq KO$ (see [Rog08, 5.3]), we have also $(M_{KU})^{hC_2} \simeq M$ (since homotopy limits commute with smashing with finite modules). By Theorem 6.1.6, there is a homotopy fixed point spectral sequence

$$E_2^{**}(M) = H^*(C_2, \pi_* M_{KU}) \Rightarrow \pi_* M,$$

which is a module spectral sequence over the homotopy fixed points spectral sequence for KU.

Claim 7.2.5. *The (non-trivial) permanent cycle $\eta \in H^1(C_2, \pi_2 KU)$ in the E_2-term of the homotopy fixed point spectral sequence of KU acts injectively on the E_r-term for M_{KU} beginning with the $(r-1)$st row.*

Proof. The groups $H^i(C_2, \mathbb{Z}[C_2] \otimes KU_*)$ vanish for $i > 0$ and $H^i(C_2, KU_*)$ looks in Adams convention as follows:



The diagonal strokes stand here for multiplication by $\eta \in H^1(C_2, \pi_2 KU)$. Furthermore, the pattern continues to the left, right and top. Thus, η operates injectively on $H^i(C_2, KU_*)$ for $i > 0$. Now, by Section 3.1, every integral C_2-representation is isomorphic to a sum of copies of $\mathbb{Z}[C_2]$, $\mathbb{Z}$ and the sign representation $\mathbb{Z}'$. As KU_* is in degrees divisible by 4 isomorphic to $\mathbb{Z}$ and in the other even degrees isomorphic to $\mathbb{Z}$, every finite-dimensional free graded KU_*-module with twisted C_2-action is isomorphic to a sum of shifts of copies of $\mathbb{Z}[C_2] \otimes KU_*$ and KU_*. Thus, the result follows for $r = 2$. Inductively, one sees that η operates injectively on E_r beginning with the $(r-1)$st row (similar to Lemma 8.3.5). □

The edge homomorphism $\pi_*M \to H^0(C_2, \pi_*M_{KU}) \subset \pi_*M_{KU}$ equals c_*. Assume that there is no indivisible element in $\mathrm{im}(c_*)$.[3] Thus, all indivisible elements in row 0 of the homotopy fixed point spectral sequence must support differentials. Hence, every element in the higher rows must support a (non-trivial) differential since they are all multiples by a power of η of row 0 elements and η operates injectively on the E_r-term beginning with the $(r-1)$st row. Hence π_*M has no torsion, which is a contradiction to the third observation.

Second proof: Let $x \in \pi_*M$ be a torsion element. Every torsion element is divisible by η since its image in π_*M_{KU} is torsion, hence zero. Therefore, we can write $x = \eta^k y$, for y non-torsion and $k \in \{1, 2\}$ maximal (since $\eta^3 = 0$). Thus, y is detected in the 0-line of the homotopy fixed point spectral sequence[4] and we assume (for contradiction) it reduces to an element in $\overline{y} \in 2H^0(C_2; \pi_*M_{KU})$.

Viewing M_{KU} as a KO-module, we get a homotopy fixed point spectral sequence computing π_*M_{KU} (out of $H^*(C_2; \pi_*(M_{KU} \wedge_{KO} KU))$), which is concentrated in the 0-line since $M_{KU} \wedge_{KO} KU \simeq \overbrace{KU[C_2]}^{n}$ for $n = \mathrm{rk}_{KU} M_{KU}$ (by Example 6.2.2). The map $r\colon M_{KU} \to M$ induces a map of spectral sequences, which equals in the 0-line the map

$$\pi_*M_{KU} \to H^0(C_2; \pi_*M_{KU}) \subset \pi_*M_{KU}$$

given by $x \mapsto x + \tau x$ (since $cr = 1 + \tau$). Clearly, $2H^0(C_2; \pi_*M_{KU})$ is in the image. Thus, there is a $y' \in \mathrm{im}(r_*) \subset \pi_*M$ such that $y - y'$ is of first filtration and $\eta y' = 0$. Hence, $\eta^k(y - y') = x$ and $y - y'$ is torsion, which is a contradiction to the maximality of k.

Therefore, y projects non-trivially to $H^0(C_2, \pi_*M_{KU})/2$. The edge morphism

$$\pi_*M \to H^0(C_2, \pi_*M_{KU}) \subset \pi_*M_{KU}$$

converges to c_*. Thus, $c(y)$ is not divisible by 2 and we can assume it generates a direct summand of π_*M_{KU}.

[3] Equivalently, indivisible in $H^0(C_2; \pi_*M_{KU})$ and in π_*M_{KU}.

[4] Indeed, else y was detected by an element in a higher row, i.e., a torsion element. This shows that $2^i y$ has arbitrary high filtration. This shows that there is a surjection $\pi_*M \to \mathbb{Z}_2$. But π_*M is finitely generated, contradicting the uncountability of $\mathbb{Z}_2$.

Third proof: We assume that every element in $\mathrm{im}(c_*)$ is divisible by 2.

By the third observation above, there is an $x \in \pi_n M_{KU}$ such that $2\rho(x) = 0$, but $\rho(x) \neq 0$. Our first goal is to show that $\rho(x)$ is of the form $\eta^2 e$. By the exactness of (7.1), we have $\rho(x) = \eta \cdot y$, for a $y \in \pi_{n-3}M$. Since clearly $2\eta y = 0$, there is a $z \in \pi_{n-1}M_{KU}$ with $\rho(z) = 2y$.

Assume $\pi_* M_{KU}$ had a (C_2-equivariant) summand of the form $\mathbb{Z}[C_2] \otimes KU_*$. Then $c_* r_*(1,0) = (1,1)$ in this summand (with respect to the basis $(1,t)$ of $\mathbb{Z}[C_2]$) and therefore, there would be an indivisible element in the image of c_*. Therefore, we can assume that $\pi_* M_{KU}$ has no such summand and, by the classification of integral C_2-representations (see Section 3.1), $\pi_* M_{KU}$ is a sum of trivial and sign representations. Hence, we can write $z = u + v$ with $\tau u = u$ and $\tau v = -v$. We have

$$2B_c c_* y = B_c c_* \rho z = B_C c_* r_* B_C^{-1}(z) = z - \tau z = 2v$$

and therefore $c_*(y) = B_C^{-1} v$ since $\pi_* M_{KU}$ is torsionfree. By our contradiction assumption, this must be divisible by 2 and we can write $c_* y = 2B_C^{-1} w$. Thus, we have $v = 2w$.

We have now that

$$\rho(u) = \rho(z - v) = \rho(z) - \rho(v) = 2y - 2\rho(w).$$

Therefore, $\rho(u)$ is divisible by 2. But $\rho(u)$ is also torsion since $c_* \rho(u) = (1+\tau)B_C^{-1}(u) = B_C^{-1}(u - \tau(u)) = 0$. This implies $\rho(u) = 0$ (since all torsion is 2-torsion). Hence, $2\rho(w) = 2y$ and thus $\rho(w) = y + d$, where d is 2-torsion. Now we have $\eta y + \eta d = \eta \rho(w) = 0$ and therefore $\rho(x) = \eta y = \eta d$. But since d is torsion, it is in the image of η: $d = \eta e$. Therefore, $\rho(x)$ is of the form $\eta^2 e$, which was our first goal.

Recall that the Toda bracket $\langle \eta, \eta^2, 2 \rangle$ equals $\xi + 2\mathbb{Z} \cdot \xi \subset \pi_4 KO$.[5] By Lemma 4.6.2, we have $\pm \xi e \in \langle \eta, \rho(x), 2 \rangle$. Thus, $c_*(\xi e) = \pm 2x'$ for an x' with $\rho(x') = \rho(x)$ by Lemma 4.6.1. Since c_* is a KO_*-module map and $c_*(\xi) = 2B_C^2$, we have $2B_C^2 c_*(e) = \pm 2x'$ or with other words: $c_*(e) = \pm B_C^{-2} x'$. Since x' is not divisible by 2 (else $\rho(x) = \rho(x')$ would be divisible by 2), $c_*(e)$ is not divisible by 2, which proves the proposition.[6]

□

7.3 Classification of *KO*-Standard Modules

Our goal in this section is the classification of relatively free *KO*-modules, recovering a result by Bousfield.

Theorem 7.3.1. *Every (finite) relatively free KO-module is a direct sum of shifts of KO, KU and KT.*

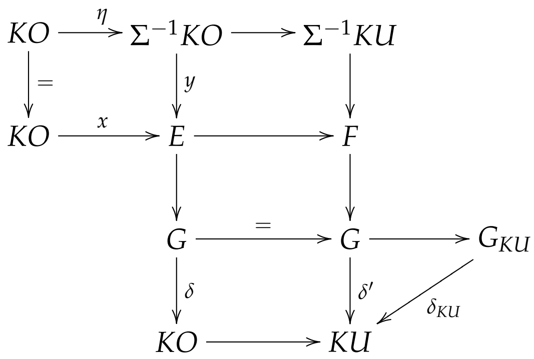
Proof. We know by the last section that every relatively free *KO*-module is a standard module. Call a module that can be written as a direct sum of shifts of *KO*, *KU* and *KT* *very standard*. We will assume for induction that all standard modules of rank $< n$ are very standard.[7]

[5] This can be shown by a straightforward computation with Massey products in the E_2-term of the Adams spectral sequence for *KO*.

[6] Note that one does not really need the computation of $\langle \eta, \eta^2, 2 \rangle$ – if it contained zero, the argument would have been even simpler.

[7] Here, the rank of a relatively free module M is defined to be the rank of $\pi_* M_{KU}$ as a $\pi_* KU$-module.

By Corollary 7.2.3, every relatively free module F of rank $n > 0$ sits in an exact triangle of the form $KO \to E \to F$ with $\operatorname{rk} E = n - 1$ and $KO \to E$ corresponding to a torsion-element $x \in \pi_* E$. In general, one has to consider a suspension of KO, but one can just shift. We can assume x to be non-zero. Every torsion element in $\pi_* E$ is divisible by η and we choose a $y \in \pi_* E$ with $\eta y = x$. Then we have by the octahedral axiom a diagram of the form



where the two columns and the upper two rows are triangles. Assume first that x is not divisible by η^2. As in the (second) proof of Proposition 7.2.1, we can choose y in a way such that $c(y)$ is a primitive vector in $\pi_* E_{KU} \cong \mathbb{Z}^?$. Therefore, the map $\Sigma^{-1}KU \xrightarrow{c_*(y)} E_{KU}$ has a section and G_{KU} a direct summand of E_{KU} of rank $n-2$ (and therefore very standard by induction). In particular, $\delta_{KU}\colon G_{KU} \to KU$ must be zero (since it is zero on homotopy groups and the source is a free module). Since $\delta'\colon G \to KU$ factors over δ_{KU}, it has also to be zero. Therefore, $F \cong G \oplus \Sigma^{-1}KU$.

If x is divisible by η^2, we can assume $E \cong \bigoplus \Sigma^{-2}KO$ since only in these summands there is a π_0-element divisible by η. Thus, $\pi_0 E \cong \mathbb{F}_2^k$ and we can lift $x \in \mathbb{F}_2^k$ to a primitive vector x' in $\mathbb{Z}^k$. We can choose a matrix $A \in GL_k(\mathbb{Z})$ with x' as first column. Its inverse defines an automorphism of E sending x to $(\eta^2, 0, \dots, 0)$. After this change of basis, it is immediate that $F \cong \Sigma^{-2}KT \oplus \bigoplus \Sigma^{-2}KO$. □

Chapter 8

Relatively Free TMF-Modules

In this chapter, we will investigate the relationship between various sub classes of relatively free TMF-modules, namely standard, hook-standard and algebraically standard TMF-modules. Everything will be implicitly 3-local; this means, for example, that we write TMF for $TMF_{(3)}$ and $\mathcal{M}$ for $\mathcal{M}_{(3)}$.

8.1 Definitions, Observations and Statement of Results

Let M be a finite TMF-module such that $M(2) := M \wedge_{TMF} TMF(2)$ is free of rank n as a $TMF(2)$-module (i.e. a *relatively free* TMF-module). By abuse of terminology, we will often call n also the *rank of* M. As before, we can associate to M a quasi-coherent $\pi_*\mathcal{O}^{top}$-module $\pi_*\mathcal{F}_M$ on $\mathcal{M}$ with $\mathcal{F}_M(U) \simeq \mathcal{O}^{top}(U) \wedge_{TMF} M$ for U a stack with an étale map to $\mathcal{M}$ (see the end of Section 4.5). If M is relatively free, this is a vector bundle, as can be seen by evaluating on $\mathcal{M}(2)$.

Definition 8.1.1. A finite TMF-module M is *algebraically standard* if the vector bundles $\pi_0\mathcal{F}_M$ and $\pi_1\mathcal{F}_M$ are standard in the sense of Definition 3.0.3, i.e., these vector bundles can be built up iteratively by extensions with line bundles.

If we can realize these extensions topologically, we call a TMF-module *(topologically) standard*. More precisely, we propose the following definition:

Definition 8.1.2. We define the notion of a finite TMF-module being (topologically) standard inductively: First, $\Sigma^k TMF$ is standard for all k. Furthermore, for M standard and $x \in \pi_k M$ torsion, the cofiber of $\Sigma^k TMF \xrightarrow{x} M$ is standard. A TMF-module is standard if it can built in finitely many steps in this way.

It is easy to see that every standard module is also algebraically standard.

The module $M(2)$ carries an S_3-action induced by the S_3-action on $TMF(2)$. We denote by $E(M)$ the set of generators $x \in \pi_*(M(2))$ of direct $TMF(2)_*$-summands which are invariant under the S_3-action. Let (by abuse of notation) denote $c\colon M \to M(2)$ the map induced by the algebra map $c\colon TMF \to TMF(2)$. We say that M *has an invariant generator* if $E(M) \cap \operatorname{im}(c_*) \neq \varnothing$. We will prove the following in Section 8.4:

Proposition 8.1.3. *If every finite TMF-module has an invariant generator, every finite TMF-module is standard. If every algebraically standard TMF-module has an invariant generator, every algebraically standard TMF-module is standard.*

The author was not able to show that every finite (algebraically standard) TMF-module has an invariant generator and therefore also not to show that every finite (algebraically standard) TMF-module is standard. Instead, we propose a weaker version of being standard:

Definition 8.1.4. We define the notion of a finite TMF-module being *hook-standard* inductively: First, $\Sigma^k TMF$ is hook-standard for all k. Furthermore, a TMF-module M is hook-standard if there are cofiber sequences

$$\Sigma^{|a|}TMF \xrightarrow{a} M \to X$$
$$\Sigma^{|x_1|}TMF \xrightarrow{x_1} X \to X'$$
$$\Sigma^{|x_2|}TMF \xrightarrow{x_2} X' \to X''$$

with X'' hook-standard, where a correspondsto a torsion element and $c_*(x_1) \in E(X)$ and $c_*(x_2) \in E(X')$.

Every standard module is hook-standard: If $a = 0$, $X = \Sigma^{|a|+1}TMF \oplus M$ and we can choose $x_1 : \Sigma^{|a|+1} \to X$ to be the inclusion of the direct summand.

Our main theorem in this chapter will be:

Theorem 8.1.5 (The hook theorem)**.** *Every algebraically standard TMF-module is hook-standard.*

Note that in principle it is possible to classify all hook-standard TMF-modules up to a certain finite rank: For rank 1, we have just suspensions of TMF. Now suppose, we have classified all hook-standard modules up to rank $(n-1)$. Given a hook standard module Z of this rank, we can choose a torsion element in $\pi_* D_{TMF} Z$, cone it off to get a module Z' of rank n. Here, we choose again a torsion element, cone it off and get a module Z''. Here, we choose a $z \in \pi_* Z''$ with $c(z) \in E(Z'')$ and get a module $D_{TMF}M$ after coning it off whose dual M is hook-standard. All hook-standard modules of rank n are built in this way.

We complement the hook theorem by the following proposition to be proven at the end of this chapter:

Proposition 8.1.6. *Every algebraically standard TMF-module of rank ≤ 3 is standard.*

We now come to the strategy of the proof of Theorem 8.1.5. An important observation (in Section 8.3) is that if we have a summand of the form $f_* f^* \mathcal{O}$ in $\pi_* \mathcal{F}_M$ (where $f\colon \mathcal{M}_0(2) \to \mathcal{M}$ denotes the usual projection map), M decomposes as $TMF_0(2) \oplus M'$. So our strategy is to enlarge M by coning off torsion elements to get such summands to kill, which we will do in Section 8.7. To succeed, it is necessary to study the torsion of M before, especially the multiplication by α and β on it, which we will do in Sections 8.5 and 8.6. At the end, we will either get an invariant generator or a "hook". This all relies on the classification of standard vector bundles and on certain algebraic preliminaries, which are presented in the next section.

In the whole proof, the following triangle is very important:

$$M \xrightarrow{c} M(2) \xrightarrow{\sigma(2)} \Sigma^4 M_\alpha \vee \Sigma^4 M_0(2) \xrightarrow{t\widetilde{\alpha}} \Sigma M \tag{8.1}$$

Here $M_\alpha := M \wedge_{TMF} TMF_\alpha$ and $M_0(2) := M \wedge_{TMF} TMF_0(2)$. It is induced by the triangle

$$TMF \xrightarrow{c} TMF(2) \xrightarrow{\sigma(2)} \Sigma^4 TMF_\alpha \vee \Sigma^4 TMF_0(2) \xrightarrow{t\widetilde{\alpha}} \Sigma TMF.$$

This in turn you get from the more well known triangle

$$TMF \to TMF_0(2) \xrightarrow{\sigma} \Sigma^4 TMF_\alpha \to \Sigma TMF$$

since $TMF(2) \cong TMF_0(2) \vee \Sigma^4 TMF_0(2)$ (see also Section 5.2).

8.2 Algebraic Preliminaries

Recall that $\mathcal{M}(2) = \operatorname{Spec} TMF(2)_*//\mathbb{G}_m$. Furthermore, we have that $\mathcal{M} = \mathcal{M}(2)//S_3$. Therefore, by Galois descent, the category of graded $TMF(2)_*$-modules with twisted S_3-action is equivalent to the category of quasi-coherent sheaves over $\mathcal{M}$. More precisely, $\Gamma(p^*\pi_*\mathcal{F}_M)$ corresponds to $\pi_0\mathcal{F}_M$, where $p\colon \mathcal{M}(2) \to \mathcal{M}$ is the projection. Note that we have $\pi_*M(2) = \pi_*\Gamma(p^*\mathcal{F}_M) \cong \Gamma(p^*\pi_*\mathcal{F}_M)$ by the descent spectral sequence since the higher cohomology of $p^*\pi_*\mathcal{F}_M$ is trivial (by the same argument as in Lemma 3.4.4). Furthermore, $(\pi_*M(2))^{S_3} \cong H^0(\mathcal{M}; \pi_*\mathcal{F}_M)$ (by Lemma 2.7.1).

We will work for the next paragraphs more generally with an arbitrary étale map $p\colon \mathcal{X} \to \mathcal{Y}$ since we do not gain by specializing at this point. Let $\mathcal{F}$ be quasi-coherent sheaf on $\mathcal{Y}$. The adjunction unit $\mathcal{F} \to p_*p^*\mathcal{F}$ induces a map

$$c_{alg}\colon \Gamma(\mathcal{F}) \to \Gamma(p_*p^*\mathcal{F}) = \Gamma(p^*\mathcal{F}),$$

corresponding to the inclusion of the S_3-invariants (see the proof of Lemma 2.7.1). Another interpretation of this map is as the morphism $\mathcal{F} \otimes \omega^{\otimes *}(\mathcal{Y}) \to \mathcal{F} \otimes \omega^{\otimes *}(\mathcal{X})$ induced by p.

The following lemma is well-known, but I was unable to find a complete and elementary proof in the literature.

Lemma 8.2.1. *For any étale map $p\colon \mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks, the functor*

$$p^*\colon \mathcal{O}_{\mathcal{X}}\text{-mod} \to \mathcal{O}_{\mathcal{Y}}\text{-mod}$$

has a left adjoint $p_!$.

Proof. We will begin by describe a left adjoint of p^* on the level of presheaves. For $\mathcal{F}$ a presheaf of $\mathcal{O}_{\mathcal{X}}$-modules, a presheaf $p_?\mathcal{F}$ of $\mathcal{O}_{\mathcal{Y}}$-modules is defined as follows: For $f\colon U \to \mathcal{Y}$ an étale map, $p_?\mathcal{F}(U,f) := \oplus_s \mathcal{F}(U,s)$, where the direct sum ranges over all maps $s\colon U \to \mathcal{X}$ such that $ps = f$. We want to prove that $p_?$ is left adjoint to p^* at the level of presheaves. For $\mathcal{G}$ a presheaf of $\mathcal{O}_{\mathcal{Y}}$-modules, define the counit $p_?p^*\mathcal{G} \to \mathcal{G}$ on an $f\colon U \to \mathcal{Y}$ by the summing map

$$\bigoplus_{s \text{ lifting of } f} \mathcal{G}(U, ps) \to \mathcal{G}(U, f)$$

(note that $ps = f$ by definition). For $\mathcal{F}$ a presheaf of $\mathcal{O}_{\mathcal{X}}$, define the unit $\mathcal{F} \to p^*p_?\mathcal{F}$ on a $t\colon U \to \mathcal{X}$ by the inclusion of the t-summand $\mathcal{F}(U,t) \to \bigoplus_{s \text{ lifting of } p \circ t} \mathcal{F}(U,s)$. It is easy to check that the transformations $p_? \to p_?p^*p_? \to p_?$ and $p^* \to p^*p_?p^* \to p^*$ are identity.

Denote the "forgetful" functor $\mathcal{O}_{\mathcal{X}}\text{-mod} \to \operatorname{Pre}_{\mathcal{X}}$ from $\mathcal{O}_{\mathcal{X}}$-modules to presheaves of $\mathcal{O}_{\mathcal{X}}$-modules by u and the sheafification by S and likewise for $\mathcal{Y}$. Define $p_!\mathcal{F}$ as $S(p_?(u\mathcal{F}))$.

Moreover, we have that $u(p^*\mathcal{G}) = p^*(u\mathcal{G})$ by definition. Since sheafification is left adjoint to u, we get that $p_!$ is left adjoint to p^*:

$$\begin{aligned}\mathcal{O}_{\mathcal{Y}}\text{-mod}(p_!\mathcal{F},\mathcal{G}) &= \mathcal{O}_{\mathcal{Y}}\text{-mod}(S(p_?(u\mathcal{F})),\mathcal{G}) \cong \mathrm{Pre}_{\mathcal{Y}}(p_?(u\mathcal{F}),u\mathcal{G}) \cong \mathrm{Pre}_{\mathcal{X}}(u\mathcal{F},p^*u\mathcal{G}) \\ &= \mathrm{Pre}_{\mathcal{X}}(u\mathcal{F},up^*\mathcal{G}) = \mathcal{O}_{\mathcal{X}}\text{-mod}(\mathcal{F},p^*\mathcal{G})\end{aligned}$$

□

Note that a lifting $U \to \mathcal{X}$ is equivalent to a section of $U \times_{\mathcal{Y}} \mathcal{X} \to U$. Let p now be a G-Galois covering (with G finite again). Then for U connected, $p_?\mathcal{F}(U) \cong \oplus_G \mathcal{F}(U)$ for p trivial over U and $p_?\mathcal{F}(U) = 0$ for every U where p is non-trivial since non-trivial Galois coverings have no sections. Since $p_*\mathcal{F}(U) \cong \oplus_G \mathcal{F}(U)$ as well for p over U trivial, we have a map $p_?\mathcal{F} \to p_*\mathcal{F}$, defined as identity for p trivial on U and 0 else, which is locally an isomorphism. Therefore the induced map $p_!\mathcal{F} \to p_*\mathcal{F}$ is also an isomorphism. Hence, for p a G-Galois covering, we have a map

$$r_{alg}\colon \Gamma(p^*\mathcal{F}) \cong \Gamma(p_*p^*\mathcal{F}) \cong \Gamma(p_!p^*\mathcal{F}) \to \Gamma(\mathcal{F}).$$

Clearly, r_{alg} is natural with respect to maps of sheaves since the counit map is a natural transformation. For the rest of this section, we abbreviate r_{alg} and c_{alg} to r and c for ease of notation.

Lemma 8.2.2. *We have the identities $rc = |G|$ and $cr = \Sigma_{g \in G} g$. Furthermore, r is surjective as a sheaf map.*

Proof. It is enough to show these statements locally since both r and c are induced by morphisms of sheaves. So we may assume that p is trivial, i.e., $\mathcal{X} = \coprod_G \mathcal{Y}$. Hence, we have $\Gamma(p^*\mathcal{F}) \cong \prod_G \Gamma(\mathcal{F})$. For every $g \in G$, the map $ps_g\colon \mathcal{Y} \to \mathcal{X} \to \mathcal{Y}$ is the identity, where s_g is the section corresponding to the element g. Therefore, the map $c\colon \Gamma(\mathcal{F}) \to \prod_G \Gamma(\mathcal{F})$ is the diagonal. Since $ps = \mathrm{id}$ for all sections $s\colon \mathcal{Y} \to \mathcal{X}$, we have that

$$r\colon \prod_G \Gamma(\mathcal{F}) \cong \bigoplus_G \Gamma(\mathcal{F}) \to \Gamma(\mathcal{F})$$

is the summing map (by the definition of the counit) and hence surjective. Therefore an element

$$x = (0, \ldots, 0, a, 0, \ldots, 0) \in \prod_G \Gamma(\mathcal{F})$$

is sent to $(a, \ldots, a) = \Sigma_{g \in G} gx$ by cr. On the other hand, an element $a \in \Gamma(\mathcal{F})$ is sent to $\Sigma_{g \in G} a = |G|a$. □

Now, we come back to the specific situation of $p\colon \mathcal{M}(2) \to \mathcal{M}$ and $G = S_3$. Note that we can view r for a quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{M}$ also as a map $\Gamma_*(p^*\mathcal{F}) \to \Gamma_*(\mathcal{F})$ by considering one degree at a time. We want to prove the following proposition:

Proposition 8.2.3. *Let E be a standard vector bundle on $\mathcal{M}$. Let furthermore $x \in \Gamma_*(E)$ be an element not in the image of $r\colon \Gamma_*(p^*E) \to \Gamma_*(E)$. Then there is a $z \in \Gamma_*(p^*E)$ such that $c(r(z) + x)$ is a generator of a direct summand of $\Gamma_*(p^*E)$ over $TMF(2)_*$.*

Proof. First, suppose we have shown the proposition for two vector bundles E_1 and E_2. Let now $E = E_1 \oplus E_2$ and $x \in \Gamma_*(E)$ outside $\mathrm{im}(r)$. We can write $x = (x_1, x_2)$ and get that $c(r(z_1) + x_1) = y_1$ or $c(r(z_2) + x_2) = y_2$ is a generator of a direct summand of $\Gamma(p^*E_1)$ and $\Gamma(p^*E_2)$ respectively for some $z_i \in \Gamma(p^*E_i)$. Hence, $(y_1, y_2) = c(r(z_1, z_2) + (x_1, x_2))$ is a generator of a direct summand of $\Gamma(p^*E)$ as well. Therefore, we can assume E in our proposition to be indecomposable.

According to Theorem 3.0.5, every standard vector bundle on $\mathcal{M}$ is a direct sum of (indecomposable) vector bundles of the form $\mathcal{O}$, E_α and $f_*f^*\mathcal{O}$ and twists of these by ω^j. Here E_α denotes the extension

$$0 \to \mathcal{O} \to E_\alpha \to \omega^{-2} \to 0$$

classified by $\alpha \in H^1(\mathcal{M}; \omega^2)$ and $f\colon \mathcal{M}_0(2) \to \mathcal{M}$ is the usual projection map. It suffices to prove the proposition for each of the listed standard indecomposables.

- Consider the case $E = \mathcal{O}$: The image of r contains the ideal I in $\Gamma_*(\mathcal{O}) \cong \mathbb{Z}_{(3)}[c_4, c_6, \Delta^{\pm 1}]$ generated by 3, c_4 and c_6. Indeed, $cr(\frac{1}{2}) = 3$, $cr(4x_2^2) = c_4$ and $cr(-32x_2^2y_2) = c_6$ by the formulas for the action of S_3 on $\Gamma(p^*\mathcal{O}) \cong \mathbb{Z}_{(3)}[x_2, y_2, \Delta^{-1}]$ in Section 2.5. It follows that the $\pm\Delta^i$ form a set of representatives for the non-zero elements in $\Gamma_*(\mathcal{O})/I$. Since $\pm\Delta^i$ is a unit in $TMF(2)_*$ and hence generates a direct summand, the result follows.

- Consider the case $E = f_*f^*\mathcal{O}$: The stack $\mathcal{M}(2) \times_{\mathcal{M}} \mathcal{M}_0(2)$ classifies elliptic curves with level-2-structure and choice of one point of exact order 2 and is hence equivalent to $\coprod^3 \mathcal{M}(2)$. This implies that the vector bundle p^*E has rank 3 and S_3 operates by interchanging the 3 factors simultaneously with the action on each factor. Since $c\colon \Gamma_*(E) \to \Gamma_*(p^*E)$ is an embedding with image $\Gamma_*(p^*E)^{S_3}$, every element in $\mathrm{im}(c)$ is of the form (a, ta, t^2a) (with respect to the above decomposition) with $t = (2\ 3\ 1) \in S_3$ and $a \in \Gamma_*(p^*\mathcal{O})^{C_2}$ (with respect to the C_2-action given by the involution $(1\ 3\ 2)$). Since the morphism $\mathcal{M}(2) \to \mathcal{M}_0(2)$ (corresponding to the choice of the first point of exact order 2) is C_2-Galois, $\Gamma_*(f_*f^*\mathcal{O}) \cong \Gamma_*(p^*\mathcal{O})^{C_2}$ and we can view a as an element in $\Gamma_*(E)$. Because $cr(\frac{1}{2}a, 0, 0) = (a, ta, t^2a)$ for $a \in \Gamma_*(f^*\mathcal{O})$, the image of c is contained in the image of cr and r is surjective. Thus, an $x \notin \mathrm{im}(r)$ as in the statement of the proposition does not exist.

- Consider the case $E = E_\alpha$: The short exact sequence

$$0 \to \mathcal{O} \to f_*f^*\mathcal{O} \xrightarrow{\sigma} E_\alpha \otimes \omega^{-2} \to 0 \tag{8.2}$$

induces a diagram of the form

$$\begin{array}{ccccc} H^0_*(\mathcal{M}; f_*f^*\mathcal{O}) & \xrightarrow{\sigma} & H^0_*(\mathcal{M}; E_\alpha \otimes \omega^{-2}) & \xrightarrow{\partial} & H^1_*(\mathcal{M}; \mathcal{O}) \\ \uparrow{\scriptstyle r^{(1)}} & & \uparrow{\scriptstyle r^{(2)}} & & \\ H^0_*(\mathcal{M}(2); p^*f_*f^*\mathcal{O}) & \longrightarrow & H^0_*(\mathcal{M}(2); p^*E_\alpha \otimes \omega^{-2}) & \longrightarrow & H^1_*(\mathcal{M}(2); p^*\mathcal{O}) = 0 \end{array}$$

First observe that $\mathrm{im}\, r^{(2)} = \mathrm{im}\, \sigma$ since both $r^{(1)}$ and the lower horizontal map are surjective. By Lemma 3.4.2, ∂ equals multiplication with the element ${}^t\widetilde{\alpha} \in \mathrm{Ext}^1(E_\alpha \otimes$

$\omega^{-2}, \mathcal{O})$ classifying (8.2). The pullback of ${}^t\widetilde{\alpha}$ along $\mathcal{O} \otimes \omega^{-2} \to E_\alpha \otimes \omega^{-2}$ equals $\pm\alpha \in H^1(\mathcal{M}; \omega^{-2})$ by Section 3.4. Thus, $\partial(u\Delta^i) = \pm u\alpha\Delta^i$ for $u \in \{0, 1, 2\}$, where we use the convention that we denote an element in $H^*_*(\mathcal{M}, \mathcal{O})$ and its image under the map in $H^*_*(\mathcal{M}, E_\alpha)$ induced by the defining map $\mathcal{O} \to E_\alpha$ by the same letter. Hence, the $u\Delta^i$ are a representing set for $\mathrm{coker}(\sigma) \cong H^0(\mathcal{M}; E_\alpha \otimes \omega^{-2}) / \mathrm{im}(r^{(2)})$. Thus, for every $x \in \Gamma_*(E)$ not in $\mathrm{im}(r^{(2)})$, we can find an $r^{(2)}(z)$ such that $x + r^{(2)}(z) = u\Delta^i$ with u a unit. We have an exact sequence

$$0 \to \Gamma_*(p^*\mathcal{O}) \to \Gamma_*(p^*E_\alpha) \to \Gamma_*(p^*\omega^{-2}) \to 0$$

since $H^1_*(\mathcal{M}; p^*\mathcal{O}) = 0$ and it splits since $\Gamma_*(p^*\omega^{-2})$ is free over $TMF(2)_*$. Thus, $u\Delta^i$ is a generator of a direct summand of $\Gamma_*(p^*E_\alpha)$. This implies the proposition.

□

Scholium 8.2.4. *For $E = \mathcal{O}$ or E_α, the cokernel of $r\colon \Gamma_*(p^*E) \to \Gamma_*(E)$ is an $\mathbb{F}_3$-vector space and the elements Δ^i, $i \in \mathbb{Z}$, form a basis. For $E = E_{\alpha,\widetilde{\alpha}}$, this cokernel is 0.*

Proof. Since $rc = 6$, $3\Gamma_*(E) \subset \mathrm{im}(r)$ and $\mathrm{coker}(r)$ is an $\mathbb{F}_3$-vector space. That the elements Δ^i generate $\mathrm{coker}(r)$ follows from the proof above. To show that the Δ^i are non-zero observe that $\Delta^i \in \Gamma_*(\mathcal{O})$ cannot be in $\mathrm{im}(r)$ since $\beta \in H^2_*(\mathcal{M}; \mathcal{O})$ operates non-trivially on it and for the same reason $\Delta^i \in \Gamma_*(E_\alpha)$ cannot be in $\mathrm{im}(r)$. The surjectivity of r in the case $E = f_*f^*\mathcal{O}$ is also contained in the proof above. □

We can also consider the map $\sigma_\alpha\colon \Gamma_*(f_*f^*\mathcal{O} \otimes E_\alpha) \to \Gamma_*(\omega^{-2} \otimes E_\alpha \otimes E_\alpha)$. We know that $E_\alpha \otimes E_\alpha \cong f_*f^*\mathcal{O} \oplus \omega^{-2}$ by Section 3.4. Therefore, the $(0, \Delta^i) \in \Gamma_*(f_*f^*\mathcal{O} \oplus \omega^{-2})$ span a representing set for $\Gamma_*(E_\alpha \otimes E_\alpha)/\ker(\alpha)$. Since α operates injectively[1] on $H^1_*(\mathcal{M}, E_\alpha)$ and multiplication by α commutes with δ, we have $\ker(\alpha) \subset \ker(\partial) = \mathrm{im}(\sigma_\alpha)$ for the boundary map

$$\partial\colon\ H^0_*(\mathcal{M}; \omega^{-2} \otimes E_\alpha \otimes E_\alpha) \to H^1_*(\mathcal{M}; E_\alpha).$$

Since the restriction of $\alpha\cdot\colon\ H^0_*(\mathcal{M}; \mathcal{O}) \to H^1_*(\mathcal{M}; \mathcal{O})$ to the span of the Δ^i is surjective, the $(0, \Delta^i)$ generate therefore the cokernel of σ_α (as an $\mathbb{F}_3$-vector space). Since the next term $H^1(\mathcal{M}; f_*f^*\mathcal{O} \otimes E_\alpha)$ in the sequence is zero, ∂ is surjective. Therefore, $\mathrm{coker}(\sigma_\alpha)$ has the same dimension as an $\mathbb{F}_3$-vector space as the span of the Δ^i. Therefore, the Δ^i form a basis for $\mathrm{coker}(\sigma_{a\alpha})$.

8.3 Low-Rank Examples and the Realification

We want to topologify the realification map r_{alg} of the last section to a map $r\colon\ p_*p^*\mathcal{O}^{top} \to \mathcal{O}^{top}$. Since

$$p_*p^*\mathcal{O}^{top} \cong f_*f^*\mathcal{O}^{top} \oplus \Sigma^4 f_*f^*\mathcal{O}^{top},$$

Lemma 5.2.2 gives us a unique map

$$r\colon\ p_*p^*\mathcal{O}^{top} \to \mathcal{O}^{top}$$

realizing the algebraic map r_{alg}.

[1] This can be seen as follows: The extension

$$0 \to E_\alpha \to f_*f^*\mathcal{O} \to \omega^{-4} \to 0$$

is classified by $\widetilde{\alpha} \in H^1(\mathcal{M}; E\alpha \otimes \omega^4)$. Since $H^1_*(\mathcal{M}; f_*f^*OO) = 0$, multiplication by $\widetilde{\alpha}$ acts injectively on α and, thus, α injectively on $\widetilde{\alpha}$.

Remark 8.3.1. Probably, the realification map $TMF(2) \to TMF \simeq TMF(2)^{hS_3}$ coincides with the transfer map, which can be defined using a form of Shapiro's lemma. Since this identification is not needed for our purposes, we abstain from a discussion.

Lemma 8.3.2. *We have* $rc = 6$ *and* $cr = \Sigma_{g \in S_3} g$*.*

Proof. These identities hold at the level of vector bundles by 8.2.2. We know that realizations of sheaf map $\pi_* p_* p^* \mathcal{O}^{top} \to \pi_* p_* p^* \mathcal{O}^{top}$ are unique, hence the second equation. The descent spectral sequence for $\mathcal{H}om_{\mathcal{O}^{top}}(\mathcal{O}^{top}, \mathcal{O}^{top})$ equals the DSS computing TMF. There are no permanent cycles in this spectral sequence in the 0-column above the 0-line; hence the first equation. □

We will need again and again the following observation:

Lemma 8.3.3. *Let M be relatively free TMF-module and $x \in \operatorname{im}(r_* \colon \pi_* M(2) \to \pi_* M)$. Then $\alpha x = \beta x = 0$.*

Proof. Let $y \in \pi_* M(2)$ such that $r_*(y) = x$. Since $M(2)$ is a free $TMF(2)$-module, $\pi_* M(2)$ is torsion-free and hence $\alpha y = \beta y = 0$. Since r is a TMF-module map, the result follows. □

Recall that we have a map $\sigma \colon M_0(2) \to \Sigma^4 M_\alpha$ given as the cofiber of $c \colon M \to M_0(2)$.[2] Note that $E(M)$ is completely in the $M_0(2)$-summand of $M(2)$ since the map $\mathcal{M}(2) \to \mathcal{M}$ factors over $\mathcal{M}_0(2)$. We can apply the realification to study σ:

Lemma 8.3.4. *Every S_3-invariant element $x \in \pi_* M_0(2) \subset \pi_* M(2)$ is mapped by σ to a 3-torsion element in $\Sigma^4 M_\alpha$.*

Proof. We have $cr(x) = \Sigma_{g \in S_3} gx = 6x$. Since 2 is invertible, this implies that $3x$ is in the image of c and, hence, $3\sigma(x) = \sigma(3x) = 0$. □

To identify the fiber of r, it will be convenient to discuss first some low-rank cases. Additionally, this will serve as an illustration of the general theory.

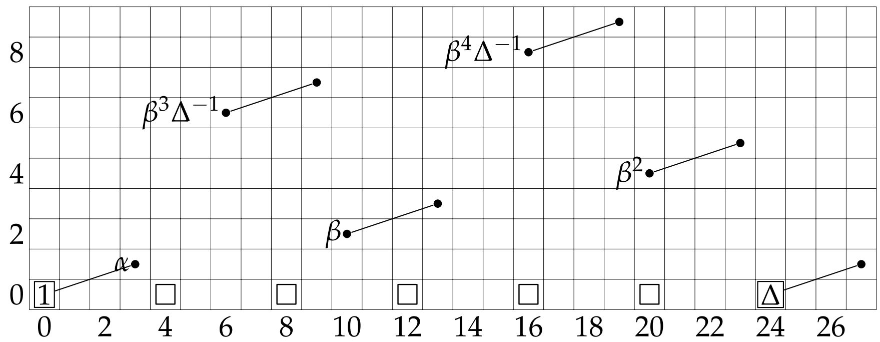
Lemma 8.3.5. *Let M be a algebraically standard TMF-module. We have an action of $\beta \in H^2(\mathcal{M}; \omega^6)$ on the DSS of M by Theorem 6.4.2, which commutes with the differentials since β is a permanent cycle in the DSS for TMF. Then β acts injectively on the E^2-term of the DSS for M beginning with the first line. In addition:*

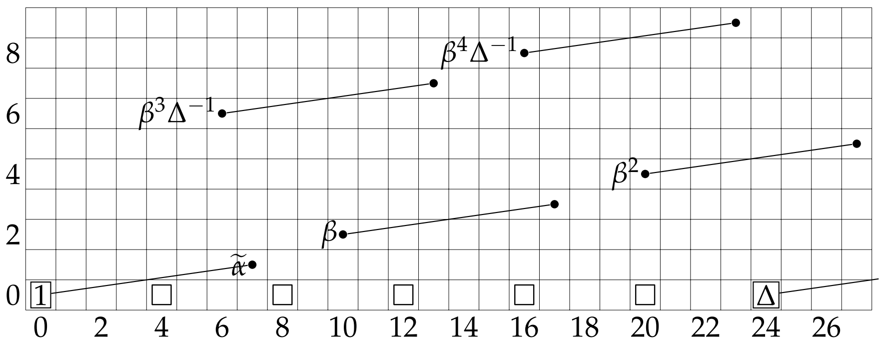
- *If $\pi_* \mathcal{F}_M$ is concentrated in even degrees, β acts injectively on odd degrees (i.e. columns) on the E^r-term of the DSS beginning with the $(r-1)$-st line.*[3]
- *If the first line consists of permanent cycles, β acts injectively on the whole E^r-term of the DSS beginning with the $(r-1)$-st line.*

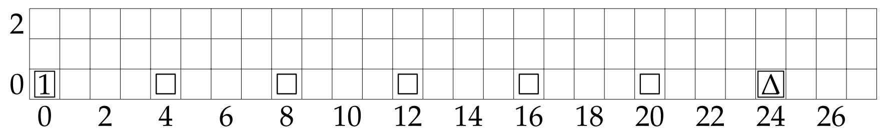
[2] We abuse here the letter c since the usual map $c \colon M \to M(2)$ factors over $M \to M_0(2)$.

[3] To act proactively against possible confusion: That $\pi_* \mathcal{F}_M$ is concentrated in even degrees means that $\pi_k \mathcal{F}_M = 0$ for k odd, where π_k denotes the sheafified homotopy group. An element in the E^2-term $H^q(\mathcal{M}; \pi_p \mathcal{F}_M)$ of the DSS is in odd degree if $p - q$ is odd.

Proof. We know that $\pi_*\mathcal{F}_M$ decomposes into a direct sum of shifts of vector bundles of the form $\pi_*\mathcal{O}^{top}$, $E_\alpha \otimes_{\mathcal{O}} \pi_*\mathcal{O}^{top}$ and $E_{\alpha,\widetilde{\alpha}} \otimes_{\mathcal{O}} \pi_*\mathcal{O}^{top}$. The cohomology of these looks as follows (where the pattern continues to the left, right and top):







This follows from the discussion in Sections 2.7 and 3.4. The injectivity of $\beta\cdot$ on E^2 beginning with the first line is now immediate. Now suppose, we have shown that β operates injectively on E^{r-1} beginning with the $(r-2)$-th line (on elements of odd degree). Now suppose $\beta\overline{a} = \beta\overline{b}$ for some $\overline{a} \neq \overline{b} \in E^r$ (in odd degrees) in line s and $s \geq r-1$. Then there are $a, b \in E^{r-1}$ reducing to $\overline{a}, \overline{b}$. Hence, there is an $x \in E^{r-1}$ with $d_{r-1}x = \beta(a-b) \neq 0$ and x is in line k with $k \geq 1$ (and of even degree). We want to show that there is a $y \in E^{r-1}$ such that $\beta y = x$: Let $x' \in E^2$ represent x. Then x' is divisible by β. Indeed, if $\pi_*\mathcal{F}_M$ is concentrated in even degrees, x' must be in every standard summand of $\pi_*\mathcal{F}_M$ of the form $\pm\Delta\beta^{k/2}$ or 0. The same holds if the first line of the DSS consists of permanent cycles since then all $\alpha\beta^l\Delta^i$ and $\widetilde{\alpha}\beta^l\Delta^i$ are permanent cycles as well and x' can be no permanent cycle. So, let $y' \in E^2$ such that $\beta y' = x'$. Suppose $d_l(y') \neq 0$ for some $l < r-1$. Then $d_l(x') = \beta d_l(y') \neq 0$ since β acts injectively beginning with $(l-1)$-st line on E^l. So, $d_l(y) = 0$ for $l < r-1$ and $x = \beta y$ for y denotes the reduction of y' to E^{r-1}. We have that $\beta d_{r-1}(y) = \beta(a-b) \in E^{r-1}$ for $d_{r-1}(y)$ and $(a-b)$ in the s-th line. Hence $d_{r-1}(y) = a-b$ and $\overline{a} = \overline{b}$.

□

Proposition 8.3.6. *If M is relatively free of $TMF(2)$-rank $n = 1$, we have $M \cong \Sigma^? TMF$. If M is of $TMF(2)$-rank 2 and $\pi_0\mathcal{F}_M = E_\alpha$, then $M \cong \Sigma^{24i} TMF_\alpha$ for some $i \in \mathbb{Z}$.*

Proof. If M is of $TMF(2)$-rank $n = 1$, we know that $\pi_*\mathcal{F}_M$ is trivial, i.e., we can assume by a shift that $\pi_0\mathcal{F}_M \cong \mathcal{O}$. Therefore, the (24-periodic) E^2-term of the DSS associated to M looks as the one for TMF.

We identify $M(2)$ with $TMF(2)$ and assume that no element of $E(M)$ is in $\mathrm{im}(c_*)$. By this contradiction assumption and Lemma 8.3.4, the $\Delta^i \in E(M)$ have to be mapped to non-trivial torsion elements y_i in even degree by σ in the exact sequence

$$\pi_*M \xrightarrow{c} \pi_*M(2) \xrightarrow{\sigma} \pi_{*-4}M_\alpha \oplus \pi_{*-4}M_0(2).$$

We can consider the y_i as lying in $\pi_{*-4}M_\alpha$ since $\pi_*M_0(2)$ is torsionfree because

$$M_0(2) \vee \Sigma^4 M_0(2) \simeq M(2).$$

We know that Δ^i in the DSS for $\mathcal{F}_M$ supports a non-zero d_{p_i}-differential: If it was a permanent cycle, the corresponding element in π_*M would map to $\Delta^i \in \pi_*M(2)$. Hence, $d_{p_i}(\beta^k\Delta^i) = \beta^k d_{p_i}(\Delta^i) \neq 0$ by Lemma 8.3.5.

Now look at the exact sequence

$$\pi_{24i-4}M \to \pi_{24i-4}M_\alpha \to \pi_{24i-8}M$$

induced by the triangle $TMF \to TMF_\alpha \to \Sigma^4 TMF$. Since no torsion element in even degree survives in M by the above argument, y_i is mapped to 0. For the same reason, it can come only from a non-torsion element in $\pi_{24i-4}M$. But $\pi_0\mathcal{F}_{M_\alpha} \cong E_\alpha$ by Lemma 4.5.12 and the injection $\mathcal{O} \to E_\alpha$ induces an injection on graded global sections. Thus every non-torsion element in π_*M maps to a non-torsion element in π_*M_α (since it is in the 0-line of the DSS). This is a contradiction and one of the Δ^i must be a permanent cycle. Thus, we get a map $\Sigma^{24i}TMF \to M$ inducing an equivalence $TMF(2) \to M(2)$. Thus, $M \cong TMF$ by the faithfulness of $TMF(2)$ (proven in Lemma 5.2.6).

The same argument works for $\pi_0\mathcal{F}_M = E_\alpha$ and we get a map $x\colon \Sigma^{24i}TMF \to M$ such that $c(x)\colon \Sigma^{24i}TMF(2) \to M(2)$ splits off a direct summand. Let Y be the fiber of x. Then $Y(2)$ has rank 1, therefore Y is equivalent to some $\Sigma^k TMF$. We know that $\pi_0\mathcal{F}_Y \cong \omega^{-2}$. Thus, we have a cofiber sequence

$$\Sigma^k TMF \xrightarrow{y} \Sigma^{24i}TMF \xrightarrow{x} M.$$

We know that y is of filtration (at least) 1 in the DSS for TMF since $\Sigma^{24i}TMF \to M$ induces an injective map $\pi_*\mathcal{F}_{\Sigma^{24i}TMF} \to \pi_*\mathcal{F}_M$. Thus, it equals $\pm\alpha\Delta^{3j}$ by Corollary 6.4.4 since else $\pi_*\mathcal{F}_M$ would split into two line bundles. Therefore, $M \cong \Sigma^{24i}TMF_\alpha$. □

The next case is that $\pi_0\mathcal{F}_M = f_*f^*\mathcal{O}$. We will treat a more general case:

Proposition 8.3.7. *Let M be a relatively free TMF-module and $\pi_0\mathcal{F}_M \cong f_*f^*\mathcal{O} \oplus Z_0$ for some vector bundle Z_0. Then there is a cofiber sequence*

$$TMF_0(2) \xrightarrow{\overline{y}} M \to Z \to \Sigma TMF_0(2)$$

such that $\pi_0\mathcal{F}_Z = Z_0$. This cofiber sequence splits.

Proof. By Lemma 5.2.2, the first statement is clear. Furthermore, the morphism

$$Z(2) \to \Sigma TMF(2) \wedge_{TMF} TMF_0(2)(2)$$

is zero on homotopy groups (since the map $\pi_*\mathcal{F}_Z \to f_*f^*\pi_*\Sigma\mathcal{O}^{top}$ is zero and $\pi_*Z(2) = (\pi_*\mathcal{F}_Z)(\mathcal{M}(2))$) and hence zero since $Z(2)$ is a projective $TMF(2)$-module. Thus, the composition

$$Z \to \Sigma TMF_0(2) \to \Sigma TMF(2) \wedge_{TMF} TMF_0(2)$$

is zero and the map $Z \to \Sigma TMF_0(2)$ factors over the first map in the triangle

$$\begin{array}{ccc} (\Sigma^4 TMF_\alpha \oplus \Sigma^4 TMF_0(2)) \wedge_{TMF} TMF_0(2) & & \Sigma TMF(2) \wedge_{TMF} TMF_0(2) \\ \downarrow {\scriptstyle {}^t\widetilde{\alpha} \wedge_{TMF} TMF_0(2)} & \nearrow & \\ \Sigma TMF_0(2). & & \end{array}$$

(See (8.1) with $M = TMF_0(2)$ for this triangle.) This map is zero since ${}^t\widetilde{\alpha}$ is torsion and both source and target are projective $TMF_0(2)$-modules. Hence, the map $Z \to \Sigma TMF_0(2)$ is zero as was to be shown. □

This implies, in particular, that we can always assume for the proof of Theorem 8.1.5 that $\pi_0(\mathcal{F}_M)$ contains no summand of the form $f_*f^*\mathcal{O}$ since we could compose the map $TMF_0(2) \to M$ with the unit map $TMF \to TMF_0(2)$ and get an invariant generator.

Now, we want to identify the fiber of r and begin by identifying the fiber of $r_{alg}\colon\ p_*p^*\mathcal{O} \to \mathcal{O}$. We have that $p_*p^*\mathcal{O}(\mathcal{M}(2)) \cong \bigoplus_{S_3} \mathcal{O}(\mathcal{M}(2))$ with diagonal S_3-action. By (the proof of) Lemma 8.2.2, r_{alg} maps on $\mathcal{M}(2)$ an element $(a_g)_{g\in S_3}$ to $\sum_{g\in S_3} a_g \in \mathcal{O}(\mathcal{M}(2))$. Recall also that $f_*f^*\mathcal{O}(\mathcal{M}(2))$ is $\bigoplus_{i=1}^3 \mathcal{O}(\mathcal{M}(2))$ with the permutation action. Sending $(a_g)_{g\in S_3}$ to $(\sum_{g:\ g(1)=i} a_g)_{i=1}^3$ defines a projection to a direct summand $p_*p^*\mathcal{O} \to f_*f^*\mathcal{O}$ such that the complement is isomorphic to $f_*f^*\mathcal{O} \otimes \omega^2$. (by Lemma 3.5.4). Since $\mathrm{QCoh}(\mathcal{M}) \simeq \widetilde{TMF(2)}_*[S_3]$-grmod by Galois descent, r_{alg} factors thus as $p_*p^*\mathcal{O} \to f_*f^*\mathcal{O} \to \mathcal{O}$, where the second map is the summing map on $\mathcal{M}(2)$. In Section 3.5, it was shown that the latter map has kernel $E_\alpha \otimes \omega^4$. Thus $\ker(r_{alg}) \cong f_*^*f^*\mathcal{O} \otimes \omega^2 \oplus E_\alpha \otimes \omega^4$.

Let X be the fiber of $\Gamma(r)\colon\ TMF(2) \to TMF$.[4] Then $\pi_*\mathcal{F}_X \cong \omega^{2+*} \otimes f_*f^*\mathcal{O} \oplus \omega^{4+*} \otimes E_\alpha$. We get by the last proposition a triangle $\Sigma^4 TMF_0(2) \to X \to Y$. One sees that $\pi_*\mathcal{F}_Y \cong \omega^{4+*} \otimes E_\alpha$. Hence, by the arguments above, $Y \cong \Sigma^{-8+24i} TMF_\alpha$. Since there is no non-zero map $\Sigma^{-8+24i} TMF_\alpha \to \Sigma^5 TMF_0(2)$ (the groups $\pi_* TMF_0(2)$ vanish in odd degrees), we have $X \cong \Sigma^{-8+24i} TMF_\alpha \vee \Sigma^4 TMF_0(2)$. The fiber $\Sigma^{-1} TMF \to X$ of $X \to TMF(2)$ can only be of the form $\widetilde{\alpha} = (\widetilde{\alpha}, 0)$ since this is the only one which fits into the long exact sequence of cohomology of the occurring vector bundles. Thus, $i = 0$ and we have a triangle

$$\Sigma^{-1} TMF \xrightarrow{\widetilde{\alpha}} \Sigma^{-8} TMF_\alpha \vee \Sigma^{2?} TMF_0(2) \xrightarrow{d} TMF(2) \xrightarrow{r} TMF,$$

which, in turn, induces a triangle

$$\Sigma^{-1} M \xrightarrow{\widetilde{\alpha}} \Sigma^{-8} M_\alpha \vee \Sigma^{2?} M_0(2) \xrightarrow{d} M(2) \xrightarrow{r} M. \tag{8.3}$$

8.4 Building Up and Tearing Down

The aim of this section is to show Proposition 8.1.3. The basic idea is to have as induction hypothesis that every (algebraically standard) TMF-module of rank smaller than n is standard and then use a invariant generators to reduce from rank n to rank $n-1$. This works

[4] This map and the induced map $M(2) \to M$ for a TMF-module M will often also be denoted by r.

in an easy way without the hypothesis of being algebraically standard. The main difficulty if we include this hypothesis is that the cokernel of a map of standard vector bundles may be not a standard vector bundle in general, which we have to deal with first.

To that purpose, recall that $TMF(2)_* \cong \mathbb{Z}_{(3)}[x_2, y_2, \Delta^{-1}]$.

Lemma 8.4.1. *The element* $1 \in TMF(2)_*$ *is not in the ideal* $(3, x_2 + y_2)$.

Proof. Assume that $1 \in (3, x_2+y_2)$. This implies that 1 is divisible by x_2+y_2 in $TMF(2)_*/3$; hence x_2+y_2 is a unit in this ring. This, in turn, implies that $(x_2+y_2)\cdot z = \Delta^k$ for some $z \in \mathbb{F}_3[x_2, y_2]$. We know that $\mathbb{F}_3[x_2, y_2]$ is factorial and, hence, x_2+y_2 is a prime element (since it is irreducible). Since $\Delta^k = 16^k x_2^{2k} y_2^{2k}(x_2-y_2)^{2k}$, the element x_2+y_2 has to divide x_2, y_2 or x_2-y_2 in $\mathbb{F}_3[x_2, y_2]$, which is clearly impossible. □

Proposition 8.4.2. *Let M be a relatively free TMF-module such that there is a $y \in \pi_k M$ with $c(y) \in E(M)$. Assume that $\pi_*\mathcal{F}_M$ has a decompositions into shifts of $\pi_*\mathcal{O}^{top}$ and $\pi_*\mathcal{O}^{top} \otimes E_\alpha$. Then there exists a $y' \in \pi_k M$ such that the cofiber of $\Sigma^k TMF \xrightarrow{y'} M$ is algebraically standard.*

Proof. For ease of notation, we assume $k = 0$. The element y corresponds to a $\overline{y} \in \Gamma(\pi_0\mathcal{F}_M)$.

First assume that $\overline{y} \in \operatorname{im}(r_{alg})$. The module $\pi_*\mathcal{F}_M(\mathcal{M}(2)) = \pi_* M(2)$ is a free $TMF(2)_*$-module. We want to show that we can choose a basis such that $c(y)$ corresponds to an element $(a_1, \ldots, a_n)$ with $a_i \in (3, x_2+y_2) \subset TMF(2)_*$. This is enough since $1 \notin (3, x_2+y_2)$ by the last lemma and this is a contradiction to the assumption that $c(y) \in E(M)$.

The vector bundle $\pi_0\mathcal{F}_M$ decomposes into a sum $\bigoplus_i \omega^{n_i} \oplus \bigoplus_j E_\alpha \otimes \omega^{m_j}$. Thus, we can show the claim just for one of the standard summands. First assume $\pi_0\mathcal{F}_M \cong \omega^j$. Since $\overline{y} \in \operatorname{im}(r_{alg})$, we know that $\overline{y}$ lies in the ideal $(3, c_4, c_6)$ (see Scholium 8.2.4). As shown in Section 2.5, $c_{alg}(c_4)$ and $c_{alg}(c_6)$ are divisible by (x_2+y_2) after reducing mod 3. For $\pi_0\mathcal{F}_M \cong \omega^j \otimes E_\alpha$, we proceed as follows: In the proof of Proposition 8.2.3, it was shown that $\operatorname{im}(r_{alg})$ coincides with the image of the map $\Gamma(f_*f^*\mathcal{O} \otimes \omega^{2+j}) \to \Gamma(E_\alpha \otimes \omega^j)$. We know that $\Gamma_*(f_*f^*\mathcal{O}) \cong \mathbb{Z}_{(3)}[b_2, b_4, \Delta^{-1}]$, where b_2 maps to $-4(x_2+y_2)$ and b_4 to $2x_2y_2$ in $\Gamma_*(p_*p^*\mathcal{O})$ (see also Section 2.5); thus, $\Gamma_*(f_*f^*\mathcal{O})$ is exactly the ring of invariant elements in $\Gamma_*(p_*p^*\mathcal{O})$ for a subgroup $C_2 \subset S_3$. The image of $\Gamma_*(f_*f^*\mathcal{O})$ in $\Gamma_*(p_*p^*f_*f^*\mathcal{O}) \cong \bigoplus_{i=1}^3 TMF(2)_*$ consists of (a, ta, t^2a) for $a \in \Gamma_*(f_*f^*\mathcal{O})$ and $t \in S_3$ an element of order 3. In Section 3.5, it was shown that $E_\alpha \otimes \omega^{-2} \cong I\mathbb{Z}_{(3)}[\zeta_3]$ (notation as in 3.5) and that the map $f_*f^*\mathcal{O} \to E_\alpha \otimes \omega^{-2}$ is induced by the quotient map $\mathbb{Z}_{(3)}[C_3] \to \mathbb{Z}_{(3)}[\zeta_3]$ (given by quotienting out the diagonal). Thus, giving $\mathbb{Z}_{(3)}[\zeta_3]$ the basis $(1, \zeta_3)$, the element $(a, ta, t^2a) \in \Gamma_*(p_*p^*f_*f^*\mathcal{O})$ is sent to $(a - t^2a, a - ta) \in \Gamma_*(p_*p^*E_\alpha)$. We can assume that a is a monomial of the form $b_2^k b_4^l$ (since Δ is invariant). This is sent to

$$\left((x_2+y_2)^k x_2^l y_2^l - (y_2-2x_2)^k (y_2-x_2)^l(-x_2)^l, (x_2+y_2)^k x_2^l y_2^l - (x_2-2y_2)^k(-y_2)^l(x_2-y_2)^l\right)$$

by the formulas in Section 2.5. Modulo three, $y_2 - 2x_2$ equals x_2+y_2, so both entries are in the ideal $(3, x_2+y_2)$, which was to be proven.

Thus, $\overline{y} \notin \operatorname{im}(r_{alg})$. This implies that its projection to one of the standard summands $\mathcal{E}$ (isomorphic to ω^j or $E_\alpha \otimes \omega^j$) is not in $\operatorname{im}(r_{alg})$. Since every element in $\operatorname{im}(r_{alg})$ is a permanent cycle, we can by Scholium 8.2.4 find an element $z \in \operatorname{im}(r)$ such that for $y' = y + z$ the projection of the reduction $\overline{y'} \in \Gamma(\pi_0\mathcal{F}_M)$ to $\mathcal{E}$ equals $\pm\Delta^{j/12}$. We have still $c(y') = E(M)$ since an element in a free module generates a direct summand if it projects to a unit in one of the summands. Thus, we get a diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \pi_0\mathcal{F}_M - \mathcal{E} & \longrightarrow & \pi_0\mathcal{F}_M - \mathcal{E} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O} & \xrightarrow{\overline{y'}} & \pi_0\mathcal{F}_M & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \\
 & & \downarrow = & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Here, the map $\pi_0\mathcal{F}_M \to \mathcal{E}$ is the projection. By the exactness of the lower two rows and the columns, the identification of the upper row follows by the Snake lemma. We have that $\mathcal{L} = 0$ if $\mathcal{E} \cong \mathcal{O}$ and $\mathcal{L} \cong \omega^{-2}$ for $\mathcal{E} \cong E_\alpha$. In both cases, $\mathcal{G}$ is standard since $\pi_0\mathcal{F}_M - \mathcal{E}$ is.

If M' is the cofiber of $\Sigma^k TMF \xrightarrow{y'} M$, then $\mathcal{G} = \pi_0\mathcal{F}_{M'}$. Thus, M' is algebraically standard since $\pi_1\mathcal{F}_{M'} \cong \pi_1\mathcal{F}_M$. □

If $\pi_*\mathcal{F}_M$ has a summand of the form $f_*f^*\pi_*\mathcal{O}^{top}$ with complement a standard vector bundle, then we can write $M = TMF_0(2) \oplus M'$ with M' algebraically standard as in Proposition 8.3.7. In particular, we can use $1 \in \pi_0 TMF_0(2)$ to get a map $TMF \to M$ whose cofiber is $\Sigma^4 TMF_\alpha \oplus M'$. Thus (using the last proposition for all cases without $f_*f^*\pi_*\mathcal{O}^{top}$-summand), we get for M algebraically standard of rank n with an invariant generator a triangle

$$\Sigma^k TMF \xrightarrow{y} M \to M'$$

such that M' is algebraically standard of rank $n-1$. Since $TMF(2)$ is faithful over TMF, this implies firstly that for $n = 1$, we have $M \cong \Sigma^k TMF$, and secondly that in general M is an extension of a rank 1 and a rank $(n-1)$-module.

Definition 8.4.3. A relatively free module X can be *built up* if there is a sequence $X_0 = 0, X_1, \ldots, X_n \cong X$ (for n the rank of X) with cofiber sequences $\Sigma^? TMF \to X_i \to X_{i+1}$. Dually, X can be *torn down* if there is a sequence of modules $X^0 = 0, X^1, \ldots, X^n = X$ with cofiber sequences $\Sigma^? TMF \to X^{i+1} \to X^i$.

Corollary 8.4.4. *If every (algebraically standard) module M has an invariant generator, every (algebraically standard) TMF-module can be torn down.*

Proposition 8.4.5. *Every module that can be torn down can be built up and vice versa. Such modules are standard modules.*

Proof. Let $X^0, \ldots, X^n = X$ be a tearing down sequence. Then define X_i as the fiber of $X^n \to$

X^{n-i}. By the octahedral axiom the left column of the following diagram is distinguished:

$$
\begin{array}{ccccc}
X_{i-1} & \longrightarrow & X^n & \longrightarrow & X^{n-i+1} \\
\downarrow & & \downarrow {\scriptstyle =} & & \downarrow \\
X_i & \longrightarrow & X^n & \longrightarrow & X^{n-i} \\
\downarrow & & \downarrow & & \downarrow {\scriptstyle =} \\
\Sigma^? TMF & \longrightarrow & X^{n-i+1} & \longrightarrow & X^{n-i}
\end{array}
$$

Clearly, $X_n = X$ and $X_0 = 0$, so X can be built up. The dual follows by the dual proof or Spanier-Whitehead-duality. The last thing to show is that for a building up sequence, the morphisms $\Sigma^? TMF \to X_i$ correspond to torsion elements x_i in $\pi_* X_i$. By the triangle 8.1 this is equivalent to $c(x_i) = 0$. So suppose we had $c(x)\colon \pi_* \Sigma^? TMF(2) \to \pi_* X_i(2)$ non-zero. This is also non-zero if we tensor with $\mathcal{Q}$, the quotient field of $TMF(2)_*$. Therefore, $(\pi_* X_i(2) \otimes \mathcal{Q})/c(x)$ has rank $i-1$. Hence, $\dim_{\mathcal{Q}}(\pi_* X_{i+1}(2) \otimes \mathcal{Q}) \leq i$, which is a contradiction. □

Thus, we proved Proposition 8.1.3.

8.5 The divisibility by β

Let M be algebraically standard of rank n and $E(M)$ be the set of invariant generators of $\pi_* M(2)$.

Assumption 8.5.1. *We assume in this whole section that M has no invariant generator, i.e. no $x \in E(M)$ is in the image of $c\colon \pi_* M \to \pi_* M(2)$.*

Under this assumption, we have the following proposition:

Proposition 8.5.2. *The restricted projection map* $\operatorname{Tors} \pi_* M \to \pi_* M/\operatorname{im}(r_*)$ *is a surjection.*

Proof. Look at the following diagram

$$
\begin{array}{ccccc}
\pi_* M(2) \pi_* \Gamma(p^* \mathcal{F}_M) & \xrightarrow{r_*} & \pi_* \Gamma(\mathcal{F}_M) = \pi_* M(2) & \xrightarrow{c_*} & \pi_* \Gamma(p^* \mathcal{F}_M) \\
\downarrow {\scriptstyle l} & & \downarrow {\scriptstyle \kappa} & & \downarrow {\scriptstyle l} \\
\Gamma(p^* \pi_* \mathcal{F}_M) & \xrightarrow{r_{alg}} & \Gamma(\pi_* \mathcal{F}_M) & \xrightarrow{c_{alg}} & \Gamma(p^* \pi_* \mathcal{F}_M)
\end{array}
$$

Here κ and l denote the edge morphisms in the descent spectral sequence for $\mathcal{F}_M$ and $p^* \mathcal{F}_M$ respectively. Note that l is an isomorphism. Let $y \in \pi_* \Gamma(\mathcal{F}_M) = \pi_* M$. Then $\kappa(y) \in \operatorname{im}(r_{alg})$, because else there is an element $a \in \Gamma(p^* \pi_* \mathcal{F}_M)$ such that $c_{alg}(\kappa(y) + r_{alg}(a))$ is in $l(E(M))$ by Proposition 8.2.3. This implies that $c_*(y + r_*(l^{-1} a)) \in E(M)$, which is a contradiction to our assumption. Therefore, we can write $\kappa(y) = r_* l(a) = \kappa r_*(a)$ for some $a \in \pi_* \Gamma(p^* \mathcal{F}_M)$. So we see that $\kappa(y - r_*(a)) = 0$. Therefore, $c_*(y - r_*(a)) = 0$ and by the exact sequence induced by the triangle 8.1 in the introduction, we have that $y - r_*(a)$ is torsion, which implies the statement. □

Corollary 8.5.3. *Let $x \in E(M) \subset \pi_* M_0(2)$. Then $\sigma(x) = \beta^k g$, $k \geq 1$, where $g \in F_0 \pi_* M_\alpha$. Here, $F_\bullet$ denotes the filtration associated to the DSS.*

Proof. Let $x \in E(M)$. By Lemma 8.3.4 and the contradiction assumption, $\sigma(x)$ is a non-zero 3-torsion element in $\pi_*\Sigma^4 M_\alpha$. Thus, $d(\sigma(x)) = 0$ and $\sigma x = \widetilde{\alpha} u_x$ for some $u_x \in \pi_* M$ (for d see the end of Section 8.3). The element u_x is only well-defined up to the image of r – therefore we can assume by the last proposition that u_x is torsion. Hence $u_x = {}^t\widetilde{\alpha} y_x$ since $c(u_x) = 0$ for some $y_x \in \pi_*\Sigma^4 M_\alpha$ by (8.1). By Lemma 5.2.1, we get that $\sigma(x) = \beta y_x$ for some $y_x \in \pi_* M_\alpha$. By the same argument, every torsion element in M_α is divisible by β and so we can repeat the process if y_x is not already in F_0. □

Recall now that on the level of vector bundles, $\sigma\colon M_0(2) \to \Sigma^4 M$ induces the map

$$\sigma_{alg}\colon \Gamma(f_* f^* \mathcal{O} \otimes \pi_* \mathcal{F}_M) \to \Gamma(E_\alpha \otimes \omega^{-2} \otimes \pi_* \mathcal{F}_M)$$

called σ in Section 8.2.

Corollary 8.5.4. *The* 0*-line of the DSS for* M_α *consists of permanent cycles.*

Proof. We will use a rank argument: Let $X \subset \Gamma(\pi_* \mathcal{F}_{M_\alpha})$ be the subgroup of permanent cycles. Then $\mathrm{im}(\sigma_{alg}) \subset X$ since the descent spectral sequence for $M_0(2)$ collapses on E^2. Define a filtration on X by setting $B_k = \{\overline{x} \in X : \beta^{k+1} x = 0 \text{ for some } x \in F_0 \pi_* M_\alpha \text{ reducing to } \overline{x}\}$. Since β operates trivially on $M_0(2)$, we have $\mathrm{im}(\sigma_{alg}) \subset B_0$. Hence X/B_0 is a subquotient of $\mathrm{coker}(\sigma_{alg})$. The latter is an $\mathbb{F}_3[\Delta^{\pm 3}]$-vector space of rank $3n$ for n the number of irreducible direct summands of $\pi_* \mathcal{F}_M$ – this is proven in the proof of Proposition 8.2.3 and at the end of Section 8.2. So, if $X \neq \Gamma_*(\pi_* \mathcal{F}_{M_\alpha})$, then X/B_0 is an $\mathbb{F}_3[\Delta^{\pm 3}$-vector space of rank smaller than $3n$. We have $3n$ invariant generators of the form Δ^j for $j \in \{0, 1, 2\}$ in the direct summands of $\pi_* M_0(2)$ and we choose a basis g_i of the $\mathbb{Z}_{(3)}$ span of these elements indexed by some index set I with $|I| = 3n$. We know that $\sigma(g_i) = \beta^{n_i} v_i$ for some $v_i \in F_0 \pi_* M_\alpha$ with n_i maximal under all choices of v_i; so there are $3n$ elements v_i. We assume that we have chosen inductively the g_i in the following way: We order I in some way. The first of the g_i is chosen to be a primitive vector in the span of the Δ^j with maximal n_i. The $(k+1)$-st g_i is chosen to be one that is part of a basis of the span of the Δ^j together with the first k elements g_i and is among these one with the maximal n_i. This insures that $\sigma(\Sigma_{j \in J} a_j g_j) = \beta^l v$ with a_j units and $v \neq 0$ always implies that $l \leq n_j$ for all j with $a_j \neq 0$.

We have $\overline{v_i} \in B_{n_i}$ since $\beta\sigma(g_i) = \sigma(\beta g_i) = 0$. Suppose, there exists an $v_i' \pi_* M_\alpha$ with the same reduction $\overline{v_i'} = \overline{v_i}$ in the zero-line, but $\beta^{n_i} v_i' = 0$. Then exists an $x \in \pi_* M_\alpha$ of higher filtration such that $v_i' = v_i - x$. Since x is torsion, it is by the (proof of the) last corollary of the form $\beta^l v$ for $v \in F_0 \pi_* M_\alpha$. Thus, $\beta^{l+n_i} v = \beta^{n_i} x = \beta^{n_i} v_i = \sigma(g_i)$ in contradiction to the maximality of n_i. Thus, $\overline{v_i} \notin B_{n_i - 1}$.

Since $\oplus_{i \geq 1} B_i / B_{i-1} \cong X/B_0$, there is a $k \in \mathbb{N}$ and $J \subset I$ such that $\overline{v_j} \in B_k - B_{k-1}$ and the $(\overline{v_j})_{j \in J}$ are linear dependent over $\mathbb{F}_3$ in B_k / B_{k-1}. That is, there exist $a_j \in \{1, -1\}$ such that $\Sigma_j a_j \overline{v_j} \in B_{k-1}$. As above, this implies $\beta^{l+k} v = \beta^k \Sigma_j a_j v_j = \sigma(\Sigma_j a_j g_j) = 0$ for some $v \in \pi_* M_\alpha$ and $l > 0$ and thus $v = 0$. Hence $\Sigma_j a_j g_j \in \mathrm{im}(c)$. But since 1 and -1 are units in $\mathbb{Z}_{(3)}$, we have $\Sigma_j a_j g_j \in E(M)$, which is a contradiction to our main contradiction assumption. □

Notation 8.5.5. We recollect the notation from the last proof for the rest of the chapter: We have an index set I of cardinality $3n$, indexing elements $g_i \in \pi_* M_0(2) \subset \pi_* M(2)$ spanning $E(M)$ in the sense that every element in $E(M)$ is of the form $\sum a_i g_i$ for $a_i \in \mathbb{Z}_{(3)}$. We have numbers n_i and elements $v_i \in F_0 \pi_* M_\alpha$ such that $\sigma(g_i) = \beta^{n_i} v_i$. The v_i reduce by the last proof to a basis $\{\overline{v_i}\}$ of $\mathrm{coker}\, \sigma_{alg}$. Note that the v_i are (thus, since $\mathrm{im}(r) = \mathrm{im}(\sigma)$ by the proof of Proposition 8.2.3) not in $\mathrm{im}(r_*)$ and can be modified by elements in $\mathrm{im}(r_*)$ so that the $\overline{v_i}$ are in the span of the elements of the form Δ^j in $H^0(\mathcal{M}; \pi_* M_\alpha)$ by Proposition 8.2.3 and the fact that $\beta \cdot \mathrm{im}(r) = 0$.

Corollary 8.5.6. *The* 1*-line of the DSS of* M *consists of permanent cycles.*

Proof. The map ${}^t\widetilde{\alpha}$ in the triangle (8.1) in the introduction induces as in Theorem 6.4.3 a morphism of descent spectral sequences, which is exactly ${}^t\widetilde{\alpha}$ on E^2. This implies that the whole first line of the descent spectral sequence in M consists of permanent cycles (which, of course, cannot be boundaries) since ${}^t\widetilde{\alpha}\colon\ \Gamma(\pi_*\mathcal{F}_{M_\alpha}) \cong \Gamma(\pi_*\mathcal{F}_M \otimes E_\alpha) \to H^1(\mathcal{M}; \pi_{*+4}\mathcal{F}_M)$ is surjective (as $H^1(\mathcal{M}; \pi_*\mathcal{F}_M \otimes f_* f^* \mathcal{O}\omega^{-2}) = 0$). □

In the rest of this section, we want first to investigate how many times an element might be divided by β and then investigate in detail how the torsion exactly looks like. Before we begin with this, we have to compute a Toda bracket:

Lemma 8.5.7. *The Toda bracket* $\langle \widetilde{\alpha}, \beta^4, 3\rangle$ *(where we view* $\widetilde{\alpha}$ *again as a map* $\Sigma^7 TMF \to TMF_\alpha$*) contains* $\pm\{3\Delta^2\}$.

Proof. We first want to check that the Toda bracket is actually defined. Since $\beta^2\alpha = 0$ in $\pi_* TMF$, we see that $\beta^2\widetilde{\alpha} \in \pi_{27} TMF_\alpha$ is mapped to zero in the exact sequence

$$\pi_* TMF \to \pi_* TMF_\alpha \to \pi_{*-4} TMF$$

and is thus the image of an element $a \in \pi_{27} TMF$. The only non-zero elements in this degree are $\pm\{\alpha\Delta\}$.[5] These are annihilated by β^2 and thus $\beta^4\widetilde{\alpha} = 0$ and the Toda bracket is defined.

The element $\beta^4\widetilde{\alpha}$ in the E^2-term of the DSS of TMF_α is a permanent cycle (since $\widetilde{\alpha}$ is in $DSS(TMF_\alpha)$ and β^4 is one in $DSS(TMF)$) and can only be hit by a d_9-differential from $\pm\Delta^2$: Column 48 in lines below 9 consists only of line zero elements and by Scholium 8.2.4 and the fact that $\mathrm{im}(r)$ consists of permanent cycles, the existence of a non-trivial differential implies a non-trivial differential from Δ^2. Using Theorem 6.4.1, we could use that Massey products converge to Toda brackets and get the result.

Alternatively, one can use the definition of the Toda bracket and sees that it suffices to prove that the lift of $\beta^4 \in \pi_{40} TMF$ in the exact sequence

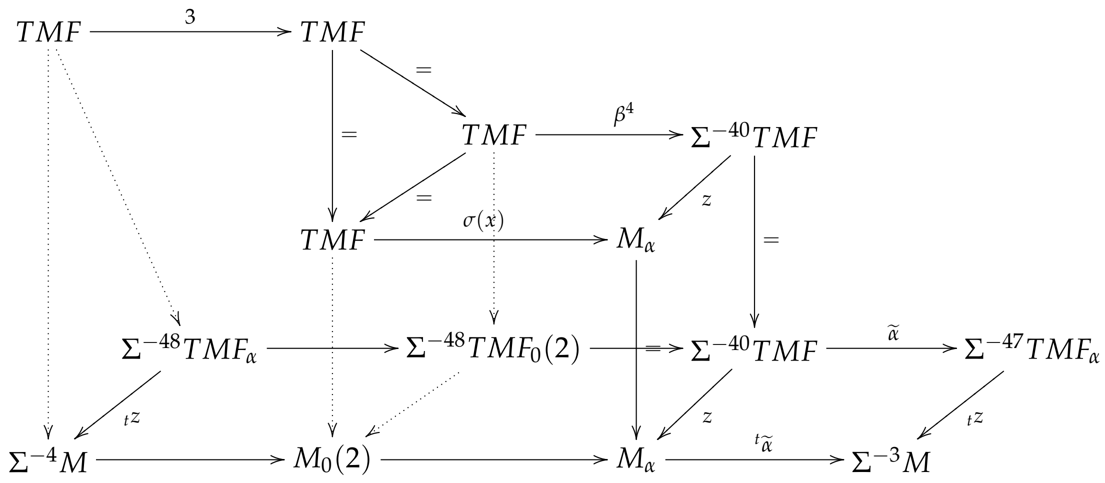
$$\pi_{48} TMF_\alpha \to \pi_{48} TMF_0(2) \to \pi_{40} TMF$$

is $\pm\Delta^2 \in \pi_{48} TMF_0(2)$. Indeed, these span the non-trivial elements in $\pi_{48} TMF_0(2)$ which are mapped trivially into the zero line of the DSS of TMF modulo the image of $\pi_{48} TMF_\alpha$ (as can be seen, for example, by an $\mathrm{im}(r)$-argument). □

Lemma 8.5.8. *All* n_i *are smaller than* 4.

[5] One can check that $\beta\widetilde{\alpha}$ is non-zero and therefore a is non-zero as well. But this is not needed for our argument.

Proof. Assume we have an $x \in E(M)$ such that $\sigma(x) = \beta^4 z$, which is automatically $\neq 0$ since else x would be in the image of c. Look at the following diagram:



Here we use the isomorphism $DTMF_\alpha = \mathrm{Hom}_{TMF}(TMF_\alpha, TMF) \cong \Sigma^{-4}TMF_\alpha$, under which ${}_tz$ corresponds to z as in Lemma 4.2.6 (with $k = 7$ and $Z = TMF_\alpha$). The Toda bracket $\langle \widetilde{\alpha}, \beta^4, 3\rangle$ contains $\{3\Delta^2\}$. Therefore, we get that $\langle {}^t\widetilde{\alpha}, \sigma(x), 3\rangle$ contains $b = {}_tz(\{3\Delta^2\})$. We have $c(b) = 3x'$ by the definition of the Toda bracket with $\sigma(x) = \sigma(x')$. The element x' is invariant (since $3x'$ is), but is not in the image of c (since $\sigma(x') \neq 0$). Hence, the corresponding element $\overline{x'}$ in $H^0(\mathcal{M}; \pi_*\mathcal{F}_M) \cong \pi_*(M(2))^{S_3}$ cannot be a permanent cycle in $DSS(M)$ and hence is not in the image of r. By Proposition 8.2.3, we can find an $y \in \pi_*M(2)$ with $c_{alg}(r_{alg}(\overline{y}) + \overline{x'})$ a generator of a direct summand. Set $x'' = x' + cr(y) \in \pi_*M(2)$. This is clearly an invariant generator. We have that $c(b + 3r(y)) = 3x''$. Furthermore, for $w := {}_tz(1) + r(\Delta^{-2}y) \in \pi_*M$, the following holds:

$$\begin{aligned} 3\Delta^2 c(w) = c(\{3\Delta^2\}w) &= 3x' + cr(\{3\Delta^2\}\Delta^{-2}y) \\ &= 3x' + cr(3y) \\ &= 3x''. \end{aligned}$$

Hence, $c(w) = \Delta^{-2}x''$, which is an invariant generator. This is a contradiction to our global contradiction hypothesis. □

We assume now that $\pi_*\mathcal{F}_M$ has only summands of the form $\mathcal{O}$ and its shifts. Our aim for the rest of the section is to understand the torsion in π_*M and π_*M_α. The arguments will resemble these of our argumentations in the low rank cases. We have that ${}^t\widetilde{\alpha}(\overline{v_i}) = \alpha\overline{w_i}$ for some elements $\overline{w_i}$ in the 0-line of the E^2-term of the descent spectral sequence of M. The $\overline{w_i}$ can be chosen to span the $\mathbb{Z}_{(3)}$-span of elements of the form Δ^i since these generate $H^0_*(\mathcal{M}; \pi_*\mathcal{F}_M)/\ker(\alpha)$ and ${}^t\widetilde{\alpha}$ is surjective onto $H^1(\mathcal{M}; \pi_*\mathcal{F}_M)$. All elements $\beta^k\alpha\overline{w_i}$ are permanent cycles since the β^k are permanent cycles in $DSS(TMF)$. The elements $\beta^k\alpha\overline{w_i}$ for $k \geq n_i$ must be boundaries since ${}^t\widetilde{\alpha}(\beta^k v_i)$ is zero. We know that the $\overline{w_i}$ support nontrivial differentials $d_{p_i}\overline{w_i}$. Hence, also $d_{p_i}(\beta^k\overline{w_i}) \neq 0$ in E^{p_i} by Lemma 8.3.5 since the 1-line consists of permanent cycles. All in all, this implies that all torsion in π_*M is detected by the $\alpha\beta^k\overline{w_i}$ for $k < n_i$. Note that this also implies that all elements of the form $\widetilde{\alpha}\beta^k\overline{v_i}$ cannot be permanent cycles in $DSS(M_\alpha)$ since $\beta^{k+1}\overline{w_i} = {}^t\widetilde{\alpha}(\widetilde{\alpha}\beta^k\overline{v_i})$ is not a permanent cycle. Therefore, the $\beta^k v_i$ span all the torsion in π_*M_α.

Suppose that some linear combination $\Sigma_{i\in I'} a_i\beta^k\overline{v_i}$ is a boundary for $k < n_i$ with $I' \subset I$ non-empty and $a_i \in \{\pm 1\}$. Since $\Sigma_i a_i \beta^k v_i \neq 0$ (by the linear independence statements in the proof of Corollary 8.5.4), $\beta^k \Sigma_i a_i v_i$ must be detected by a permanent cycle of the form $\Sigma_j b_j \beta^m \overline{v_j}$. Assume $\beta^k\Sigma_i a_i v_i = \beta^m \Sigma_j b_j v_j$ and set n to be the maximum of the n_i for $i \in I'$. Then $\sigma(\Sigma_{i \text{ with } n_i = n} a_i g_I) = \beta^n \Sigma_i a_i v_i = \beta^{n+m-k}\Sigma_j b_j v_j$. As in the proof of Corollary 8.5.4, this implies $m = k$, which is not true. Thus, $\beta^k\Sigma_i a_i v_i \neq \beta^m\Sigma_j b_j v_j$ and their difference x is detected a permanent cycle of the form $\Sigma_\nu c_{nu}\overline{v_{nu}}$. As before, $x \neq \Sigma_\nu c_{nu}\overline{v_{nu}}$ and is detected by a permanent cycle of even higher filtration and so on. Since the filtration is bounded by the last lemma, at some point we get an equality, which implies a contradiction as before. Thus, $\Sigma_{i\in I'} a_i\beta^k\overline{v_i}$ is no boundary for $k < n_i$ with $I' \subset I$ non-empty and $a_i \in \{\pm 1\}$.

Suppose now that some linear combination $\Sigma_{i\in I'} a_i \beta^{k_i} v_i$ is in the image of σ for $k_i < n_i$ with $I' \subset I$ non-empty and $a_i \in \{\pm 1\}$. Then $\Sigma_i a_i \beta^k \overline{v_i}$ has to be a boundary for k the minimum of the k_i with $a_i \neq 0 \in \mathbb{F}_3$. Thus, $\Sigma_i a_i\beta^k\overline{v_i} = 0$ in E^2, which implies $\Sigma_i a_i \beta^k v_i = 0$. Thus, arguing as in Corollary 8.5.4, $\Sigma_i a_i \overline{v_i} \in B_{k-1}$, which is a contradiction to that the $\overline{v_i}$ are linear independent in $H^0(\mathcal{M}; \pi_*\mathcal{F}_{M_\alpha})/B_{k-1}$. Thus $\Sigma_i a_i \beta^{k_i} v_i$ cannot be in $\mathrm{im}(\sigma)$.

Since we know thus that the $\mathbb{F}_3$-span of the $\beta^k v_i$ for $k < n_i$ gets mapped injectively into the torsion of $\pi_* M$ by ${}^t\widetilde{\alpha}$, we know by rank comparison that no $\alpha\beta^k\overline{w_i}$ is a boundary. We set $\{\alpha\beta^k w_i\} := {}^t\widetilde{\alpha}(\beta^k v_i)$, which is detected by $\alpha\beta^k\overline{w_i}$ and is therefore in strict filtration $2k+1$.

All in all, we have thus proven the following proposition:

Proposition 8.5.9. *Let $\pi_*\mathcal{F}_M$ have only summands of the form $\mathcal{O}$ and its shifts. Then the torsion of $\pi_* M$ is an $\mathbb{F}_3$-vector space with basis given by $\{\alpha\beta^k w_i\}$ with $k < n_i$ and $i \in I$. The torsion of $\pi_* M_\alpha$ is an $\mathbb{F}_3$-vector space with basis given by $\beta^k v_i$ with $k \leq n_i$ and $i \in I$.*

Warning 8.5.10. Similar to $\{\alpha\Delta\} \in \pi_{27}TMF$, the notation $\{\alpha\beta^k w_i\}$ does not entail that this element is divisible by α. But it is true that $\beta^k\{\alpha w_i\} = \{\alpha\beta^k w_i\}$.

8.6 Multiplication by α

Assumption 8.6.1. *Assume that $\pi_*\mathcal{F}_M$ has only summands of the form $\mathcal{O}$ and its shifts. Furthermore, assume again that M has no invariant generator.*

We use the notation of the last section concerning the v_i, $\{\alpha\beta^k w_i\}$ and n_i. Furthermore, we denote by $F_n = F_n\pi_* M$ the filtration coming from the descent spectral sequence and by $S_n\pi_*M$ the stratum $F_n\pi_*M - F_{n+1}\pi_*M$. The main result of this section is now the following:

Proposition 8.6.2. *There exists always an element x in $S_1\pi_*M$ such that $\alpha x = 0$.*

Proof. The proof will be by contradiction, so we assume that $\alpha x \neq 0$ for all $x \in S_1\pi_*M$. We know already from the last section that $n_i \leq 4$ for all i. The proof has now two parts. First we exclude the case that some $n_i \leq 2$. Finally, we lead the case that all n_i equal 3 to a contradiction.

We get a short exact sequence associated to

$$\Sigma^3 M \xrightarrow{\alpha} M \xrightarrow{i} M_\alpha \xrightarrow{p} \Sigma^4 M$$

of the form

$$0 \to \pi_* M/(\operatorname{im}\alpha) \to \pi_* M_\alpha \to \ker(\alpha) \to 0.$$

We will show that this restricts to a short exact sequence

$$0 \to \{\{\alpha\beta^k w_i\}\}_{\mathbb{F}_3}^{k=0,\dots n_i-1}/(\operatorname{im}\alpha) \to \{\beta^k v_i\}_{\mathbb{F}_3}^{k=0,\dots n_i} \to (\{\{\alpha\beta^k w_i\}\}_{\mathbb{F}_3}^{k=0,\dots n_i-1})_{\alpha=0} \to 0.$$

Here $()_{\alpha=0}$ denotes the elements where multiplication by α is zero. In addition note that the $\mathbb{F}_3$-spans run over all $i \in I$.

The first map restricts since all torsion in $\pi_* M_\alpha$ is spanned by the $\beta^k v_i$ as shown in the last section; it is automatically injective. The elements v_i map to torsion because the $\overline{v_i}$ get mapped to 0 in the spectral sequence since they are in the span of the elements of the form Δ^j and therefore come from M. Hence, the second map restricts.

Suppose an element $z \in \{\beta^k v_i\}$ is in the image of i. Since

$$\{\{\alpha\beta^k w_i\}\}_{\mathbb{F}_3}^{k=0,\dots n_i-1} \subset \pi_* M \to \pi_* M/\operatorname{im}(r_*)$$

is a surjection by Proposition 8.5.2, we can write $z = i(x+y)$, where $x \in \{\alpha\beta^k w_i\}\}$ and $y \in \operatorname{im}(r_*)$. Since by Corollary 8.5.4 the whole 0-line of the DSS of the fiber of $r\colon M(2) \to M$ consists of permanent cycles, $\operatorname{im}(r_*)$ is completely detected by $\operatorname{im}(r_{alg})$ in the 0-line. Since $\beta^k\overline{v_i} \notin i_*(\operatorname{im}(r_{alg}))$, it follows $y = 0$ and we have exactness in the middle term.

If $p_*(x)$ is torsion, then either x is torsion or the reduction $\overline{x} \in \Gamma_*(\pi_*\mathcal{F}_{M_\alpha})$ maps to zero in $\Gamma_*(\pi_*\mathcal{F}_{\Sigma^4 M})$. We know that the $\overline{v_i}$ and $\overline{i_*(\operatorname{im}(r_*))} = i_*(\operatorname{im}(r_{alg}))$ span $\ker(\Gamma_*(\pi_*\mathcal{F}_{M_\alpha}) \to \Gamma_*(\pi_*\mathcal{F}_{\Sigma^4 M})) \cong \operatorname{im}(i_*)$ by Scholium 8.2.4. Since $p_* i_*(\operatorname{im}(r_*)) = 0$ in $\pi_* M$ and all torsion in $\pi_* M_\alpha$ is spanned by the $\beta^k v_i$, we have $p_*(x) = p_*(x')$ for some x' in the span of the $\beta^k v_i$. This proves exactness of the above short exact sequence.

Define $l := \dim_{\mathbb{F}_3[\Delta^{\pm 3}]}(\operatorname{im}(\alpha))$. Since $\operatorname{im}(\alpha) = \operatorname{im}(\alpha|_{\text{-tors}\,\pi_* M})$ (since -tors $\pi_* M$ surjects to $\pi_* M/\operatorname{im}(r_*)$), we see that

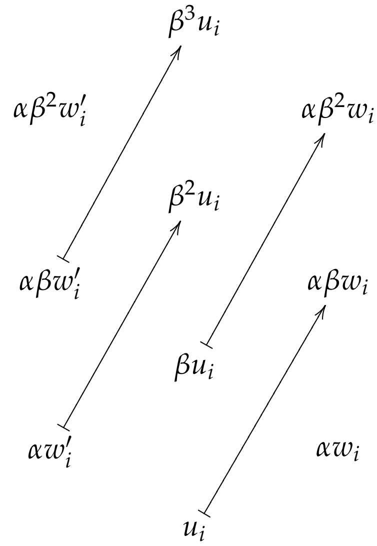
$$\Sigma_i(n_i+1) = 2\Sigma_i n_i - 2l.$$

This is equivalent to

$$2l + 3n = \Sigma_i n_i$$

since $|I| = 3n$ for n the rank of M. We know that all $n_i \leq 3$. Assume that $n_i < 3$ for one i. Then we see that $l < |I|$. Since there are $|I|$ elements $\{\alpha w_i\}$, we have $\alpha \sum_{j\in J} a_j\{\alpha w_j\} = 0$ for suitable $a_j \in \{1,2\}$ and non-empty $J \subset I$, which would imply the proposition.

Now, we are in the situation that all $n_i = 3$ and $l = |I|$. Furthermore, we still assume that α acts non-trivially on all non-zero elements of strict filtration 1. Thus, $\operatorname{im}(\alpha) = \alpha \cdot S_1\pi_* M$ for rank reasons. Suppose that $\alpha x \neq 0$ for x of filtration greater than 1. Then $\alpha x = \alpha y$ for a $y \in S_1\pi_* M$. Thus, $\alpha(y - x) = 0$, which is not possible since $y - x \in S_1\pi_* M$. Thus, α acts trivially on all elements of higher filtration. Hence, we know that $\beta\alpha x = \alpha\beta x = 0$ for $x \in \pi_* M$. Thus, multiplication by α has image in strict filtration 5. More precisely, for rank reasons, it determines an isomorphism $F_1\pi_* M/F_2\pi_* M \to F_5\pi_* M/F_6\pi_* M$. Since $\alpha\{\alpha\beta w_i\} = \alpha\beta\{\alpha w_i\} = 0$, we must have $\{\alpha\beta w_i\} = p_*(\beta^k u_i)$ with u_i of strict filtration 0 in $\pi_* M_\alpha$. Because p_* preserves filtration, $k \leq 1$. If $k = 1$, then $\beta p_*(u_i) = \{\alpha\beta w_i\}$, hence $p_*(u_i) = \{\alpha w_i\}$ and thus $\alpha\{\alpha w_i\} = 0$, which is a contradiction to our assumption. Therefore, $\{\alpha\beta w_i\} = p_*(u_i)$. We see that $p_*(\beta^2 u_i) = 0$. For similar reasons as above, $\beta^2 v = i_*(\{\alpha w_i'\})$ for some $\{\alpha w_i'\}$ in strict filtration 1; indeed, if $\beta^2 v$ is the image of an element of higher filtration, βv is in $\mathrm{im}(i_*)$, but $\beta\{\alpha\beta w_i\} \neq 0$. Thus we get the following picture of a part of the exact sequence induced by $M \to M_\alpha \to \Sigma^4 M$:



Note furthermore that we can write $\{\alpha\beta^2 w_i\} = \alpha(\alpha w_i'')$.

By Lemma 4.6.2, we see that $\langle \alpha, \{\alpha\beta^2 w_i\}, \beta^2\rangle$ contains $\{\alpha\Delta\}\{\alpha w_i''\}$ (since $\langle \alpha, \alpha, \beta^2\rangle$ contains $\{\alpha\Delta\}$) and we know from the picture above that $\beta\{\alpha w_i'\} \in \langle \alpha, \{\alpha\beta^2 w_i\}, \beta^2\rangle$. The indeterminacy is $\beta^2\pi_{*-20}M + \alpha\pi_{*-3}M \subset F_5\pi_*M$. Hence $\beta\{\alpha w_i'\} = \{\alpha\Delta\}\{\alpha w_i''\}$ in $F_3/F_4 \cong F_3/F_5$.

Suppose that the $\Sigma a_i\{\alpha w_i'\} = 0$. Taking i_*, it follows $\beta^2\Sigma a_i u_i = \Sigma a_i\beta^2 u_i = 0$. The kernel of multiplication by β^2 on strict filtration 0 in π_*M_α is contained in $\mathrm{im}(r_*)$. Thus $\Sigma a_i u_i \in \mathrm{im}(r_*)$ and $\Sigma a_i\{a_i w_i\} = p_*(\Sigma a_i u_i) \in \mathrm{im}(r_*)$, which cannot be since $\mathrm{im}(r_*)$ contains no torsion (as noted above). Thus $a_i = 0$ for all i and the $\{\alpha w_i'\}$ are linearly independent. Thus, also the $\beta\{\alpha w_i'\}$.

Hence, multiplication by $\{\alpha\Delta\}$ is a surjective map from $F_1/F_2 = F_1/F_3$ to $F_3/F_4 = F_3/F_5$ and thus, by a dimension count, an isomorphism. But this isomorphism commutes with multiplication by β. Therefore, since multiplication is an isomorphism between F_1/F_2 and the F_3/F_4 and the F_3/F_4 and the F_5/F_6, multiplication by $\{\alpha\Delta\}$ is also an isomorphism between F_3/F_4 and F_5/F_6. This is obviously a contradiction since the square of $\{\alpha\Delta\}$ is zero as $\pi_{54}TMF = 0$. □

8.7 Enlargement and Shrinking

We know that $\pi_*\mathcal{F}_M$ has no $f_*f^*\mathcal{O}$-summand. Our strategy in this section is to enlarge our module M by coning off elements of first filtration to produce $f_*f^*\mathcal{O}$-summands, which can then be killed. This works in an easy way if we have an E_α-summand in $\pi_*\mathcal{F}_M$. If we have no E_α-summand, we get in general only a hook and no invariant generator.

So, suppose first that $\pi_0\mathcal{F}_M$ has an E_α-summand.[6] Furthermore assume that M has no invariant generator. Then we know that every element in the first line of the descent spectral sequence survives by Corollary 8.5.6, especially $\widetilde{\alpha}_{(0)}$ in the direct summand $H^1_*(\mathcal{M}; E_\alpha)$

[6] If some other $\pi_n\mathcal{F}_M$ has a summand of the form E_α, we can deal with this the same way by shifting.

of $H^1_*(\mathcal{M};\pi_*\mathcal{F}_M)$. Take the map $\Sigma^7 TMF \to M$ representing this $\widetilde{\alpha}_{(0)}$. We get a cofiber sequence

$$\Sigma^7 TMF \xrightarrow{\widetilde{\alpha}_{(0)}} M \to X \to \Sigma^8 TMF.$$

This corresponds to a short exact sequence

$$0 \to \pi_*\mathcal{F}_M \to \pi_*\mathcal{F}_X \to \pi_*\Sigma^8\mathcal{O}^{top} \to 0,$$

which corresponds again to the Ext-class $\widetilde{\alpha}_{(0)} \in \mathrm{Ext}^1(\omega^{-4}, \pi_0\mathcal{F}_M)$ by Corollary 6.4.4. That this is short exact can be seen as follows: The DSS of $\Sigma^{-7}M$ is equivalent to the DSS for $\mathcal{H}om(\Sigma^7\mathcal{O}^{top}, \mathcal{F}_M)$ and thus the map $\widetilde{\alpha}_{(0)}$ has filtration 1. Thus, it is send by the edge homomorphism

$$\begin{array}{c} [\Sigma^7 TMF, M] \cong \pi_0\Gamma(\mathcal{H}om(\Sigma^7\mathcal{O}^{top}, \mathcal{F}_M)) \\ \big\downarrow \\ \mathrm{Hom}(\pi_*\Sigma^7\mathcal{O}^{top}, \pi_*\mathcal{F}_M) \cong \Gamma_0(\mathcal{H}om(\pi_*\Sigma^7\mathcal{O}^{top}, \pi_*\mathcal{F}_M)) \end{array}$$

to 0.

Thus $\pi_*\mathcal{F}_X$ contains a summand of the form $f_*f^*\mathcal{O}$. As in Proposition 8.3.7, we get a split map $\overline{y}\colon TMF_0(2) \to X$, which kills the $f_*f^*\mathcal{O}$-summand in $\pi_*\mathcal{F}_X$. Denote its cofiber by Y and the composition $M \to X \cong TMF_0(2) \oplus Y \xrightarrow{\mathrm{pr}_2} Y$ by g. Then g induces a surjective map $\pi_*\mathcal{F}_M \to \pi_*\mathcal{F}_Y$ with kernel $E_\alpha \otimes \pi_*\mathcal{O}^{top}$. Thus $\pi_*\mathcal{F}_{\mathrm{fib}(g)} \cong E_\alpha \otimes \pi_*\mathcal{O}^{top}$ and $Y \cong \Sigma^{24l} TMF_\alpha$ by Proposition 8.3.6. The element $1 \in \pi_{24l} TMF_\alpha$ maps to a $z \in \pi_{24l} M$ with $c(z) \in E(M)$. Thus, an M with an E_α-summand has always an invariant generator.

We can therefore assume that $\pi_*\mathcal{F}_M$ is a direct sum of shifts of $\pi_*\mathcal{O}^{top}$ and we assume again that M has no invariant generator. We want to play the same game as above. Choose a non-zero element $\alpha_{(0)} \in \pi_*M$ in filtration 1 such that $\alpha\alpha_{(0)} = 0$. The reduction $\overline{\alpha_{(0)}} \in H^1(\mathcal{M};\pi_*\mathcal{F}_M)$ is of the form $\alpha \cdot 1_{(0)}$ for some $1_{(0)} \in \Gamma(\pi_*\mathcal{F}_\mathcal{M})$ and by a shift, we assume that $v \in \Gamma(\pi_0\mathcal{F}_\mathcal{M})$. Since $\alpha \cdot \mathrm{im}(r_{alg}) = 0$, we can by Proposition 8.2.3 furthermore assume that the corresponding map $\pi_*\mathcal{O}^{top} \to \pi_*\mathcal{F}_\mathcal{M}$ is the inclusion of a direct summand and we call it the 0-*summand*. We get a cofiber sequence

$$\Sigma^3 TMF \xrightarrow{\alpha_{(0)}} M \to X \to \Sigma^4 TMF.$$

The (induced) 0-summand of X is of the form E_α and in first line of $DSS(X)$ we have elements $\Delta^i\widetilde{\alpha}$. Suppose one of these survives the descent spectral sequence. Then we have a map $\Sigma^k TMF \to X$ whose cofiber is a TMF-module Z of the form $TMF_0(2) \oplus Y$ as above. The fiber of the map

$$M \to X \to Z \cong TMF_0(2) \oplus Y \xrightarrow{\mathrm{pr}_2} Y$$

has rank 1 and is therefore isomorphic to $\Sigma^l TMF$ for some $l \in \mathbb{Z}$ by Proposition 8.3.6. The image z of $1 \in \pi_l\Sigma^l TMF$ in $\pi_l M$ satisfies $c_*(z) \in E(M)$. Thus, we can assume that none of the $\Delta^i\widetilde{\alpha}$ is a permanent cycle. Suppose that y is another element in the first line of the DSS of X projecting to the 0- summand as $\Delta^i\widetilde{\alpha}$. Then y can also be no permanent cycle since every element projecting to 0 in the 0-summand is in the image of $DSS(M) \to DSS(X)$ and therefore a permanent cycle.

Since $\alpha\alpha_{(0)} = 0$, there is an element $x \in \pi_7 X$ which is sent to $\alpha \in \pi_7\Sigma^4 TMF$. Since $\widetilde{\alpha} \in E^2(DSS(X))$ does not survive, x must live in filtration 0. The 0-summand has no elements in this degree and filtration. Therefore the projection of $\overline{x}$ to the 0-summand is zero. By Proposition 8.2.3, x can even be chosen such that $c(x) \in E(X)$ since outside the 0-summand $\mathrm{im}(r_*)$ maps to 0 in $\pi_*\Sigma^4 TMF$. Since X is algebraically standard, we can argue as in Proposition 8.4.2 that we can modify x by $\mathrm{im}(r_*)$ even in a way such that the cokernel of $\pi_*\Sigma^7\mathcal{O}^{top} \to \pi_*\mathcal{F}_X$ is standard.

Consider the cofiber sequence $\Sigma^7 TMF \xrightarrow{x} X \to X'$. Then $\pi_*\mathcal{F}_{X'}$ contains still a summand of the form E_α and is algebraically standard of $TMF(2)$-rank n. Therefore, we can apply the results of the beginning of the section and see that X' has an invariant generator, more precisely an $x \in \pi_{|x|}X'$ such that Cofiber($\Sigma^{|x|}TMF \to X'$) has rank one less than M. This provides a "hook" for M and the main theorem follows inductively:

Theorem 8.7.1. *Every algebraically standard module is hook-standard.*

We still have to show that every algebraically standard TMF-module M of rank ≤ 3 is standard. By Section 8.4, it is enough to show that every such module has an invariant generator. So, suppose that M has no invariant generator. Thus, we get a cofiber sequence

$$\Sigma^3 TMF \xrightarrow{\alpha_{(0)}} M \to X \to \Sigma^4 TMF.$$

and an $x \in \pi_7 X$ as above (reinstancing these shifting conventions). Furthermore, $\pi_*\mathcal{F}_M$ is a sum of shifts of $\pi_*\mathcal{O}^{top}$. We fix an element $1_{(0)} \in \Gamma_*(\mathcal{F}_M)$ such that $\alpha 1_{(0)}$ detects $\alpha_{(0)}$. Suppose that $d_5^M(1_{(0)}) = \alpha\beta^2\Delta^{-1}1_{(0)}$. Then

$$\begin{aligned} d_5^M(\Delta^2 1_{(0)}) &= d_5^{TMF}(\Delta^2)1_{(0)} + \Delta^2 \cdot d_5^M(1_{(0)}) \\ &= -\alpha\beta^2\Delta 1_{(0)} + \alpha\beta^2\Delta 1_{(0)} \\ &= 0 \end{aligned}$$

If $d_5^M(1_{(0)}) = -\alpha\beta^2\Delta^{-1}1_{(0)}$, we can do the same argumentation with Δ instead of Δ^2. Thus, we cannot have for all $i \in \{0,1,2\}$ that $d_5^M(\Delta^i 1_{(0)}) = \pm\alpha\beta^2\Delta^{i-1}1_{(0)}$. A non-zero differential in $DSS(M)$ can only be of length 3, 5, 7 or 9 (as can be seen in the argumentation at the end of Section 8.5) and the $\Delta^i 1_{(0)}$ must support non-zero differentials since otherwise M would have an invariant generator. Thus, $H^{2k+1}(\mathcal{M}; \omega^k \otimes \pi_*\mathcal{F}_M)$ consists not only of $\alpha\beta^2 1_{(0)}\mathcal{F}_3$ for $1 \leq k \leq 4$. Checking dimension, this yields that $\pi_*\mathcal{F}_M$ has an (additional) summand of the form $\pi_*\Sigma^k\mathcal{O}^{top}$ for $k = 0, 4, 10$ or 14 (for $k = 0$ this means that we have two summands of the form $\pi_*\mathcal{O}^{top}$).

The element x reduces to an $\overline{x} \in \Gamma(\pi_7\mathcal{F}_X)$ not in $\mathrm{im}(r_{alg})$. Since $\Gamma(\pi_*\mathcal{F}_M) \to \Gamma(\pi_*\mathcal{F}_X)$ is an isomorphism in odd degrees, $\overline{x}$ is the image of an element $\overline{x}'$ in $\Gamma(\pi_*\mathcal{F}_M)$ not in $\mathrm{im}(r_{alg})$. Thus, $\pi_*\mathcal{F}_M$ has a summand of the form $\pi_*\Sigma^7\mathcal{O}^{top}$. Arguing for $\overline{x}'$ as for $1_{(0)}$ above, we get that $\pi_*\mathcal{F}_M$ has an (additional) summand of the form $\pi_*\Sigma^k\mathcal{O}^{top}$ for $k = 7, 11, 17$ or 21. Thus, $\pi_*\mathcal{F}_M$ has rank at least 4 and it follows that every algebraically standard module of rank ≤ 3 has an invariant generator and is thus standard.

Chapter 9

Examples and Application

In this chapter, we will present first an infinite family of indecomposable standard $TMF_{(3)}$-modules. Next, we will consider $\mathbb{CP}^\infty \wedge TMF$ and $BU(2) \wedge TMF$ and the analogous modules also for connective tmf. At last, we will depict the rank 1 and 2 (algebraically) standard $TMF_{(3)}$-modules.

9.1 An Infinite Family of Modules

In this section, we will again localize at 3 and write TMF for $TMF_{(3)}$.

Roughly the example of an infinite family is the following: Consider

$$C(\beta^3, \beta^4, \beta^3, \ldots, \beta^4, \beta^3) \text{ and } C(\beta^3, \beta^4, \beta^3, \ldots, \beta^4).$$

These exist since $\langle \beta^3, \beta^4, \beta^3, \ldots, \beta^4, \beta^3 \rangle$ and $\langle \beta^3, \beta^4, \beta^3, \ldots, \beta^3, \beta^4 \rangle$ lie in $\pi_k TMF$ with $k = 70$ or $k = 29$ mod 72 and these groups are zero. If one of these modules split into two standard modules, it would have two invariant generators (in the sense of the last chapter). The second generator would have to lift from a torsion element somewhere – which is not possible for degree reasons.

More precisely define $X_1 = TMF$ and $x_1 \in \pi_{30} TMF$ to be β^3. Now assume that X_k has been defined and also $x_k \in \pi_{30} X_k$ if k is odd or $x_k \in \pi_{71} X_k$ if k is even. Furthermore, we assume inductively that $\pi_{70} X_k = 0$ and $\pi_{29} X_k = 0$. Define $X_{k+1} = \mathrm{Cone}(\Sigma^{|x_k|} TMF \to X_k)$. First consider the case that k is odd. Then we have an exact sequence

$$\pi_{71} X_k \to \pi_{71} X_{k+1} \to \pi_{71} \Sigma^{31} TMF \to \pi_{70} X_k.$$

This implies that there is a lift of $\beta^4 \in \pi_{71} \Sigma^{31} TMF$ to $\pi_{71} X_{k+1}$, which we define to be x_{k+1} (any choice is possible). Furthermore, we see that $\pi_{70} X_{k+1} = 0$ since $\pi_{70} X_k = 0$ and $\pi_{39} TMF = 0$. The same way, we see that $\pi_{29} X_{k+1} = 0$ since $\pi_{29} X_k = 0$ and $\pi_{70} TMF = 0$.

Now consider the case that k is even. Then we have an exact sequence

$$\pi_{30} X_k \to \pi_{30} X_{k+1} \to \pi_{30} \Sigma^{72} TMF \to \pi_{29} X_k.$$

This implies that there is a lift of $\beta^3 \in \pi_{30} \Sigma^{72} TMF$ to $\pi_{30} X_{k+1}$, which we define to be x_{k+1} (again, any choice is possible). Furthermore, we see that $\pi_{70} X_{k+1} = 0$ since $\pi_{70} X_k = 0$ and $\pi_{70} TMF = 0$. The same way, we see that $\pi_{29} X_{k+1} = 0$ since $\pi_{29} X_k = 0$ and $\pi_{29} TMF = 0$.

Before we go on, we want to define an invariant of TMF-modules. For a TMF-module M, consider $\pi_* M / \operatorname{im}(r_*)$. This is an $\mathbb{F}_3[\Delta^{\pm 3}]$-vector space since $rc = 6$. Set now

$$d(M) := \dim_{\mathbb{F}_3[\Delta^{\pm 3}]} \left(F_0 \pi_* M / (\operatorname{im}(r_*) + F_1 \pi_* M)\right),$$

where $F_\bullet$ denotes the filtration of the descent spectral sequence.

Lemma 9.1.1. *If $\pi_* \mathcal{F}_M$ consists of a direct sum of shifts of the structure sheaf, then $d(M) > 0$. Furthermore, d sends direct sums to sums.*

Proof. Let $x \in \pi_* M$ be an element such that $c(x) \in \pi_* M(2)$ generates a direct $TMF(2)_*$-summand of $\pi_* M(2)$ (this exists by the TMF-extension theorem).[1] The element $c(x)$ corresponds to a tuple $(a_1, a_2, \ldots, a_n)$ if we choose a basis for $M(2)_*$. If $d(M) = 0$, x is in the submodule $\bigoplus(3, c_4, c_6) \subset H^0(\mathcal{M}; \pi_* \mathcal{F}_M)$ since $\operatorname{im}(r_{alg}) = (3, c_4, c_6)$ by Scholium 8.2.4. If we reduce modulo 3, we see that all a_i are divisible by $(x_2 + y_2)$ by the formulas in Section 2.5. But if $c(x)$ generates a direct summand, there must be a linear combination $\lambda_1 a_1 + \cdots + \lambda_n a_n = 1$. By Lemma 8.4.1, the element $1 \in \pi_* TMF(2)$ is not in $(3, x_2 + y_2)$, so we have a contradiction. Hence, $d(M) > 0$ for all relatively free modules with $\pi_* \mathcal{F}_M$ being a direct sum of shifts of the structure sheaf. □

Proposition 9.1.2. *The TMF-modules X_k are not decomposable in the homotopy category of TMF-modules into TMF-standard modules. If an X_k decomposes, it decomposes into two algebraically standard modules of which exactly one is standard.*

Proof. For contradiction, let $X_k \cong A \oplus B$ for some k with A and B non-zero. We want to show that $\pi_* \mathcal{F}_A$ and $\pi_* \mathcal{F}_B$ are sums of shifts of $\pi_* \mathcal{O}^{top}$: We know that $\pi_* \mathcal{F}_{X_k}$ decomposes into an even part $\bigoplus \pi_* \mathcal{O}^{top}$ and an odd part $\bigoplus \pi_* \Sigma^{31} \mathcal{O}^{top}$ (using Corollary 6.4.4), which can be treated separately. It is enough to show that every direct summand $\mathcal{E}$ of $\bigoplus \mathcal{O} = \bigoplus \pi_0 \mathcal{O}^{top}$ is again a direct sum of the form $\bigoplus \mathcal{O}$. We know that $\Gamma(\mathcal{E})$ is a projective $\Gamma(\mathcal{O})$-module. Thus, $\Gamma(\mathcal{E})$ is a free $\mathbb{Z}_{(3)}[j]$-module by Seshadri's Theorem, a special case of Serre's conjecture (see [Lam06], II.6.1). Choose a basis $(a_1, \ldots, a_n)$ of $\Gamma(\mathcal{E})$ as a $\mathbb{Z}_{(3)}[j]$-module and consider the associated morphism $f\colon \bigoplus_{i=1}^n \mathcal{O} \to \mathcal{E}$. For a complement $\mathcal{G}$ of $\mathcal{E}$ in $\pi_0 \mathcal{F}_{X_k} \cong \bigoplus \mathcal{O}$, we can do the same and get a morphism $g\colon \bigoplus \mathcal{O} \to \mathcal{G}$. The morphism

$$f \oplus g\colon \bigoplus \mathcal{O} \to \mathcal{E} \oplus \mathcal{G} \xrightarrow{\cong} \bigoplus \mathcal{O}$$

is an isomorphism on Γ, hence of the vector bundles. Therefore, also f is an isomorphism and $\mathcal{E}$ is free (since $0 = \operatorname{coker}(f \oplus g) = \operatorname{coker}(f) \oplus \operatorname{coker}(g)$).

Thus, $d(A)$ and $d(B)$ are greater than 0 and the quantity $d(X_k)$ had to be at least 2. We want to prove by induction that $d(X_k) = 1$. This is obviously true for $k = 1$. The E^2-term of the DSS shows that X_k can have "generators" (that is, elements in $F_0 \pi_* X_k / F_1 \pi_* X_k$ which are not in $\operatorname{im}(r_*)$) only in dimensions $0, 24, 48, 31, 55$ and 7. It is easy to check that neither TMF nor $\Sigma^{31} TMF$ have any torsion there. So, given a generator x in $\pi_* X_{k+1}$, it has to map to some element y of (strict) filtration 0 in $\pi_* TMF$ or $\pi_* \Sigma^{31} TMF$. Now note that $X_{k+1}(2)$ splits into $X_k(2)$ and (a suspension of) $TMF(2)$ and therefore every element in $\operatorname{im}(r_*)$ in TMF has a lift to an element in X_{k+1} (which lies also in $\operatorname{im}(r_*)$). The $\mathbb{Z}_{(3)}[\Delta^{\pm 3}]$-module $F_0 \pi_* TMF / F_1 \pi_* TMF$ is generated by $\operatorname{im}(r_*)$ and 1 by Scholium 8.2.4. Therefore, we can subtract from x an element z in $\operatorname{im}(r_*)$ and it maps (up to a unit) to 1 or 0 in TMF_*.

[1] Here, we use the same notation as before and denote by $M(2)$ the $TMF(2)$-module $M \wedge_{TMF} TMF(2)$

But 1 cannot lift. Therefore, $x = i_*(x') + z$, where $x' \in \pi_* X_k$ is of strict filtration 0 and $i_*\colon \pi_* X_k \to \pi_* X_{k+1}$ is the map given by the construction of X_{k+1}. Since x is not in $\mathrm{im}(r_*)$, x' cannot be in $\mathrm{im}(r_*)$. Hence, x' is a generator and generators are by induction unique in $\pi_* X_k$ up to the image of r. Therefore, generators in $\pi_* X_{k+1}$ are unique up to multiplication by units and addition of $(\mathrm{im}(r_*) + F_1 \pi_* X_{k+1})$ and $d(X_{k+1}) = 1$ follows.

This implies that there is no splitting of one of the X_k into standard modules. □

Note that the proof also excludes stable splittings, i.e., isomorphisms $X_k \oplus D \cong A \oplus B \oplus D$ with $A, B \neq 0$ standard modules, since d respects sums.

9.2 Computing the Vector Bundle Associated to a Space

Let X be a finite spectrum such that $MU \wedge X$ is a free MU-module. Since $TMF(2)$ is Landweber exact, we have that $TMF(2) \wedge_{TMF_{(3)}} (TMF_{(3)} \wedge X)$ is $TMF(2)$-free. Indeed $\pi_* TMF(2) \wedge X = MU_*(X) \otimes_{MU_*} TMF(2)_*$. Therefore, $M_X := TMF_{(3)} \wedge X$ is a relatively free $TMF_{(3)}$-module. Thus, $\pi_* \mathcal{F}_{M_X}$ is a vector bundle (in some sense, it is the totality of all elliptic homology theories evaluated at X). A similar argument can be made at other primes.

The question we want to pose is: How can one determine the vector bundle on the moduli stack of elliptic curves associated to X for well-known spaces like $X \cong \mathbb{CP}^n$? The strategy is like follows: $MU_* X$ has the structure of a MU_*MU-comodule (with explicit formulas), corresponding to a quasi-coherent sheaf on $\mathcal{M}_{FG}$. For an elliptic curve E over a ring R with automorphism group G, the formal group $\hat{E}$ gives rise to a morphism $\operatorname{Spec} R//G \to \mathcal{M}_{FG}$, factoring over $\mathcal{M}$, and we can pull the quasi-coherent sheaf back along this map to do concrete calculations.

Proposition 9.2.1. *Let K be a (MU_*, MU_*MU)-comodule with coaction map ψ and let E be an elliptic curve over a ring R with chosen formal coordinate z. Furthermore, let s be an automorphism of E, sending z to $z + a_1 z^2 + a_2 z^3 + \cdots$ with $a_i \in R$. Let $\mathcal{F}_K$ be the quasi-coherent sheaf associated to K on $\mathcal{M}_{FG}$ and $F\colon \operatorname{Spec} R \to \operatorname{Spec} R//\langle s \rangle \to \mathcal{M}_{FG}$ the morphism classifying $\hat{E}$ and let $f\colon MU_* \to R$ be classifying $(\hat{E}, z)$.*

Then $\Gamma(F^*\mathcal{F}_K) \cong K \otimes_{MU_*} R$ and $s \cdot (x \otimes 1) = \sum x_i \otimes f(P_i)(a_1, a_2, \dots)$ for $\psi(x) = \sum x_i \otimes P_i$, $P_i \in MU_*[b_1, b_2, \dots]$.

Proof. By Proposition 2.6.6 and the discussion in Section 2.8, the action of s on $\mathcal{F}_K(\operatorname{Spec} R) \cong K \otimes_{MU_*} R$ is given by the action of the power series $z + a_1 z^2 + \cdots \in H(\operatorname{Spec} R)$ on $\mathcal{F}_K(\operatorname{Spec} R)$ (for H as in Section 2.8). The correspondence between H-action and MU_*MU-comodule structure implies that $(z + a_1 z^2 + \cdots) \cdot (x \otimes 1) = \sum x_i \otimes f(P_i)(a_1, a_2, \dots)$. □

At the prime 3, we consider the elliptic curve E with the equation $y^2 = x^3 - x$ over $\mathbb{F}_3$. We choose the automorphism s, mapping $y \mapsto y$ and $x \mapsto x + 1$, generating the group C_3. The coordinate transformation $z = -\frac{x}{y}, w = -\frac{1}{y}$ sends the neutral element $(0, \infty)$ to $(0, 0)$. In this coordinates, s has the form $z \mapsto z + w, w \mapsto w$. Note that $x = \frac{z}{w}$ and $y = -\frac{1}{w}$. The equation $y^2 = x^3 - x$ becomes transformed to

$$\begin{aligned} \frac{1}{w^2} &= \frac{z^3}{w^3} - \frac{z}{w} \\ \Leftrightarrow w &= z^3 - zw^2 \end{aligned}$$

We get

$$\begin{aligned} w &= z^3 - zw^2 = z^3 - z(z^3 - zw^2)^2 = z^3 - z^7 - z^5w^2 - z^3w^4 \\ &= \cdots = z^3 - z^7 + z^{11} - z^{15} + z^{19} \cdots \end{aligned}$$

This gives a formal expression for w in terms of z. Probably the pattern continues, but it won't be important for our purposes. This implies that s is given in formal coordinates by

$$z \mapsto z + w = z + z^3 - z^7 + z^{11} - z^{15} + z^{19} \cdots .$$

To apply Proposition 9.2.1 to $X = \mathbb{CP}^n$, we have to recall its (MU_*, MU_*MU)-comodule structure. The Atiyah–Hirzebruch spectral sequence for $\mathbb{CP}^n$ collapses and so we have $\widetilde{MU}_*(\mathbb{CP}^n) \cong MU_*\{\beta_i\}_{i=1,\ldots,n}$.

Theorem 9.2.2 ([Ada74], Proof of II.11.3)**.** *The coaction map*

$$\psi\colon\ \widetilde{MU}_*(\mathbb{CP}^n) \to MU_*MU \otimes_{MU_*} \widetilde{MU}_*(\mathbb{CP}^n)$$

is given by

$$\psi(\beta_i) = \sum_{0 \le j \le i} (\sum_{0 \le k} b_k)^j_{i-j} \otimes \beta_j.$$

Here, the lower index $i-j$ denotes the degree of the term (where $|b_k| = k$) and $b_0 = 1$.

We can easily deduce from this also the comodule structure for $\mathbb{HP}^n$. The map

$$p\colon\ \mathbb{CP}^{2n+1} \cong S^{2n+3}/U(1) \to S^{2n+3}/Sp(1) \cong \mathbb{HP}^n$$

is surjective on (MU_*-)homology. Set $\gamma_i = p_*\beta_{2i}$. We get the comodule structure for $\mathbb{HP}^n$ by replacing β_{2i} by γ_i and ignoring odd degree classes.

As noted above, we have for s the coefficients $a_2 = 1, a_6 = -1, \ldots$. Thus, we have by Proposition 9.2.1, up to terms of degree lower then $6k-4$, the equations (in $\mathbb{F}_3\{\beta_i\}_{i=1,\ldots n}$):

$$\begin{aligned} s \cdot \beta_{6k} &= \beta_{6k} + \binom{6k-2}{1}\beta_{6k-2} + \binom{6k-4}{2}\beta_{6k-4} = \beta_{6k} + \beta_{6k-2} + \beta_{6k-4} \\ s \cdot \beta_{6k-2} &= \beta_{6k-2} + \binom{6k-4}{1}\beta_{6k-4} = \beta_{6k-2} - \beta_{6k-4} \\ s \cdot \beta_{6k-4} &= \beta_{6k-4} \end{aligned}$$

This subquotient representation corresponds therefore to the matrix $\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Changing the basis to $(\beta_{6k-4}, -\beta_{6k-2}, \beta_{6k-2} + \beta_{6k})$, we get the matrix $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = J_3$. As shown in Section 3.5, the only standard vector bundle on $\mathcal{M}_{(3)}$ inducing J_3 is $f_*f^*\mathcal{O}$. The vector bundles $\pi_0\mathcal{F}_{M_{\mathbb{CP}^n}}$ and $\pi_0\mathcal{F}_{M_{\mathbb{HP}^n}}$ are both standard as can be deduced from their cell structure. This implies that both $\pi_0\mathcal{F}_{M_{\mathbb{CP}^n}}$ and $\pi_0\mathcal{F}_{M_{\mathbb{HP}^n}}$ have $f_*f^*\mathcal{O}$ as a subquotient and, hence, as a summand. Thus, by induction, $\pi_0\mathcal{F}_{M_{\mathbb{HP}^n}}$ is a sum of summands of the form

$f_*f^*\mathcal{O}$ and we have $\mathbb{HP}^n \wedge TMF_{(3)}$ is a sum of summands of the form $TMF_0(2)$ by Proposition 8.3.7 if $3|n$ and has else a rest of at most rank 2.[2] To get a similar conclusion for $\mathbb{CP}^n$, we have also to consider the odd degree parts. Here, we have (modulo parts of degree lower than $6k-7$)

$$\begin{aligned} s\cdot\beta_{6k-3} &= \beta_{6k-3} + \binom{6k-5}{1}\beta_{6k-5} + \binom{6k-7}{2}\beta_{6k-7} = \beta_{6k-3}+\beta_{6k-5}+\beta_{6k-7} \\ s\cdot\beta_{6k-5} &= \beta_{6k-5} + \binom{6k-7}{1}\beta_{6k-7} = \beta_{6k-5}-\beta_{6k-7} \\ s\cdot\beta_{6k-7} &= \beta_{6k-7} \end{aligned}$$

Hence, all odd degree parts beginning with β_5 split also in $f_*f^*\mathcal{O}$-summands. Furthermore, we have

$$\begin{aligned} s\cdot\beta_3 &= \beta_3 + \binom{1}{1}\beta_1 = \beta_3+\beta_1 \\ s\cdot\beta_1 &= \beta_1, \end{aligned}$$

hence one J_2-summand, corresponding to an E_α-summand on vector bundles. The module $TMF_{(3)} \wedge \mathbb{CP}^\infty$ without all its $TMF_0(2)$-parts is of rank 2 and has E_α as its vector bundle and has therefore to be a shift of TMF_α.

All in all, we conclude that $TMF_{(3)} \wedge \mathbb{CP}^\infty$ decomposes into a sum of shifts of TMF_α and infinitely many copies of $TMF_0(2)$.

Corollary 9.2.3. *A homotopy commutative and homotopy associative $TMF_{(3)}$-algebra R is complex orientable iff $\alpha\cdot 1 = 0$ in π_*R.*

Proof. Recall that a complex orientation is a class in $R^2(\mathbb{CP}^\infty)$ restricting to the standard generator $1 \in R^2(\mathbb{CP}^1) \cong R^2(S^2) \cong \pi_0 R$. The above discussion shows that the map $TMF_{(3)} \wedge \mathbb{CP}^1 \to TMF_{(3)} \wedge \mathbb{CP}^\infty$ factors as

$$TMF_{(3)} \wedge \mathbb{CP}^1 \to Z \to TMF_{(3)} \wedge \mathbb{CP}^3 \to TMF_{(3)} \wedge \mathbb{CP}^\infty$$

for a $TMF_{(3)}$-module $Z \cong \Sigma^? TMF_\alpha$ such that $Z \to TMF_{(3)} \wedge \mathbb{CP}^\infty$ is the inclusion of a direct summand; thus, we have also a factorization

$$R^2(\mathbb{CP}^\infty) \to R^2(\mathbb{CP}^3) \to [Z, \Sigma^2 R]_{TMF_{(3)}} \to R^2(\mathbb{CP}^1).$$

Hence, it is enough to show that $1 \in R^2(\mathbb{CP}^1)$ has a lift to $R^2(\mathbb{CP}^3)$. Since $\eta = 0$ at the prime 3, we have $\mathbb{CP}^2 \cong \mathbb{CP}^1 \vee S^4$ at 3. Thus, we have a cofiber sequence

$$S^5 \to S^2 \vee S^4 \to \mathbb{CP}^3.$$

The map $S^5 \to \mathbb{CP}^1 \vee S^4$ is non-zero stably at 3, since the Steenrod power operation $\mathcal{P}^1$ is non-zero on $\mathbb{CP}^3$, thus the map is stably equivalent to $(\pm\alpha_1, 0)$ (where we identify $\mathbb{CP}^1$ with S^2 again). Thus, 1 lifts to $\mathbb{CP}^3$ exactly iff $\alpha\cdot 1 = 0$ in π_*R. □

[2]To get around the question whether the complement of $f_*f^*\mathcal{O}$ is a standard vector bundle again, one can argue as follows: Since $f_*f^*\mathcal{O}$ splits off from $\pi_0\mathcal{F}_{M_{\mathbb{HP}^n}}$, the representation J_3 splits off from the C_3-representation. Arguing as above, we get an additional J_3-summand. This implies, since $\pi_0\mathcal{F}_{M_{\mathbb{HP}^n}}$ is standard, that $\pi_0\mathcal{F}_{M_{\mathbb{HP}^n}}$ has at least two $f_*f^*\mathcal{O}$-summands. These split off again, so we can argue as before and get $\lfloor \frac{n}{3} \rfloor$ summands of the form J_3 in the C_3-representation and, hence, $\lfloor \frac{n}{3} \rfloor$ summands of the form $f_*f^*\mathcal{O}$ split off from $\pi_0\mathcal{F}_{M_{\mathbb{HP}^n}}$.

The following question remains open:

Question 9.2.4. *Are there finite CW-complexes X such that $TMF_{(3)} \wedge X$ is a relatively free indecomposable $TMF_{(3)}$-module of arbitrary high-rank?*

Computing the associated (standard) vector bundles on the moduli stack of elliptic curves is here of little help since the indecomposable ones have rank bounded by 3 (as shown in Theorem 3.0.5).

The situation is much more interesting for $p = 2$, but our results are less complete. Again, we try to detect big indecomposable summands in $\mathbb{CP}^n \wedge TMF_{(2)}$ (or $BU(k) \wedge TMF_{(2)}$) by studying an associated representation.

More precisely, we consider the elliptic curve E given by the equation $y^2 + y = x^3$ over $\mathbb{F}_4$. As noted in Section 3.5, there is a subgroup of the automorphism group of E isomorphic to the quaternion group Q with 8 elements. As before, for an X with free MU-homology, we can set $M_X = X \wedge TMF_{(2)}$ and get an associated vector bundle $\pi_0\mathcal{F}_{M_X}$. Via E, we get then an associated Q-representation R_X over $\mathbb{F}_4$ as in Section 3.5. If R_X has an indecomposable summand of dimension k, then $\pi_0\mathcal{F}_{M_X}$ has an indecomposable summand of dimension $\geq k$ and likewise X has an indecomposable $TMF_{(2)}$-module summand of $TMF(2)$-rank $\geq k$. We are using here the theorem of Krull–Remak–Schmidt, which says that an artinian and noetherian module has an (essentially) unique decomposition into indecomposable summands.

We will summarize now a few computations we did with Magma (more precisely described in Appendix A). Using Proposition 9.2.1, we calculated decompositions of R_X for $X = \mathbb{CP}^n$. At the beginning, the dimension of the biggest indecomposable summand is increasing quickly, with a summand of dimension 7 for $n = 8$. The first summand of dimension 8 appears at $n = 16$. Contrary to what might be expected, there is no summand of dimension 9 at $n = 32$ and it is unclear if the dimensions of the indecomposable summands stay bounded or not.

The second series of computations concerns $X = BU(2)$. Recall that $MU_*(BU(2)) = MU_*[c_1, c_2]$. We define R_n to be the subquotient of R_X corresponding to polynomials in the c_i of degree $\leq n$. Note that this provides lower bounds on the size of indecomposable summands in the same way as above. Here, we get that R_5 is an indecomposable $\mathbb{F}_4[Q]$-module of rank 15, but for higher n, the rank of the biggest indecomposable summand of R_n is smaller in the range we computed.

9.3 The Connective Case

We have focused our attention so far mainly on modules over TMF. But also modules over connective tmf are worth considering; even if one is, at the end, only interested in TMF-modules as the following proposition shows:

Proposition 9.3.1. *Every standard TMF-module M is of the form $TMF \wedge_{tmf} M_0$ for a tmf-module M_0.*

Proof. We prove this by induction. For rank 0, this is clear. Now assume, we have proven the statement for rank n and M is a standard TMF-module of rank $n+1$. There is then a standard TMF-module N of rank n together with a map $\Sigma^k TMF \to N$ (representing a torsion element $x \in \pi_* N$) whose cofiber is isomorphic to M. Choose a tmf-module N_0 such that $TMF \wedge_{tmf} N_0 \cong N$. Since $TMF \cong tmf[\Delta^{-1}]$, we have $\pi_* N \cong \pi_* N_0[\Delta^{-1}]$ and there is an element x_0 in $\pi_* N_0$ such that $x_0 \mapsto \Delta^{3l} x$ under the morphism $N_0 \to N$ induced by $tmf \to TMF$. Since TMF is Δ^3-periodic, we can assume $l = 0$. Thus, we have that $TMF \wedge_{tmf} \mathrm{Cofiber}(\Sigma^k tmf \xrightarrow{x_0} N_0) \cong M$. □

The study of tmf-modules is in certain aspects more accessible by the fact that ordinary homology comes here to our help. The following is partially based on ideas from [Hil07], although we will prefer to do our computations in *co*homology.

We will work at the prime 3 in this section and set $tmf = tmf_{(3)}$ and $H = H\mathbb{F}_3$. By taking coconnective cocover and then reducing, we get a morphism $tmf \to H$, which is a ring map and induces, hence, a tmf-module structure on H. Set $C = C(\alpha, \widetilde{\alpha}) \simeq \Sigma^{-4}\mathbb{H}\mathbb{P}^3$ and let $V(1)$ be the Toda–Smith complex where 3 and v_1 are zero. As in [Hil07], we have a cofiber sequence

$$\Sigma^8 tmf \wedge C \wedge V(1) \to tmf \wedge C \wedge V(1) \to H \to \Sigma^9 tmf \wedge C \wedge V(1)$$

Mapping into H in tmf-mod, we get a diagram

$$\begin{array}{ccccc} H^*(\Sigma^9 C \wedge V(1)) & \longrightarrow & H^*_{tmf} H & \longrightarrow & H^*(C \wedge V(1)) \\ \downarrow & & \downarrow u & & \downarrow \\ H^*(tmf) \otimes H^*(\Sigma^9 C \wedge V(1)) & \longrightarrow & H^* H & \longrightarrow & H^*(tmf) \otimes H^*(C \wedge V(1)) \end{array}$$

Here, we use the notation $H^*_{tmf} M = [M, H]^*_{tmf}$ for a tmf-module M. The right square consists of isomorphisms in degrees smaller than 8. In particular, the element $\mathcal{P}^1 \in H^* H$ lifts to $H^*_{tmf} H$.

The Hopf algebra $H^*_{tmf} H$ acts on $H^*_{tmf} M$ for every tmf-module M via

$$H^*_{tmf} H \otimes H^*_{tmf} M \cong \pi_* \left(\mathrm{Hom}_{tmf}(H, H) \wedge_H \mathrm{Hom}_{tmf}(M, H)\right) \to \pi_* \mathrm{Hom}_{tmf}(M, H) = H^*_{tmf} M$$

as composition. For $M = tmf \wedge X$, we have $H^*_{tmf} M \cong H^* X$ and $H^*_{tmf} H$ acts via u and the usual action of the Steenrod algebra. This gives us the following:

Proposition 9.3.2. *Let X be a CW-complex of finite type such that $H^* X$ has an indecomposable graded $\mathbb{F}_3[\mathcal{P}^1]$-summand of rank n. If we write $tmf \wedge X \cong M_1 \oplus \cdots \oplus M_k$ as a sum of tmf-modules, then there is an i such that $H^*_{tmf} M_i$ has rank at least n.*

Proof. Since the indecomposable summand can only be in finitely many degrees, we can assume X to be finite. Then $H^* X$ is a noetherian and artinian $\mathbb{F}_3[\mathcal{P}^1]$-module. Thus, the decomposition into indecomposables is (essentially) unique by the theorem of Krull–Remak–Schmidt. Hence, one of the modules M_i must contain the indecomposable summand of rank n. □

Example 9.3.3. As a warm-up, we begin with $X = \mathbb{CP}^\infty$. We have $H^*(\mathbb{CP}^\infty) \cong \mathbb{F}_3[c_1]$. By the axioms for Steenrod operations, we have

$$\begin{aligned} \mathcal{P}^1 c_1 &= c_1^3 \\ \mathcal{P}^1(c_1^2) &= 2c_1^4 \\ \mathcal{P}^1(c_1^n) &= n! c_1^{n+2} = 0 \text{ for } n \geq 3 \end{aligned}$$

Example 9.3.4. Now consider $X = BU(2)$. We have $H^*(BU(2)) \cong \mathbb{F}_3[c_1, c_2]$. Via the map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to BU(2)$, we get an embedding $\mathbb{F}_3[c_1, c_2] \hookrightarrow \mathbb{F}_3[x, y] \cong H^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty)$ sending c_1 and c_2 to the elementary symmetric polynomials $x + y$ and xy, respectively. We have

$$\mathcal{P}^1(xy) = x^3y + xy^3 = (x+y)^2xy + x^2y^2,$$

hence $\mathcal{P}^1(c_2) = c_1^2c_2 + c_2^2$.

Claim 9.3.5. *The graded sub-$\mathbb{F}_3\langle\mathcal{P}^1\rangle$-module of $H^*(BU(2))$ generated by c_1c_2 is indecomposable of infinite rank.*

Proof. Since it is at most 1-dimensional in every degree, the only thing we need to show is $(\mathcal{P}^1)^k(c_1c_2) \neq 0$ for every k. We begin with preliminary calculations, everywhere assuming $n \geq 3$:

$$\begin{aligned}
\mathcal{P}^1(c_1^nc_2) &= c_1^{n+2}c_2 + c_1^nc_2^2 \\
\mathcal{P}^1(c_2^2) &= -c_1^2c_2^2 - c_2^3 \\
\mathcal{P}^1(c_1^3c_2^3) &= 0 \\
\mathcal{P}^1(c_1^{n+2}c_2 + c_1^nc_2^2) &= c_1^{n+4}c_2 + c_1^{n+2}c_2^2 - c_1^{n+2}c_2^2 - c_1^nc_2^3 = c_1^{n+4}c_2 - c_1^nc_2^3 \\
(\mathcal{P}^1)^2(c_1^{n+2}c_2 + c_1^nc_2^2) &= c_1^{n+6}c_2 + c_1^{n+4}c_2^2
\end{aligned}$$

Now we come to the calculation of the iterated Steenrod operation on c_1c_2:

$$\begin{aligned}
\mathcal{P}^1(c_1c_2) &= c_1^3c_2 + c_1(c_1^2c_2 + c_2^2) = c_1c_2^2 - c_1^3c_2 \\
(\mathcal{P}^1)^2(c_1c_2) &= (c_1^3c_2^2 - c_1^3c_2^2 - c_1c_2^3) - c_1^5c_2 - c_1^3c_2^2 = -c_1c_2^3 - (c_1^5c_2 + c_1^3c_2^2) \\
(\mathcal{P}^1)^3(c_1c_2) &= -c_1^3c_2^3 - (c_1^7c_2 - c_1^3c_2^3) \\
(\mathcal{P}^1)^4(c_1c_2) &= -(c_1^9c_2 - c_1^7c_2^2) \\
(\mathcal{P}^1)^{4+2k}(c_1c_2) &= -(c_1^{9+4k}c_2 + c_1^{7+4k}c_2^2)
\end{aligned}$$

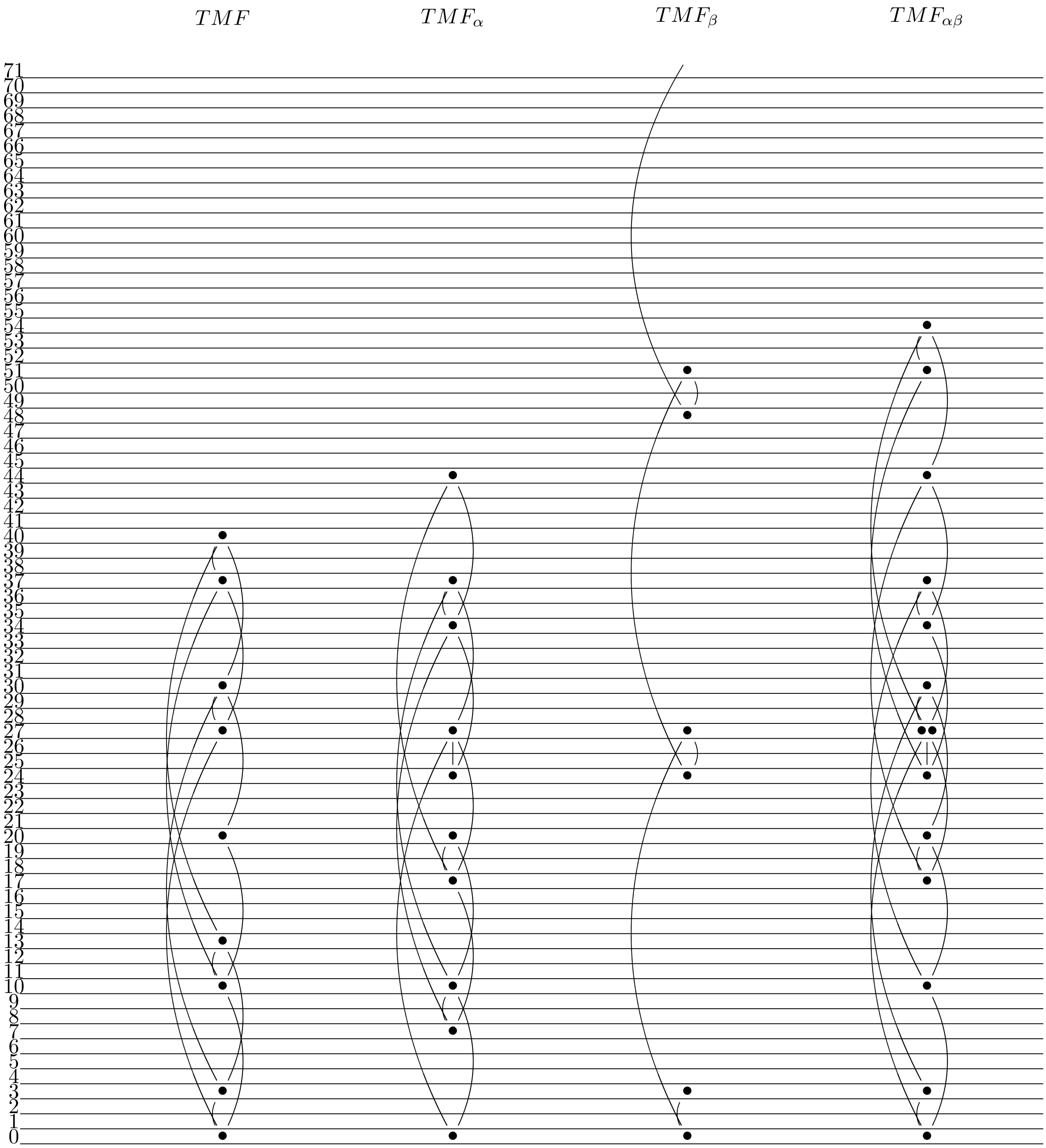
The last step is by induction, using the computations before. This term is obviously non-zero. □

It follows that $BU(2)$ does not decompose into tmf-modules M_i such that $\dim H^*_{tmf}M_i < \infty$ for all i. Using finite skeleta of $BU(2)$ one obtains finite spectra X such that $X \wedge tmf$ has indecomposable summands of arbitrary high cohomology-rank. Note that for tmf-standard modules (which are defined analogously to TMF-standard modules), the cohomology rank seems to be closely related to the rank defined by the number of times one cones off a torsion element. Indeed, $\alpha \in \pi_3 tmf$ is the only torsion element in $\pi_* tmf$ that induces a non-trivial morphism in H_*^{tmf} by Figure 1 of [Hil07]. Note also that for all finite skeleta X of $BU(2)$, the TMF-module $TMF \wedge X$ is relatively free.

9.4 Low-Rank Examples

We want to present some examples of $TMF_{(3)}$-modules. Since we are mostly interested in torsion, we depict just $\pi_*M/\operatorname{im}(r_*)$ in the pictures, where every • stands for one $\mathbb{F}_3$. The (bend) vertical lines allude to non-zero multiplication by α, β or $\{\alpha\Delta\}$, depending on their length.

The computations of these low rank examples are straightforward (using triple Toda brackets). Note that TMF_x denotes the cone of the map $\Sigma^{|x|}TMF_{(3)} \to TMF_{(3)}$ corresponding to an element $x \in \pi_*TMF_{(3)}$.



TMF
TMF_α
TMF_β
$TMF_{\alpha\beta}$
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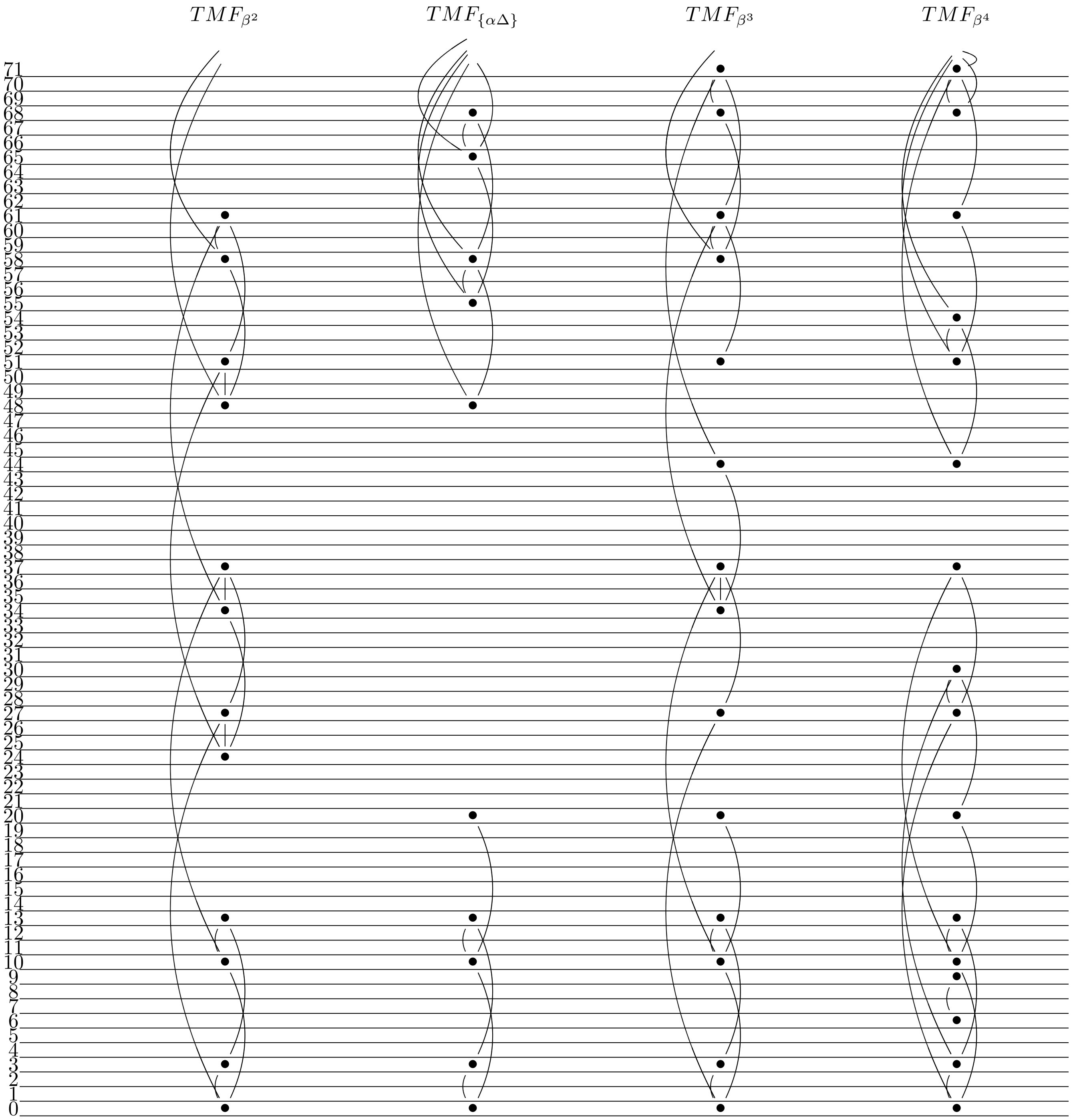
TMF_{β^2}
$TMF_{\{\alpha\Delta\}}$
TMF_{β^3}
TMF_{β^4}
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8
7
6
5
4
3
2
1
0

Appendix A

MAGMA Computations

The author used the following Magma program for computations of vector bundles associated to $BU(n)$ at the prime 2:

```
K := GF(4); // The field with 4 elements
e := One(K);
E := EllipticCurve([0,0,e,0,0]); // y^2 + y = x^3
AutomorphismGroup(E); // The quaternion group of automorphisms of E
Automorphisms(E);

l := 36;

R<z> := PowerSeriesRing(K,l+2);
i := Automorphisms(E)[5];
j := Automorphisms(E)[7]; // Choosing two generators of the quaternion group.
Fi<z> := FormalGroupHomomorphism(i,l+2);
Fj<z> := FormalGroupHomomorphism(j,l+2);
Li := [Coefficient(Fi ,n+1): n in [1..l]];
Lj := [Coefficient(Fj ,n+1): n in [1..l]];

P := PolynomialRing(K, [1..l]);
g1:=1;
for x in [1..l] do
g1 := g1+P.x;
end for;

g2:=1;
for x in [1..Floor(l/2)] do
g2 := g2+P.x;
end for;
g3:=1;
for x in [1..Floor(l/3)] do
g3 := g3+P.x;
end for;
```

```
g4:=1;
for x in [1..Floor(l/4)] do
g4 := g4+P.x;
end for;

function g(a,b) //Computing parts of the coaction of MU_*MU on MU_*(BU(n))
if b ge 0 then
if b le l/4 then
return HomogeneousComponent(g4^a,b);
else
if b le l/3 then
return HomogeneousComponent(g3^a,b);
else
if b le l/2 then
return HomogeneousComponent(g2^a,b);
else
if a eq 2 then
return HomogeneousComponent(g2^2,b);
else
if a eq 4 then
return HomogeneousComponent(g4^4,b);
else
return HomogeneousComponent(g1^a,b);
end if;
end if;
end if;
end if;
end if;
else
return Zero(P);
end if;
end function;

procedure mat(n)
MLi := [Evaluate(g(a,b-a), Li): a, b in [1..n]];
Mi := GL(n,K) ! MLi;
MLj := [Evaluate(g(a,b-a), Lj): a, b in [1..n]];
Mj := GL(n,K) ! MLj;
Mi;
Mj;
end procedure;

C<imag> := ComplexField();
G := MatrixGroup<2, C | [[imag, 0, 0, -imag], [0, 1, -1, 0]]>; // The quaternion
group again
```

```
procedure indec(n,k) //Computes the decomposition of (the part of) the quater-
nion group representation associated to BU(k) (corresponding to polynomials
of degree ≤ n in the homology of BU(k)) into indecomposables
MLi := [Evaluate(g(a,b-a), Li): a, b in [1..n]];
Mi := GL(n,K) ! MLi;
MLj := [Evaluate(g(a,b-a), Lj): a, b in [1..n]];
Mj := GL(n,K) ! MLj;
M := GModule(G, [Mi, Mj]);
Decomposition(SymmetricPower(M,k));
end procedure;

procedure listindec(m,k)
for n in [1..m] do
indec(n,k);
end for;
end procedure;

procedure listindecb(m,k)
for n in [1..m] do
indec(m,n);
end for;
end procedure;
```

One can compute the dimensions of the indecomposable summands of the representation associated to $\mathbb{CP}^n$ up to $n = 33$ by **listindec(33,1)**. The output is the following:

```
GModule of dimension 1 over GF(2^2) // These are the dimensions of the in-
decomposable summands

GModule of dimension 2 over GF(2^2)

GModule of dimension 1 over GF(2^2),
GModule of dimension 2 over GF(2^2)

GModule of dimension 4 over GF(2^2)

GModule of dimension 5 over GF(2^2)

GModule of dimension 6 over GF(2^2)

GModule of dimension 1 over GF(2^2),
GModule of dimension 6 over GF(2^2)

GModule of dimension 1 over GF(2^2),
GModule of dimension 7 over GF(2^2)

GModule of dimension 2 over GF(2^2),
```

GModule of dimension 7 over GF(2^2)

GModule of dimension 3 over GF(2^2),
GModule of dimension 7 over GF(2^2)

GModule of dimension 4 over GF(2^2),
GModule of dimension 7 over GF(2^2)

GModule of dimension 5 over GF(2^2),
GModule of dimension 7 over GF(2^2)

GModule of dimension 6 over GF(2^2),
GModule of dimension 7 over GF(2^2)

GModule of dimension 7 over GF(2^2),
GModule of dimension 7 over GF(2^2)

GModule of dimension 1 over GF(2^2),
GModule of dimension 7 over GF(2^2),
GModule of dimension 7 over GF(2^2)

GModule of dimension 1 over GF(2^2),
GModule of dimension 7 over GF(2^2),
GModule of dimension 8 over GF(2^2)

GModule of dimension 2 over GF(2^2),
GModule of dimension 7 over GF(2^2),
GModule of dimension 8 over GF(2^2)

GModule of dimension 3 over GF(2^2),
GModule of dimension 7 over GF(2^2),
GModule of dimension 8 over GF(2^2)

GModule of dimension 4 over GF(2^2),
GModule of dimension 7 over GF(2^2),
GModule of dimension 8 over GF(2^2)

GModule of dimension 5 over GF(2^2),
GModule of dimension 7 over GF(2^2),
GModule of dimension 8 over GF(2^2)

GModule of dimension 6 over GF(2^2),
GModule of dimension 7 over GF(2^2),
GModule of dimension 8 over GF(2^2)

GModule of dimension 7 over GF(2^2),
GModule of dimension 7 over GF(2^2),

GModule of dimension 8 over GF(2^2)

GModule of dimension 1 over GF(2^2),
GModule of dimension 7 over GF(2^2),
GModule of dimension 7 over GF(2^2),
GModule of dimension 8 over GF(2^2)

GModule of dimension 1 over GF(2^2),
GModule of dimension 7 over GF(2^2),
GModule of dimension 8 over GF(2^2),
GModule of dimension 8 over GF(2^2)

GModule of dimension 2 over GF(2^2),
GModule of dimension 7 over GF(2^2),
GModule of dimension 8 over GF(2^2),
GModule of dimension 8 over GF(2^2)

GModule of dimension 3 over GF(2^2),
GModule of dimension 7 over GF(2^2),
GModule of dimension 8 over GF(2^2),
GModule of dimension 8 over GF(2^2)

GModule of dimension 4 over GF(2^2),
GModule of dimension 7 over GF(2^2),
GModule of dimension 8 over GF(2^2),
GModule of dimension 8 over GF(2^2)

GModule of dimension 5 over GF(2^2),
GModule of dimension 7 over GF(2^2),
GModule of dimension 8 over GF(2^2),
GModule of dimension 8 over GF(2^2)

GModule of dimension 6 over GF(2^2),
GModule of dimension 7 over GF(2^2),
GModule of dimension 8 over GF(2^2),
GModule of dimension 8 over GF(2^2)

GModule of dimension 7 over GF(2^2),
GModule of dimension 7 over GF(2^2),
GModule of dimension 8 over GF(2^2),
GModule of dimension 8 over GF(2^2)

GModule of dimension 1 over GF(2^2),
GModule of dimension 7 over GF(2^2),

GModule of dimension 7 over GF(2^2),
GModule of dimension 8 over GF(2^2),
GModule of dimension 8 over GF(2^2)

GModule of dimension 1 over GF(2^2),
GModule of dimension 7 over GF(2^2),
GModule of dimension 8 over GF(2^2),
GModule of dimension 8 over GF(2^2),
GModule of dimension 8 over GF(2^2)

GModule of dimension 2 over GF(2^2),
GModule of dimension 7 over GF(2^2),
GModule of dimension 8 over GF(2^2),
GModule of dimension 8 over GF(2^2),
GModule of dimension 8 over GF(2^2)]

One can compute the dimensions of the indecomposable summands of the representation associated to the "n-part" of $BU(2)$ up to $n = 12$ by **listindec(12,2)**. The output is the following:

GModule of dimension 1 over GF(2^2)

GModule of dimension 3 over GF(2^2)

GModule of dimension 1 over GF(2^2),
GModule of dimension 2 over GF(2^2),
GModule of dimension 3 over GF(2^2)

GModule of dimension 1 over GF(2^2),
GModule of dimension 9 over GF(2^2)

GModule of dimension 15 over GF(2^2)

GModule of dimension 1 over GF(2^2),
GModule of dimension 6 over GF(2^2),
GModule of dimension 6 over GF(2^2),
GModule of dimension 8 over GF(2^2)

GModule of dimension 1 over GF(2^2),
GModule of dimension 1 over GF(2^2),
GModule of dimension 6 over GF(2^2),
GModule of dimension 6 over GF(2^2),
GModule of dimension 6 over GF(2^2),
GModule of dimension 8 over GF(2^2)

GModule of dimension 1 over GF(2^2),
GModule of dimension 6 over GF(2^2),

GModule of dimension 6 over GF(2^2),
GModule of dimension 7 over GF(2^2),
GModule of dimension 8 over GF(2^2),
GModule of dimension 8 over GF(2^2)

GModule of dimension 3 over GF(2^2),
GModule of dimension 6 over GF(2^2),
GModule of dimension 6 over GF(2^2),
GModule of dimension 6 over GF(2^2),
GModule of dimension 8 over GF(2^2),
GModule of dimension 8 over GF(2^2),
GModule of dimension 8 over GF(2^2)

GModule of dimension 5 over GF(2^2),
GModule of dimension 6 over GF(2^2),
GModule of dimension 6 over GF(2^2),
GModule of dimension 6 over GF(2^2),
GModule of dimension 8 over GF(2^2),
GModule of dimension 8 over GF(2^2),
GModule of dimension 8 over GF(2^2),
GModule of dimension 8 over GF(2^2)

GModule of dimension 4 over GF(2^2),
GModule of dimension 6 over GF(2^2),
GModule of dimension 6 over GF(2^2),
GModule of dimension 8 over GF(2^2),
GModule of dimension 8 over GF(2^2),
GModule of dimension 8 over GF(2^2),
GModule of dimension 8 over GF(2^2),
GModule of dimension 8 over GF(2^2),
GModule of dimension 10 over GF(2^2)

GModule of dimension 3 over GF(2^2),
GModule of dimension 6 over GF(2^2),
GModule of dimension 6 over GF(2^2),
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Appendix B

List of Notation

We want to collect here some notation, appearing again and again in this thesis.

Algebraic Geometry and Ring Theory:

- $\mathcal{M}$: The moduli stack of elliptic curves, often localized at the prime 3 (see Section 2.4).
- $\mathcal{M}(n)$: The moduli stack of elliptic curves with level-n-structure; for $n = 2$ often localized at the prime 3. We have maps $f : \mathcal{M}_0(2) \to \mathcal{M}[\frac{1}{2}]$, $p : \mathcal{M}(2) \to \mathcal{M}[\frac{1}{2}]$ and $q : \mathcal{M}(4) \to \mathcal{M}[\frac{1}{2}]$ (see Section 2.5).
- $\mathcal{O}$: The structure sheaf of a stack, often on $\mathcal{M}$ (see Section 2.3 and Section 2.5).
- ω: A line bundle given by a grading, usually on $\mathcal{M}$.
- E_α: The unique non-split rank 2 standard vector bundle on $\mathcal{M}_{(3)}$ (see Section 3.4).
- $E_{\alpha,\widetilde{\alpha}}$: The unique non-split rank 3 standard vector bundle on $\mathcal{M}_{(3)}$. It is isomorphic to $f_*f^*\mathcal{O}$ (see Section 3.4).
- α: A non-trivial element in $H^1(\mathcal{M}_{(3)};\omega^2)$ (see Section 2.7).
- β: A non-trivial element in $\beta \in H^2(\mathcal{M}_{(3)};\omega^6)$ (see Section 2.7).
- $\widetilde{R[G]}$: Given a commutative ring R and a group G acting on R via ring maps, we define the *twisted group ring* $\widetilde{R[G]}$ additively as $\bigoplus_{g\in G} Rg$ (where g is just a symbol) with multiplication given by

  $$(r_1g_1)\cdot(r_2g_2) = (r_1g_1(r_2))(g_1g_2).$$

  The category of modules over $\widetilde{R[G]}$ is equivalent to R-modules with twisted R-linear action by G.

Group Theory:

- C_n: The cyclic group of order n.

- S_n: The symmetric group of order n. We view elements of S_n as maps $\{1,\ldots,n\} \to \{1,\ldots,n\}$. Our notation of elements in S_n is slightly non-standard. For example, by $(2\ 3\ 1)$ we denote the element in S_3 sending 1 to 2, 2 to 3 and 3 to 1.

Homotopy Theory:

- KO and KU: The (commutative ring) spectra of real and complex K-theory.
- TMF: The (commutative ring) spectrum of topological modular forms (see Section 5.1).
- $TMF_0(2)$ and $TMF(n)$: Variants of TMF with level structures (see Section 5.2).
- $\mathcal{F}_M$: A sheaf on $\mathcal{M}$ associated to a TMF-module M (see the end of Section 4.5).
- DSS: The descent spectral sequence (see Section 4.4 and Section 6.4). For a TMF-module M, we often denote the descent spectral sequence for $\mathcal{F}_M$ by $DSS(M)$.
- $F_n\pi_*M$: The filtration associated to $DSS(M)$.
- α: The element in $\pi_3 TMF_{(3)}$ detected by $\alpha \in H^1(\mathcal{M}_{(3)};\omega^2)$ in the DSS of $TMF_{(3)}$ (see Section 5.1).
- β: The element in $\pi_{10} TMF_{(3)}$ detected by $\beta \in H^2(\mathcal{M}_{(3)};\omega^6)$ in the DSS of $TMF_{(3)}$ (see Section 5.1).
- TMF_α: The cone of the map $\Sigma^3 TMF_{(3)} \to TMF_{(3)}$, given by multiplication by α (see Section 5.2).
- $TMF_{\alpha,\widetilde{\alpha}}$: The cone of a map $\Sigma^7 TMF_{(3)} \to TMF_\alpha$; equivalent as an $TMF_{(3)}$-module to $TMF_0(2)$.
- $\widetilde{R[G]}$: Given a commutative ring spectrum R and a group G acting on R via ring maps, $\widetilde{R[G]}$ is the *twisted group ring* defined in Section 6.1. The category of modules over $\widetilde{R[G]}$ is equivalent to R-modules with twisted R-linear action by G.

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Zusammenfassung

Diese Dissertation beschäftigt sich mit Modulspektren über reeller K-Theorie KO und topologischen Modulformen TMF.

Bousfield hat in [Bou90] einen Funktor π_*^{CRT} von KO-Moduln in eine gewisse abelsche Kategorie CRT-mod definiert, der im folgenden Sinne Isomorphismusklassen detektiert: Sind für zwei KO-Moduln M und N die Objekte $\pi_*^{CRT}(M)$ und $\pi_*^{CRT}(N)$ isomorph, so sind auch M und N isomorph in der Homotopiekategorie von KO-Moduln. Wir geben in dieser Arbeit einen neuen Zugang zu diesem Satz, basierend auf einer Klassifikation der relativ freien KO-Moduln. Dazu nennen wir einen KO-Modul M *relativ frei*, wenn $M \wedge_{KO} KU$ ein freier Modul über komplexer K-Theorie KU ist. Der Vergleich zur deutlich einfacheren Theorie der KU-Moduln erlaubt dann alternative Beweise des Satzes von Bousfield. Bousfield hat die Theorie von KO-Moduln dann in einem weiteren Schritt auf Fragen über K-lokale Spektren angewendet. Diese Richtung haben wir in der vorliegenden Arbeit aber noch nicht weiter verfolgt.

Während für das Studium von KO-Moduln die Theorie der integralen Darstellungen der zyklischen Gruppe C_2 ein wichtiges Werkzeug ist, ist für das Studium von TMF-Moduln die Theorie der quasi-kohärenten Garben und Vektorbündel auf dem Modulstack von elliptischen Kurven $\mathcal{M}$ entscheidend. Dazu erinnere ich daran, dass TMF selbst als die globalen Schnitte einer gewissen Garbe $\mathcal{O}^{top}$ von kommutativen Ringspektren auf $\mathcal{M}$ definiert ist. Man sieht leicht, dass man so jedem TMF-Modul erst einen quasi-kohärenten $\mathcal{O}^{top}$-Modul und durch Anwenden des Homotopiegrupppenfunktors dann eine quasi-kohärente Garbe auf $\mathcal{M}$ zuordnen kann. Eines der Ergebnisse dieser Arbeit ist eine Äquivalenz zwischen den ∞-Kategorien quasi-kohärenten $\mathcal{O}^{top}$-Moduln und TMF-Moduln zu zeigen, zumindets an Primzahlen größer als 2.

Beschränken wir uns immer noch auf Primzahlen größer 2, so ergibt die Betrachtung von Levelstrukturen von Niveau 2 eine TMF-Algebra $TMF(2)$, deren Homotopiegruppen sehr einfache Gestalt haben. Analog zur K-Theorie nennen wir einen TMF-Modul M *relativ frei/projektiv*, wenn $M \wedge_{TMF} TMF(2)$ ein freier/projektiver $TMF(2)$-Modul ist. Wir können jeden TMF-Modul in zwei Schritten durch einen relativ projektiven auflösen. Wichtig ist, dass die quasi-kohärente Garbe auf $\mathcal{M}$, die einem relativ freien Modul zugeordnet wird, ein Vektorbündel ist.

Während die Klassifikation von Geradenbündeln auf $\mathcal{M}$ wohlbekannt ist, erscheint die Klassifikation von Vektorbündel schwieriger, selbst wenn 2 invertiert ist. Wenn wir uns auf Vektorbündel, die als iterierte Extensionen von Geradenbündeln beschränken, gelingt in dieser Arbeit eine Klassifikation: Die einzigen solchen unzerlegbaren Vektorbündel sind von Rang 1, 2 und 3. Wenn das einem TMF-Modul M zugeordnete Vektorbündel solchermaßen aus Extensionen entsteht, nennen wir M *algebraisch von Standard-Typ*. Ein wesentliches Ziel dieser Arbeit ist das Verständnis dieser Moduln.

Die einfachste Klasse von algebraischen Standard-Moduln sind TMF-Moduln, die durch iteriertes Abkegeln von Torsionselementen aus TMF entstehen, sogenannte *Standard-Moduln*. Jeder algebraische Standard-Modul vom Rang ≤ 3 ist (an der Primzahl 3) von dieser Form. Wir zeigen für allgemeinen Rang eine leicht schwächere Form dieses Satzes. Diese erlaubt es, prinzipiell gesehen, algebraische Standard-Moduln bis zu jedem beliebigen endlichen Rang zu klassifizieren.

Eine vollständige Klassifikation selbst von Standard-Moduln über TMF scheint jedoch ein sehr schwieriges Unterfangen zu sein. Wir konstruieren eine unendliche Folge von solchen, die nicht in Standard-Moduln von kleinerem Rang zerfallen. Dies zerschlägt einerseits die Hoffnung für eine ähnlich einfache Theorie wie für KO, zeigt aber andererseits auch auf, dass TMF-Moduln weitaus reichhaltiger sind als KO-Moduln.