Exercise 8.1
The idea of this exercise is to apply the exact relations (7.13) and (7.14) for Gaussian (i.e. quadratic) integrals to eqs. (8.2) and (8.8), which indeed contain only quadratic integrals. As mentioned in the exercise the first step is to expand the fields $\phi_{j,\alpha}$ in terms of the eigenstates given by eq. (6.3), giving

$$
\phi_{j,\alpha}(x) = \sum_n \phi^{n,j,\alpha}_n(x),
$$

(1)

with the complex conjugated relation for $\phi^*_{j,\alpha}$. This results in

$$
(\phi_j | \phi_j) \equiv \sum_\alpha \int dx \phi_{j,\alpha}(x) \phi_{j,\alpha}(x) = \sum_{n,\alpha} \phi^*_{n,j,\alpha} \phi_{n,j,\alpha}
$$

(2)

$$
(\phi_j | \phi_{j-1}) = \sum_{n,\alpha} \phi^*_{n,j,\alpha} \phi_{n,j-1,\alpha}
$$

(3)

$$
H[\phi_j, \phi_{j-1}] = \sum_{n,\alpha} (\epsilon_{n,\alpha} - \mu) \phi^*_{n,j,\alpha} \phi_{n,j-1,\alpha},
$$

(4)

with $\epsilon_{n,\alpha} = \epsilon_n + \epsilon_\alpha$. To obtain (4), we substituted (1) into (8.5) and since we are dealing with an ideal gas, the second line of (8.5) is absent. Next we substitute (8.8) into (8.2), where we rename in (8.2) $\phi$ as $\phi_M$. Together with the condition $\phi_M = \pm \phi_0$, we get for the partition sum $Z$

$$
Z = \int \prod_{j=1}^M \prod_n \phi^*_{n,j,\alpha} \phi_{n,j,\alpha} \exp \left\{ - \sum_{j=1}^M \sum_{n,\alpha} \left( \phi^*_{n,j,\alpha} \phi_{n,j,\alpha} \right) \left[ -1 + (\epsilon_{n,\alpha} - \mu) \frac{\Delta \tau}{\hbar} \right] \phi^*_{n,j,\alpha} \phi_{n,j-1,\alpha} \right\}.
$$

(5)

To this expression we can apply (7.12) for fermions or (7.13) for bosons. Let’s visualize the shape of our matrix $A$. First of all, note that $A$ is diagonal in $n$ and $\alpha$, since only variables with the same $n$ and $\alpha$ give a contribution to the integral. Since we know that the determinant of a diagonal matrix is just the product of the diagonal elements, this means that the determinant of $A$ becomes $\det[A] = \prod_{n,\alpha} A^{n,\alpha}_{j,j'}$. Now, what does this reduced matrix $A^{n,\alpha}_{j,j'}$ look like? It is not diagonal in $j$, since in eq. (5) also variables with $j$ and $j-1$ give a contribution to the integral. This actually leads to the following shape
for $A^{n,\alpha}_{jj'}$

$$A^{n,\alpha}_{jj'} = \begin{pmatrix} 1 & 0 & 0 & \ldots & \pm c_{n,\alpha} \\ c_{n,\alpha} & 1 & 0 & 0 \\ 0 & c_{n,\alpha} & 1 & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & c_{n,\alpha} & 1 \end{pmatrix}, \quad (6)$$

with $c_{n,\alpha} = -1 + (\epsilon_{n,\alpha} - \mu) \Delta \tau / \hbar$. The element right above in the matrix comes from the term $\phi_{n,1,\alpha}^* \phi_{n,0,\alpha}$ in eq. (5) and the condition $\phi_{n,0,\alpha} = \pm \phi_{n,M,\alpha}$. It is easy to determine the determinant of $A^{n,\alpha}_{jj'}$, which gives

$$\det[A^{n,\alpha}_{jj'}] = 1 + (-1)^{M+1} (\pm 1) c_{n,\alpha}^M = 1 \mp (1 - (\epsilon_{n,\alpha} - \mu) \beta / M)^M, \quad (7)$$

where we used $\Delta \tau = \hbar \beta / M$. As a result applying eq. (7.12)/(7.13) to eq. (5) yields for fermions/bosons

$$Z = \prod_{n,\alpha} [1 \mp (1 - (\epsilon_{n,\alpha} - \mu) \beta / M)^M]^{\mp 1}, \quad (8)$$

where the upper (lower) sign corresponds to bosons (fermions). Taking the limit $M \to \infty$ and using the hint in the exercise, we finally obtain

$$Z = \prod_{n,\alpha} [1 \mp e^{-\beta(\epsilon_{n,\alpha} - \mu)}]^{\mp 1}, \quad (9)$$

which is exactly the same result as in eq. (8.16).

**Exercise 8.2**

In the lecture notes it is explained how equation (8.23) can be proven by using contour integration. Now, we have to prove a similar relation, where the only difference is the minus sign in front of $i\omega_n$, which we will show to lead to the extra $-1$ on the right.

If $C$ is a contour enclosing only the imaginary axis as in fig. 8.2, then Cauchy’s theorem (or maybe better: the residue theorem) tells us that

$$\lim_{\eta \downarrow 0} \frac{1}{2 \pi i} \int_C \frac{e^{\eta z}}{z - (\epsilon - \mu) / \hbar \epsilon^{h \beta z}} \frac{\pm 1}{\mp 1} = \lim_{\eta \downarrow 0} \frac{1}{\hbar \beta} \sum_n \frac{e^{i\omega_n \eta}}{-i\omega_n - (\epsilon - \mu) / \hbar}, \quad (10)$$

where the upper signs correspond to bosons and the lower signs correspond to fermions. Note that for bosons we have to sum over the even Matsubara frequencies, whereas for fermions we have to sum over the odd Matsubara frequencies. Relation (10) comes about because the function $\pm 1 / (e^{h \beta z} \mp 1)$ has first order poles on the imaginary axis at the positions $z = in\pi / h\beta = i\omega_n$ with $n$ even/odd. The residue of the function on the left-hand-side of eq. (10) at its first order poles are given by

$$\lim_{z \to i\omega_n} \left( z - i\omega_n \right) \frac{e^{\eta z}}{-z - (\epsilon - \mu) / \hbar \epsilon^{h \beta z}} = \frac{1}{\hbar \beta - i\omega_n - (\epsilon - \mu) / \hbar}, \quad (11)$$
And so, eq. (10) follows directly from the residue theorem.

Next, we can add freely the curves $C'$ as shown in figure (8.2), since the integral over these curves with infinite radius vanishes as explained in the lecture notes on page 95. As a result we have now two closed contours, both half circles, just like in figure (8.2). We call the left half circle $C_L$ and the right half circle $C_R$. This yields

$$\lim_{\eta \downarrow 0} \frac{1}{2\pi i} \int_C \frac{e^{\eta z}}{z - (-\epsilon - \mu)/\hbar e^{\hbar \beta z}} \frac{\pm 1}{\mp 1} = \lim_{\eta \downarrow 0} \frac{1}{2\pi i} \int_{C_L} \frac{e^{\eta z}}{z - (-\epsilon - \mu)/\hbar e^{\hbar \beta z}} \frac{\pm 1}{\mp 1} + \lim_{\eta \downarrow 0} \frac{1}{2\pi i} \int_{C_R} \frac{e^{\eta z}}{z - (-\epsilon - \mu)/\hbar e^{\hbar \beta z}} \frac{\pm 1}{\mp 1}$$

(12)

Note that only one of the contours encloses a pole, namely at $z = (-\epsilon - \mu)/\hbar$, the other contour integral gives 0. Using Cauchy’s theorem (or the residue theorem) again, we find

$$\lim_{\eta \downarrow 0} \frac{1}{2\pi i} \int_C \frac{e^{\eta z}}{z - (-\epsilon - \mu)/\hbar e^{\hbar \beta z}} \frac{\pm 1}{\mp 1} = -\lim_{\eta \downarrow 0} \frac{\pm e^{-\eta(-\epsilon - \mu)/\hbar}}{e^{-\beta(-\epsilon - \mu)} \mp 1} \mp 1 \mp \frac{1}{e^{\beta(-\epsilon - \mu)} \mp 1} - 1, \quad (13)$$

where the extra minus signs in the first line comes from the fact that the contours $C_L$ and $C_R$ are clockwise instead of counterclockwise.

**Imaginary time path integral**

1. $t \to -i\tau \Rightarrow dt \to -id\tau \Rightarrow \mathcal{T}(t) \propto \dot{x}(t)^2 \to -\mathcal{T}(\tau)$. Hence, $iS = i \int dt (\mathcal{T}(t) - \mathcal{V}) \to \int d\tau (-\mathcal{T}(\tau) - \mathcal{V}) = -S_E$.

2. With the Wick rotation, the Euler-Lagrange equation becomes $\frac{\partial \mathcal{L}(\tau)}{\partial x} - \frac{d}{d\tau} \frac{\partial \mathcal{L}(\tau)}{\partial \dot{x}(\tau)} = 0$, where $\mathcal{L}(\tau) = \frac{1}{2}m\ddot{x}(\tau)^2 + \frac{1}{2}m\omega^2 x(\tau)^2$; and thus, $\omega^2 x(\tau) = \ddot{x}(\tau)$. Solving this equation yields $x(\tau) = C_1 \cosh \omega \tau + C_2 \sinh \omega \tau$. Plugging in the boundary conditions $x(\tau_a) = x_a$ and $x(\tau_b) = x_b$, we arrive, after some lengthy algebra, at

$$x(\tau) = \frac{x_b \sinh (\omega (\tau - \tau_a)) - x_a \sinh (\omega (\tau - \tau_b))}{\sinh (\omega (\tau_b - \tau_a))}; \quad (14)$$

substituting this into the action then reads (0.2).

3. We have

$$\mathcal{Z} = \sum_n e^{-\beta E_n} = \text{Tr} e^{-\beta H} = \sum_n \langle \phi_n | e^{-\beta H} | \phi_n \rangle. \quad (15)$$

Using (0.5) and the completeness relation, we then get

$$\mathcal{Z} = \sum_n \langle \phi_n | e^{-\beta H} | \phi_n \rangle = \sum_n \int dx dx' \langle \phi_n | x \rangle \langle x | e^{-\beta H} | x' \rangle \langle x' | \phi_n \rangle = \int dx \langle x | e^{-\beta H} | x \rangle. \quad (16)$$
4. The idea of this exercise is to observe the relation between the imaginary-time propagator and the partition function $Z$, that is,

$$ Z = \int_{x_o}^{x, \hbar \beta} \mathcal{D}x(\tau) \exp \left\{ -\frac{1}{\hbar} \int_0^{\hbar \beta} d\tau \left[ \frac{m}{2} \left( \frac{dx}{d\tau} \right)^2 + V(x) \right] \right\} = \int dx \langle x | e^{-\beta H} | x \rangle. \tag{17} $$

Substituting our expression for the propagator and the Euclidean action then gives

$$ Z = \sqrt{\frac{m \omega}{2\pi \hbar \sinh (\omega (\tau_b - \tau_a))}} \int dx \exp \left( -\frac{m \omega x^2}{\hbar \sinh (\omega \hbar \beta)} [\cosh (\omega \hbar \beta) - 1] \right) \tag{18} $$

$$ = -\frac{1}{\sqrt{2}} \left( \cosh (\hbar \beta \omega) - 1 \right) = \frac{1}{2 \sinh \left( \frac{\hbar \beta \omega}{2} \right)}. $$

Additional remarks about "the Jacobian"

Before looking at eq. (8.20), where there is an infinite coordinate transformation involved, we’ll look at a simpler problem to illustrate the difference between bosonic transformation of variables and fermionic transformation of Grassmann variables. In the text below eq. (8.20) there is already a short explanation about the difference between these two types of transformations. Here we’ll show the difference again, via a slightly different approach, namely the exact relations (7.12) and (7.13), which we have already proven before. From these relations we have that

$$ \int \frac{d\phi^* d\phi}{(2\pi i)^{(1\pm 1)/2}} \exp \{-\phi^* \phi\} = 1 \tag{19} $$

Now, let’s perform the following change of variables

$$ \phi = c \psi \quad , \quad \phi^* = c^* \psi^*. \tag{20} $$

Since changing the variables shouldn’t change the outcome of the integrals, we have that

$$ \int \frac{d\psi^* d\psi}{(2\pi i)^{(1\pm 1)/2}} J \exp \{-|c|^2 \psi^* \psi\} = J |c|^2 = 1, \tag{21} $$

where the upper (lower) signs corresponds to bosons (fermions) and $J$ is a factor (or in general: function) that accounts properly for the change of variables. For ordinary complex variables (i.e. bosons) we all know this factor, it is the Jacobian of the transformation, which immediately gives $J = |c|^2$ for the transformation (20), in agreement with eq. (21). However, we see that for Grassmann variables this transformation factor $J$ should be $1/|c|^2$, which is precisely the inverse of the factor that we have for the bosonic case. So in general, we have that changing the variables in the fermionic case gives an inverse Jacobian compared to the bosonic case. This confirms and explains again the $1/(\hbar \beta)^{\pm 1}$ factor in equation (8.20).
However, in eq. (8.20) we have an infinite transformation and the Jacobian be-
comes an infinite matrix, whose elements are easily determined, but whose determinant
is rather cumbersome to calculate. So it is better to calculate it indirectly from other
relations that we have already proven. This is actually already done by us and the lecture
notes. In exercise 8.1 we derived the exact result for an ideal quantum gas. From eq.
(8.21) and (8.22), we see that the same result is obtained using the Matsubara expansion
method, as it should, since we only have performed a change of variables. However, in
getting to (8.21) we explicitly used the Jacobian as given in (8.20) and any other Jaco-
bian would have led to a different and incorrect result. Therefore, the Jacobian in (8.20)
is the only one that can be correct.