Exercise 1
We’re given the definition of annihilation operator:

$$\psi_{n\alpha}|...N_{n\alpha}...\rangle = (-1)^{M_{n\alpha}} \sqrt{N_{n\alpha}} |...N_{n\alpha} - 1...\rangle$$  \hspace{1cm} (1)

where \(n\alpha\) denote different quantum states. One may get a hermitian-conjugate equation:

$$\langle...N_{n\alpha}...|\psi_{n\alpha}^+ = (-1)^{M_{n\alpha}} \sqrt{N_{n\alpha}} |...N_{n\alpha} - 1...\rangle$$  \hspace{1cm} (2)

Now one may act with this expression on the state \(|...N_{n\alpha} - 1...\rangle\):

$$\langle...N_{n\alpha}...|\psi_{n\alpha}^+ |...N_{n\alpha} - 1...\rangle = (-1)^{M_{n\alpha}} \sqrt{N_{n\alpha}}$$  \hspace{1cm} (3)

Here one has used that the states \(\langle...N_{n\alpha}...|\) are orthonormal. One sees that contraction of state \(\langle...N_{n\alpha}...|\) and \(\psi_{n\alpha}^+ |...N_{n\alpha} - 1...\rangle\) gives non-zero result. Since all the states with different \(N_{n\alpha}\) are orthogonal to each other it is only possible when

$$\psi_{n\alpha}^+ |...N_{n\alpha} - 1...\rangle \propto |...N_{n\alpha}...\rangle$$  \hspace{1cm} (4)

From (3) one easily finds the coefficient of proportionality, thus

$$\psi_{n\alpha}^+ |...N_{n\alpha} - 1...\rangle = (-1)^{M_{n\alpha}} \sqrt{N_{n\alpha}} |...N_{n\alpha}...\rangle$$  \hspace{1cm} (5)

This derivation was true for values \(N_{n\alpha} \geq 1\). For bosons, any \(N_{n\alpha}\) is allowed thus one changes \(N_{n\alpha}\) to \(N_{n\alpha} + 1\) and obtains for any \(N_{n\alpha} \geq 0\)

$$\psi_{n\alpha}^+ |...N_{n\alpha}...\rangle = \sqrt{1 + N_{n\alpha}} |...N_{n\alpha} + 1...\rangle$$  \hspace{1cm} (6)

For fermions, the above derivation holds for \(N_{n\alpha} = 1\). Thus one changes \(N_{n\alpha}\) to \(1 - N_{n\alpha}\) in order to account for the Fermi statistics:

$$\psi_{n\alpha}^+ |...N_{n\alpha}...\rangle = (-1)^{M_{n\alpha}} \sqrt{1 - N_{n\alpha}} |...1 + N_{n\alpha}...\rangle$$  \hspace{1cm} (7)

Exercise 2
The aim of this exercise is to prove the equivalence of first and second quantization in the particular case of two-body potential \(V\) acting on two-particle states. We will work in the momentum space and for simplicity omit spin degrees of freedom. The matrix elements of the two-body potential are given in the first quantization as following:

$$\langle k_1k_2|V|k_3k_4\rangle_{FQ} = \int \int dx_1dx_2 \overline{\Psi}_{k_1k_2}(x_1,x_2)V(x_1 - x_2)\Psi_{k_3k_4}(x_1,x_2)$$  \hspace{1cm} (8)

where \(\Psi_{k_1k_2}(x_1,x_2)\) are normalized two-particle wave-functions in the first quantization formalism:

$$|k_1k_2\rangle_{FQ} = \Psi_{k_1k_2}(x_1,x_2) = \frac{1}{\sqrt{2}}(\chi_{k_1}(x_1)\chi_{k_2}(x_2) \pm \chi_{k_1}(x_2)\chi_{k_2}(x_1))$$  \hspace{1cm} (9)
where $\chi_k(x)$ are orthonormal set of single-particle wave-functions. $+$ stands for bosonic case and $-$ for fermionic.

$$
\langle k_1k_2|V|k_3k_4 \rangle_{FQ} = \frac{1}{2} \int dx_1dx_2 \left( \overline{\chi_{k_1}(x_1)}\chi_{k_2}(x_2)V(x_1 - x_2)\chi_{k_3}(x_1)\chi_{k_4}(x_2) \right.
+ \chi_{k_2}(x_1)\chi_{k_1}(x_2)V(x_1 - x_2)\chi_{k_4}(x_1)\chi_{k_3}(x_2)
\pm \chi_{k_2}(x_1)\chi_{k_1}(x_2)V(x_1 - x_2)\chi_{k_3}(x_1)\chi_{k_4}(x_2)
\left. \pm \chi_{k_1}(x_1)\chi_{k_2}(x_2)V(x_1 - x_2)\chi_{k_4}(x_1)\chi_{k_3}(x_2) \right) 
$$

(10)

$$
= \frac{1}{2} \left( V_{k_1k_2;k_3k_4} + V_{k_2k_1;k_4k_3} \pm V_{k_1k_2;k_4k_3} \pm V_{k_2k_1;k_3k_4} \right) 
$$

(11)

Where Fourier components are

$$
V_{k_1k_2;k_3k_4} = \int dx dx' \overline{\chi_{k_1}(x)}\chi_{k_2}(x')V(x - x')\chi_{k_3}(x)\chi_{k_4}(x')
$$

(12)

Of course, matrix elements should not depend on specific representation. So now we need to calculate $\langle k_1k_2|V|k_3k_4 \rangle$ in the second quantization and compare them with (12) (the result should be the same). Two-particle states are given now simply as

$$
|k_1k_2\rangle_{SQ} = \psi_{k_1}^+\psi_{k_2}^+|0\rangle
$$

(13)

and second-quantized representation of operator $V$ is given as

$$
V_{SQ} = \frac{1}{2} \sum_{kk'pp'} V_{kk';pp'}\psi_{k}^+\psi_{k'}^+\psi_{p}^+\psi_{p'}
$$

(14)

Now we calculate its matrix elements:

$$
\langle k_1k_2|V|k_3k_4 \rangle_{SQ} = \frac{1}{2} \sum_{kk'pp'} V_{kk';pp'}\langle 0|\psi_{k_1}\psi_{k_2}\psi_{k_3}^+\psi_{k_4}^+\psi_{p}^+\psi_{p'}\psi_{k_4}^+\psi_{k_3}^+|0\rangle 
$$

(15)

We use that $\psi_{k_1}\psi_{k_2}^+ = \psi_{k_2}^+\psi_{k_1} + \delta_{kk'}$, $\epsilon = \pm 1$, so

$$
\langle 0|\psi_{k_1}\psi_{k_2}\psi_{k_2}^+\psi_{p}^+\psi_{p'}\psi_{k_3}^+\psi_{k_4}^+|0\rangle = \langle 0|\psi_{k_1}(\epsilon\psi_{k_2}^+\psi_{k_2} + \delta_{kk'}\psi_{p}^+\psi_{p'(k_3 + \epsilon\psi_{k_3}^+\psi_{p'}\psi_{k_4}^+|0\rangle
$$

$$
= \epsilon\delta_{p'k_3}\langle 0|\psi_{k_1}\psi_{k_2}^+\psi_{k_2}^+\psi_{k_3}^+\psi_{k_4}^+|0\rangle + \delta_{kk'}\delta_{p'k_3}\langle 0|\psi_{k_1}\psi_{k_2}^+\psi_{p}^+\psi_{p'}\psi_{k_4}^+|0\rangle
$$

$$
= \epsilon\delta_{p'k_3}\delta_{k_1k}\delta_{p'k_4}\delta_{k_3k_4} + \epsilon\delta_{k_2k'}\delta_{p'k_4}\delta_{k_1k}\delta_{p'k_3} + \delta_{k_2k'}\delta_{k_1k}\delta_{p'k_4}\delta_{p'k_3} + \delta_{k_1k}\delta_{k_2k'}\delta_{p'k_3}\delta_{p'k_4} + \delta_{k_1k}\delta_{k_2k'}\delta_{p'k_3}\delta_{p'k_4}
$$

(16)

and

$$
= \frac{1}{2} \sum_{kk'pp'} V_{kk';pp'}\langle 0|\psi_{k_1}\psi_{k_2}\psi_{k_2}^+\psi_{k_1}^+\psi_{p}^+\psi_{p'}|0\rangle
$$

(17)

and

$$
= \frac{1}{2} \left( V_{k_1k_2;k_3k_4} + V_{k_2k_1;k_4k_3} \pm V_{k_1k_2;k_4k_3} \pm V_{k_2k_1;k_3k_4} \right) \equiv \langle k_1k_2|V|k_3k_4 \rangle_{FQ} 
$$

(18)

Thus we proved the equality of the matrix elements in the both first- and second quantization. To prove the complete equivalence of first and second quantization, one needs to prove the same equality for all $N$-particle states; this is not required here.
Exercise 3
Here we need to prove equality of matrix elements of one-body potential in both representations; we proceed similarly to the previous exercise: we deal with one-particle states $|k\rangle$ and matrix elements $\langle k|A|k'\rangle$ and we have to prove that $\langle k|A|k'\rangle_{SQ} = \langle k|A|k'\rangle_{FQ}$.

Exercise 4
We start from the commutation relations in momentum representation

$$[\psi_{k\alpha}, \psi^{+}_{k'\alpha'}]_{\pm} = \delta_{kk'}\delta_{\alpha\alpha'}, \quad [\psi_{k\alpha}, \psi_{k'\alpha'}]_{\pm} = 0$$  \hspace{1cm} (19)

In real space

$$\psi_{\alpha}(x) = \sum_{k} e^{ikx} \psi_{k\alpha}$$  \hspace{1cm} (20)

Hence

$$[\psi_{\alpha}(x), \psi^{+}_{\alpha'}(x')]_{\pm} = \sum_{kk'} \delta_{kk'}\delta_{\alpha\alpha'}e^{ikx-ik'x'} = \delta_{\alpha\alpha'} \sum_{k} e^{ik(x-x')} = \delta(x-x')\delta_{\alpha\alpha'}$$  \hspace{1cm} (21)

$$[\psi_{\alpha}(x), \psi_{\alpha}(x')]_{\pm} = \sum_{kk'} [\psi_{k\alpha}, \psi_{k'\alpha'}]_{\pm} e^{ikx-ik'x'} = 0$$  \hspace{1cm} (22)