Itinerant Ferromagnetism

In a Fermi system, itinerant ferromagnetism is a result of the competition between the kinetic energy and repulsive interactions. The word “itinerant” reflects the fact that the magnetism is due to spins that are delocalized, as opposed to localized spins in, e.g., the Ising or Heisenberg models.

\[ \hat{H} = \sum_{k,\alpha} \epsilon_k \hat{\psi}_{k,\alpha}^\dagger \hat{\psi}_{k,\alpha} + \frac{V_0}{V} \sum_{K,k,k',\sigma} \hat{\psi}_{K/2+k',\sigma}^\dagger \hat{\psi}_{K/2-k',\sigma}^\dagger \hat{\psi}_{K/2-k,\sigma} \hat{\psi}_{K/2+k,\sigma} . \]  

Let us first convince ourselves of the fact that for the same total number of particles, the polarized Fermi gas has a larger kinetic energy than the unpolarized Fermi gas, where the Hamiltonian of the system is again given by (1), but now with \( V_0 > 0 \).

(a) Consider the unpolarized state \( |\Psi_{\text{up}}\rangle = \prod_k \hat{\psi}_{k,\uparrow}^\dagger \hat{\psi}_{k,\downarrow}^\dagger |0\rangle \), where \( \prod' \) denotes the restricted product of all states below the Fermi energy. Calculate its total kinetic energy in terms of the total density \( n \) in 3d, assuming \( \epsilon_k = \frac{\hbar^2 k^2}{2m} \). Note that for the total density we have \( n = n_\uparrow + n_\downarrow \), where \( n_\alpha \) is the density of atoms in spin state \( |\alpha\rangle \). Remember that

\[ \sum_k \to V \int \frac{d^3 k}{(2\pi)^3} . \]  

(b) Next, consider the fully-polarized state \( |\Psi_p\rangle = \prod_k \hat{\psi}_{k,\uparrow}^\dagger |0\rangle \). Calculate its kinetic energy in terms of the total density \( n = n_\uparrow \). Note that the Fermi energy is now different from the previous exercise.

(c) We define the magnetization by \( \hat{m} \equiv \sum_{k,\alpha,\alpha'} \hat{\psi}_{k,\alpha}^\dagger \hat{\sigma}_{\alpha\alpha'} \hat{\psi}_{k,\alpha'} / V \), where \( \hat{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \) is a vector of Pauli matrices. What is the magnetization for the above two states?

(d) Calculate the interaction energy for the polarized and unpolarized state. Show that the polarized state has no interaction energy. Explain why.

(e) Consider now the state \( |\Psi\rangle = \prod_k \hat{\psi}_{k,\uparrow}^\dagger \prod_k \hat{\psi}_{k,\downarrow}^\dagger |0\rangle \), where the products over \( k, k' \) are restricted by the requirement that the density of \( \uparrow \) atoms is \( n_\uparrow \) and the density of \( \downarrow \) atoms is \( n_\downarrow \). Give the total energy as a function of \( n_\uparrow \) and \( n_\downarrow \). (Hint: show that \( E_{\text{int}} = V_0 V n_\uparrow n_\downarrow \))

(f) Using this result, show that the system becomes polarized (the unpolarized state becomes unstable) if \( V_0 D(\epsilon_F) > 1 \), where \( D(\epsilon_F) \) is the density of states of a single spin specie at the Fermi level. This is the Stoner criterion. Remember that

\[ D_\alpha(\epsilon) \equiv \frac{\partial n_\alpha}{\partial \epsilon} . \]