The total action consists of three parts, \( S_{\text{total}}[\psi, \bar{\psi}, u] = S_{\text{el}}[\psi, \bar{\psi}] + S_{\text{ph}}[u] + S_{\text{el-\text{ph}}}[\psi, \bar{\psi}, u] \), where

\[
S_{\text{el}}[\psi, \bar{\psi}] = \int_0^\beta d\tau \int_0^L dx \bar{\psi} \left( \partial_\tau - \frac{1}{2m} \partial_x^2 - \mu \right) \psi 
\]

\[
S_{\text{ph}}[u] = \frac{\rho}{2} \int_0^\beta d\tau \int_0^L dx \left\{ (\partial_\tau u)^2 + c^2 (\partial_x u)^2 \right\} 
\]

\[
S_{\text{el-\text{ph}}}[\psi, \bar{\psi}, u] = g \int_0^\beta d\tau \int_0^L dx \bar{\psi} \psi \partial_x u. 
\]

(a) In order to find the effective action for the bosonic field \( u(x, \tau) \) we have to integrate out the fermionic degrees of freedom in the partition function,

\[
Z = \int d[\bar{\psi}]d[\psi]d[u] \ e^{-S_{\text{total}}[\bar{\psi}, \psi, u]} 
\]

\[
= \int d[u] e^{-S_{\text{ph}}[u]} \int d[\bar{\psi}]d[\psi] \ e^{-S_{\text{el}}[\bar{\psi}, \psi] - S_{\text{el-\text{ph}}}[\psi, \bar{\psi}, u]} 
\]

\[
= \int d[u] e^{-S_{\text{ph}}[u]} \int d[\bar{\psi}]d[\psi] \ e^{-\int dx \tau \bar{\psi} \left[ \partial_\tau - \frac{\partial^2_x}{2m} - \mu + g \partial_x u \right] \psi} 
\]

\[
= \int d[u] e^{-S_{\text{ph}}[u]} e^{\text{Tr} \left[ \ln (-G^{-1}) \right]} = \int d[u] e^{-S_{\text{eff}}[u]} 
\]

where

\[
S_{\text{eff}}[u] \equiv S_{\text{ph}}[u] - \text{Tr} \left[ \ln (-G^{-1}) \right] 
\]

and

\[
-G^{-1} \equiv -G_0^{-1} + g \partial_x u \equiv \partial_\tau - \frac{\partial^2_x}{2m} - \mu + g \partial_x u. 
\]

The trace is here defined as

\[
\text{Tr}[A] \equiv \int_0^L dx \int_0^\beta d\tau A(x, x, \tau). 
\]

We can write the effective action in powers of the small coupling constant \( g \) by Taylor expanding the logarithm,

\[
\ln(1 - x) = -\sum_{n=1}^\infty \frac{x^n}{n}, 
\]

and thus find that

\[
\text{Tr} \left[ \ln (-G^{-1}) \right] = \text{Tr} \left[ \ln \left( -G_0^{-1} (1 - G_0 g \partial_x u) \right) \right] 
\]

\[
= \text{Tr} \left[ \ln (-G_0^{-1}) \right] - \sum_{n=1}^\infty \frac{g^n}{n} \text{Tr} \left[ (G_0 \partial_x u)^n \right]. 
\]
Then the effective action becomes

$$S_{\text{eff}}[u] = S_{ph}[u] + \frac{g^2}{2} \text{Tr} \left[ (G_0 \partial_x u)^2 \right] + \mathcal{O}(g^3) \quad (14)$$

because the first order term is zero, $\text{Tr} [G_0 \partial_x u] = 0$. In order to evaluate the second order term we have to perform a Fourier transformation of the fields,

$$u(x, \tau) = \frac{1}{\sqrt{\beta L}} \sum_{k,n} u(k, i\Omega_n) e^{i k x - i n \Omega \tau} \quad (15)$$

$$\partial_x u(x, \tau) = \frac{1}{\sqrt{\beta L}} \sum_{k,n} i k u(k, i\Omega_n) e^{i k x - i n \Omega \tau} \quad (16)$$

$$G_0(x - x', \tau - \tau') = \frac{1}{\beta L} \sum_{k,n} G_0(k, i\omega_n) e^{i k (x - x') - i \omega_n (\tau - \tau')} \quad (17)$$

$$G_0(k, i\omega_n) \equiv \frac{1}{i \omega_n - k^2/2m + \mu}, \quad (18)$$

where $\omega_n \equiv (2n + 1)\pi/\beta$ and $\Omega_n \equiv 2n\pi/\beta$ are the fermionic and bosonic Matsubara frequencies.

Using translational invariance of the Green’s function, $G(x, \tau; x', \tau') = G(x - x', \tau - \tau')$ we find that

$$\text{Tr} \left[ (G_0 \partial_x u)^2 \right] \quad (19)$$

$$= \int_0^L dx_1 dx_2 \int_0^\beta d\tau_1 d\tau_2 G_0(x_1, \tau_1; x_2, \tau_2) \partial_x u(x_2, \tau_2) G_0(x_2, \tau_2; x_1, \tau_1) \partial_x u(x_1, \tau_1)$$

$$= \frac{1}{\beta^3 L^3} \sum_{k_1, k_2, k_3, k_4} \int_0^L dx_1 dx_2 \int_0^\beta d\tau_1 d\tau_2 \left\{ G_0(k_1, i\omega_n) i k_2 u(k_2, i\Omega_m) G_0(k_3, i\omega_n) \times \right.$$}

$$\left. i k_4 u(k_4, i\Omega_m) e^{i (k_1 - k_3 + k_4) x_1} e^{i (k_1 + k_2 - k_3 - k_4) x_2} e^{i (-\omega_n - \omega_n - \Delta) \tau_1} e^{i (\omega_n - \Omega_m) \tau_2} \right\}$$

Now, using that

$$\int_0^\beta dx e^{i k x} = L \delta_{k, 0}, \quad \int_0^\beta d\tau e^{i \omega_n \tau} = \beta \delta_{\omega_n, 0}, \quad (20)$$

we obtain

$$\text{Tr} \left[ (G_0 \partial_x u)^2 \right] \quad (21)$$

$$= -\frac{1}{\beta L} \sum_{k_1, k_3, n_1, n_2} \{(k_1 - k_3)(k_3 - k_1) G_0(k_1, i\omega_n) G_0(k_3, i\omega_n) \times$$

$$u(k_1 - k_3, i(\omega_n - \omega_n)) u(k_3 - k_1, i(\omega_n - \omega_n)) \}$$

$$= \frac{1}{\beta L} \sum_{q, k_3, m_3} q^2 G_0(k_3 - q, i(\omega_n - \Omega_m)) G_0(k_3, i\omega_n) u(q, -i\Omega_m) u(q, i\Omega_m)$$

$$= -\sum_{q, m} q^2 |u(q, i\Omega_m)|^2 \left( -\frac{1}{\beta L} \sum_{k,n} G_0(k - q, i(\omega_n - \Omega_m)) G_0(k, i\omega_n) \right)$$

$$= -\sum_{q, m} q^2 |u(q, i\Omega_m)|^2 \chi(q, i\Omega_m).$$
In the fourth line we introduced \( q \equiv k_3 - k_1 \) and \( \Omega_m \equiv \omega_{n_3} - \omega_{n_1} \) and we used that \( u(-q, -i\Omega_m) = u(q, i\Omega_m)^* \) for the real field \( u \) in the fifth line. In the last step we used that \( q^2|u(q, i\Omega_m)|^2 \) is invariant under simultaneous change of sign for \( q \) and \( m \).

\[
S_{\text{eff}}[u] = S_{ph}[u] - \text{Tr} \left[ \ln (-G_0^{-1}) \right] - \frac{g^2}{2} \sum_{q,m} q^2|u(q, i\Omega_m)|^2 \chi(q, i\Omega_m).
\]

Fourier transforming \( S_{\text{eff}}[u] \) with

\[
\partial_x u(x, \tau) = \frac{1}{\beta L} \sum_{k,n} (-i\Omega_n)u(k, i\Omega_n)e^{ikx-i\Omega_n\tau} \tag{22}
\]

\[
\int_0^\beta d\tau \int_0^L dx \left[ \partial_x u(x, \tau) \right]^2 = \frac{1}{\beta L} \sum_{k,k',n,n'} (-i\Omega_n)(-i\Omega_n')u(k, i\Omega_n)u(k', i\Omega_n')e^{i(k+k')x-i(\Omega_n+\Omega_n')\tau}
\]

\[
= \frac{\beta L}{\beta L} \sum_{k,n} \Omega_n^2|u(k, i\Omega_n)|^2
\]

\[
\int_0^\beta d\tau \int_0^L dx \left[ \partial_x u(x, \tau) \right]^2 = \sum_{k,n} e^2k^2|u(k, i\Omega_n)|^2,
\]

and omitting the constant term \( \text{Tr} \left[ \ln (-G_0^{-1}) \right] \), yields

\[
S_{\text{eff}}[u] = \sum_{q,n} \left[ \frac{\mu}{2}(\Omega_n^2 + e^2k^2) - \frac{g^2}{2}q^2 \chi(q, i\Omega_n) \right]|u(q, i\Omega_n)|^2. \tag{23}
\]

(b) An existence of a saddle-point of the \( S_{\text{eff}}[u] \) means that the system is not stable at the point and has some directions of instability. In our case it is enough to demonstrate that \( S_{\text{eff}}[u_0 \cos(2k_Fx + \varphi)] < S_{\text{eff}}[u_0] \), i.e. the system is not stable to the formation of a charge-density wave (CDW).

- \( u(x, \tau) = u_0 \cos(2k_Fx + \varphi) \)

\[
u(k, i\Omega_n) = \frac{1}{\sqrt{\beta L}} \int_0^\beta d\tau \int_0^L dx u(x, \tau)e^{-ikx+i\Omega_n\tau}
\]

\[
= \frac{1}{\sqrt{\beta L}} \int_0^\beta e^{i\Omega_n\tau} d\tau \int_0^L dx u_0 \cos(2k_Fx + \varphi)e^{-ikx}
\]

\[
= \frac{1}{2} \sqrt{\frac{\beta}{L}} \delta_{\Omega_n,0} \int_0^L dx u_0 \left[ e^{i(2k_F-k)x+i\varphi} + e^{-i(2k_F+k)x-i\varphi} \right]
\]

Therefore,

\[
u(k, i\Omega_n) = \frac{1}{2} u_0\delta_{\Omega_n,0}\sqrt{\beta L} \left( e^{i\varphi}\delta_{k,2k_F} + e^{-i\varphi}\delta_{k,-2k_F} \right) \tag{25}
\]

and

\[
|u(k, i\Omega_n)|^2 = \frac{\beta L}{4} u_0^2\delta_{\Omega_n,0}(\delta_{k,2k_F} + \delta_{k,-2k_F}) \tag{26}
\]
Substitution into Eq. (??) yields

\[ S_{\text{eff}}[u_0 \cos(2k_Fx + \varphi)] = \sum_{q,n} \left[ \frac{\rho}{2} (\Omega_n^2 + c^2 k^2) - \frac{g^2 q^2}{2} \chi(q, i\Omega_n) \right] \frac{\beta L}{4} u_0^2 \delta_{\Omega_n,0} (\delta_{q,2k_F} + \delta_{q,-2k_F}). \]

This can be rewritten as

\[ S_{\text{eff}}[u_0 \cos(2k_Fx + \varphi)] = \left[ \frac{\rho}{2} c^2 - \frac{g^2}{2} \chi(2k_F, 0) \right] \frac{\beta L}{4} u_0^2 (2k_F)^2 + \left[ \frac{\rho}{2} c^2 - \frac{g^2}{2} \chi(-2k_F, 0) \right] \frac{\beta L}{4} u_0^2 (-2k_F)^2. \]

Since \( \chi(-q,0) = \chi(q,0) \), the expression simplifies to

\[ S_{\text{eff}}[u_0 \cos(2k_Fx + \varphi)] = \left[ \rho c^2 - g^2 \chi(2k_F, 0) \right] \beta L k_F^2 u_0^2. \]

\[ u(x, \tau) = u_0 \]

\[ u(k, i\Omega_n) = u_0 \sqrt{\beta L} \delta_{\Omega_n,0} \delta_{k,0} \]  

\[ u_0 \sqrt{\beta L} \delta_{\Omega_n,0} \delta_{k,0} \]  

\[ u(k, i\Omega_n) = u_0 \sqrt{\beta L} \delta_{\Omega_n,0} \delta_{k,0} \]  

Substitution into Eq. (??) yields

\[ S_{\text{eff}}[u_0] = 0. \]  

It follows then that in case \( S_{\text{eff}}[u_0 \cos(2k_Fx + \varphi)] < S_{\text{eff}}[u_0] \) should be \( \rho c^2 < g^2 \chi(2k_F, 0) \) and the transition to CDW is favorable

\[ g^2 \chi(2k_F, 0) = g^2 \frac{\ln(\beta \omega_D)}{4\pi v_F} > \rho c^2. \]  

Therefore, at critical temperature

\[ \ln(\beta \omega_D) = \frac{4\pi v_F \rho c^2}{g^2} \Rightarrow \beta \omega_D = e^{\frac{4\pi v_F \rho c^2}{g^2}} \]  

we finally obtain

\[ T_c = \omega_D e^{-\frac{4\pi v_F \rho c^2}{g^2}}. \]  

\[ T_c(g) = \omega_D e^{-\frac{4\pi v_F \rho c^2}{g^2}} \]  

one notice that the function has an essential singularity at \( g = 0 \) and, therefore, its Laurent series has infinitely many negative degree terms in \( g \). Thus, one can not recover the function \( T_c(g) \) by summing any finite amount of terms obtained from a perturbation theory around the point \( g = 0 \). The result we obtained is connected to the BCS theory of superconductivity in a sense that both the formation of a charge-density wave and the BCS condensation happen due to a Fermi surface instability. As a result of this instability in the BCS case the Cooper pairs are formed, whereas in the case of a CDW dimerization takes place (atoms in a lattice experience periodic displacement, such that the lattice period doubles). In addition, both phenomena come from electron-electron interactions mediated by phonons.