SFT Exercises Week 10

Charge-Density-Wave Instability

The Peierls instability is a phenomenon in one-dimensional crystals comprised of ions and electrons. It manifests itself in the appearance of an electron charge-density-wave (CDW), i.e. periodically modulated electron charge density in space. In such a way, apart from the underlying periodical lattice of ions, a new periodical structure appears, with a period different from the period of the ion lattice.

Let us consider an action for a one dimensional gas of spinless electrons

\[ S_{el}[\psi, \bar{\psi}] = \int_0^\beta d\tau \int_0^L dx \bar{\psi} \left( \partial_\tau - \frac{1}{2m} \partial_x^2 - \mu \right) \psi \]  

where \( L \) is the linear size of the system, \( \beta = 1/T \), with \( T \) being the temperature, in the system of units where \( k_B = 1, \hbar = 1 \). Electrons also interact with ions in the one-dimensional lattice. The action for the system of ions is given by

\[ S_{ph}[u] = \frac{\rho}{2} \int_0^\beta d\tau \int_0^L dx \left\{ (\partial_\tau u)^2 + c^2 (\partial_x u)^2 \right\} \]  

where \( u(x, \tau) \) denotes the static bosonic displacement field (describing phonons) and \( \rho > 0 \) is the density of ions and \( c \) is the sound velocity. The coupling of electrons to the lattice vibrations of the lattice of ions is described by

\[ S_{el-ph}[\psi, \bar{\psi}, u] = g \int_0^\beta d\tau \int_0^L dx \bar{\psi} \psi \partial_x u \]  

And the full interacting theory is described by the action

\[ S = S_{el} + S_{ph} + S_{el-ph} \]  

We cannot solve it exactly and will rely instead on perturbation theory.

a) As a first step, integrate out the fermionic degrees of freedom \( \psi \) and thereby obtain an effective action \( S_{eff}[u] \) for the displacement field \( u(x, \tau) \). Assuming that the electron-phonon coupling constant \( g \) is small, expand the action up to second order in \( u \). You will find that the coefficient by \( g^2 |u(q, i\Omega_n)|^2 \) in momentum space involves the density-density response function

\[ \chi(q, i\Omega_n) = -\frac{1}{2\beta L} \sum_{km} [G_0(k + q, i\omega_m + i\Omega_n)G_0(k, i\omega_m) + G_0(k - q, i\omega_m - i\Omega_n)G_0(k, i\omega_m)] \]

with \( \Omega_n = 2\pi n/\beta \) and \( \omega_n = (2n + 1)\pi/\beta \) being respectively bosonic and fermionic Matsubara frequencies.

b) Now you have to find a saddle-point of the effective action \( S_{eff}[u] \). One may look for a homogeneous displacement field, i.e. \( u(x, \tau) \equiv u_0 \). However, here it is not necessarily the best solution. Show that the static solution \( u_0(x, \tau) \equiv u_0 \cos(2k_F x + \omega_0 t) \)
\( \varphi \) is energetically favorable (i.e., \( S[u = u_0 \cos(2k_Fx + \varphi)] < S[u = 0] \)) below a certain critical temperature \( T_c \). Thus, at low temperatures, the system is unstable towards the formation of a static sinusoidal lattice distortion. Calculate the critical temperature \( T_c \), by using the following approximation for the response function \( \chi(2k_F, 0) \approx \ln(\beta \omega_D)/(4\pi v_F) \), where \( \omega_D \) is the Debye frequency and \( v_F = \pi n_e/m_e \) is the Fermi velocity; \( k_F = m_e v_F \) is the Fermi momentum, \( n_e \) the density of electrons.

c) If you have already obtained an answer for \( T_c \) as a function of \( g \), can you in principle obtain it directly by means of a perturbation theory in \( g \)? If not, why? What is the connection between the result you obtained and the BCS theory of superconductivity? What is the period of the lattice?