Pergamon

Stud. Hist. Phil. Mod. Phys., Vol. 30, No. 4, pp. 543–545, 1999

Printed in Great Britain
1355-2198/99 $ - see front matter

The Equivalence Myth of Quantum Mechanics (Addendum)

*F. A. Muller**

This addendum non-trivially strengthens one of the six claims of my two-part paper 'The Equivalence Myth of Quantum Mechanics', published in this journal in 1997. © 1999 Elsevier Science Ltd. All rights reserved.

In Muller (1997a, 1997b), I analysed the inter-theoretic relations between matrix mechanics, wave mechanics and quantum mechanics, and defended six claims. Claim IIIB, asserting the 'S-equivalence' of matrix mechanics and wave mechanics, appears however to be based on *too weak* a rigourous construal of what Schrödinger meant by 'equivalence' in his celebrated equivalence-proof (1926). This construal of Schrödinger's notion of the equivalence (S-equivalence) crucially involved (besides other things we gloss over in the present addendum becuse they remain untouched) the equinumerosity of the set of all bases of $L^2(\mathbb{R})$ and the set of all pairs $\langle P, Q \rangle$ of canonical matrices. I proved this by an appeal to Von Neumann's Unitary-Uniqueness Theorem (UUT) (1997b, p. 234). In private conversation, N.P. Landsman asked me how to understand the relevance of UUT to the matter at hand. Reflection on this question has led me to conclude that UUT is irrelevant, because one easily proves the mentioned equinumerosity without any appeal to it, by means of a simple cardinality argument (elementary exercise). I did appeal to UUT because of a remark of G. G. Emch's (1983, pp. 336–337), who stated that Schrödinger's equivalence-'proof' is *a precursor of* UUT. This remark of Emch's is incorrect and misleading; see Muller (1998, p. 372, fn. 528) for an explanation.

The historically more faithful construal of Schrödinger's intentions is to require for *proper* S-equivalence (in addition to the things we gloss over) that *moment mapping M* (Muller, 1997, eqn (39)) *exists and is bijective*. The

(*Received 28 February 1999; Revised 15 April 1999*)
* Department for Foundations and History of the Natural Sciences and Mathematics, Utrecht University, PO Box 80.000, 3508 TA Utrecht, The Netherlands (*e-mail*: f.a.muller@phys.uu.nl).

PII: S1355-2198(99)00016-7

equinumerosity of M's domain and range (S-equivalence) entails the existence of *some* bijection between these sets, but we must require that M qualifies as such a bijection (proper S-equivalence), otherwise the moment problem becomes irrelevant and this would scandalise Schrödinger's notion of equivalence entirely (1997a, p. 57). (For comparison: in Muller (1998, p. 366) we simply call 'S-equivalence' what we call here 'proper S-equivalence' and ignore there altogether what we call here, and called in Muller (1997a, 1997b), 'S-equivalence'.)

The moment mapping M relates a basis $\{\phi_j\}$ of $L^2(\mathbb{R})$ to a canonical pair $\langle P, Q\rangle$ (and their polynomials) via the canonical wave operator $\hat{Q}$ and the inner-product-integrals of $L^2(\mathbb{R})$:

$$\langle \phi_j | \hat{Q}^n \phi_j \rangle = \int_{-\infty}^{+\infty} \phi_j^*(q)\, q^n \phi_j(q)\, dq = (Q^n)_{jj}, \tag{1}$$

and *mutatis mutandis* for $\hat{P}$. As soon as some member of $\{\phi_j\}$ does *not* lie in the domain of the linear wave operator ring generated by $\hat{P}$ and $\hat{Q}$, *i.e.* not in Schwarz-space $\mathscr{S}(\mathbb{R})$, then for some power $\hat{Q}^n$ or some power $\hat{P}^m$, the integral (1) or the integrand of $\langle \phi_j | \hat{P}^m \phi_j \rangle$, respectively, is ill-defined. By implication, M does not exist with as its domain the set of all wave bases. So the very least we need in order to refute the proper S-equivalence of matrix mechanics and wave mechanics is some solution of Schrödinger's equation (with self-adjoint Hamiltonian) that does not lie inside $\mathscr{S}(\mathbb{R})$.

All $l = 0$ solutions of the hydrogen atom with its most simple Hamiltonian (*i.e.* Schrödinger's of 1926: $\hat{P}^2/2m + V_c(r)$, where V_c is the Coulomb-potential, in $L^2(\mathbb{R}^3, r^2 dr \sin\theta\,(d\theta\, d\varphi))$, are not smooth because the derivatives of their radial wavefunctions have a singularity at $r = 0$, see Schiff (1995, p. 86) (I thank N.P. Landsman for bringing this to my attention). Further, in the example of an arbitrary finite potential well, some eigenfunctions of the Hamiltonian have a discontinuity, or their derivatives have one, or both have one (*ibid*., pp. 32–33). In the example of a square potential well, discontinuities occur in the second derivative of the eigenfunctions at the edges, as one easily verifies (*ibid*., p. 36). *Hence proper S-equivalence is refuted.*

But that is not the end of our story. We really should drop the requirement of smoothness and allow for piece-wise smooth wave bases, say, because all that matters is *that we obtain matrices*, for which the existence of the definite integrals suffices. Furthermore, most potentials in physics have singularities and the eigenfunctions of the corresponding Hamiltonians are by consequence never smooth, so smoothness of wave bases is too restrictive for physics anyway. We cannot tinker with the other requirement for a wave basis to live in $\mathscr{S}(\mathbb{R})$, namely falling off faster than any power in q, because if some member of some basis $\{\phi_j\}$ for $L^2(\mathbb{R})$ does not fall off faster than some power, n say, then the integral (1) with integrand containing q^n will diverge.

Hence a more inclusive set than Schwarz-space is called for, $\mathscr{S}^+(\mathbb{R})$ say, consisting of exactly those wave functions which are piece-wise smooth and fall off faster than any power.

In order to refute S^+-equivalence, we have to find some solution of Shchrödinger's equation which is not even piece-wise smooth. In fact, one can consider pretty pathological, arguably 'unphysical', potentials, such as a harmonic oscillator plus a bounded, nowhere differentiable function (such as the famous Weierstraß-function). Then the groundstate is nowhere twice-differentiable and therefore not piece-wise smooth. (For an overview of *Schrödinger-operators*, $\Delta + V$, see Simon (1982) and references therein.) *This refutes even proper* S^+*-equivalence.*

To conclude, anyone who claims that today matrix mechanics and wave mechanics, as we currently know these theories, are 'equivalent' in Schrödinger's intended sense (proper S-equivalence), or in some reasonably weakened sense (such as proper S^+-equivalence), is not stating a mathematically established fact, but is perpetuating the Equivalence Myth of quantum mechanics.

References

Emch, G. G. (1984) *Mathematical and Conceptual Foundations of 20th Century Physics* (Amsterdam: North-Holland).

Muller, F. A. (1997a) 'The Equivalence Myth of Quantum Mechanics. Part I', *Studies in the History and Philosophy of Modern Physics* **28,** 35–61.

Muller, F. A. (1997b) 'The Equivalence Myth of Quantum Mechanics. Part II', *Studies in the History and Philosophy of Modern Physics* **28,** 219–247.

Muller, F. A. (1998) *Structures for Everyone : Proofs and Contemplations in the Foundations and Philosophy of Physics and Mathematics* (Amsterdam: Gerits & Son).

Schiff, L. I. (1995) *Quantum Mechanics*, 2nd edn (New York: McGraw–Hill).

Simon, B. (1982) 'Schrödinger Semigroups', *Bulletin of the American Mathematical Society* **7,** 447–526.