Exercise 1, 12 September 2006

\( \varphi_0(x) \) is the lowest energy eigenfunction of the harmonic oscillator Hamiltonian (3.44):

\[
\varphi_0(x) = \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} \exp\left( -\frac{m\omega x^2}{2\hbar} \right)
\]  

(1)

It is known that all other eigenfunctions \( \varphi_n(x) \equiv |n\rangle \) with energies \( \hbar\omega(n + 1/2) \) can be created by applying the bosonic creation operator \( a^+ \) on the vacuum state \( |0\rangle \equiv \varphi_0(x) \):

\[
|n\rangle = \frac{(a^+)^n|0\rangle}{\sqrt{n!}}
\]  

(2)

Bosonic operators \( a, a^+ \) can be expressed through canonical coordinates \( x, p \) as given in equation (3.51).

1) Check that

\[
\langle n|p|n\rangle = \langle n|x|n\rangle = 0
\]

2) Let us define

\[
\Delta x \equiv x - \langle n|x|n\rangle, \quad \Delta p = p - \langle n|p|n\rangle
\]

Calculate

\[
\langle n|\Delta x^2|n\rangle \langle n|\Delta p^2|n\rangle
\]  

(3)

What is the connection between this calculation and the Heisenberg’s uncertainty principle?

3) In the previous case, you have calculated \( \langle x^2 \rangle, \langle p^2 \rangle \). From these calculations, find the average values of kinetic and potential energy.