Exercise 1. Show that the Green’s function in (8.67) is related to the exact atomic Green’s function (8.69) by differentiating (8.65) with respect to $J, J^\ast$.

Exercise 2. Differentiating (8.71) once with respect to $I$ and setting $I = 0$ one obtains
\[
\langle \phi^\ast(x, \tau)\phi(x, \tau) \rangle_Z[0].
\]
On the other hand it is equal to the first derivative (functional) of (8.74) at $I = 0$. Thus,
\[
\int dx' V^{-1}(x - x')\langle \kappa(x', \tau) \rangle = \langle \phi^\ast(x, \tau)\phi(x, \tau) \rangle.
\] (1)

The expression for \[
\langle \phi^\ast(x, \tau)\phi(x, \tau)\phi^\ast(x', \tau)\phi(x', \tau) \rangle
\]
is obtained analogously by differentiating (8.71) twice with respect to $I$ and setting $I = 0$. After multiplying both sides of the resulting equation by $V(x - x'')V(x' - x''')$ and integrating over $x'', x'''$ one arrives at (8.78).

Exercise 3: Fermi gas with an population imbalance

a) Hubbard-Stratonovich (HS) transformation

i) The Hartree-type HS transformation for a general interaction in real space is as follows (no spin)
\[
1 = \int d[\rho] \exp \left\{ \frac{1}{2\hbar} \int d\tau dxdx' \left( \rho(x, \tau) - \int dx'' V(x - x'')\phi^\ast(x'', \tau)\phi(x'', \tau) \right) 
\right. 
\]
\[
\left. V^{-1}(x - x') \left( \rho(x', \tau) - \int dx'' V(x' - x'')\phi^\ast(x'', \tau)\phi(x'', \tau) \right) \right\}. 
\] (2)

Note the structure in the exponent: $x$-dependent vector on the left, a matrix that is $x, x'$-dependent in the middle and an $x'$-dependent vector on the right. Note also that in this formulation everything is already diagonal in imaginary time. I could also consider a general dependence of the interaction on imaginary time. Then I would get the same structure for the $\tau, \tau'$-dependence as for the $x, x'$-dependence.

In this particular exercise we have a diagonal interaction in real space, that is $V(x - x') = V_0\delta(x - x')$ which allows to do most of the integrals in the above equation analytically. As result, taking into account the hint, we get for the above
\[
1 = \int d[\rho] \exp \left\{ \frac{1}{2\hbar} \int d\tau dx \left( \rho(x, \tau) - V_0[\phi^\ast_\uparrow(x, \tau)\phi_\uparrow(x, \tau) + \phi^\ast_\downarrow(x, \tau)\phi_\downarrow(x, \tau)] \right) 
\right. 
\]
\[
\left. V_0^{-1}(\rho(x, \tau) - V_0[\phi^\ast_\uparrow(x, \tau)\phi_\uparrow(x, \tau) + \phi^\ast_\downarrow(x, \tau)\phi_\downarrow(x, \tau)]) \right\}. 
\] (3)
Note three things here. First of all, in real space the delta function is the unity matrix and thus also it’s own inverse. We have

\[ \int dx'' \delta(x - x'') \delta^{-1}(x'' - x') = \delta(x - x') = \int dx'' \delta(x - x'') \delta(x'' - x') \]  
(4)

showing that indeed \( \delta^{-1}(x - x') = \delta(x - x') \). This means \( V^{-1}(x - x') = V_0^{-1} \delta(x - x') \), which we have used in obtaining (3). Second, note that because we are dealing with fermions we have that \( \phi^*_{\sigma}(x, \tau) \phi^*_{\sigma}(x, \tau) = \phi_{\sigma}(x, \tau) \phi_{\sigma}(x, \tau) = 0 \), so that the terms in (3) of the form \( \phi^*_{\sigma}(x, \tau) \phi^*_{\sigma}(x, \tau) \phi_{\sigma}(x, \tau) \phi_{\sigma}(x, \tau) \) are automatically zero, which we want. Finally, note that in eq. (3) we omitted explicitly writing the trace of the logarithm of the inverse interaction, because this leads to a constant shift in the path integral over \( \phi_{\sigma} \) and is therefore irrelevant for the rest of the exercise.

Substituting eq. (3) in the partition sum, we see that all fourth order terms are zero or cancel, and we get for the action

\[ S[\phi^*_{\sigma}, \phi_{\sigma}, \rho] = \sum_{\sigma = 1, -1} \int \! d\tau d\mathbf{x} \left\{ \phi^*_{\sigma}(\mathbf{x}, \tau) \left[ \hbar \partial_{\mathbf{x}} - \frac{\hbar^2 \nabla^2}{2m_{a}} + \rho(\mathbf{x}, \tau) - \mu \right] \phi_{\sigma}(\mathbf{x}, \tau) - \frac{\rho(\mathbf{x}, \tau)^2}{4V_0} \right\}. \]  
(5)

ii) Use the following HS transformation in the coordinate-free representation

\[ \int d[\Delta] d[\Delta^*] e^{(\Delta - V \phi_1 \phi_1^* \mid V^{-1} \mid \Delta - V \phi_1 \phi_1^*)} = 1 \]  
(6)

This shorthand notation actually means for a general interaction in real space

\[ 1 = \int d[\Delta^*] d[\Delta] \exp \left\{ \int \! d\tau d\mathbf{x} \left( \Delta^*(\mathbf{x}, \tau) - \int \! dx'' V(\mathbf{x} - x'') \phi^*_1(\mathbf{x}'', \tau) \phi^*_1(\mathbf{x}'', \tau) \right) \right. \\
\left. \int \! dx'' V^{-1}(x - x') \left( \Delta(x', \tau) - \int \! dx'' V(x' - x'') \phi_1(\mathbf{x}'', \tau) \phi_1(\mathbf{x}'', \tau) \right). \right\} \]  
(7)

or in the case of the interaction \( V(\mathbf{x} - \mathbf{x}') = V_0 \delta(\mathbf{x} - \mathbf{x}') \) it simplifies to

\[ 1 = \int d[\Delta^*] d[\Delta] \exp \left\{ \int \! d\tau d\mathbf{x} \left( \Delta^*(\mathbf{x}, \tau) - V_0 \phi^*_1(\mathbf{x}, \tau) \phi^*_1(\mathbf{x}, \tau) \right) \right. \\
\left. V_0^{-1} \left( \Delta(\mathbf{x}, \tau) - V_0 \phi_1(\mathbf{x}, \tau) \phi_1(\mathbf{x}, \tau) \right) \right\}, \]  
(8)

where again we omit the trace of the logarithm because it is irrelevant for our purposes. Note that eq. (6) leads to

\[ \langle \Delta(\mathbf{x}, \tau) \rangle = \int \! dx'' V(\mathbf{x} - x'') \langle \phi_1(\mathbf{x}'', \tau) \phi_1(\mathbf{x}'', \tau) \rangle = V_0 \langle \phi_1(\mathbf{x}, \tau) \phi_1(\mathbf{x}, \tau) \rangle, \]  
(9)

which is easy to understand if we think about the analogy with \( \langle x \rangle = x_0 \) for a simple gaussian distribution \( e^{A(x-x_0)^2} \). With this HS transformation the fourth order interaction
term in the action cancels and the resulting action is

\[ S[\phi^*, \phi, \rho] = \int d\tau dx \left\{ \sum_{\sigma=\perp, \parallel} \phi^*_\sigma(x, \tau) \left[ \frac{\hbar^2 \nabla^2}{2m} - \mu \right] \phi_\sigma(x, \tau) + \Delta^*(x, \tau) \phi_\perp(x, \tau) \phi_\parallel^*(x, \tau) + \frac{|\Delta(x, \tau)|^2}{V_0} \right\} \] (10)

This can be rewritten in the form of eq. (5) in the question with \( G^{-1} \) given by

\[ -\hbar G^{-1} = \begin{pmatrix} -hG_{0\perp}^{-1}(x, \tau; x', \tau') & \Delta(x, \tau) \delta(x - x') \delta(\tau - \tau') \\ \Delta^*(x, \tau) \delta(x - x') \delta(\tau - \tau') & hG_{0\parallel}^{-1}(x', \tau'; x, \tau) \end{pmatrix} \] (11)

The 'meaning' of a HS-transformation is maybe a little bit vaguely formulated, so we’ll just start chatting away. In general, the HS transformation is a convenient way to get the correct order parameter into your action, so that typically the order parameter you want to study determines the HS transformation you do. From this perspective, the Hartree-like HS from part i), i.e. \( \langle \rho \rangle = V_0 \langle \phi^*_\parallel \phi_\parallel + \phi^*_\perp \phi_\perp \rangle \), would be suitable to study the phase transition from zero density of particles to a nonzero density of particles, since the density of particles is given by \( \langle \phi^*_\sigma \phi_\sigma \rangle \). However, although the transition from zero to a nonzero density of particles is formally a phase transition, it is a rather trivial one (the zero particle case is pretty uninteresting), and usually experimentally not so relevant. Therefore, the main advantage of the Hartree-like HS is that doing mean-field theory in \( \rho \) (after the HS) gives you precisely the Hartree approximation (indeed that’s why we call it the Hartree-like HS). Another way to think about this HS transformation is the following: if we draw the corresponding Feynman-diagram of the term \( \rho \phi^*_\sigma \phi_\sigma \) that we get into the action due to the HS transformation (see figure a), then we see that \( \rho \) couples to a creation and an annihilation of a particle. Since the annihilation of a particle can also be seen as the creation of a hole, one can say that this HS transformation can be used to study particle-hole excitations.
On the other hand, the HS from part ii), introduces the order parameter \( \langle \Delta \rangle = V_0 \langle \phi \phi \rangle \) into our system. This order parameter is very interesting and lateron we will see that it is the order parameter from BCS theory that explains superconductivity, and that it can be physically interpreted as a condensate of Cooper pairs. If you draw the Feynman diagram corresponding to the term \( \Delta \phi \phi^* \) into our action after the HS transformation, then we see that \( \Delta \) couples to the creation operator of two particles with different spin. Therefore one can say that this HS can be used to study particle-particle excitations.

b) Poles of the Green’s function.

We use the following Fourier transforms

\[
\mathcal{G}^{-1}(\mathbf{x}, \tau; \mathbf{x}', \tau') = \sum_n \sum_k \mathcal{G}^{-1}(k, i\omega_n) \frac{e^{ik(x-x')}}{V} \frac{e^{-i\omega_n(\tau-\tau')}}{\hbar \beta},
\]

\[
\left( \phi^*_1(\mathbf{x}, \tau), \phi_1(\mathbf{x}, \tau) \right) = \sum_n \sum_k \left( \phi^*_1(k, n), \phi_1(-k, -n) \right) \frac{e^{-ikx}}{\sqrt{V}} \frac{e^{i\omega_n \tau}}{\sqrt{\hbar \beta}},
\]

with

\[
-h\mathcal{G}^{-1}(\mathbf{x}, \tau; \mathbf{x}', \tau') = \begin{pmatrix}
-h\mathcal{G}^{-1}_{0,1}(\mathbf{x}, \tau; \mathbf{x}', \tau') & \Delta_0 \delta(\mathbf{x} - \mathbf{x}') \delta(\tau - \tau') \\
\Delta_0^* \delta(\mathbf{x} - \mathbf{x}') \delta(\tau - \tau') & \hbar \mathcal{G}^{-1}_{0,1}(\mathbf{x}', \tau'; \mathbf{x}, \tau)
\end{pmatrix}
\]

and therefore

\[
-h\mathcal{G}^{-1}(k, n) = \begin{pmatrix}
-i\hbar \omega_n + \epsilon_k - \mu_\uparrow & \Delta_0 \\
\Delta_0^* & -(i\hbar \omega - \epsilon_{-k} - \mu_\downarrow)
\end{pmatrix}.
\]

The term left above is just the well-known expression for the non-interacting Green’s function in frequency and momentum space \(-h\mathcal{G}^{-1}_0(k, n)\) as can be found at various places in Stoof’s script. The term right below is similar with the differences coming from the reversed order of the arguments in eq. (14). The off-diagonal terms are particularly simple, since the Fourier-transform of a \( \delta \)-function is 1. Also note that \( \omega_{-n} = -\omega_n \) and \( \epsilon_{-k} = \epsilon_k \). The fermionic part of the action then becomes in Fourier space

\[
S[\phi^*_\sigma, \phi_\sigma] = \frac{-\hbar}{V \hbar \beta} \sum_{k, n} \left( \phi^*_\uparrow(k, n), \phi_\downarrow(-k, -n) \right) \mathcal{G}^{-1}(k, n) \begin{pmatrix}
\phi_\uparrow(k, n) \\
\phi^*_\downarrow(-k, -n)
\end{pmatrix},
\]

which can only be obtained by inserting eqs. (12) and (13) into the action, which is eq. (5) on the exercise sheet. Then you get a sum over three different momenta and three different frequencies and two integrals over real space and two over imaginary time. Each integral over space (imaginary time) results in a Kronecker delta in momenta (frequency) space. As a result, we end up with a final sum over 1 momentum and 1 frequency which is the expression given above. To get the poles of the Green’s function, we have to get the zero’s of the inverse Green’s function. That is we will do the analytic continuation.
Figure 2: a) Dispersion for the case $h < |\Delta_0|$ b) Dispersion for the case $h > |\Delta_0|$

$i\hbar\omega_m \rightarrow \hbar\omega_k$ and demand that the determinant of the two-by-two Green’s function matrix is zero. By writing

$$-\hbar G^{-1}(k, n) = \begin{pmatrix} (\hbar\omega_k - \hbar) + (\epsilon_k - \mu) & \Delta_0 \\ \Delta^*_0 & (\hbar\omega_k - \hbar) - (\epsilon_k - \mu). \end{pmatrix}$$

we see that the determinant is zero if

$$(\hbar\omega_k + \hbar)^2 = (\epsilon_k - \mu)^2 + |\Delta_0|^2,$$

which has the solutions

$$\hbar\omega_{k,\pm} = -\hbar \pm \sqrt{(\epsilon_k - \mu)^2 + |\Delta_0|^2}.$$  

The dispersions for the up and down quasiparticles are shown for the two cases ($h < |\Delta_0|$ and $h > |\Delta_0|$) in Figure 2. Note that these dispersions of the quasiparticles are excitation spectra for the system. In the case that $h < |\Delta_0|$ the dispersion is always larger than zero, meaning that the quasiparticle excitation spectrum is gapped, as is the usual case for BCS theory (as we will also see later). However, if $h > |\Delta_0|$ then the quasiparticle spectrum for the up quasiparticles, which is the so-called majority species, is not gapped anymore. This means that we can lower the energy of the system by populating these quasiparticle states, rather than having only a condensate of Cooper-pairs. As a result the ground state of this system consist of both up quasiparticles and Cooper pairs and is therefore called a polarized superfluid.