An Axiomatic Approach to Physical Systems

Januari 2004

Summary

Mereology is generally considered as a formal theory of the part-whole relationship concerning material bodies, such as planets, pickles and protons. We argue that mereology can be considered more generally as axiomatising the concept of a ***physical system****, such as a planet in a gravitation-potential, a pickle in heart-burn and a proton in an electro-magnetic field. We design a theory of sets and physical systems by extending standard set-theory (*ZFC*) with mereological axioms. The resulting theory deductively extends both 'Mereology' of Tarski & Lèsniewski as well as the well-known 'calculus of individuals' of Leonard & Goodman. We prove a number of theorems and demonstrate that our theory extends* ZFC *conservatively and hence equiconsistently. We also erect a model of our theory in* ZFC*.*

The lesson to be learned from this paper reads that not only is a marriage between standard set-theory and mereology logically respectable, leading to a rigorous vindication of how physicists talk about physical systems, but in addition that sets and physical systems interact at the formal level both quite smoothly and non-trivially.

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0.0 Overview

This paper subdivides into three large Sections, one small closing Section and one Appendix. In Section 0, we prepare ourselves in all respects to take off; we motivate the present paper, and we delve into the heuristics, desiderata and requirements of a mereological theory we shall found and develop in the subsequent Section (Section 1). In Section 2, we answer various meta-theoretical questions, notably the relation between our theory and standard set-theory (ZFC). In the small final Section 3, we verify which desiderata and requirements of Section 0 our theory fulfils. The Appendix contains proofs of Lemmas and Theorems.

The current Section is further organised as follows We address the question why we should conduct mereological investigations at all (Subsection 0.1). We formulate and motivate a list of requirements which our theory of physical systems must satisfy (Subsection 0.3), after a few heuristic considerations concerning the concept of a physical system (Subsection 0.2). We end this Section by reporting the most important logical results about the mereological theories known to the author (Subsection 0.4)

0.1 Motivation

A substantial part if not the overwhelming majority of physicists holds the view that, *first*, physics is about physical systems, ranging from galaxies swirling majestically in the cosmic void via colliding gas molecules in a piston to unobservable red bottom quarks locked up in baryons; *secondly*, that we can legitimately ascribe properties and interrelationships to physical systems on the basis of successful physical theories; and, *thirdly*, that physics without the application of mathematics is inconceivable. To tell a story about the epistemic fruits of the enterprise of physics without mentioning physical systems might be possible, if only because no story is impossible to tell for an imaginative mind, but it would be a tall story, an astronomically tall piece of fiction.

In the 1950s and 1960s, P. Suppes, influenced by the emergence of Tarskian model theory, revolutionised philosophy of science by proclaiming that *scientific theories* should not be considered as deductive closures of sentences in some formal language, as the logical-positivist philosophers construed theories, but as sets of Bourbakian *set-structures* in the domain of discourse of standard, *Zermelo-Fraenkel set-theory* (ZFC). Suppes' Slogans: *To axiomatise a scientific theory is to define a set-theoretical predicate.* And: *No meta-mathematics for the philosophy of science but mathematics!*[1] Set-theory takes care of the application of mathematics to physics, as well as of the concept of a 'set-structure'.[2] We therefore do not delve into this

[1]See Sneed [1979] for the first treatise in the Suppes Tradition; and see Muller [1998, pp. 253-303] for an exposition and overview of these matters, with examples, for finger-pointing to the inconsistencies in Sneed [1979], and for references to Suppes' original publications which we have omitted here; for ZFC, see Fraenkel [1973, p. 22], Lévy [1979] or Muller [1998, pp. 34-44].

[2]Ssee Muller [1998, pp. 98-122] for an accessible exposition of Bourbaki's concept of a set-structure and his

topic. The topic we address in this paper is that of physical systems and how to incorporate them formally in the domain of discourse of ZFC.

Lists of examples of physical systems can easily be produced. But only knaves and fools would claim they can think of a general *definiens* for the concept of a physical system in terms of concepts which are philosophically less problematic than the *definiendum*. Therefore we adopt, in the spirit of Hilbert, a direct approach to physical systems: we axiomatise them and thus obtain a general theory of them. The meaning of a concept is generally determined by how it is used and can be used (Wittgenstein). To erect a formal theory is to lay down those rules explicitly. When our formal theory of mereolgy manifestly governs the behaviour of the concept of a physical system and a variety of related concepts in the practice of physics, then we have done the best we can to elucidate the meaning these concepts have. This formal theory will also justify the use of the concept of a *physical structure*, which stands in contrast to the concept of a *mathematical structure*, because our formal theory incorporates, besides pure sets, also physical systems (we gloss over this here; see Muller [1998, p. 276]). We then are able to characterise physical theories as *sets of physical structures*.

Conclusion: we want a formal theory of physical systems that extends standard set-theory.

0.2 Heuristics

When besides sets we adopt physical systems as primitive elements to inhabit the **U**niverse of theoretical discourse, call it **U**, we need to ask ourselves whether we need more primitive notions and what they are. Which primitive attributes and relations to adopt for physical systems? We need to answer these questions before we can lay down a formal language to formulate the theory.

Now, the axioms of ZFC postulate, besides the *existence* of sets (conditionally on *given* sets: Paring, Separation, Union, Power, Replacement, Choice, or unconditionally: Transfinity), also certain *attributes* that all sets share (Extensionality, Regularity). Remarkably few attributes, one should observe, and this is exactly why there are so many sophisticated set-theoretical statements which turn out to be undecidable — of which the continuum hypothesis is the paradigm. Nonetheless the axioms of ZFC suffice to found all of mathematics (save the theory of large categories), which we take to be even more remarkable. Are there any attributes or relationships which are plausibly shared by all physical systems according to *all* successful

view of mathematics as the study of set-structures, including a critical discussion.

physical theories? We can think of the following three features.

1. Physical systems are always situated in space-time: they occupy certain regions of space, tiny, small, large or huge, over a tiny, small, large or huge period of time. *Existence of physical systems* is nothing but *persistence in space over time.* Physical systems existing 'above and beyond space' or 'outside time' smells like nonsense. Talk of the existence of physical systems without presupposing or even mentioning space or time, might be conceivable, but it would not be physics as we know it.
2. Some general attributes of, and relationships between physical systems from the practice of physics are the following ones: physical systems can be *identical*, they can be *subsystems* of other physical systems, they can have something *in common* with other physical systems, they can be composed with other physical systems to form *composite physical systems*, they can be *elementary* in that they have no genuine subsystems.
3. Physical systems are characterised by *physical magnitudes*, the numerical determination of which, experimentally, theoretically or computationally, virtually characterises the collective activity of physics. A physical system without at least one physical magnitude pertaining to it is no physical system at all. Theorising about physical systems without mentioning physical magnitudes might be *possible* but, again, it would not be physics as we know it.

A few comments are in order.

Ad. 1. Feature 1 confronts us with the concepts of *space* and *time*, which are as old as physics, mathematics and philosophy itself. Should they be introduced as primitive notions in a founding theory of physical theories? J. Earman [1989] has exemplified that all space-time conceptions, ranging from Aristotle and Leibniz via Galilei and Newton to Mach and Einstein, can be smoothly represented as differentiable manifold structures. This makes plausible the adoption of *space-time points* as primitive elements in our theoretical universe of discourse **U** and *define* particular differentiable manifold structures such that the base sets of these manifold structures consist of the primitive space-time points. (This would actually be the most faithful rigourous construal of what physicists do: they think of the base sets of these manifolds as *space-time* and their members as *space-time points*, not as pure sets of pure sets, as we are forced to say when working in pure set-theory.) No axioms then need be adopted to govern the use of space-time points. But then, as far a space and time is concerned, this comes down to accepting entities in **U** about which nothing further is assumed, save that they are *not sets.* ZFC is known to remain consistent when such non-set elements are invited for which no additional axioms are adopted (see Fraenkel [1973, p. 24]). So we could make this move at any point we feel like it and pass the minimal requirement of logical respectability (consistency). Whether *this* can be done *conservatively*, we have not been able to find in the literature on set theory — but we shall prove it in passing and thereby achieve a status of higher logical respectability than when having proved 'mere' consistency.

A positive reason for *not* adopting axioms to govern space-time points is there are so many different physical theories of space-times (cf. Earman [1989]), which must all find a home in our domain of discourse **U** when we want to contrue physical theories as sets of physical structures, just as all theories of mathematics find a home there (again, save the theory of large categories). Further, at the cutting edge of theoretical physics it is believed that the very concepts of space and time as we know them will 'break down at the Planck-scale'. Finally, there is quantum mechanics, where the relations between physical systems and space-time is not straightforward. All the more reasons to introduce at most bare space-time points only and not to endow them *axiomatically* with all kinds of attributes and relationships.

Ad. 2. The italicised notions of feature 2 occur in physics 'at the heuristic level', as the saying goes; they are never axiomatised in physical theories and rarely present in symbolism, in marked contrast to the other physical notions, *viz.* space, time and physical magnitudes, which occur in *physical theories as certain mathematical entities.* In *mereological theories* (unknown to most physicists and mathematicians), one talks about 'bodies' and adopts a 'part-whole' relation as primitive, in terms of which the following notions can be defined (sometimes with the aid of the concept of a set and the membership-relation): a 'composition' of bodies (mereological sum or fusion), 'part of a body' (via the part-whole relation), 'atomic body' (body without genuine parts) and 'common part of several bodies' (most encompassing part all bodies share). We shall show that by adopting a primitive relation of 'being a subsystem of', analogous to the mereological part-whole relation, all these mereological notions can be carried over to a general theory of physical systems.

The use of composite systems and subsystems is particularly endemic in quantum mechanics, where the state space (a Hilbert-space) of a composite physical system is the direct-product space of the state spaces of the composing subsystems; and the state space of each is a subspace of the state space of the composite system. Hence the relations between the various physical systems are reflected by the relations between their state spaces.

Ad. 3. Physical magnitudes are standardly 'represented' in physics by purely mathematical entities, which can all be constructed in ZFC. We therefore leave them be.

These considerations lead us to answer the questions we raised in the beginning of this Subsection: besides sets, only physical systems will be primitive elements in **U**, and besides the membership-relation, only a primitive dyadic relation of subsystemhood. So we know in which formal language we shall express ourselves. This brings us to the next question, addressed in the next Subsection: which axioms to adopt?

0.3 Requirements

The requirements we list below, and motivate below the list, are based on a few reflections as to how the concepts of physical system, subsystem and composite system are used in physics.

(1) *Subsystems.* The subsystem-relation is reflexive and transitive but not connective, and the proper subsystem-relation is irreflexive, transitive but neither symmetric nor connective.

(2) *Identity.* There is an identity-relation for physical systems which is an equivalence-relation and which obeys the requirement of Substitutivity; identical physical systems share all their subsystems; and if two physical systems are each other's subsystem, then they are identical.

(3) *Composite Physical Systems.* When we compose physical systems into a composite system, these physical systems are subsystems of it and it does not have a subsystem which has nothing in common with every the subsystems we started with; and when we delete some of the composing systems we never obtain a more encompassing physical system.

(4) *Consistency.* Our theory must demonstrably extend ZFC consistently and preferably conservatively.

We can imagine to get on with our lives when our mereological theory is *progressive* over ZFC (*i.e.* not conservative), but our intention is not to mess with ZFC in any way: we want to leave everything on the purely mathematical side untouched (4).

The other three, mereological requirements (1)-(3) border on being self-evident. Concerning requirement (1), we can conceive of a subsystem-relation and a proper one in exact analogy to the subset-relation ($\subseteq$) and the proper-subset-relation ($\subset$) of $\mathcal{L}_\in$. For example, a proton is a *proper* subsystem of a Helium-atom but the Helium-atom is not a *proper* subsystem of itself. The requirements of reflexivity of the relations of subsystemhood and irreflexivity of the relation of proper subsystemhood express this. Further, if some neutron is a subsystem of the nucleus of some gold atom and the gold atom is a subsystem of some wedding ring, then the neutron is also a subsystem of the wedding ring; when the Earth is a subsystem of the solar system, which in turn is a subsystem of the Milky Way, then the Earth is a subsystem of the Milky Way; *etc.* So the subsystem-relation must be transitive. We remark that a *symmetric* proper subsystem-relation would be mereology gone mad: my toe is a proper physical subsystem of my foot but my foot surely is not a proper physical subsystem of my toe. A connected subsystem-relation is also mereology gone mad: my toe and Mars are not subsystems of each other — both are subsystems of the solar system.

To require of an identity-relation (2) that it is an equivalence-relation which meets Substitutivity is standard. The further requirement that identical physical systems have the same physical subsystems is plausible, although we should not forget that it expresses our extensional attitude towards physical systems: every physical system is determined by its subsystems, as every set is determined by its members. Just as there is no more to a set but its members, there is no more to a physical system but its subsystems.

That we ought to be able to compose physical systems into *composite* physical systems is evident (3). For consider, we can compose three quarks into a baryon, a proton say, and compose the proton with a lepton, an electron say, to yield a Hydrogen atom (H); then we can compose two Hydrogen atoms into a Hydrogen gas molecule (H_2); then we combine it with an Oxygen atom (O) to obtain a water molecule (H_2O); we can make about 10^{21} water molecules form a droplet; droplets form an ocean; *etc.* These compositions are never such that physical systems are created out of thin air during the composition; the composite system does not have subsystems which do not 'overlap' with all of the subsystems we started with. The only trouble that arises for such a 'conservation of physical systems when making compositions' occurs when we enter the realm of relativistic quantum field theory. According to this theory it is possible to create particle/anti-particle pairs just by bringing two electric charges close to each other, or by bringing closely together two neutral, conducting plates in a vacuum (Casimir effect), or by accelerating a piece of matter in a vacuum (Unruh effect). One would never consider these physical processes as bringing about composite systems out of 'loose' systems that persist in the composition. Besides composition, one could therefore consider *creation* and *annihilation* as other primitive concepts, but we shall not do so here. Furthermore, in some creation-processes the created physical systems are *virtual*; think of interaction-bosons in a Feynman diagram, they do not exist in the same sense as the other mentioned pieces of matter. This is however an issue in current philosophy of physics. It therefore seems wise to remain *pro tem* neutral as to how these issues finally will be understood best.

The requirements about a theory of physical systems we listed above, perhaps save (4), are admittedly not the most challenging ones. But remember that it is up to the particular *physical theories* to tell us all kinds of things of particular physical systems, just as mathematical theories tell us all kinds of things about certain combinations of sets — notably *set-structures*, which characterise mathematical theories. Our mereological theory must remain completely general and apply to *all* physical systems, just as ZFC applies to all sets. And just as ZFC provides a home for the subject-matter of all theories of mathematics, our mereological extension must provide a home for all theories of physics, many of which contradict each other.

0.4 Extant Mereological Theories

Well-known mereological theories are the following two. First *Mereology* (with a capital M) of the Polish logician Stanislaus Leŝniewski (who introduced the name 'mereology' — Greek for 'the science of parts'), simplified and introduced in the Anglo-American world by Tarski [1956, pp. 24-29]; we abbreviate the theory by LT, from Lèsniewski and Tarski'. Secondly, there is the *Calculus of Individuals* (abbreviated by CI) of H. Leonard and N. Goodman [1940]. Both Tarski and Leonard & Goodman worked in the context of *Principia Mathematica* and employed its currently outdated notation. In contrast to Leŝniewski, who took *objects* and a *part-whole* relation as the mereological primitives, Leonard & Goodman took *individuals* (Russell's name for non-classes) and the *distinct-from* relation as primitives. P. Simons [1987, p. 54] has asserted that the theories LT and CI are identical but he has neither proved it nor provided a reference to a proof. Further, we have H.C. Bunt's so-called *Ensemble Theory* (BET), described in Bunt [1985]; and there is Simons' book [1987], which is the most comprehensive overview to date of mereology (although it does not refer to Bunt [1985]), but is disappointing from both a logical as well as a physical point of view (for a motivation of these judgements, see Muller [1998, p. 195]). The primitive elements inhabiting the domain of discourse of BET are, as the name suggests, *ensembles*. Besides the part-whole relation, Bunt also adopts a relation called 'is a unique element of' (*unicle*) as a primitive dyadic predicate. Sets are defined as 'discrete' ensembles, in which case the part-whole relation boils down to the familiar subset-relation. Set-membership, as in $X \in Y$, Bunt [1985, p. 106] defines as Y having a part of which X is a unicle. The set-theoretical analogue of this assertion reads:

$$X \in Y \;\longleftrightarrow\; \exists\, Z \subseteq Y : Z = \{X\}\ .$$

The 'objects' of LT and the 'individuals' of CI turn out to have much in common with Bunt's 'continuous' ensembles, but differ from them in that they need not allow for infinite part-whole chains (*op. cit.*, p. 261).

Lewis [1987] has a theory of sets, classes and individuals that is very much like Bunt's and we therefore ignore; the only mereological axioms that Lewis adopts are those of LT. The difference between Bunt and Lewis is that Bunt writes from a formal-semantic perspective and combines informal explanation with formal expression, whereas Lewis' perspective is strictly metaphysical and indecently informal. Another difference is that Bunt investigates his theory meta-mathematically whereas Lewis does no such thing. Yet a final difference is that Bunt proves a lot theorems, whereas Lewis is doing little with them, save recovering the axioms of ZFC. These differences, taken together with the fact that the present journal is not the appropriate place for a thorough analysis of 'the metaphysics' of Lewis' theory (which it putatively merits), form the justification for leaving Lewis' theory aside.

With regard to pressing questions from a logical perspective, we have been able to collect the following results. LT has been proved consistent relative to the theory of the real numbers (cf. Simons [1987, p. 110]). Since both LT and CI employ the concept of a class and the membership-relation, axioms governing these notions are needed to let LT and CI take off. We replace these axioms by the ones of ZFC. Bunt [1985, pp. 275-288] proved that BET extends ZF (which is ZFC without the Axiom of Choice) and CI, and that BET is equiconsistent with ZF. Summary of results:

$$\mathsf{BET} \vdash \mathsf{CI} \wedge \mathsf{ZF} \quad \text{and} \quad \mathrm{Con}(\mathsf{ZF}) \longleftrightarrow \mathrm{Con}(\mathsf{BET}) \longrightarrow \mathrm{Con}(\mathsf{CI})\ . \tag{1}$$

The questions of whether LT, CI and our theories ZFCM and $\mathsf{ZFCM}_{\mathfrak{p}}$ extend ZFC *conservatively*, as well as whether our theories are consistent relative to ZF, will be answered during the course of this paper. The answers are: our theories ZFCM and $\mathsf{ZFCM}_{\mathfrak{p}}$ will extend both CI and LT consistently, which means that everything advanced in favour of the axioms of CI and LT can also be advanced in favour of our theory ZFCM; and both ZFCM and $\mathsf{ZFCM}_{\mathfrak{p}}$ will extend

ZFC conservatively and hence equiconsistently We consider the mereological theories CI and LT too weak to be interesting as a theory of physical systems. The relation between our theories and Bunt's BET turns out to be more complicated. About Lewis' theory no meta-mathematical results are known to this author, save that ZFC is relatively consistent to his theory because it entails ZFC by construction; see Lewis [1991, pp. 100-107]. Lewis does not raise the issue of conservativeness.

1 Mereological Investigations

1.0 The Language of Physical Systems

Formal Language. Already in 1908, Zermelo [1908, § 1] talked about "sets and objects" rather than of sets only; and in 1930 he baptised the non-set elements **primordial elements** (German: *Urelementen*; in set theory primordial elements are also referred to as 'atoms' (Fraenkel, Jech) or by the German-English hybrid 'urelements'). As announced, we are going to call the primordial elements **physical systems**. Then every inhabitant of the domain of discourse is either a set or a physical system. We consider $\mathcal{L}_\in$, the usual 1st-order formal language of ZFC, to be extended with Quinean *virtual sets*, often misleadingly called 'proper classes'; we denote them by bold-faced capital letters (the Kunen-convention).[3]

We use the following variables and names:

$$\begin{aligned} &\textit{set-variables} && : A, A', B, B', \ldots, X, X', Y, Y', Z, Z', \mathcal{N}, \mathcal{P}, \mathcal{P}', \mathcal{R}, \mathcal{R}' , \\ &\textit{virtual set-names} && : \mathbf{C}, \mathbf{D}, \mathbf{D}_0, \mathbf{D}_1, \mathbf{P}, \mathbf{S}, \mathbf{U}, \mathbf{V}, \mathbf{V}_{\text{ZFC}} , \\ &\textit{physical system-variables} && : a, a', b, b', \ldots, x, x', y, y', z, z' , \end{aligned} \tag{2}$$

and occasionally with double accents. The names of the virtual sets will be defined as we go along. Other standard symbols for *names*, such as $\varnothing$, $\mathbb{N} \equiv \omega$ (finite ordinals), $\mathbb{R}$, $\aleph_\omega$, $\beth_\alpha$, *etc.*, are considered to be defined as usual. We use $\#X$ for the cardinal number of set X; and $\mathfrak{a}, \mathfrak{p}$ as cardinal-number variables ranging over virtual set $\mathbf{C}$ of all cardinals. The set of *terms* of the language $\mathcal{L}_\sqsubseteq \supset \mathcal{L}_\in$, which is the extension of $\mathcal{L}_\in$ we are in the process of defining, consists of all variables (2). Besides the primitive dyadic membership-relation ($\in$), we introduce the *subsystem-relation* ($\sqsubseteq$) between physical systems as primitive:

$$\begin{aligned} &X \in Y && : X \ \textit{is a member of} \ Y , \\ &a \sqsubseteq b && : a \ \textit{is a subsystem of} \ b . \end{aligned} \tag{3}$$

In $\mathcal{L}_\in$, $\ulcorner X \in Y \urcorner$ and $\ulcorner X \in \mathbf{V} \urcorner$ are the only type of *atomic sentences* (besides $\perp$). In $\mathcal{L}_\sqsubseteq$ we have in addition sentences $\ulcorner a \sqsubseteq b \urcorner$, $\ulcorner a \in X \urcorner$ and $\ulcorner a \in \mathbf{P} \urcorner$ as three other types of atomic sentences, for all variables (2). The set $\text{SENT}(\mathcal{L}_\sqsubseteq)$ of all *sentences*, *etc.* have their usual definitions.

[3]See Lévy [1979], § 1.4 for this notational device. The first of the three auxiliary axioms Lévy adopts to govern virtual sets is however redundant, see Muller [1998, p. 409].

We could allow for ⌜$a \sqsubseteq X$⌝ and then adopt a blanket axiom asserting that no physical system is a subsystem of some set. Little of substance would be gained by this move. Similarly, one could allow even for ⌜$X \sqsubseteq Y$⌝, in particular if one reads $\sqsubseteq$ as the part-whole relation, because a subset can be considered as a part of another set. This is why Lewis opens his book [1991, p. vii] with the sentence: "There is more mereology in set theory than we usually think." We could allow for this too and then simply *define* $X \sqsubseteq Y$ as $X \subseteq Y$. Again, little would be gained by such a move.

Language $\mathcal{L}_{\sqsubseteq}$ is 1st-order; its deductive apparatus coincides with that of $\mathcal{L}_{\in}$, for which we take the usual System of Natural Deduction. We further use ambiguously ⌜$\equiv$⌝ for sentence- and term-definition.

The Axioms of set-theory in $\mathcal{L}_{\sqsubseteq}$. Recall the axioms of ZFC:

$$\mathsf{ZFC}: \ \mathsf{Ext},\ \mathsf{Un},\ \mathsf{Pow},\ \mathsf{Transf},\ \mathsf{F},\ \mathsf{Reg},\ \mathsf{C}\,. \tag{4}$$

The axioms of ZFC now also apply to physical systems, in so far as applicable. We shall spell out the amended axioms formally later (in Section 2); for our present purposes a pedestrian look suffices.

The Axiom of Extensionality (Ext) also applies to sets of physical systems: if two sets, X and Y say, have identical members, these members being sets, physical systems or both of them, then the two sets are also identical: $X = Y$ (identity for physical systems will be defined shortly). The Power Axiom (Pow) only applies to sets, it does not apply to physical systems because they have no *subsets*, but the subsets mentioned by Pow may contain physical systems. The Axioms of Union (Un) and Choice (C) only apply to sets of sets, but again the last-mentioned sets may in turn contain physical systems. Replacement (F) allows us now *also* to replace sets in some set with physical systems, and conversely, or physical systems with other physical systems. Zermelo's Separation (Sep), which is a theorem in ZFC, will also be a theorem in our theory; it allows us also to separate a set of physical systems of another set Z of physical systems by means of any predicate $\varphi(\cdot, X, x)$ with one free physical system-variable, where X and x stand for any number of set- and physical system-parameters, respectively:

$$\forall X,\ \forall x,\ \exists P,\ \forall a\,\big(a \in P \ \longleftrightarrow\ \varphi(a, X, x) \wedge a \in Z\big)\,. \tag{5}$$

Then $P \subseteq Z$. Also the separation from mixed sets is possible. These axioms of ZFC show some minor interplay between sets and physical systems.

1.1 The Domain of Mereological Discourse

We can proceed for now with ZFCM, our mereological extension of ZFC, without knowing what the mereological axioms are, because for the considerations below they are largely irrelevant. We have called $\mathbf{U}$ the domain of discourse of ZFCM; it is a virtual set:

$$\mathsf{ZFCM} \ \vdash\ \forall X(X \in \mathbf{U}) \ \wedge\ \forall a(a \in \mathbf{U}) \ \wedge\ \forall Y(Y \neq \mathbf{U})\,. \tag{6}$$

In sets such as $\{a\}$, $\wp\{a, b, c\}$, $\{a, \varnothing\}$. *etc.* we encounter physical systems, which means that $\mathbf{U}$ has besides physical systems and pure sets lots of other sets. Let $\mathbf{V}$ the virtual set of all *pure* sets (shortly to be defined rigourously) and $\mathbf{P}$ the virtual set of all **P**hysical systems:

$$\mathsf{ZFCM} \vdash \forall a (a \in \mathbf{P}) \ \wedge \ \forall Y (Y \notin \mathbf{P}) \, . \tag{7}$$

$\mathbf{S} \equiv \mathbf{U} \backslash \mathbf{P}$ is the virtual set of all **S**ets. Set $\mathbf{P}$ may turn out to be a non-virtual set; in that case we denote it by $\mathcal{P}$. $\mathbf{V}_{\text{ZFC}}$ is the domain of discourse of ZFC; it is not defined in the context ZFCM but in that of ZFC. For the sake of convenience, we summarise this notation:

$$\begin{aligned}
\mathbf{V}_{\text{ZFC}} &\equiv \text{ the domain of discourse of ZFC} \, , \\
\mathbf{U} &\equiv \text{ the domain of discourse of ZFCM, the Universe of discourse} \, , \\
\mathbf{P} &\equiv \text{ the virtual set of all Physical systems in } \mathbf{U} \, , \\
\mathbf{S} &= \text{ the virtual set of all Sets in } \mathbf{U} \, , \\
\mathbf{V} &= \text{ the virtual set of all pure sets in } \mathbf{U} \\
\mathbf{C} &= \text{ the virtual set of all cardinal numbers} \, .
\end{aligned} \tag{8}$$

A few obvious relations between all these virtual sets:

$$\mathbf{S} \cap \mathbf{P} = \varnothing \, , \quad \mathbf{V} \cap \mathbf{P} = \varnothing \, , \quad \mathbf{S} \cup \mathbf{P} = \mathbf{U} \, , \quad \mathbf{V} \cup \mathbf{P} \subset \mathbf{S} \, , \quad \mathbf{C} \subset \mathbf{V} \subset \mathbf{S} \subset \mathbf{U} \, . \tag{9}$$

To define the virtual set of all pure sets ($\mathbf{V}$), we need a theorem of grounded induction. Recall that a binary relation $\lessdot$ is *grounded* on set X iff X has a member, $B \in X$ say, such that no member of X preceeds it: $\neg(Y \lessdot B)$ for all $Y \in X$ (see Lévy [1979, pp. 42-44]):

Principle of Grounded Induction (ZF). *Let $\lessdot$ be some grounded relation and let $\varphi(\cdot) \in$* SENT$(\mathcal{L}_{\sqsubseteq})$ *be some sentence with one free set-variable. If for every Y it holds that $\varphi(Y)$ is implied by $\varphi(X)$ for all $\lessdot$-predecessors of Y, then $\varphi(\cdot)$ holds for all sets*:

$$\forall Y \big(\forall X \, [X \lessdot Y \longrightarrow \varphi(X)] \longrightarrow \varphi(Y) \big) \ \longrightarrow \ \forall Z : \varphi(Z) \, . \tag{10}$$

Parameters can be added to $\varphi(\cdot)$ at will.

Since the membership-relation ($\in$) is grounded (due to the presence of Regularity in ZFC), one speaks in the case of using $\in$ and Principle (10) in a proof of 'a proof by $\in$-induction'. A corollary of the Principle of Grounded Induction (10) is the existence of a unique virtual function F on the domain of discourse (*op. cit.*, pp. 63-67) that sends every set Y to the set of the images of all its $\lessdot$-predecessors:

$$F : Y \mapsto F(Y) = \{ F(X) \, | \, X \lessdot Y \} \, , \tag{11}$$

whose virtual domain is $\mathbf{V}_{\text{ZFC}}$ in ZFC and $\mathbf{S}$ in ZFCM (we do not write this virtual function bold-faced, as $\mathbf{F}$, because for every X, $\mathbf{F}(X)$ is a normal, non-virtual set). When for $\lessdot$ the membership-relation $\in$ is chosen, we speak in the case of (11) of 'a definition by $\in$-recursion'.

In $\mathcal{L}_{\sqsubseteq}$ we define by $\in$-recursion set Y to be **pure** iff Y contains no physical systems and all its set-members are pure:

$$\mathsf{Pure}(Y) \equiv Y \cap \mathbf{P} = \varnothing \ \wedge \ \forall X \in Y : \mathsf{Pure}(X) \ , \tag{12}$$

where it is to be noted that all members of Y precede Y in the sense of $\in$. Virtual set $\mathbf{V}$ contains by definition exactly the pure sets:

$$\mathbf{V} \equiv \{X \in \mathbf{S} \mid \mathsf{Pure}(X)\} \ . \tag{13}$$

Evidently $\varnothing$ is a pure set. For physical systems, $\mathsf{Pure}(\cdot)$ is not defined, but it can be defined so as to make $\mathsf{Pure}(a)$ always false; for example as follows:

$$\mathsf{Pure}(a) \equiv \bot \ . \tag{14}$$

Then the impurity of physical systems becomes a theorem of logic:

$$\vdash \ \forall a : \neg\mathsf{Pure}(a) \ . \tag{15}$$

Definition (12) of purity carries over to $\mathcal{L}_{\in}$ when we delete the first conjunct. By $\in$-induction one proves the following expected

Theorem. *In standard set-theory all sets are pure*:

$$\mathsf{ZFC} \ \vdash \ \forall Y \in \mathbf{V}_{\text{ZFC}} : \mathsf{Pure}(Y) \ . \tag{16}$$

In the light of def. (13), which entails that $\mathbf{V} \subseteq \mathbf{V}_{\text{ZFC}}$, Theorem (16) says: $\mathbf{V} = \mathbf{V}_{\text{ZFC}}$.

A virtual function we shall employ is the unique function (11) generated by the grounded relation $\in$ on *all sets* in $\mathbf{U}$:

$$F_{\in} : \mathbf{S} \to \mathbf{V}, \quad Y \mapsto F_{\in}(Y) \ \equiv \ F_{\in}[Y] \ \equiv \ \{F_{\in}(X) \in \mathbf{V} \,|\, X \in Y\} \ . \tag{17}$$

This function sends a set Y to the set of the images of its set-members; $F_{\in}$ ignores the physical system-members in Y: $F_{\in}(Y) = F_{\in}(Y \backslash \mathbf{P})$. Obviously $F_{\in}(\varnothing) = \varnothing$. For future reference, we prove two theorems.

Theorem. *The virtual function* $F_{\in}$ *is the identity on the domain of discourse of* ZFC:

$$F_{\in} : \mathbf{V}_{\text{ZFC}} \rightarrowtail\!\!\!\twoheadrightarrow \mathbf{V}_{\text{ZFC}}, \quad Z \mapsto F_{\in}(Z) \ \equiv \ F_{\in}[Z] \ \equiv \ \{F_{\in}(X) \in \mathbf{V}_{\text{ZFC}} \,|\, X \in Z\} \ = \ Z \ . \tag{18}$$

Proof. It is evident that function $F_{\in}$ is defined literally everywhere: $\mathbf{V}_{\text{ZFC}}$ is its virtual domain. We have to prove that $F_{\in}(Z) = Z$ for all $Z \in \mathbf{V}_{\text{ZFC}}$, which we do by $\in$-induction (10). The Induction-Assumption is that $F_{\in}(X) = X$ (IA) for all $X \in Y$. We now have to prove that also $F_{\in}(Y) = Y$. By definition, $F_{\in}(Y) = \{F_{\in}(X) \in \mathbf{V}_{\text{ZFC}} \mid X \in Y\}$. Then $F_{\in}(Y) = \{X \in \mathbf{V}_{\text{ZFC}} \mid X \in Y\}$ by (IA), which is Y. So $F_{\in}(Y) = Y$. The Grounded Induction-Theorem (10) then gives us the desired result. $\square$

So in ZFC we have that all sets are pure (16) and $F_\in$ is the identity on $\mathbf{V}_{\text{ZFC}}$ (18). Then in ZFCM, which extends ZFC deductively and where $F_\in$ can be considered with domain $\mathbf{S}$ (17), we expect that restricted to the virtual set $\mathbf{V} \subset \mathbf{S}$ of all pure sets, $F_\in$ is also the identity and that consequently its range consists of pure sets only. This we prove next.

Theorem. *In* ZFCM *the virtual function* $X \mapsto F_\in(X)$ *is* (*i*) *the identity on the virtual set* $\mathbf{V}$ *of all pure sets and* (*ii*) *a surjection from* $\mathbf{S}$ *onto* $\mathbf{V}$*; this implies that* $F_\in$ *sends sets to pure sets only and that all pure sets are reached in this fashion*:

$$\mathsf{ZFCM} \vdash F_\in : \mathbf{V} \rightarrowtail\!\!\!\twoheadrightarrow \mathbf{V},\ X \mapsto X \ \wedge\ F_\in : \mathbf{S} \rightarrowtail\!\!\!\longrightarrow \mathbf{V}\ . \tag{19}$$

Proof. (*i*) We can repeat the proof of Theorem (18) in ZFCM by replacing $\mathbf{V}_{\text{ZFC}}$ with $\mathbf{V}$ — simply notice that the presence of the conjunct $\ulcorner X \cap \mathbf{P} = \varnothing \urcorner$ in the definition of purity in $\mathcal{L}_{\sqsubseteq}$ (12) does not spoil the proof.

(*ii*) We observe first that on $\mathbf{S}$, rather than restricted to $\mathbf{V}$, the function $F_\in$ is *not* injective (hence not the identity), due to, for instance $F_\in(Y) = F_\in(Y \backslash \mathbf{P})$, and $F_\in(P) = \varnothing$ for every set P of physical systems ($P \subset \mathbf{P}$).

Let $X \in \mathbf{S}$ be some set. Function $F_\in$ sends X to $F_\in(X) = \{F_\in(A) \,|\, A \in X\}$, which by definition only contains sets, *i.e.* $F_\in(X) \cap \mathbf{P} = \varnothing$. By $\in$-induction one proves immediately that $F_\in(X)$ is pure.

Is every pure set, *i.e.* every member of $\mathbf{V}$, reached by $F_\in$ from $\mathbf{S}$? Yes, because $F_\in$ is surjective already on domain $\mathbf{V}$, as we have proved in (*i*), and enlarging the domain from $\mathbf{V}$ to $\mathbf{S} \supset \mathbf{V}$ does not destroy its surjectivity. $\square$

In the light of Theorem (19), part (*ii*), function $F_\in$ will also be referred to as the *purifier*. On $\mathbf{S}/\mathbf{V}$, $F_\in$ is not the identity.

Finally, we address the relation between $\mathbf{V}$ and $\mathbf{V}_{\text{ZFC}}$, the domain of discourse of ZFC. Domain $\mathbf{V}_{\text{ZFC}}$ is somehow located in $\mathbf{U}$ because ZFCM will deductively extend ZFC: every set which exists according to ZFC, exists also according to ZFCM; and every theorem about sets in $\mathbf{V}_{\text{ZFC}}$ according to ZFC is also a theorem of ZFCM. Further, Theorem (16) tells us that all sets in ZFC are pure. Thus we have in ZFCM that $\mathbf{V}_{\text{ZFC}} \subseteq \mathbf{V}$. If we can prove that every pure set which exists according to ZFCM also exists according to ZFC; and if we further can prove that every theorem in ZFCM about pure set $X \in \mathbf{V}$ can also be proved in ZFC, *i.e.* if we can prove that ZFCM is conservative over ZFC, then the pure sets of ZFCM coincide with the sets of ZFC, which are all pure. Hence the question of whether ZFCM extends ZFC conservatively can be posed symbolically as follows:

$$\mathsf{ZFCM} \overset{?}{\vdash} \mathbf{V}_{\text{ZFC}} = \mathbf{V}\ . \tag{20}$$

Section 2 is devoted to erasing the question-mark.

1.2 Mereological Axioms

In this Subsection we proceed as follows. First we provide some considerations about the axiom to be adopted, then we formulate the axiom in $\mathcal{L}_{\sqsubseteq}$. In passing a few definitions are given and a few elementary theorems are proved.

1.2.0 Plenitude vs. Parsimony

In the first mereological axiom we are going to guarantee that we have physical systems in $\mathbf{U}$ and that we have enough of them in order to characterise physical theories. In ZFC, the Axiom of Transfinity postulates the existence of a denumerably infinite set, for which usually the set ω of all finite Von Neumann ordinals is taken (the natural numbers: $\mathbb{N} \equiv \omega$). The other axioms of ZFC, notably the Power Axiom, the Union Axiom and the Replacement Axiom Schema, create the entire cumulative hierarchy of all finite and infinite sets out of ω. This procedure plausibly cannot be imitated for physical systems, for there seem to be no operations on physical systems analogous to making union-sets and power-sets. We can *split* physical systems, such as when cutting bread or throwing an atomic bomb, but the splitting ends when we reach the elementary particles. If one wishes to indulge in the possibility, *contra* everything we know about the physical reality, that elementary particles can be split *ad infinitum*, then although one obtains more and more physical systems by splitting, this is 'more' in the sense of cardinal number; one does not obtain 'more encompassing' physical systems, as in the case the cumulative hierarchy of all sets. On the contrary, the split physical systems become less and less encompassing. Besides splitting, we can *compose* physical systems into a composite system. But when we have obtained the composite physical system, there seems to be nothing more left to do. For suppose we commence with a set $\mathcal{P}$ of physical systems such it contains all the subsystems of every of its members (if $a \sqsubseteq b \in \mathcal{P}$, then $a \in \mathcal{P}$), and we make composite systems each of which is built from physical systems in $\mathcal{P}$, *i.e.* from the members of some subset of $\mathcal{P}$. When we have all of them, what is there left to do? When we want to compose any two of these newly obtained composite systems, say the ones which result from composing the members of $P' \subseteq \mathcal{P}$ and $P'' \subseteq \mathcal{P}$, say, then arguably we already have obtained this composite system when we composed all physical systems in $P' \cup P''$, which is a subset of $\mathcal{P}$. If we thus begin by assumption with set $\mathcal{P}$ of physical systems with cardinality $\mathfrak{p} \in \mathbf{C}$, and throw all possible compositions of members of $\mathcal{P}$ into another set, $\mathcal{P}_{\sqcup}$ say, we never shall have more than

$$\#(\mathcal{P} \cup \mathcal{P}_{\sqcup}) \leqslant \#\mathcal{P} + \#\mathcal{P}_{\sqcup} \leqslant \mathfrak{p} + 2^{\mathfrak{p}} \tag{21}$$

physical systems. Question: how large should we choose $\mathfrak{p}$?

Physical systems consisting of an arbitrary but finite number of particles are considered in many physical theories. So we need at least as many physical systems as there are natural numbers. Physical systems which have an *infinite* number of subsystems are considered in physics, *viz.* the Coleman-Hepp model of the measurement process in quantum mechanics, which consists of a denumerable number of spin systems, and models of the electro-magnetic field consisting of an denumerable number of harmonic oscillators. So we can postulate that $\mathfrak{p} = \beth_0 \equiv \aleph_0$. Then we have the following number of physical systems:

$$\#(\mathcal{P} \cup \mathcal{P}_\sqcup) \leqslant \beth_0 + \beth_1 = \beth_1 \,, \tag{22}$$

where, to recall, $\beth_{\alpha+1}$ is recursively defined as $2^{\beth_\alpha}$, for all ordinal numbers (standardly denoted by small Greek letters). How *many* physical systems we then are assuming to exist has become undecidable because of the undecidability of the Continuum Hypothesis — but $\mathcal{P} \cup \mathcal{P}_\sqcup$ remains a *set* in **S**, that much is certain. Confronted with this problem one can respond that we are prepared to live with this undecidability because we have to live with the undecidability of the Continuum Hypothesis in ZFC anyway.

So far we have been assuming that we start with a *set* of physical systems, then we obtain a *set* of all physical systems present in **U**. But there is no *set* of all sets, so why should there be a *set* of all physical systems? Well, in contrast to mathematics, where *some* branches need the plenitude of *all* sets ($\mathbf{V}_{\text{ZFC}}$), such as in cardinal arithmetic and category theory, no branch of physics stands in need of an indefinitely accumulating extravaganza of physical systems. This motivates the adoption of not just a *set*, but even some *small set* of physical systems — *small* that is, in comparison with how enormously large sets can be in $\mathbf{V}_{\text{ZFC}}$. So we arrive at our previous conclusion of setting $\mathfrak{p}$ equal to $\beth_0$ or perhaps to $\beth_1$.

Thus there are two kinds of axioms conceivable that will guarantee the existence of physical systems in **U**: a Parsimony Axiom and a Plenitude Axiom. Below $\mathfrak{p} \in \mathbf{C}$ is an arbitrary cardinal number, that remains fixed in the theory once a value is chosen.

Parsimony Axiom (Pars($\mathfrak{p}$))**.** *There exists a set of at least cardinality* $\mathfrak{p} \in \mathbf{C}$ *that contains only physical systems*:

$$\exists\, X \left(\#X \geqslant \mathfrak{p} \;\wedge\; \forall Y : Y \notin X \right) . \tag{23}$$

Stated in virtual terms: there is a $X \subset \mathbf{P}$ such that $\#X \geqslant \mathfrak{p}$. Alternatively we simply postulate there is a *virtual* set **P** of all physical systems, thus equinumerous to the virtual set **V** of all pure sets. This will pleasingly guarantee that we never run out of physical systems in our domain of discourse **U**.

Plenitude Axiom (Plen)**.** *There are as many physical systems as there are pure sets: for some dyadic predicate* $\varphi(\cdot,\cdot)$ *of* $\mathcal{L}_\sqsubseteq$:

$$\forall\, X,\ \exists!\, b \left(\mathsf{Pure}(X) \wedge \varphi(b, X) \right) \;\wedge\; \forall\, a,\ \exists!\, Y \left(\mathsf{Pure}(Y) \wedge \varphi(a, Y) \right) , \tag{24}$$

or stated in virtual terms: $\mathbf{P} \sim \mathbf{V}$, *where* $\sim$ *stands for equinumerosity, the meta-mathematical definition of which is spelled out in the formal sentence* (24).

In this fashion physics does not have to defer mathematics in terms of cardinality. When we write ZFCM, the Plenitude Axiom (Plen) is adopted, and when we write $\mathsf{ZFCM}_\mathfrak{p}$, the Parsimony Axiom Pars($\mathfrak{p}$) is adopted instead. Most theorems we shall prove are theorems in both ZFCM and $\mathsf{ZFCM}_\mathfrak{p}$ because their proofs will not depend on which of these blanket axioms one adopts, so *from the point of vantage of the deductive development of the theory*, there are no decisive arguments in favour of Plen or $\mathsf{Pars}(\mathfrak{p})$.

The next theorem establishes the expected relation between ZFCM and $\mathsf{ZFCM}_{\mathfrak{p}}$.

Proposition. *The Plenitude Axiom* (Plen) *entails there is a set of physical systems of every cardinality*:

$$\mathsf{ZFC} \;\vdash\; \mathsf{Plen} \longrightarrow \forall\, \mathfrak{p} \in \mathbf{C} : \mathsf{Pars}(\mathfrak{p}) \; . \tag{25}$$

Proof. Since ZFCM will deductively extend ZFC, the theory of cardinality of ZFC is also present in ZFCM. Every cardinal number $\mathfrak{a} \in \mathbf{C}$ is a pure set of cardinality equal to $\mathfrak{a}$. Assume Plen. We can replace every member of $\mathfrak{a}$ with a physical system from $\mathbf{P}$. According to the Axiom of Replacement this results in a set of physical systems of cardinality $\mathfrak{a}$. There are always enough physical systems available because every set, including $\mathfrak{a}$, is *minumerous* to $\mathbf{P}$ (*i.e.* no surjections from the set onto $\mathbf{P}$). □

At gun-point, we shall choose for a theory with Parsimony Axiom $\mathsf{Pars}(\beth_0)$, because of the extravaganza-argument we advanced *supra*. In the light of Proposition (25), everything proved in this parsimonious theory will be a theorem in all deductively stronger theories, notably in ZFCM. But since there will be no waving of guns in this paper, we consider ourselves free to make a different choice any time we feel like it.

1.2.1 Subsystem Axioms

Pre-Ordering Axiom. Although we shall also be able to *prove* the reflexivity of $\sqsubseteq$ for all physical systems except for the so-called 'elementary' ones, we assume the reflexivity for all physical systems nonetheless.[4]

Pre-Ordering Axiom (PreOrd). *The subsystem-relation is a pre-ordering, i.e. it is a reflexive and transitive relation on the physical systems*:

$$\forall\, a \big((a \sqsubseteq a) \;\wedge\; \forall\, b, c \, (a \sqsubseteq b \sqsubseteq c \;\longrightarrow\; a \sqsubseteq c) \big) \; . \tag{26}$$

The subsystem-relation is not required to be connective, which is good. For think of a quark (q, say) and a neutrino (n, say); neither is a subsystem of the other nor are they identical (to repeat, identity for physical systems will be defined shortly):

$$q \not\sqsubseteq n \;\wedge\; n \not\sqsubseteq q \;\wedge\; q \neq n \, ,$$

from which it follows that $\sqsubseteq$ is *not connective*:

$$\neg \big(\forall\, a, b \, (a \sqsubseteq b \;\vee\; b \sqsubseteq a \;\vee\; a = b) \big) \; . \tag{27}$$

[4] Such moves are familiar from ZFC: the Axiom of Choice is assumed for all sets, whereas it can be proved for all finite sets and even for a number of infinite sets (all infinite ordinals, all infinite sets containing singleton-sets only). Apparently the official canon of mathematical aesthetics disfavours axioms that postulate sets conditional on particular types of sets, even if this introduces deductive redundancies.

Identity. In ZFC, the Axiom of Extensionality says that two sets are identical if each is a subset of the other (the converse is a theorem of logic). Mereology obtains a similar extensional character when we simply *define* two physical systems to be **identical** iff each one is a subsystem of the other:

$$a = b \equiv a \sqsubseteq b \;\wedge\; b \sqsubseteq a\,. \tag{28}$$

All mereologists seem satisfied by relegating identity to the background logic, *viz.* Tarski [1956], Leonard & Goodman [1940], Simons [1987] — Bunt [1985, p. 56, 235] laudably also adopts def. (28) but then needs an additional identity axiom for his more elaborate system (BET). We relegate the identity to the logic, because Logic is, properly conceived, concerned with relations between sentences, not with relations between the elements of the domain of discourse, *that* is what the *theory* is supposed to be about, not *Logic*. Moreover, if identity is relegated to Logic, an additional axiom schema is needed to fix identity as an equivalence relation that obeys Substitutivity. Further, def. (28) then has to be adopted too as an *additional* mereological axiom. The more you postulate, the less you have to prove, but the challenge lies of course in assuming little and proving much. Since we adopt def. (28) and do not *assume* anything about the identity between physical systems, we are committed to prove that def. (28) is an adequate definition of identity in that the following theorem holds (see the Appendix).

Identity Theorem. *The identity-relation between physical systems defined as mutual subsystemhood is* (*a*) *an equivalence-relation that* (*b*) *obeys Substitutivity.*

Proper Subsystems. A physical subsystem is a *proper subsystem* of, or *properly included in* another system ($\sqsubset$) iff it is a subsystem of this other system but is not identical to it:

$$a \sqsubset b \equiv a \sqsubseteq b \;\wedge\; a \neq b\,. \tag{29}$$

Evidently, a physical system is a subsystem of another iff it is either a proper subsystem of it or identical to it:

$$\mathsf{PreOrd} \vdash a \sqsubseteq b \;\longleftrightarrow\; (a \sqsubset b \;\sqcup\; b = a)\,, \tag{30}$$

where $\sqcup$ is exclusive disjunction (either-or). Remember that an ordering is by definition *partial* iff it is reflexive, anti-symmetric and transitive; and it is *strict* iff it is a-symmetric and transitive. We easily prove the following

Theorem. *The subsystem-relation* ($\sqsubseteq$) *is a partial ordering and the proper subsystem-relation* ($\sqsubset$) *is an irreflexive strict ordering.*

Proof. The reflexivity and transitivity of $\sqsubseteq$ is an axiom (PreOrd). Half of def. (28) states the anti-symmetry of $\sqsubseteq$, which also implies the a-symmetry of $\sqsubset$. The proper subsystem-relation obviously is transitive too. Hence the ordering $\sqsubseteq$ is partial and $\sqsubset$ is strict. The irreflexivity of $\sqsubset$ follows from the reflexivity of $\sqsubseteq$ and (29): its reflexivity ($a \sqsubset a$) would imply $a \neq a$, whereas $=$ is reflexive. □

Subsystem-Set Axiom. In ZFC the existence of the power-set $\wp X$ of every set X is assumed (Pow). In analogy with Pow we assume, following Bunt's BET, the existence of a set of all subsystems of every physical system. In ZFCM, we cannot separate this set from **P** because **P** is not a set; in $\mathsf{ZFCM_p}$ we *can* separate this set from $\mathcal{P}$, but we have no guarantee that $\mathcal{P}$ contains all subsystems of every $a \in \mathcal{P}$; therefore we have to adopt the

Subsystem-Set Axiom (Subs). *For every physical system there exists a set that contains exactly its subsystems.*

$$\forall\, a,\ \exists\, X,\ \forall\, b\, (b \sqsubseteq a \ \longleftrightarrow\ b \in X)\ . \tag{31}$$

The Axiom of Extensionality secures the uniqueness of this set of subsystems. We shall refer to this set as the **subsystem set** of a physical system and denote it by:

$$\wp a \ \equiv\ \{\, b \mid b \sqsubseteq a \,\}\ . \tag{32}$$

The Subsystem-Set Axiom thus says that every physical system has a subsystem-set.

We can also define the set of *all* physical systems in $\mathcal{P}$ and *all their* subsystems:

$$\mathcal{P}_{\sqsubseteq} \ \equiv\ \bigcup\{\wp a \mid a \in \mathcal{P}\} \ =\ \{a \in \mathcal{P} \mid \exists\, b \in \mathcal{P} : a \sqsubseteq b\}\ , \tag{33}$$

which includes $\mathcal{P}$ due to the reflexivity of $\sqsubseteq$: $\mathcal{P} \subseteq \mathcal{P}_{\sqsubseteq}$. The Union Axiom guarantees its existence. Evidently also $\mathcal{P}_{\sqsubseteq} \subseteq \mathcal{P}$. Then:

$$\mathsf{ZFCM_p} \ \vdash\ \mathcal{P} = \mathcal{P}_{\sqsubseteq}\ . \tag{34}$$

Before we can formulate the next mereological axiom, we first have to introduce more definitions, in particular the notion of 'physically nothing'.

Null Physical Systems. In analogy with a set having no members ($\varnothing$), we introduce a physical system being 'null'. In ZFC, $\varnothing$ is defined as the set having no members, which definition is of no use here because physical systems have no members. But the following observation is: $\varnothing$ is the only set which has no proper subsets. Similarly we define a **null physical system** as a system which has no proper subsystems:

$$\mathsf{Null}(a) \equiv \neg\big(\,\exists\, b :\ b \sqsubset a\,\big)\ . \tag{35}$$

The *definiens* is logically identical to the assertion that every subsystem of a is identical to a, and therefore a null physical system is included in all *its* subsystems. This is what we also have in ZFC for the empty set: $X = \varnothing$ iff $\forall\, Y \subseteq X : Y = X$. We also have that all subsets of $\varnothing$ are empty; this carries over to the null physical systems of ZFCM.

Theorem. *All subsystems of a null physical system are null:*

$$\mathsf{PreOrd}\ \vdash\ \big(b \sqsubseteq a \wedge \mathsf{Null}(a)\big)\ \longrightarrow\ \mathsf{Null}(b)\ . \tag{36}$$

Proof. (Bunt [1985, p. 236] gives here a rather complicated proof.) Premise: let a be null and let $b \sqsubseteq a$. Then by virtue of (35): $b = a$. On the basis of the Identity Theorem we then conclude that b is null too. $\square$

In $\mathsf{ZFCM}_{\mathfrak{p}}$, the Separation Schema (5) gives us a unique set of all and only null physical systems *in* $\mathcal{P}$:

$$\mathcal{N} \equiv \{a \mid \mathsf{Null}(a)\} \subseteq \mathcal{P} . \tag{37}$$

Whether we are in ZFCM or $\mathsf{ZFCM}_{\mathfrak{p}}$, we are not yet able to prove whether there are null physical systems at all, *i.e.* whether the set (37) is empty or not.

Genuine Systems and Elementary Systems. The motivation for having the concept of a null physical system is analogous to having the concept of an empty set in set-theory, namely that its presence enhances the expressive power of the language and streamlines the deductive edifice; null systems are however not particularly interesting entities in themselves. We define a physical system to be **genuine**, denoted by $\mathsf{Gen}(\cdot)$, iff it is not null:

$$\mathsf{Gen}(a) \equiv \neg\mathsf{Null}(a) . \tag{38}$$

We further introduce the dyadic relations of genuine subsystemhood, denoted by $\sqsubseteq\!\!\!\cdot$, and of proper genuine subsystemhood, denoted by $\sqsubset\!\!\!\cdot$:

$$a \sqsubseteq\!\!\!\cdot\, b \equiv a \sqsubseteq b \wedge \mathsf{Gen}(a) \quad \text{and} \quad a \sqsubset\!\!\!\cdot\, b \equiv a \sqsubset b \wedge \mathsf{Gen}(a) . \tag{39}$$

Then a is a genuine proper subsystem of b iff a is a genuine subsystem of, but not identical to b (theorem of logic):

$$\vdash\ a \sqsubset\!\!\!\cdot\, b \ \longleftrightarrow\ (a \neq b \wedge a \sqsubseteq\!\!\!\cdot\, b) . \tag{40}$$

Definition: a physical system is **elementary particle** iff it is genuine and has no subsystems besides itself, *i.e.* iff it has no proper genuine subsystems:

$$\mathsf{ElemPart}(a) \equiv \neg\big(\exists\, b : b \sqsubset\!\!\!\cdot\, a\big) . \tag{41}$$

A null physical system has by definition no proper subsystems, and therefore also has no genuine subsystems; so without the genuine-conjunct in the *definiens* of $\mathsf{ElemPart}(\cdot)$, which is in the definition of $\sqsubset\!\!\!\cdot$ (39), null systems would count as elementary particles. This move makes it redundant to add the clause 'except for null systems' all the time. An elementary particle has only itself as a genuine subsystem (the proof is trivial):

$$\mathsf{PreOrd}\ \vdash\ \mathsf{ElemPart}(a)\ \longleftrightarrow\ \big(\forall\, b \sqsubseteq a\,[\mathsf{Null}(b) \vee b = a]\ \wedge\ \neg\mathsf{Null}(a)\big) . \tag{42}$$

From the definitions given so far we immediately deduce that every physical system is either an elementary particle or a null system or possesses a genuine subsystem.

Notice the analogue between elementary particles and singleton sets: $\{X\}$ has no non-empty proper subsets either.

Overlap Axiom. We need some condition for two physical systems to meet in order to infer that one is a subsystem of the other, otherwise we shall hardly be able to prove anything. First we define for physical systems the analogues of the set-theoretical notions of sharing members and being disjoint. Definitions: two physical systems **overlap** (symbol: $\circ$) iff they have a genuine subsystem in common; and they are **distinct** (symbol: $\rfloor\!\!\lceil$) iff the only subsystems they share are null:[5]

$$
\begin{aligned}
a \circ b \quad &\equiv \quad \exists\, c\,(c \mathrel{\dot\sqsubseteq} a \;\wedge\; c \mathrel{\dot\sqsubseteq} b)\,, \\
a \mathbin{\rfloor\!\!\lceil} b \quad &\equiv \quad \forall\, c\,\big((c \sqsubseteq a \;\wedge\; c \sqsubseteq b) \longrightarrow \mathsf{Null}(c)\big)\,.
\end{aligned}
\tag{43}
$$

Notice that $\circ$ is symmetric and reflexive and that $\rfloor\!\!\lceil$ is symmetric and irreflexive, but neither of them is transitive; so these dyadic predicates are not ordering relations. We deduce directly, by virtue of the reflexivity of $\sqsubseteq$, that a genuine physical system overlaps all its subsystems because then every subsystem is itself a common subsystem:

$$
\mathsf{PreOrd} \;\vdash\; a \mathrel{\dot\sqsubseteq} b \;\longrightarrow\; a \circ b\,. \tag{44}
$$

Two physical systems are distinct iff they do not overlap (theorem of logic):

$$
\vdash\; a \mathbin{\rfloor\!\!\lceil} b \;\longleftrightarrow\; \neg(a \circ b)\,. \tag{45}
$$

Evidently null physical systems are distinct from every genuine physical system and they do not overlap with any genuine physical system because they are themselves not genuine.

Suppose we have physical systems a and b, and suppose that all physical systems that overlap a also overlap b. Then it seems inconceivable for a *not* to be included in b, because *if* a were *not* included in b, then plausibly a would have a proper genuine subsystem, u say, not included in b, which means that u trivially overlaps a *but not* b, in contradiction to our supposition. So it is plausible to have this as an axiom:

Overlap Axiom (Olap)**.** *If physical system c overlaps all subsystems of physical system b, then b is a subsystem of c*:

$$
\forall\, a\,(a \mathrel{\dot\sqsubseteq} b \longrightarrow a \circ c) \;\longrightarrow\; b \sqsubseteq c\,. \tag{46}
$$

Now we have a sufficient condition for subsystemhood. Being *genuine* is needed for c, otherwise all physical systems will turn out to be subsystems of another, because we shall prove that null systems are subsystems of every physical system. Bunt [1985, p. 239] argued that when faced with elementary particles, Olap is not helpful as a supplier of a sufficient condition to deduce subsystemhood, for the following reason. Suppose a is an elementary particle (41). In order to know whether subsystems of a overlap with some other system, b say, we have to know whether a is a subsystem of b because an elementary particle has only itself as a non-null subsystem. But that is exactly what we are trying to establish in the first place! (Bunt remedies this situation by adding a disjunct to the antecedent of Olap, which however appeals to his primitive notion of 'unicle' we do not possess.)

[5] The symbolic notation here is Leonard & Goodman's [1940]; it is standard (see Simons [1987], *passim*). Our definitions are however neither Leonard & Goodman's nor Leśniewski & Tarski's, because we take the null system into account — something they have overlooked, it seems.

Bunt (*ibid.*) proved the reflexivity of $\sqsubseteq$ on the basis of Olap. This proof carries over from his BET to our theory (M) for the null systems and for systems that have a genuine proper part, but not for elementary particles, which are governed in BET by a special axiom we have not adopted. So strictly speaking, in PreOrd, we only had to assume the reflexivity of $\sqsubseteq$ for elementary particles, not for all physical systems.

The converse of the Overlap Axiom is easily established, as we shall show next.

Theorem. *Every physical system overlaps all physical systems that overlap its subsystems*:

$$a \sqsubseteq b \;\longrightarrow\; \forall\, c\,(c \circ a \longrightarrow c \circ b)\;. \tag{47}$$

Proof. Let $a \sqsubseteq b$ and let c be a physical system that overlaps a. Then there is a genuine physical system, d say, which is a subsystem of both a and c. By virtue of the transitivity of $\sqsubseteq$, it is then also a subsystem of b. So d is a common genuine subsystem of c and b, *i.e.* c overlaps b. □

1.2.2 Composite Physical Systems

In our discussion of the Plenitude Axiom we talked about *composite* physical systems. In physics, the concept of a subsystem comes in tandem with the concept of a composite system (or *fusion*, or *mereological sum*). We introduce, following Tarski [1958, p. 26], the **physical composition** p of a set P of physical systems such that all members of P are subsystems of p and no subsystem of p is distinct from all members of P:

$$\begin{aligned} \mathsf{PhysComp}(p, P) \;\equiv\; & (1)\;\; \forall\, b \in P : \, b \sqsubseteq p \quad \wedge \\ & (2)\;\; \forall\, c\,\big(\, c \,\dot\sqsubseteq\, p \;\longrightarrow\; \exists\, b \in P : c \circ b \,\big)\;. \end{aligned} \tag{48}$$

Conjunct (1) is obvious; conjunct (2) captures that p does not include redundant subsystems: p has as subsystems only the members of P *or* those systems that are compositions of members of P, which last disjunct is expressed by saying that (2) every genuine subsystem of p overlaps with some member of P.

Do identical sets of physical systems give rise to identical compositions? Only if a set of physical systems has one physical composition, because with the Identity Theorem for sets (substitutivity of set-identity, to be specific), we deduce immediately:

$$\forall\, P, P' \subset \mathbf{P}\,\big[\,P = P' \;\longrightarrow \forall\, p\,\big(\mathsf{PhysComp}(p, P) \longleftrightarrow \mathsf{PhysComp}(p, P')\big)\,\big]\;. \tag{49}$$

In an 'extensional, anti-holistic' slogan: identical building blocks, identical building; there is nothing more to the whole than the composition of its parts. The proofs of the functional character and the least-inclusiveness of composition require more work and are relegated to the next Subsection. Anticipating the first result however, we shall sometimes denote the composition by $\ulcorner \bigsqcup P \urcorner$ (until the first result is proved, consider $\bigsqcup P$ as a variable running over the set-extension of $\mathsf{PhysComp}(\cdot, P)$.)

We introduce as a notational abbreviation:

$$a \sqcup b \;\equiv\; \sqcup \{a, b\}\ . \tag{50}$$

If we were to adopt composition as a dyadic operation $\langle a, b\rangle \mapsto a \sqcup b$ on P, as (50) would suggest, it would send members of $P \times P$ to members of P, rather than subsets of P into $\mathbf{P}$, *i.e.* $P \mapsto \sqcup P$. We then could only make compositions of a *finite* number of physical systems (by successive application). Definition (48) applies however to sets of physical systems of arbitrary cardinality — and therefore is a superior notion of composition.

The fifth axiom asserts the existence of the composite system of every set of physical systems, somewhat in analogy to the Union Axiom of ZFC.

Composition Axiom (Comp). *Every set P of physical systems has a composition*:

$$\forall\, P \subset \mathbf{P},\ \exists\, p : \mathsf{PhysComp}(p, P)\ . \tag{51}$$

In this axiom we see both sets, physical systems, the membership- and the subsystem-relation occurring: it is an axiom where set-theory and mereology cooperate non-trivially. Observe that $p \in \mathbf{P}$ need not be a member of $P \subset \mathbf{P}$.

Continuous and Reducible Physical Systems. Besides elementary particles, Bunt [1981, pp. 63-64] defined two other types of physical systems. Definition: a physical system is **continuous** iff it is genuine and all its genuine parts have a genuine proper part:

$$\mathsf{Cont}(a) \;\equiv\; \mathsf{Gen}(a) \;\wedge\; \forall\, b \mathrel{\dot\sqsubseteq} a,\ \exists\, c :\ c \mathrel{\dot\sqsubset} b\ . \tag{52}$$

Evidently continuous physical systems generate a infinite chain of genuine proper subsystems, so that each part of a continuous system is itself continuous. Further, a continuous system is not elementary and neither are its subsystems. The counterpart notion of a continuous physical system is a system built up from elementary particles. Definition: a physical system is **reducible** iff it is the composition of the elementary particles it has a subsystems:

$$\mathsf{Reduc}(a) \;\equiv\; a = \sqcup \big\{ b \mid b \sqsubseteq a \;\wedge\; \mathsf{ElemPart}(b) \big\}\ . \tag{53}$$

Evidently all elementary particles trivially are reducible. Continuous and reducible physical systems never overlap, because if they did, they would have a common subsystem which is both continuous and reducible, and this is not possible. Then a physical system is continuous iff it is irreducible. Bunt proves that being continuous and reducible are invariant under the formation of compositions (*ibidem*).

Now we can rigourously formulate the Leucippus-Democritus thesis of Atomism, an idea which has conquered physics as no other idea from ancient philosophy of nature.

Axiom of Atomism (Atoms). *All physical systems are reducible to elementary particles*:

$$\forall\, a,\ \exists\, c :\ c = \sqcup \big\{ b \sqsubseteq a \mid \mathsf{ElemPart}(b) \big\}\ . \tag{54}$$

The Axiom of Subsystems (31) implies that if some physical system b has elementary particles as subsystems, then these elementary particles exist. But it may still be the case that among the subsystems of all subsystems of b there are no elementary particles. The Axiom of Atomism (54) — with 'atomism' taken in its etymological sense — excludes this possibility: there are no continuous physical subsystems, all physical systems ultimately are built from elementary particles.

1.2.3 Axiomatics

Our notation is as follows (recall that LT stands for Leśniewski & Tarski. Below Trns stands for the transitivity of $\sqsubseteq$, which is included in our PreOrd (26):[6]

$$\begin{aligned} \mathsf{LT} &: \mathsf{Trns},\ \mathsf{Comp},\ \text{Identity Axiom (28)}\ , \\ \mathsf{MA} &: \mathsf{Plen},\ \mathsf{PreOrd},\ \mathsf{Subs},\ \mathsf{Olap},\ \mathsf{Comp},\ \mathsf{Atoms}\ . \end{aligned} \tag{55}$$

The combinations of M with ZFC then are:

$$\mathsf{ZFM}:\ \mathsf{ZF},\ \mathsf{M}\ , \qquad \text{and} \qquad \mathsf{ZFCM}:\ \mathsf{ZFC},\ \mathsf{M}\ , \tag{56}$$

where it is left implicit that in the case of ZFCM the axioms of ZFC are adjusted as we outlined earlier. Recall that LT relegates identity to the background logic; one then has to adopt def. (28) as an additional axiom. The parsimonious versions are as follows:

$$\begin{aligned} \mathsf{M}_{\mathfrak{p}} &: \mathsf{Pars}(\mathfrak{p}),\ \mathsf{PreOrd},\ \mathsf{Subs},\ \mathsf{Olap},\ \mathsf{Comp},\ \mathsf{Atoms}\ ; \\ \mathsf{ZFCM}_{\mathfrak{p}} &: \mathsf{ZFC},\ \mathsf{M}_{\mathfrak{p}}\ . \end{aligned} \tag{57}$$

For the sake of convenience we list the axioms of M and $\mathsf{M}_{\mathfrak{p}}$ in one place:

$$\begin{aligned} \mathsf{Pars}(\mathfrak{p}) &: \exists\, X \subset \mathbf{P} : \#X = \mathfrak{p}\ . \\ \mathsf{Plen} &: \mathbf{P} \sim \mathbf{V}\ . \\ \mathsf{PreOrd} &: \forall\, a,b,c \big((a \sqsubseteq a)\ \wedge\ (a \sqsubseteq b \sqsubseteq c\ \longrightarrow\ a \sqsubseteq c) \big)\ . \\ \mathsf{Subs} &: \forall\, a,\ \exists\, S,\ \forall\, b\, (b \sqsubseteq a\ \longleftrightarrow\ b \in S)\ . \\ \mathsf{Olap} &: \forall\, a,b \left[\forall\, c \left(c \sqsubseteq\!\!\cdot\, a \longrightarrow (c \circ b) \right)\ \longrightarrow\ a \sqsubseteq b \right] . \\ \mathsf{Comp} &: \forall\, P \subset \mathbf{P},\ \exists\, a \left(\forall\, b \in P : b \sqsubseteq a\ \wedge\ \forall\, c (c \sqsubseteq\!\!\cdot\, a\ \longrightarrow\ \exists\, d \in P : c \circ d) \right) . \\ \mathsf{Atoms} &: \forall\, a,\ \exists\, c:\ c = \bigsqcup \left\{ b \mid b \sqsubseteq a\ \wedge\ \mathsf{ElemPart}(b) \right\} . \end{aligned} \tag{58}$$

1.3 Deductive Development

In the previous Subsection we mentioned and proved a few simple theorems; we now mention a few more to get a clear idea of what a theory of physical systems amounts to. Proofs of *known* theorems are only provided when it is not clear whether and how proofs from other mereological systems, in particular BET, carry over to M; many theorems presented here are

[6] Cf. Simons [1987, p. 54].

however novel. All theorems are theorems in both ZFCM and $\mathsf{ZFCM_p}$, which we shall therefore not explicitly mention all the time unless there is some specific reason to do so.

Null Physical Systems Revisited. The analogy of null physical systems with the empty set will reach its completion right now. Recall that so far we proved that all subsystems of null systems are themselves null (36) and we considered a set of exactly the null systems (37).

Theorem. (*i*) *Null physical systems are subsystems of every other physical system; and* (*ii*) *there exists at most one null physical system.*

Proof. (*i*) Let a be a null physical system and let b be some physical system. On the basis of Olap (46), we can conclude that $a \sqsubseteq b$ if we can establish the antecedent of the following conditional:

$$\forall c \left(c \mathrel{\dot{\sqsubseteq}} a \longrightarrow c \circ b\right) \longrightarrow a \sqsubseteq b \,.$$

Since a is null, all its subsystems are null; so the first antecedent $c \mathrel{\dot{\sqsubseteq}} a$ is always false; then the first conditional is always true, which is what we had to establish.

(*ii*) Let a and b be null. Then by (*i*) they are each other's subsystems. Hence they are identical. So all null systems are identical, there is at most one null physical system. □

This theorem implies that the set-extension (37) of $\mathsf{Null}(\cdot)$ is empty or a singleton set. We next prove that it is a singleton set.

Theorem. *There exists one null physical system; it is the composition of nothing.*

Proof. The existence of the composition of nothing, *i.e.* $\bigsqcup \varnothing$, follows from the Composition Axiom (51), with $P = \varnothing$. We prove the composition is null. The *definiens* of $\bigsqcup \varnothing$ (48) with clause (1) expanded reads:

$$\begin{aligned}
&(1)\quad \forall b \left(b \in \varnothing \longrightarrow b \sqsubseteq \textstyle\bigsqcup \varnothing\right) \quad \wedge \\
&(2)\quad \forall c \left(c \mathrel{\dot{\sqsubseteq}} \textstyle\bigsqcup \varnothing \longrightarrow \exists\, b \in \varnothing : c \circ b\right) .
\end{aligned}$$

Notice that conjunct (1) holds vacuously because the antecedent $b \in \varnothing$ is always false. For the same reason, the consequent of conjunct (2) is always false. Then the antecedent of (2) is false: c is not a genuine sybsystem of $\bigsqcup \varnothing$, that is, c is null or is not a subsystem of $\bigsqcup \varnothing$. Hence *all* subsystems of $\bigsqcup \varnothing$ are null, which holds in particular for $\bigsqcup \varnothing$. Thus $\bigsqcup \varnothing$ must be null. So there exists at least one null system. Since we have proved already (see above) there is at most one null system, part (*ii*), we conclude there exists exactly one null physical system. □

Notice that we have not invoked that sets of physical systems has a single composition. If there were to exist more than one composition of nothing, they would all be null according to the proof above. So we have proved that the physical systems in $\varnothing$ have a single physical composition, $\bigsqcup \varnothing$. Since there exists a unique null physical system, we can baptise it: $\boldsymbol{\emptyset}$. We then have: $\mathsf{Null}(a) \longleftrightarrow a = \boldsymbol{\emptyset}$ and $\mathsf{Gen}(b) \longleftrightarrow a \neq \boldsymbol{\emptyset}$.

Composition Theorems. We promised that the composition $\sqcup P$ of a set P of physical systems is the least inclusive one. The proof of this theorem requires some preliminary work.

Theorem. *Physical system a is identical (i) to the physical composition of the singleton set $\{a\}$, and (ii) to the physical composition of its subsystem set $\wp a$:*

$$\sqcup\{a\} = a \;\wedge\; \sqcup\wp a = a\ . \tag{59}$$

Proof. (i) Let a be a physical system. Then the set $\{a\}$ exists by virtue of Replacement and the existence of $\{\varnothing\}$, and its composition $\sqcup\{a\}$ exists by virtue of Comp. The definition of $\mathsf{PhysComp}(a, \{a\})$ reads:

$$(1)\quad \forall\, b \in \{a\} : b \sqsubseteq a \quad \wedge$$

$$(2)\quad \forall\, c\,\big(c \mathrel{\dot\sqsubseteq} a \longrightarrow \exists\, b \in \{a\} : c \circ b\big)\ ,$$

which is what we have to prove. Since a is the only member of $\{a\}$ and $\sqsubseteq$ is reflexive, conjunct (1) holds. For conjunct (2) to hold, we have to prove that all genuine subsystems of a overlap a because a is the only member of $\{a\}$, *i.e.* we have to prove that a and every *genuine* subsystem $c \mathrel{\dot\sqsubseteq} a$ have a genuine subsystem in common:

$$\exists\, b\,\big(b \sqsubseteq c \;\wedge\; b \sqsubseteq a \;\wedge\; \mathsf{Gen}(b)\big)\ .$$

For every c we can choose *it* for b, because the reflexivity of $\sqsubseteq$ then yields the first conjunct, the transitivity of $\sqsubseteq$ yields together with $c \sqsubseteq a$ and $c \sqsubseteq c$ the second conjunct, and the third conjunct holds because c is genuine by assumption.

(ii) To prove that $\sqcup\wp a = a$ we have to prove that

$$(1)\quad \forall\, b \in \wp a : b \sqsubseteq a \quad \wedge$$

$$(2)\quad \forall\, c\,\big(c \mathrel{\dot\sqsubseteq} a \longrightarrow \exists\, b \in \wp a : c \circ b\big)\ .$$

Conjunct (1) holds because $\wp a$ contains by definition exactly the subsystems of a. For conjunct (2) we have to find, for every genuine subsystem c of a an overlapping member $b \in \wp a$. We can always choose c for b, because $c \in \wp a$ due to $c \sqsubseteq a$, and because the overlap-relation is symmetric. □

For the special sets of physical systems $\wp a$ and $\{a\}$, Theorem (59) demonstrates that their compositions are equal. Now we are in a position to demonstrate the following

Single Composition Theorem. *The physical systems in some set $P \subset \mathbf{P}$ compose to a single composite physical system.*

Proof. The cases for which P is empty or a singleton set have been established above. We now consider the case when P has at least two members.[7] Suppose v and w are any two of the possible multitude of compositions of all the members of P:

$$\mathsf{PhysComp}(v, P) \wedge \mathsf{PhysComp}(w, P)\ . \tag{60}$$

[7]The author has not been able to shorten Bunt's proof of his ensemble-version of this theorem; we reproduce it here with minor modifications for the sake of completeness. See Bunt [1985, p. 245].

We prove they are identical: $v = w$ (the converse holds as a result of Substitutivity).

Since P has two members, v and w have both at least two subsystems, namely the members of P. At least one of them is genuine, because P can contain at most one null physical system. Let e be any genuine subsystem of w: $e \dot{\sqsubseteq} w$. Then by definition P has a member, d say, that overlaps with e: $e \circ d$. This means d and e have some genuine common subsystem; call it c. Then $c \sqsubseteq d$ $(*)$ and $c \sqsubseteq e$ $(\blacktriangle)$

For v we also have that $d \sqsubseteq v$, because $d \in P$ and by definition v includes all members of P. By transitivity and $(*)$ we then have that $c \sqsubseteq v$. From this and $(\blacktriangle)$ we also have that e and v overlap. So we have proved that any genuine subsystem $e \dot{\sqsubseteq} v$ overlaps w. Then the Overlap Axiom (Olap) permits us to deduce that $v \sqsubseteq w$.

The proof for $w \sqsubseteq v$ is a notational variant of the previous argument. Hence $w = v$. $\square$

Every set P of physical systems has a single composition, which permits us to write $\sqcup P$ for it. But is it the *least inclusive* one, as was the erstwhile motivation for the second conjunct in the definition of composition (48)? An affirmative answer is provided by the

Supremum Theorem. *The physical composition of the members of set P of physical systems is the least inclusive physical system which has all members of P as its subsystems*:

$$\forall P \subset \mathbf{P}, \forall a \big(\forall b \in P : b \sqsubseteq a \longrightarrow \sqcup P \sqsubseteq a \big) . \qquad (61)$$

Proof. The proof proceeds by distinguishing three cases. (i) Let $P = \varnothing$. Then $\sqcup P = \text{ø}$ $(\triangle)$. Let b be such that it has all members of P as subsystems. Because P has no members, b has no subsystems either, save ø, because ø is a subsystem of *every* system. From $\text{ø} \sqsubseteq b$ and $(\bullet)$ we deduce that $\sqcup P \sqsubseteq b$.

(ii) Let P have one member, a say. Then $\sqcup P = \sqcup \{a\} = a$. Let b be such that all members of P are its subsystems. Since P has only one member, we have that $a \sqsubseteq b$. Because $\sqcup P = a$, we conclude that $\sqcup P \sqsubseteq b$.

(iii) This part of the proof is again similar to Bunt's [1981, p. 247]. Let P have at least two members. Let e be any genuine subsystem of $\sqcup P$: $e \sqsubseteq \sqcup P$. ($\sqcup P$ has at least one genuine subsystem in case it has exactly two members, so there always is such a e.) Then by definition there is some $d \in P$ that overlaps with e, *i.e.* there is some c such that $c \sqsubseteq d$ $(*)$ and $c \sqsubseteq e$ $(\blacktriangle)$.

Next let a be any physical system such that all members of P are subsystems of a. Then $d \sqsubseteq a$. By transitivity and $(*)$ we then have that $c \sqsubseteq a$. From this and $(\blacktriangle)$ we also have that e and a overlap. So we have proved that any genuine subsystem $e \sqsubseteq u$ overlaps with a. Then Olap permits us to deduce that $\sqcup P \sqsubseteq a$. $\square$

We now can write down result (49) as follows:

$$P = P' \longrightarrow \sqcup P = \sqcup P' . \qquad (62)$$

Observe that the converse fails: $\sqcup \{a\} = \sqcup \wp a$ but $\{a\} = \wp a$ only holds for elementary particles and null systems, in all other cases: $\wp a \neq \{a\}$.

A very useful result, because it gives us another sufficient condition for subsystemhood, is the following

Corollary.[8] *The physical composition of every subset P' of a set P of physical systems is a subsystem of the physical composition of P:*

$$P \subset P' \longrightarrow \sqcup P \sqsubseteq \sqcup P' . \tag{63}$$

Proof. Let P and P' be sets of physical systems and let $P \subset P'$. The unique composite system $\sqcup P'$ has by definition all members of P' as subsystems so by implication it also has all members of P as subsystems. Theorem (61) informs us that $\sqcup P$ is the least inclusive physical system that includes all members of P. Hence $\sqcup P \sqsubseteq \sqcup P'$. □

The converse of Corollary (63) fails, as the following corollary testifies.

Corollary. *All subsets of the subsystem set $\wp b$ of b have the same composite system, provided these subsets contain b:*

$$\forall P \subseteq \wp b \, (b \in P \longrightarrow \sqcup P = b) . \tag{64}$$

Proof. Let $P \subseteq \wp b$ and $b \in P$. Then

$$\{b\} \subseteq P \subseteq \wp b .$$

Invoking Corollary (63) we deduce that:

$$\sqcup \{b\} \sqsubseteq \sqcup P \sqsubseteq \sqcup \wp b .$$

With Theorem (59) we then derive that b and $\sqcup P$ are each other's subsystems, which by definition means they are identical. □

The failure of the converse of Corollary (63) is readily illustrated by an example from physics. Let P contain an α-particle and two electrons, and let P' contain two protons and two neutrons. Since $\sqcup P'$ is a Helium nucleus and $\sqcup P$ is a Helium atom, we have that $P' \sqsubset P$, but $P' \not\subset P$ because $P \cap P' = \varnothing$. So our mereological theorems give a rigorous vindication of this elementary truth from physics.

Finally we prove a theorem that shows how the physical composition and the set-theoretical union mesh.

Theorem. *The composition of the union of two sets of physical systems is identical to the composition of their compositions.*

$$\forall P, P' \subseteq \mathbf{P} : \ \sqcup (P \cup P') = (\sqcup P) \sqcup (\sqcup P') . \tag{65}$$

[8] Leonard & Goodman assert this theorem without proof [1940, p. 49], I.55. The proof is 'trivial' once the Single Composition Theorem has been proved, a theorem that Leonard & Goodman do not even care to mention. Trivial? (This does not seem to have been noticed before.)

Proof. We show that the composition of all physical systems in $P \cup P'$, which exists uniquely, is identical to the composition of the two physical systems $\sqcup P$ and $\sqcup P'$, which also exists uniquely. First we introduce an abbreviation:

$$u \equiv (\sqcup P) \sqcup (\sqcup P') \equiv \sqcup \{ \sqcup P, \sqcup P' \} .$$

We have to prove the following two clauses of the *definiens* of 'u is a physical composition of all systems in $P \cup P'$', that is, of $\mathsf{PhysComp}(u, P \cup P')$ (48):

$$(1) \quad \forall b \in P \cup P' : b \sqsubseteq u \quad \wedge$$

$$(2) \quad \forall c \big(c \sqsubseteq u \longrightarrow \exists b \in P \cup P' : c \circ b \big) .$$

This proves the Theorem because the physical systems in $P \cup P'$ have a single composition.

(1) Let $b \in P \cup P'$. Then by definition of $\sqcup P$ and $\sqcup P'$, b is a subsystem of $\sqcup P$ or $\sqcup P'$. Because these compositions are members of $\{ \sqcup P, \sqcup P' \}$, b is a subsystem of u. The transitivity of $\sqsubseteq$ then yields that $b \sqsubseteq u$.

(2) Let c be an arbitrary genuine subsystem of u. We have to prove there exists some $b \in P \cup P'$ such that $c \circ b$, *i.e.* there exists some d such that $d \sqsubseteq c$ and $d \sqsubseteq b$. We prove this by *reductio ad absurdum*. The *reductio* assumption is that for all $b \in P \cup P'$, c is distinct from b, *i.e.* for all d it holds that:

$$d \sqsubseteq c \ \wedge \ d \sqsubseteq b \ \longrightarrow \ d = \varnothing . \qquad \text{(RA)}$$

Then this must hold in particular when we substitute b for d:

$$b \sqsubseteq c \ \wedge \ b \sqsubseteq b \ \longrightarrow \ b = \varnothing .$$

The second conjunct holds, so $b \not\sqsubseteq c$ or $b = \varnothing$. If $b = \varnothing$, then $b \sqsubseteq c$, because $\varnothing$ is a subsystem of every physical system, in contradiction to $b \not\sqsubseteq c$. Hence b cannot be the null physical system. But then $b \not\sqsubseteq c$. Since $c \sqsubseteq u$, we have by transitivity that $b \not\sqsubseteq u$, in contradiction to clause (1). So (RA) must go: all $b \in P \cup P'$ overlap c. □

Critical Remark Concerning Compositions. The Composition Axiom (Comp) postulates the existence of a composite system for *every* set of physical systems; moreover, the Single Composition Theorem asserts there is only one. Simons [1987, p. 111] submits, following Leonard & Goodman, that the motivation for Comp derives from an analogy with ZFC. Extensional mereology postulates that the members of *every* set of physical systems can be composed to another physical system *because* all kinds of 'ill-sorted' sets can also be collected into a set, *viz.*

$$\big\{ \mathbb{R}^{2000}, \ \aleph_{2000}, \ \omega^{2000}, \ \wp^{2000}\mathbb{Q}, \ 2000, \ \beth_{2000} \big\} .$$

This analogy with set-theory is however cripple, because in the domain of discourse of set-theory even 'well-sorted' elements, *viz.* all functions, all relations, all groups, all infinite ordinal numbers, all Peano structures, *etc.* can only be collected into a set at the pain of inconsistency (see Muller [1998, p. 535]). It is a piece of fiction that all 'well-sorted' elements can be collected into a set, hence the analogy for postulating the existence of a composition of *every* set of members is gone.

Further, the composition of physical systems is supposed to be something over and above merely collecting them in a set, *i.e.* it is supposed to form some 'ontological whole', in contrast to a mere 'combinatorial whole', *i.e.* a set; $a \sqcup b$ is not the same as $\{a, b\}$. Consider a barbican, a binnacle, a beetle and a boomerang. What ontological quagmire constitutes their composite system when this is supposed to be something more than just a four-element set? To assert that *they* form a *composite physical system* may stretch this concept beyond breaking point. Hence a *weakening* of the Composition Axiom is not implausible.

Related is the issue, also not raised so far as we can tell, of whether it is reasonable to have that *identical* sets of physical systems give rise to *identical* composite physical systems. Consider a set of two protons (p^+), two neutrons (n) and two electrons (e^-). What is their composition? A Helium atom (He) or a Deuterium gas molecule (D_2)? The manner of composing these particles physically matters. Suppose we first press two neutrons and two protons sufficiently close together for the strong interaction to come into play. We then obtain a Helium nucleus (He^{2+}). Next we let two electrons gently enter the vincinity of this nucleus. They are attracted by the positive nucleus to form a Helium atom (He). Now we first press a neutron and a proton together (np^+) and bring an electron in its vicinity. We obtain a Deuterium atom (D). We repeat the process with the remaining three particles. We obtain two Deuterium atoms. We can combine them by some other chemical process into a Deuterium gas molecule (D_2). So indeed composition is not associative, as ZFCM says it is, *if* composition is to correspond to anything in physical reality, because a Deuterium atom is not a Helium atom, they are similar but distinct physical systems with distinct physical and chemical properties — Deuterium behaves chemically just like Hydrogen (think of D_2O, heavy water), not like Helium, which is chemically inept. Think further of organic molecules, which consist mostly of Carbon, Hydrogen, Nitrogen and Oxygen. The spatial arrangement of the atoms in the molecule are crucial for the properties of the ensuing organic molecule and the kind of organic material that results. Physical systems do generally *not* determine their composition, they determine it in conjunction with *how* the composition is physically produced.

This critical remark is just to emphasise that the mereological treatment of physical systems in utter generality is only the beginning for finding out what physical reality is like. Further structure has to imposed upon them in order to become informative about 'physical reality' and to make empirically testable assertions. This further structure is exactly what *physical theories* are supplying. Since these are captured by set-theoretical predicates (Subsection 0.1), the presence of physical systems can now be employed to make better characterisations of physical theories, in whose structures now physical systems occur. To give but one example, quantum mechanics is characterised as a vast set of physical structures of which the following is but one member:

$$\left\langle \mathrm{H},\ \widehat{H},\ \psi,\ \{m, e\} \right\rangle , \tag{66}$$

where H is a Hydrogen atom (yer physical system), $\widehat{H}$ the Schrödinger Hamiltonian (a selfadjoint operator acting on the Hilbert-space $L^2(\mathbb{R}^3)$ of square-integrable complex functions representing the physical magnitude energy, $\psi \in L^2(\mathbb{R}^3)$ is some stationary state that H is prepared in, and $m \in \mathbb{R}^+$ and $e \in \mathbb{R}^-$ are the electron's mass and charge, respectively. H is a composite physical system of an electron (e^-) and a proton (p^+), which is the atomic nucleus in this case: $\mathrm{H} = \mathrm{e}^- \sqcup \mathrm{p}^+$. (See Muller [1998, pp. 293-306] for more examples of this kind.)

To elaborate a bit further of how a stronger notion of a physical composition can be had in ZFCM that makes sense, rather than nonsense, of physics, without introducing novel primitive notions, we submit the following sentences. Suppose we are in the process of characterising classical mechanics as a sets of physical structures. Consider the composite system of the Earth (mass $M \in \mathbb{R}^+$) and the Moon (mass $m \in \mathbb{R}^+$). For any other physical system to qualify as the composite system Earth-Moon, it better be the one which has a mass $M + m$. The definition of a composite physical system in the context of classical mechanics should be in agreement with this. But when we move to the special theory of relativity, this constraint on composition needs to be revised, because the mass of a composite system may be smaller than the sum of the masses of the composing physical systems — some of the 'missing mass', call it μ, is then perhaps converted into binding energy, due to Einstein's mass-energy law $E = \mu c^2$.

Conditions for Subsystemhood. The Overlap Axiom (Olap) provides us with a sufficient condition to infer subsystemhood. We now prove two theorems that provide criteria (conditions which are sufficient and necessary) for subsystemhood. The first one highlights the extensional character of mereology.

Theorem. *Two physical systems are identical iff they share all their subsystems*:

$$a = b \;\longleftrightarrow\; \wp a = \wp b\ . \tag{67}$$

Proof. [$\longrightarrow$] By definition, the identity of a and b means they are each other's subsystems: $a \sqsubseteq b$ and $b \sqsubseteq a$. Let $c \sqsubseteq a$. Then by transitivity of $\sqsubseteq$ and by $a \sqsubseteq b$ we infer that $c \sqsubseteq b$. Hence

$$a \sqsubseteq b \;\longrightarrow\; \forall\, c\,(c \sqsubseteq a \longrightarrow c \sqsubseteq b)\ .$$

Similarly one proves that:

$$b \sqsubseteq a \;\longrightarrow\; \forall\, d\,(d \sqsubseteq b \longrightarrow d \sqsubseteq a)\ .$$

The conjunction of these results, together with the definition of identity and choosing c for d in the second result, establishes $\wp a = \wp b$ for $a = b$.

[$\longleftarrow$] Suppose that $\wp a = \wp b$. Then by virtue of Theorem (49) their compositions coincide: $\bigsqcup \wp a = \bigsqcup \wp b$. Theorem (59) gives us the equalities $\bigsqcup \wp a = a$ and $\bigsqcup \wp b = b$. Hence $a = b$. □

Leonard & Goodman [1940] took the distinctness-relation (43) as primitive and defined subsystemhood in terms of it. We have their definition as a

Theorem. *One physical system is a subsystem of another physical system iff every physical system which is distinct from the other is also distinct from the first.*

$$a \sqsubseteq b \;\longleftrightarrow\; \exists\, c\,\big(c \rfloor b \;\longrightarrow\; c \rfloor a\big)\ . \tag{68}$$

Proof. [$\longrightarrow$] We distinguish exhaustively three cases.

(i) Let b be the null physical system and let a be such that $a \sqsubseteq b$. Then $a = b = \emptyset$. For every c it holds that the only subsystems it has in common with a and b is $\emptyset$; hence both antecedent and consequent of the necessary condition for $a \sqsubseteq b$ (68) hold by definition for *all* c, and therefore for some c; so the conditional also holds.

(ii) Let a be the null physical system and let b be a genuine system such that $a \sqsubseteq b$, *i.e.* $a \mathrel{\dot\sqsubseteq} b$. In this case the consequent of the necessary condition for $a \sqsubseteq b$ is again true, because every genuine c is distinct from $\emptyset = a$, and therefore the conditional is also true.

(iii) Let a and b be genuine physical systems such that $a \sqsubseteq b$. Let e be some physical system that overlaps with a (such a system always exist: a is an example; if a is not elementary, there are more). Then according to the premise there is a genuine system, c say, such that $c \mathrel{\dot\sqsubseteq} e$ ($*$) and $c \mathrel{\dot\sqsubseteq} a$ ($\blacktriangle$). To show that e also overlaps with b, we have to find a physical system that is a common genuine subsystem of e and b. Such a system is c, which is a genuine subsystem of e ($*$), and of b because of $a \sqsubseteq b$ and ($\blacktriangle$). Hence we have proved that:

$$\forall e \, (e \circ a \;\longrightarrow\; e \circ b) \,, \tag{69}$$

which is logically identical to what we have establish due to theorem of logic (45).

[$\longleftarrow$] (i′, ii′) In the case that a is null, we established in (i) and (ii) above that the necessary condition of (68) holds. We also have that $a \sqsubseteq b$, which yields the theorem because a biconditional holds if both its conditions hold.

(iii′) We assume that a and b are genuine and that all systems distinct from b are also distinct from a, which is again logically identical to (69). Among the genuine systems that overlap with a are all subsystems of a:

$$\forall e \, (e \circ a \;\longrightarrow\; e \circ b) \,.$$

With the Overlap Axiom (46) we conclude from this that $a \sqsubseteq b$. ☐

As an immediate corollary we have that two systems are identical iff they overlap with the same systems:

$$a = b \;\longleftrightarrow\; \forall e \, (e \circ a \;\longleftrightarrow\; e \circ b) \,. \tag{70}$$

1.4 The Physical Universe

We describe how every subsystem set gives rise to an atomistic Boolean lattice; to achieve this we first need to define a few more mereological concepts.

Overlapping. Besides composing physical systems out of the members of a set $P \subset \mathbf{P}$, we can also look at which physical subsystems these members have in common. The least encompassing subsystem they all share would then be the mereological analogue of set-theoretical intersection. So we define the **physical overlapping** of the physical systems in set P as the physical system that is a common subsystem of all members of P, and it includes all common subsystems of all members of P:

$$\mathsf{PhysOlap}(a, P) \equiv \begin{array}{l}(1) \quad \forall b \in P : a \sqsubseteq b \quad \wedge \\ (2) \quad \forall b \big(\forall c \in P\,(b \circ c) \longrightarrow b \sqsubseteq a\big)\ .\end{array} \tag{71}$$

A sequence of theorems for $\sqcap$ analogous to $\sqcup$ can now be proved (which we shall leave as an exercise for the reader).

Single Overlap Theorem. *For every set $P \subset \mathbf{P}$ of physical systems there exists a single overlapping which is the most inclusive physical system that includes all common subsystems of the members of P.*

Bunt's proof carries over from BET to ZFCM and $\mathsf{ZFCM}_\mathfrak{p}$ [1981, p. 252]. The existence of a single overlapping makes the following notation possible for $\mathsf{PhysOlap}(a, P)$: $a = \sqcap P$. Evidently two physical systems are distinct iff their overlapping is the null system:

$$a \,\rfloor\!\!\lceil\, b \longleftrightarrow a \sqcap b = \varnothing\ , \quad \text{where } a \sqcap b \equiv \sqcap\{a,\, b\}\ . \tag{72}$$

Whenever $\varnothing \in P$, one has that $\sqcap P = \varnothing$; in particular, because every physical system includes $\varnothing$, we have that for every a:

$$\sqcap \wp a = \varnothing\ . \tag{73}$$

Completion. The final notion we introduce is that of completion, as the analogue of the set-theoretical complement. Definition: the **completion** of physical system $a \in P$, denoted by $\overline{a}$ (again anticipating there is only one), is the composition of all physical systems in P which do not overlap with a:

$$\overline{a} \equiv \sqcup\big\{b \in P \mid b \,\rfloor\!\!\lceil\, a\big\}\ . \tag{74}$$

The existence of a single $\overline{a}$ is an immediate consequence of the existence of the Single Composition Theorem. One further deduces that (see Bunt [1985, pp. 254-256]):

$$a \sqcup \overline{a} = \sqcup P\ . \tag{75}$$

Similar in the way the power-set $\wp X$ of every set X gives rise to a Boolean subset lattice, does the subsystem set $\wp a$ of every physical system give rise to a Boolean subsystem lattice; Bunt's proof [1981, pp. 257-259] carries over from his BET to our M because his notion of a 'unicle' is not involved in it (Tarski [1956, pp. 333-334] conjectured the contents of Bunt's Theorem in the context of LT).

Bunt's Boolean Theorem. *For every physical system u, the subsystem set $\wp u$ gives rise to a Boolean lattice*:

$$\big\langle\, \wp u,\ \sqsubseteq,\ \overline{(\cdot)},\ \sqcup,\ \sqcap,\ \{\varnothing, u\}\,\big\rangle\ . \tag{76}$$

If u is reducible, which is the case in ZFCM due to Atomism (54), the Boolean lattice becomes what is known as an *atomistic* Boolean lattice. One could further consider to define in $\mathsf{ZFCM}_\mathfrak{p}$ **physical universe** as the composition of all physical systems in $\mathcal{P} \cup \mathcal{P}_\sqcup$.

2 Meta-Mereological Investigations

2.0 Relation to Extant Mereological Theories

Before we begin our meta-mathematical investigations concerning ZFCM, we relate both mereological theories ZFCM and $\mathsf{ZFCM}_{\mathfrak{p}}$ to: (b) Mereology of Lesniewski & Tarski's (LT), (c) the Calculus of Individuals of Leonard & Goodman (CI) and (d) the Ensemble Theory of Bunt (BET). But first of all (a) we compare our theories ZFCM and $\mathsf{ZFCM}_{\mathfrak{p}}$ with each other.

(a) The relations between ZFCM and $\mathsf{ZFCM}_{\mathfrak{p}}$ are as expected (the proof is trivial).

Theorem. *The theory* ZFCM *deductively extends theory* $\mathsf{ZFCM}_{\mathfrak{p}}$.

An immediate consequence of the theorem above is the following

Corollary. *For every cardinal number* $\mathfrak{p} \in \mathbf{C}$*, theory* $\mathsf{ZFCM}_{\mathfrak{p}}$ *is consistent relative to theory* ZFCM.

This means that as far as consistency and conservativeness of $\mathsf{ZFCM}_{\mathfrak{p}}$ and ZFCM is concerned with respect to ZFC, we can restrict our attention to ZFCM.

(b) For simplicity we consider CI (see below for the formal definition) and TL (55) to be defined against the same background set-theory (ZF), which we leave implicit. The 'individuals' of CI as well as the 'objects' of LT we can identify with our genuine physical systems because both CI and LT make do without a null individual and a null object, respectively. Then the various clauses in our definitions for systems being genuine can be deleted, because in the domains of discourse of CI and LT all physical systems are genuine (we are not going to spell this out).

Theorem. *Theory* ZFCM *deductively extends theory* LT:

$$\mathsf{ZFCM} \vdash \mathsf{LT}\,. \tag{77}$$

Proof. See the definitions of ZFCM (56) and LT (55). □

(c) For the Calculus of Individuals we have the same

Theorem. *Theory* ZFCM *deductively extends theory* CI:

$$\mathsf{ZFCM} \vdash \mathsf{CI}\,. \tag{78}$$

Proof. The language of CI, call it $\mathcal{L}(\mathsf{CI})$, is the same as $\mathcal{L}_{\sqsubseteq}$ but its primitive vocabulary differs, for it takes the distinctness-relation ($\rfloor\!\lceil$) as primitive. The relation of subsystemhood is defined in $\mathcal{L}(\mathsf{CI})$ as in our Theorem (68):

$$a \sqsubseteq b \;\longleftrightarrow\; \exists\, c\,\big(\, c \rfloor\!\lceil b \longrightarrow\; c \rfloor\!\lceil a\,\big)\,. \tag{68}$$

The overlap-relation is in $\mathcal{L}(\mathsf{CI})$ defined in the same manner as we have defined it (43). But the definition of composition (or 'fusion', as Leonard & Goodman call it) is different. Definition:

u is a **fusion** of the physical systems in set P iff exactly all systems distinct from u are distinct from all members in P, no more no less.

$$\mathsf{Fu}(u, P) \;\equiv\; \forall\, a\big(\, a \int u \;\;\longleftrightarrow\;\; \forall\, b \in P : a \int b\,\big)\;. \tag{79}$$

The first axiom of CI states that two physical systems a and b overlap iff they are not distinct:

$$\mathsf{CI1}\;:\; a \circ b \;\longleftrightarrow\; \neg\big(a \int b\big)\;. \tag{80}$$

This is Theorem (44) of ZFCM. The second axiom of CI states that two physical systems, a and b say, are identical if they are each other's subsystems:

$$\mathsf{CI2}\;:\; (a \sqsubseteq b \;\wedge\; b \sqsubseteq a) \;\longrightarrow\; a = b\,. \tag{81}$$

This is half our definition (28), hence trivially a theorem of ZFCM. The third and last axiom of CI states the existence of a fusion of every non-empty set of physical systems:

$$\mathsf{CI3}\;:\; \forall\, P \subset \mathbf{P},\; \exists\, a\,\big(\, P \neq \varnothing \;\longrightarrow\; \mathsf{Fu}(a, P)\,\big)\;. \tag{82}$$

Bunt [1981, p. 296] proved that the physical composition $\sqcup P$ satisfies the *definiens* of a fusion:

$$\forall\, a\,\big(a \int \sqcup P \;\longleftrightarrow\; \forall\, b \in P :\; \sqcup P \int b\big)\;, \tag{83}$$

from which, in combination with the Composition Axiom (Comp), axiom CI3 (82) follows.

Hence all three axioms of CI1-CI3 are theorems of ZFCM. □

(d) The relation between Bunt's Ensemble Theory (BET) and ZFCM is not so straightforward, due to the Axiom of Plenitude (Plen) and the Axiom of Atomism (Atoms), which are not axioms of BET. This suggests it is unknown whether BET is an extension of ZFCM. If axiom Plen and Atoms were deleted from ZFCM, to obtain ZFCM′ say, BET would be an extension of ZFCM′, because all axioms of ZFCM′ are also axioms of BET, save Olap, which strictly speaking is a theorem in BET:

$$\mathsf{BET} \;\vdash\; \mathsf{PreOrd} \wedge \mathsf{Subs} \wedge \mathsf{Comp} \wedge \mathsf{Olap}\;.$$

Bunt established the equiconsistency of BET and ZF [1981, pp. 269-274]. Because no cardinal upperbound on the number of primordial elements in BET is forthcoming, BET does not extend $\mathsf{ZFCM}_{\mathfrak{p}}$. Conversely, neither ZFCM nor $\mathsf{ZFCM}_{\mathfrak{p}}$ is an extension of BET either, because the axioms of BET include all axioms of ZFCM and $\mathsf{ZFCM}_{\mathfrak{p}}$, save Plen and $\mathsf{Pars}(\mathfrak{p})$, respectively, but include also axioms governing elementary particles that permits one to prove things about elementary particles which are unavailable in ZFCM. To summarise,

$$\mathsf{BET} \;\nvdash\; \mathsf{ZFM}\;,\quad \mathsf{ZFM} \;\nvdash\; \mathsf{BET}\;,\quad \mathsf{BET} \;\nvdash\; \mathsf{ZFM}_{\mathfrak{p}}\;,\quad \mathsf{ZFCM}_{\mathfrak{p}} \;\nvdash\; \mathsf{BET}\;. \tag{84}$$

Further on we shall get the result of equiconsistency between BET and ZFCM for free (p. 51).

2.1 Overview in Illustrated Prose

The principal results of the present Section are that both ZFCM and ZFCM_p are conservative hence equiconsistent deductive extensions of ZFC. In this Subsection we present an overview in illustrated prose of the formal proofs that follow in the next Subsections. We proceed with our strongest theory (ZFCM), because if we have proved its consistency relative to and its conservativeness over ZFC, then the same results follow for ZFCM_p.

What we are going to do, expressed succinctly, is to make a model of ZFCM, called $\mathbf{D}$, inside the domain of discourse $\mathbf{V}_{\text{ZFC}}$ of ZFC, which is isomorphic to $\mathbf{U}$. Figure 1 depicts the proof (p. 52). By virtue of the Model Existence Lemma (a 1st-order theory is consistent iff it has a model), we then have proven the consistency of ZFCM relative to ZFC. Since ZFC deductively extends ZFCM_p, it follows that ZFCM_p is also consistent relative to ZFC.

For the sake of perspicuity we write the universe of discourse of ZFCM, with its primitive membership- and subsystem-relation as follows:[9]

$$\Big\langle\, \mathbf{U},\ \mathbf{S},\ \mathbf{P},\ \in,\ \sqsubseteq \Big\rangle\ . \tag{85}$$

Then we can write the structure of our model of ZFCM that we erect in ZFC as follows:

$$\Big\langle\, \mathbf{D},\ \mathbf{D}_0,\ \mathbf{D}_1,\ \in',\ \sqsubseteq' \Big\rangle\ . \tag{86}$$

The virtual set $\mathbf{D}_0$ consists of every set X of ZFCM, which are collected in $\mathbf{S}$, labelled with 0, after being purified: $X \mapsto \langle 0, F_\in(X)\rangle$, where $F_\in$ is the purifier (17). Then $\mathbf{D}_0 \subset \mathbf{V}$. The purification is necessary to ensure that all physical systems are deleted from X, and from its set-members, and from the set-members of the set-members of X, *etc.*, because $\mathbf{V}$ is by definition the range of $F_\in$ on domain $\mathbf{S}$ (19). The virtual set $\mathbf{D}_1$ consists of all pure sets of ZFCM but now labelled with 1: $\langle 1, F_\in(A)\rangle$, where A, the referent of physical system a, is the image of the bijection between $\mathbf{P}$ and $\mathbf{V}$ (the Plenitude Axiom guarantees there is one) — then A is already pure but we add it for symmetry (remember that we proved $F_\in$ to be a bijection from $\mathbf{V}$ onto itself (19)). So $\mathbf{D}_1$ will model the virtual set $\mathbf{P}$ of all physical systems. Virtual set $\mathbf{D}$ is the union of $\mathbf{D}_0$ and $\mathbf{D}_1$:

$$\mathbf{D} \equiv \mathbf{D}_0 \cup \mathbf{D}_1\ , \tag{87}$$

just as $\mathbf{U} = \mathbf{S} \cup \mathbf{P}$ (9). The membership-relation between $\in$ the sets in $\mathbf{S}$ ($X \in Y$) and between the physical systems in $\mathbf{P}$ and sets in $\mathbf{S}$ ($a \in Y$) is modelled by relation $\in'$ between the members of $\mathbf{D}_0$ and between the members of $\mathbf{D}_1$ and $\mathbf{D}_0$, defined as ordinary membership between the unlabelled members:

$$\begin{aligned} \langle 0, X\rangle \in' \langle 0, Y\rangle \quad &\equiv \quad X \in Y\ , \\ \langle 1, A\rangle\ \in' \langle 0, Y\rangle \quad &\equiv \quad A \in Y\ . \end{aligned} \tag{88}$$

[9]Formally speaking, these expressions are nonsense; consider them as abbreviations of what we assert in prose around them.

Finally, the subsystem-relation $\sqsubseteq$ between the physical systems in $\mathbf{P}$ is modelled by relation $\sqsubseteq'$ between the members of $\mathbf{D}_1$, defined as the subset-relation between its unlabelled members:

$$\langle 1, A \rangle \sqsubseteq' \langle 1, B \rangle \;\equiv\; A \subseteq B \,. \tag{89}$$

So far we have not left $\mathbf{U}$ of ZFCM yet: $\mathbf{D} \subset \mathbf{U}$ (86) is therefore called an *inner model* of ZFCM. But we shall prove that we can simply replace $\mathbf{V}$ with $\mathbf{V}_{\text{ZFC}}$ and have the same model $\mathbf{D}$ of ZFCM inside $\mathbf{V}_{\text{ZFC}}$. This yields the consistency of ZFCM relative to ZFC (p. 47).

Next we prove that $\langle \mathbf{D}_0, \in' \rangle$ and $\langle \mathbf{V}_{\text{ZFC}}, \in \rangle$ are *isomorphic* (142). Then every theorem proved in ZFCM about pure sets can already be proved in ZFC about *its* sets, which are all pure. This is essentially the conservativeness of ZFCM over ZFC (147).

We just have explained everything in semantic, model-theoretic terms; but in the formal proofs we shall employ Tarski's concepts of a *set-translation* and a *set-interpretation*. We then remain entirely on the syntactic level. What we shall do reads in the terminology of set-translations as follows: we translate directly the language $\mathcal{L}_{\sqsubseteq}$ into $\mathcal{L}_{\in}$ by means of (88) and (89) and prove that all the translated axioms of ZFCM are theorems of ZFC. We then have *interpreted* ZFCM in ZFC, which yields the consistency of ZFCM. The conservativeness of ZFCM over ZFC will be proved by means of a similar translation: we prove that the relativisation of every theorem of ZFCM formulated in $\mathcal{L}_{\in}$ to the virtual set $\mathbf{V}$ of all pure sets is already a theorem in ZFC (a relativisation is a special case of a translation). This is the conservativeness of ZFCM over ZFC (147): $\mathbf{V} = \mathbf{V}_{\text{ZFC}}$.

2.2 Consistency by Interpretation

All theorems mentioned *infra* are theorems in ZFC expressed in $\mathcal{L}_{\in}$, which we shall therefore not mention each time. We first define the set-translation (Subsection 2.2.1); subsequently we establish the consistency of ZFCM relative to ZFC (Subsection 2.2.2) and the conservativeness (Subsection 2.3).

2.2.1 Set-Translation

The Domain of Translation. We first locate a virtual set $\mathbf{D} \equiv \mathbf{D}_0 \cup \mathbf{D}_1$ inside $\mathbf{V}_{\text{ZFC}}$ (8) that serves as the domain of interpretation; every inhabitant of $\mathbf{U}$ will have a not necessarily unique counter-inhabitant in $\mathbf{D} \subset \mathbf{V}_{\text{ZFC}}$.

$$\mathbf{D}_0 \;\equiv\; \{\langle 0, X \rangle \mid X \in \mathbf{V}_{\text{ZFC}}\} \quad \text{and} \quad \mathbf{D}_1 \;\equiv\; \{\langle 1, A \rangle \mid A \in \mathbf{V}_{\text{ZFC}}\} \,. \tag{90}$$

Notice that $\mathbf{D}_0 \cap \mathbf{D}_1 = \varnothing$ and $(\mathbf{D}_0 \cup \mathbf{D}_1) \subset \mathbf{V}_{\text{ZFC}}$. Virtual sets $\mathbf{D}_0$ and $\mathbf{D}_1$ are just copies of $\mathbf{V}_{\text{ZFC}}$, with the labels $0 \equiv \varnothing$ and $1 \equiv \{\varnothing\}$ added to every set. Virtual set $\mathbf{D}_0$ will contain the set-translates of $\mathbf{U}$ and set $\mathbf{D}_1$ will contain the physical system-translates of $\mathbf{U}$. When we interpret $\text{ZFCM}_{\mathfrak{p}}$, we can replace the virtual set $\mathbf{D}_1$ with a real set:

$$D_1 \;\equiv\; \{\langle 1, \alpha \rangle \mid \alpha \in 2^{\mathfrak{p}}\} \,, \tag{91}$$

where the members α of cardinal number $2^{\mathfrak{p}}$, which is itself a set of cardinality $2^{\mathfrak{p}}$, are ordinal numbers. (All proofs below then can be repeated after a 'typographical isomorphism' has been performed on them.)

Definition of the Set-Translation. We start by translating the variables. All set-variables of $\mathcal{L}_{\sqsubseteq}$, for now restricted to Roman letters occurring at the end of the alphabet with and without primes: $R, S, T, U, V, W, X, Y, Z, X', Y''$, *etc.*, are sent to the same variables of $\mathcal{L}_{\in}$. All physical system-variables of $\mathcal{L}_{\sqsubseteq}$ are sent to Roman letters occurring at the beginning of the alphabet: a to A, a' to A', b to B, c to C, *etc.* (we use accordingly in $\mathcal{L}_{\sqsubseteq}$ for physical system-variables only small italics from the beginning of the alphabet: a, b, c, d, e). The atomic sentences we translate as follows:

$$\begin{aligned} (X \in Y)^{\mathbf{D}} &\equiv \exists\, X', Y' \big(X = \langle 0, X'\rangle \wedge Y = \langle 0, Y'\rangle \wedge X' \in Y'\big)\ ; \\ (a \in Y)^{\mathbf{D}} &\equiv \exists\, A', Y' \big(A = \langle 1, A'\rangle\ \wedge Y = \langle 0, Y'\rangle \wedge A' \in Y'\big)\ ; \\ (a \sqsubseteq b)^{\mathbf{D}} &\equiv \exists\, A', B' \big(A = \langle 1, A'\rangle\ \wedge B = \langle 1, B'\rangle \wedge A' \subseteq B'\big)\ , \end{aligned} \tag{92}$$

and by the very definition of a translation the absurdity is a translation-invariant: $\perp^{\mathbf{D}}$ is $\perp$. The translation by definition 'commutes' with all the sentential connectives as a result of the following translation of the indicative conditional: for any $\boldsymbol{\varphi}, \boldsymbol{\psi} \in \text{SENT}(\mathcal{L}_{\sqsubseteq})$:

$$(\boldsymbol{\varphi} \longrightarrow \boldsymbol{\psi})^{\mathbf{D}} \equiv \boldsymbol{\varphi}^{\mathbf{D}} \longrightarrow \boldsymbol{\psi}^{\mathbf{D}}\ , \tag{93}$$

because all sentential connectives and negation are defined in terms of $\perp$ (invariant under translations) and the indicative conditional. The universal quantifiers translate as follows:

$$\begin{aligned} \big(\forall\, X : \boldsymbol{\varphi}(X)\big)^{\mathbf{D}} &\equiv \forall\, X \in \mathbf{D} : \boldsymbol{\varphi}^{\mathbf{D}}(X)\ , \\ \big(\forall\, a : \boldsymbol{\psi}(a)\big)^{\mathbf{D}} &\equiv \forall\, A \in \mathbf{D}\ : \boldsymbol{\psi}^{\mathbf{D}}(A)\ . \end{aligned} \tag{94}$$

It is easy to see that for set-variables quantification comes down for X to run over $\mathbf{D}_0 \subset \mathbf{D}$ and for physical system-variables quantification comes down for A to run over $\mathbf{D}_1 \subset \mathbf{D}$. The translations for sentences with existential quantifiers follow from the definition $\exists\, X : \boldsymbol{\varphi}(X) \equiv \neg\forall\, X : \neg\boldsymbol{\varphi}(X)$ and the translations above. The atomic sentences with the virtual sets $\mathbf{P}$ of all physical systems, $\mathbf{S} \equiv \mathbf{U}\backslash\mathbf{P}$ of all sets, and $\mathbf{V}$ of all pure sets are translated as follows:

$$\begin{aligned} (a \in \mathbf{P})^{\mathbf{D}} &\equiv A \in \mathbf{D}_1 &&\longleftrightarrow \exists\, A' : A = \langle 1, A'\rangle\ , \\ (X \in \mathbf{S})^{\mathbf{D}} &\equiv X \in \mathbf{D}_0 &&\longleftrightarrow \exists\, X' : X = \langle 0, X'\rangle\ , \\ (X \in \mathbf{V})^{\mathbf{D}} &\equiv X \in \mathbf{D}_0 &&\longleftrightarrow \exists\, X' : X = \langle 0, X'\rangle\ . \end{aligned} \tag{95}$$

We shall however in general avoid mentioning virtual sets in the formal expressions of the axioms of ZFCM. By 'proof over the length of $\boldsymbol{\varphi}$' we now conclude to have defined the set-translation:

$$(\cdot)^{\mathbf{D}} : \text{SENT}(\mathcal{L}_{\sqsubseteq}) \rightarrow \text{SENT}(\mathcal{L}_{\in})\ , \quad \boldsymbol{\varphi} \mapsto \boldsymbol{\varphi}^{\mathbf{D}}\ . \tag{96}$$

Observe we can consider $\mathbf{D}_0 \cup \mathbf{D}_1$ as the range of the following function on $\mathbf{U}$:

$$F : \mathbf{U} \rightarrow \mathbf{V}_{\text{ZFC}}\ , \quad \begin{cases} a \mapsto \langle 1, F_{\in}(A)\rangle \\ X \mapsto \langle 0, F_{\in}(X)\rangle \end{cases} \tag{97}$$

Since $F_\in$ is the purifier (17), we also have that $F[\mathbf{U}] = \mathbf{D} \subset \mathbf{V}$ when considered as a function inside $\mathbf{U}$.

2.2.2 Set-Interpretation

We know how to translate. In the Appendix we have proved a number of Lemmas in order to simplify translated expressions on the basis of ZFC. Now we can turn our attention to the

Set-Interpretation Theorem. *Standard set-theory* (ZFC) *interprets the mereological extension* ZFCM *via virtual set* $\mathbf{D} \equiv \mathbf{D}_0 \cup \mathbf{D}_1 \subset \mathbf{V}_{\text{ZFC}}$ (90)*, denoted as follows*:

$$\mathbf{D} : \mathsf{ZFC} \rhd \mathsf{ZFCM}\ . \tag{98}$$

Proof. To demonstrate that $\mathbf{D}$ gives rise to an interpretation of ZFCM in ZFC, we demonstrate that the translated axioms of ZFCM are theorems of ZFC. This suffices because the deductive apparatus of ZFC and ZFCM are identical (1st-order predicate logic).

Proof of the Axiom of Extensionality. We formulate Extensionality as follows:

$$\mathsf{Ext}^{\mathbf{D}} \ \longleftrightarrow\ \big(\forall\, X, Y\, (X \subseteq Y \ \wedge\ Y \subseteq X) \longrightarrow\ X = Y\big)^{\mathbf{D}}\ . \tag{99}$$

This is the same as:

$$\forall\, X, Y \in \mathbf{D}_0\, \big((X \subseteq Y)^{\mathbf{D}} \ \wedge\ (Y \subseteq X)^{\mathbf{D}} \ \longrightarrow\ (X = Y)^{\mathbf{D}}\big)\ .$$

Let $X, Y \in \mathbf{D}_0$. Then there are X' and Y' such that $X = \langle 0, X' \rangle$ and $Y = \langle 0, Y' \rangle$. We assume the antecedent and prove the consequent.

By virtue of Lemma (164), we derive from the antecedent that $X' \subseteq Y'$ and $Y' \subseteq X'$. Extensionality then permits us to deduce that $X' = Y'$. Then also $X = Y$. Lemma (171) says this is sufficient for $(X = Y)^{\mathbf{D}}$. □

Proof of the Power Axiom. The translation of Power:

$$\mathsf{Pow}\ \equiv\ \forall\, X,\ \exists\, P,\ \forall\, Y\, (Y \in P\ \longleftrightarrow\ Y \subseteq X)\,, \quad \textit{i.e.}\quad \forall\, X,\ \exists\, P\, (P = \wp X)\ ,$$

reads as follows (Y may contain physical systems):

$$\mathsf{Pow}^{\mathbf{D}}\ \longleftrightarrow\ \forall\, X \in \mathbf{D}_0,\ \exists\, P \in \mathbf{D}_0,\ \forall\, Y \in \mathbf{D}_0\, \big((Y \in P)^{\mathbf{D}}\ \longleftrightarrow\ (Y \subseteq X)^{\mathbf{D}}\big)\ . \tag{100}$$

This is what we have to prove.

Let $X \in \mathbf{D}_0$. Then there is some X' such that $X = \langle 0, X' \rangle$. We now have to find some P such that (100) holds.

Claim: $\quad P\ =\ \big\langle 0,\ \wp X' \big\rangle\ .$

So we claim that this P, which exists according to Power, is the P for which (100) holds. Evidently $\langle 0, \wp X' \rangle \in \mathbf{D}_0$. Let Y be an arbitrary member of $\mathbf{D}_0$. Then there is some Y' such that $Y = \langle 0, Y' \rangle$ $(*)$.

[$\longrightarrow$] We assume $(Y \in \langle 0, \wp X' \rangle)^{\mathbf{D}}$. The translation yields that $Y' \in \wp X'$, *i.e.* $Y' \subseteq X'$ ($\blacktriangle$). The translation of $(Y \subseteq X)^{\mathbf{D}}$ is by virtue of Lemma (164):

$$\exists\, Y'', X''\, (Y = \langle 0, Y'' \rangle \wedge X = \langle 0, X'' \rangle \wedge Y'' \subseteq X'') \ .$$

Clearly there is such some Y'' and some X'': Y' ($*$) and X', respectively, because then $Y' \subseteq X'$ holds ($\blacktriangle$).

[$\longleftarrow$] We assume $(Y \subseteq X)^{\mathbf{D}}$, which gives again: $Y' \subseteq X'$, *i.e.* $Y' \in \wp X'$. To prove $(Y \in \langle 0, \wp X' \rangle)^{\mathbf{D}}$, we have to find a Y'' such that:

$$Y = \langle 0, Y'' \rangle \ \wedge \ Y'' \in \wp X' \ .$$

This is met when we choose Y' for Y''. $\square$

Proof of the Union Axiom. The Union Axiom reads in $\mathcal{L}_{\sqsubseteq}$:

$$\begin{aligned} \mathsf{Un} \equiv \ \ \forall\, Z,\ \exists\, U \big[& \forall\, X \big(X \in U \ \longleftrightarrow \ \exists\, Y\, (X \in Y \wedge Y \in Z) \big) \ \wedge \\ & \forall\, a \big(a \in U \ \longleftrightarrow \ \exists\, Y\, (a \in Y \wedge Y \in Z) \big) \big] \ . \end{aligned} \tag{101}$$

We have to prove its translation $\mathsf{Un}^{\mathbf{D}}$:

$$\begin{aligned} & \forall\, Z \in \mathbf{D}_0,\ \exists\, U \in \mathbf{D}_0 \\ & \quad \big[\forall\, X \in \mathbf{D}_0 \big((X \in U)^{\mathbf{D}} \ \ \longleftrightarrow \ \exists\, Y \in \mathbf{D}_0 \, [(X \in Y)^{\mathbf{D}} \ \wedge \ (Y \in Z)^{\mathbf{D}})] \big) \\ & \quad \wedge \ \forall\, A \in \mathbf{D}_1 \big((a \in U)^{\mathbf{D}} \ \ \longleftrightarrow \ \exists\, Y \in \mathbf{D}_0 \, [(a \in Y)^{\mathbf{D}} \ \wedge (Y \in Z)^{\mathbf{D}}] \big) \big] \ . \end{aligned} \tag{102}$$

Let $Z \in \mathbf{D}_0$. Then there is some Z' such that $Z = \langle 0, Z' \rangle$. Now we have find some $U \in \mathbf{D}_0$ such that (102) holds.

Claim: $\ U \ = \ \big\langle 0,\ \bigcup Z' \big\rangle$.

The set $\bigcup Z'$ exists according to the Union Axiom. Then indeed $U \in \mathbf{D}_0$. We concentrate on the first line of (102). Let X be an arbitrary member of $\mathbf{D}_0$. Then there is some X' such that $X = \langle 0, X' \rangle$.

[$\longrightarrow$] The premise $(X \in U)^{\mathbf{D}}$ gives us that $X' \in \bigcup Z'$ ($\bullet$). We now have to find some $Y \in \mathbf{D}_0$ such that, in the light of the Claim that $U' = \bigcup Z'$:

$$\big(\langle 0, X' \rangle \in Y \big)^{\mathbf{D}} \ \wedge \ \big(Y \in \langle 0, Z' \rangle \big)^{\mathbf{D}} \ ,$$

i.e. we have to find some Y' such that $X' \in Y'$ and $Y' \in Z'$. The existence of such a Y' is guaranteed by the Union Axiom, because of ($\bullet$). The $Y \in \mathbf{D}_0$ we had to find is then of course $\langle 0, Y' \rangle$.

[$\longleftarrow$] We assume there is some $Y \in \mathbf{D}_0$ such that:

$$\big(\langle 0, X' \rangle \in Y \big)^{\mathbf{D}} \ \wedge \ \big(Y \in \langle 0, Z' \rangle \big)^{\mathbf{D}} \ ,$$

i.e. there is some Y' such that $Y = \langle 0, Y' \rangle$, $X' \in Y'$ and $Y' \in Z'$ for arbitrary X'. Then $X' \in \bigcup Z'$. We have to prove that $\big(X \in \langle 0, \bigcup Z' \rangle\big)^{\mathbf{D}}$, which is, using that $X = \langle 0, X' \rangle$, the same as $X' \in \bigcup Z'$ and this we concluded in the previous sentence.

The proof of the second conjunct of (102) is notational variant of the proof above: replace $X = \langle 0, X' \rangle$ with $A = \langle 1, A' \rangle$, the bound quantification for X over $\mathbf{D}_0$ with A over $\mathbf{D}_1$, and unbound quantification for X' with A'. □

Proof of the Axiom of Transfinity. We consider Transfinity in Zermelo's [1908] original formulation:

$$\mathsf{Transf} \;\equiv\; \exists\, Z\, \big((\varnothing \in Z) \;\wedge\; \forall\, X (X \in Z \;\longrightarrow\; \{X\} \in Z)\big)\,. \tag{103}$$

Let us call this set Z_0. We have to prove $\mathsf{Transf}^{\mathbf{D}}$, which is

$$\begin{aligned} \exists\, Z \in \mathbf{D}_0 \big[\; &\text{(i)} \quad (\varnothing \in Z)^{\mathbf{D}} \;\wedge \\ &\text{(ii)} \quad \forall\, X \in \mathbf{D}_0 \big((X \in Z)^{\mathbf{D}} \;\longrightarrow\; \exists\, S \in \mathbf{D}_0 \big((S = \{X\})^{\mathbf{D}} \wedge (S \in Z)^{\mathbf{D}}\big)\big)\,\big]\,. \end{aligned} \tag{104}$$

We have to find some $Z \in \mathbf{D}_0$ such that both these conjuncts hold for it.

Claim: $\quad Z = \langle 0,\, Z_0 \rangle\,.$

Then indeed $Z \in \mathbf{D}_0$.

(i) For the first conjunct we appeal to Lemma (178):

$$\exists\, Z' \,(Z = \langle 0, Z' \rangle \;\wedge\; \varnothing \in Z')\,.$$

According to the Claim this Z' is Z_0, and according to Transf (103) Z_0 contains $\varnothing$.

(ii) Let X be an arbitrary member of $\mathbf{D}_0$ for which the antecedent $(X \in Z)^{\mathbf{D}}$ holds. This means, by virtue of our Claim, that there is some $X' \in Z_0$ such that $X = \langle 0, X' \rangle$. According to Transf then also $\{X'\} \in Z_0$ (▲). Now we have to find some $S \in \mathbf{D}_0$ such that:

$$\big(S = \{X\}\big)^{\mathbf{D}} \;\wedge\; (S \in Z)^{\mathbf{D}}\,. \tag{105}$$

We prove that the following choice for S holds:

Claim: $\quad S \;=\; \big\langle 0,\, \{X'\}\big\rangle\,.$

The conjunct $(S \in Z)^{\mathbf{D}}$ of (105) is immediate, because we Claim this now amounts to asserting that $\{X'\} \in Z_0$ (▲). The other conjunct is also immediate in the light of Lemma (182): $(S = \{X\})^{\mathbf{D}}$ implies $S' = \{X'\}$ when $S = \langle 0, S' \rangle$. □

Proof of the Axiom of Regularity. Regularity says that every set has a set member disjoint with it, unless it is empty or contains physical systems only. (In the last-mentioned case it makes no sense to speak of intersections with members; for this reason adding physical systems to a regular set does not spoil its regularity.)

$$\mathsf{Reg} \;\equiv\; \forall\, X \left((\exists\, Y \in X) \;\longrightarrow\; \exists\, Y\, (Y \in X \;\;\wedge\;\; Y \cap X = \varnothing) \right) . \tag{106}$$

We prove the following translation $\mathsf{Reg}^{\mathbf{D}}$:

$$\begin{aligned} &\forall\, X \in \mathbf{D}_0 \left[(\exists\, Y : Y \in X)^{\mathbf{D}} \right. \\ &\qquad\qquad \left. \longrightarrow \exists\, Y \in \mathbf{D}_0 \left((Y \in X)^{\mathbf{D}} \;\wedge\; (Y \cap X = \varnothing)^{\mathbf{D}} \right) \right] . \end{aligned} \tag{107}$$

Let X be an arbitrary member of $\mathbf{D}_0$; then there is some X' such that $X = \langle 0, X' \rangle$ ($\bullet$). The Premise is the antecedent of (107). Its translation reads:

$$\exists\, Y \in \mathbf{D}_0,\; \exists\, X'', Y' \left(X = \langle 0, X'' \rangle \wedge Y = \langle 0, Y' \rangle \wedge Y' \in X'' \right) . \tag{108}$$

This X'' must coincide with X' ($\bullet$). So X' has a set member (Y'), it is not empty: $X' \neq \varnothing$ ($\star$).

To prove the consequent of (107), we have to find some $Y \in \mathbf{D}_0$ such that

$$(Y \in X)^{\mathbf{D}} \;\;\wedge\;\; (Y \cap X = \varnothing)^{\mathbf{D}} . \tag{109}$$

Since X' contains a set ($\star$), it has, according to Regularity (in ZFC), also a set-member disjoint with X'. Call this member R'. Then $R' \cap X' = \varnothing$ ($\blacktriangledown$) and $R' \in X'$.

Claim: $\quad Y \;=\; \langle 0,\; R' \rangle$.

Obvious is that $\langle 0, R' \rangle = Y \in \mathbf{D}_0$, and equally obvious is that the first conjunct of (109) is met, because it translates into $Y' \in X'$, and $Y' = R'$ according to the Claim.

The second conjunct of (109) we translate as follows:

$$\begin{aligned} (Y \cap X = \varnothing)^{\mathbf{D}} \quad &\longleftrightarrow \quad \left(\forall\, Z\, (Z \in Y \longrightarrow Z \notin X) \right)^{\mathbf{D}} , \\ &\equiv \quad \forall\, Z \in \mathbf{D}_0 \left((Z \in Y)^{\mathbf{D}} \longrightarrow \neg (Z \in X)^{\mathbf{D}} \right) . \end{aligned} \tag{110}$$

Let Z be an arbitrary member of $\mathbf{D}_0$; then there is some Z' such that $Z = \langle 0, Z' \rangle$. The indicative conditional translates as follows (we take immediately into account that $X = \langle 0, X' \rangle$, $Y = \langle 0, R' \rangle$ (Claim) and $Z = \langle 0, Z' \rangle$):

$$Z' \in R' \;\longrightarrow\; Z' \notin X' .$$

If $Z' \in R'$, then indeed $Z' \notin X'$, because $R' \cap X' = \varnothing$ ($\blacktriangledown$). $\square$

Proof of the Axiom of Choice. The Axiom of Choice focusses on (non-empty) families of disjointed sets, which in turn may contain physical systems, and ignores physical systems; Choice says that every such family admits a choice set:

$$\begin{aligned} \mathsf{C} \;\equiv\; \forall\, X \,\big(\; &\exists\, Y \in X : Y \neq \varnothing \;\wedge\; \textstyle\bigcap X = \varnothing] \longrightarrow \\ &\qquad \exists\, C\, [\mathsf{ChoiceSet}(C, X) \;\wedge\; C \neq \varnothing] \,\big) , \end{aligned} \tag{111}$$

where the choice set of X is defined in $\mathcal{L}_{\sqsubseteq}$ as follows:

$$\begin{aligned}\mathsf{ChoiceSet}(C,X) \;\equiv\;& \big(C \subseteq \bigcup X\big) \wedge \\ & \forall\, V \in X\big((\exists! W \in V : W \in C) \sqcup (\exists! b \in V : b \in C)\big)\;,\end{aligned} \tag{112}$$

where $\sqcup$ stands for exclusive disjunction (either-or). We have to prove that the following translation holds:

$$\begin{aligned}\mathsf{C}^{\mathbf{D}} \;\equiv\; & \forall\, X \in \mathbf{D}_0 \Big(\big[\exists\, Y \in \mathbf{D}_0 \big((Y \in X)^{\mathbf{D}} \wedge \neg(\varnothing \in X)^{\mathbf{D}}\big) \wedge \big(\bigcap X = \varnothing\big)^{\mathbf{D}}\big] \\ & \longrightarrow\; \exists\, C \in \mathbf{D}_0 \left[\mathsf{ChoiceSet}^{\mathbf{D}}(C,X) \;\wedge\; \neg(C = \varnothing)^{\mathbf{D}}\right]\Big)\;,\end{aligned} \tag{113}$$

Let X be an arbitrary member of $\mathbf{D}_0$; then there is some X' such that $X = \langle 0, X'\rangle$. The Premise is the antecedent of (113). Let Y be the member of $\mathbf{D}_0$, so that $Y = \langle 0, Y'\rangle$ for some Y', for which the first two conjuncts of the antecedent hold. Then $Y' \in X'$ and, by virtue of Lemma (178), we have:

$$\varnothing \notin X' \neq \varnothing\;. \tag{114}$$

The third conjunct of the Premise is more elaborate; we translate as follows:

$$\begin{aligned}\big(\bigcap X = \varnothing\big)^{\mathbf{D}} \;\longleftrightarrow\;& \left[\forall\, Z, Y \left((Z \in X \wedge Y \in X) \longrightarrow \neg \exists\, U\,(U \in Y \;\wedge\; U \in Z)\right)\right]^{\mathbf{D}}, \\ \longleftrightarrow\;& \forall\, Z, Y \in \mathbf{D}_0 \left[\big((Z \in X)^{\mathbf{D}} \wedge (Y \in X)^{\mathbf{D}}\big) \;\longrightarrow \right. \\ & \qquad \left. \forall U \in \mathbf{D}_0 \left(\neg(U \in Z)^{\mathbf{D}} \vee \neg(U \in Y)^{\mathbf{D}}\right)\right]\;.\end{aligned} \tag{115}$$

To see what relevant information we can extract from this Premise, let Z and Y be arbitrary members of $\mathbf{D}_0$; then there is some Z' and some Y' such that $Z = \langle 0, Z'\rangle$ and $Y = \langle 0, Y'\rangle$. The translations of the antecedent of (115) will give us in the by now familiar fashion that $Z' \in X'$ and $Y' \in X'$. If this is the case, the translation of the consequent will give us that for every $U \in \mathbf{D}_0$, there is a U' such that $U = \langle 0, U'\rangle$ and $U' \notin Z'$ or $U' \notin Y'$. By virtue of Lemma (159), this disjunction then holds universally. In other words, all set members of X' are disjoint, *i.e.* X' is a disjointed family of sets. Earlier we ascertained that $\varnothing \notin X' \neq \varnothing$ (114). Then by virtue of the Axiom of Choice, X' admits a non-empty choice set, $C' \subseteq \bigcup X'$ say.

We now have to prove the existence of some $C \in \mathbf{D}_0$ such that the consequent of (113) holds. As might be expected by now, we

Claim: $\;C \;=\; \langle 0,\, C'\rangle\;.$

Then $C \in \mathbf{D}_0$. When we apply Lemma (175) to the Claim, we quickly conclude that the second conjunct of the consequent of (113) is met:

$$C \in \mathbf{D}_0 \wedge C \neq \langle 0, \varnothing\rangle \;\longrightarrow\; \neg(C = \varnothing)^{\mathbf{D}}\;.$$

The translation of the first conjunct of the antecedent of (113) breaks again in two conjuncts, due to (112):

$$\begin{aligned}\mathsf{ChoiceSet}^{\mathbf{D}}(C,X) \;\longleftrightarrow\;& \big(C \subseteq \bigcup X\big)^{\mathbf{D}} \;\wedge \\ & \left[\forall\, V \in X\big((\exists! W \in V : W \in C) \sqcup (\exists! b \in V : b \in C)\big)\right]^{\mathbf{D}}\;.\end{aligned} \tag{116}$$

We consider the first conjunct of (116):

$$\begin{aligned}
\left(C \subseteq \bigcup X\right)^{\mathbf{D}} \quad &\longleftrightarrow \quad \left(\forall Y : Y \in C \longrightarrow Y \in \bigcup X\right)^{\mathbf{D}} , \\
&\equiv \quad \forall Y \in \mathbf{D}_0 \left((Y \in C)^{\mathbf{D}} \longrightarrow \left(\exists Z (Y \in Z \wedge Z \in X)\right)^{\mathbf{D}}\right) , \\
&\longleftrightarrow \quad \forall Y \in \mathbf{D}_0 \left((Y \in C)^{\mathbf{D}} \longrightarrow \exists Z \in \mathbf{D}_0 : (Y \in Z)^{\mathbf{D}} \wedge (Z \in X)^{\mathbf{D}}\right) .
\end{aligned} \tag{117}$$

Let $Y \in \mathbf{D}_0$; then there is some Y' such that $Y = \langle 0, Y' \rangle$. On the basis of the antecedent of (117), which evidently yields that $Y' \in C'$, we have to prove the consequent: we have to find some $Z \in \mathbf{D}_0$ such that both conjuncts of this consequent hold. We first translate them (taking into account what we know about Y and X):

$$\exists Z' \left(Z = \langle 0, Z' \rangle \wedge Y' \in Z' \wedge Z' \in X'\right) .$$

The last two conjuncts tell us that $Y' \in \bigcup X'$. This is indeed the case due to $Y' \in C'$ and $C' \subseteq \bigcup X'$, which conjunctively guarantee the existence of Z'. The Z we have to find is then simply: $Z = \langle 0, Z' \rangle$.

We finally consider the second conjunct of (116); its translation reads:

$$\begin{aligned}
\forall V \in \mathbf{D}_0 \big[\ & (V \in X)^{\mathbf{D}} \longrightarrow \\
\text{(i)} \ & \exists W \in \mathbf{D}_0,\ \forall Y \in \mathbf{D}_0 \big((W \in V)^{\mathbf{D}} \wedge (W \in C)^{\mathbf{D}} \wedge \\
& \big((Y \in C)^{\mathbf{D}} \wedge (Y \in V)^{\mathbf{D}}\big) \longrightarrow (Y = W)^{\mathbf{D}}\big) \ \sqcup \\
\text{(ii)} \ & \exists B \in \mathbf{D}_1,\ \forall D \in \mathbf{D}_1 \big((b \in V)^{\mathbf{D}} \wedge (b \in C)^{\mathbf{D}} \wedge \\
& \big((d \in C)^{\mathbf{D}} \wedge (d \in V)^{\mathbf{D}}\big) \longrightarrow (d = b)^{\mathbf{D}}\big) \big] .
\end{aligned} \tag{118}$$

The second lines in the exclusive disjuncts express the uniqueness of B (the translate of b) and W. Let V be an arbitrary member of $\mathbf{D}_0$; then there is some V' such that $V = \langle 0, V' \rangle$. The antecedent of the overall conditional then comes down to $V' \in X'$. We have to find either some $W \in \mathbf{D}_0$ such that (i) holds or some $B \in \mathbf{D}_1$ such that (ii) holds.

Let us now suppose, for the moment, that either there is such a $W \in \mathbf{D}_0$ or such a $B \in \mathbf{D}_1$ in order to see what we have to prove. Choosing Y arbitrarily from $\mathbf{D}_0$ for (i) and D from $\mathbf{D}_1$ for (ii), and applying Lemma (171), what we have to prove then is this:

$$\begin{aligned}
&\big[W' \in V' \ \wedge \ W' \in C' \ \wedge \ \big((Y' \in C' \ \wedge \ Y' \in V') \ \longrightarrow \ Y = W\big)\big] \ \sqcup \\
&\big[B' \in V' \ \wedge \ V \in C' \ \wedge \ \big((D' \in C' \ \wedge \ C' \in V') \ \longrightarrow \ C' = W\big)\big] .
\end{aligned} \tag{119}$$

The existence of either such a W' or B' is guaranteed by our Claim for every set $V' \in X'$, because C' is a choice set of X'. The desired $B \in \mathbf{D}_1$ or $W \in \mathbf{D}_0$ we have to find are then simply $W = \langle 0, W' \rangle$ or $B = \langle 1, B \rangle$. □

Proof of the Axiom of Replacement. In $\mathcal{L}_{\sqsubseteq}$, Replacement ($\mathsf{F}$) says that for every set D (of Domain) and any functional predicate $\mathsf{Func}(\cdot,\cdot)$ on D, there exists a set R which contains exactly those elements Y for which $\mathsf{Func}(X,Y)$ holds for all $X \in D$:

$$\begin{aligned} &\forall P,\ \forall c,\ \forall D \Big[\forall Y, Z \big(\forall X [\, X \in D \wedge \mathsf{Func}(X,Y) \wedge \mathsf{Func}(X,Z) \,] \longrightarrow\ Y = Z \big) \\ &\longrightarrow \exists R,\ \forall Y \big(Y \in R \ \longleftrightarrow\ \exists X [\, X \in D \wedge \mathsf{Func}(X,Y)] \,\big) \Big] , \end{aligned} \tag{120}$$

where it is left implicit in $\mathsf{Func}(\cdot,\cdot)$ that $P \equiv P_1, \ldots, P_n$ are n set-parameters occurring free in $\mathsf{Func}(X,Y)$ other than X and Y, and that similarly $c \equiv c_1, \ldots, c_m$ are m physical system-parameters. The first line expresses that dyadic predicate $\mathsf{Func}(\cdot,\cdot)$ is *functional*; the second line declares that all elements Y related to all elements $X \in D$ (Domain) by $\mathsf{Func}(X,Y)$ can be collected in a set R (of Range). This is Replacement when the variables in $\mathsf{Func}(X,Y)$ are set-variables. In $\mathcal{L}_{\sqsubseteq}$ this must be enlarged with an axiom schema wherein the variables are physical system-variables:

$$\begin{aligned} &\forall P,\ \forall c,\ \forall D \Big[\forall b, c \big(\forall a [\, a \in D \wedge \mathsf{Func}(a,b) \wedge \mathsf{Func}(a,c) \,] \longrightarrow\ b = c \big) \\ &\longrightarrow\ \exists R,\ \forall b \big(b \in R \ \longleftrightarrow\ \exists a [\, a \in D \wedge \mathsf{Func}(a,b)] \,\big) \Big] . \end{aligned} \tag{121}$$

Then there are still two other variants, for functional predicates that relate sets to physical systems, *viz.* $\mathsf{Func}(X,b)$, or physical systems to sets, *viz.* $\mathsf{Func}(a,Y)$, and finally combinations of the mentioned four; we do not spell all of them out. We prove one of these four types of axiom schemata, namely the one for sets (120), because the proofs of the other schemata will be notational variants of it (replace 0 with 1, $\mathbf{D}_0$ with $\mathbf{D}_1$, X' with A', *etc.*).

We translate (120). For arbitrary $P, D \in \mathbf{D}_0$ and $p \in \mathbf{D}_1$ we have to prove its translation $\mathsf{F}^{\mathbf{D}}$:

$$\begin{aligned} &\forall Y, Z \in \mathbf{D}_0 \big(\forall X [\, (X \in D)^{\mathbf{D}} \wedge \mathsf{Func}^{\mathbf{D}}(X,Y) \wedge \mathsf{Func}^{\mathbf{D}}(X,Z) \,] \longrightarrow\ (Y = Z)^{\mathbf{D}} \big) \longrightarrow \\ &\exists R \in \mathbf{D}_0,\ \forall Y \in \mathbf{D}_0 \big((Y \in R)^{\mathbf{D}} \ \longleftrightarrow\ \exists X \in \mathbf{D}_0 [\, (X \in D)^{\mathbf{D}} \wedge \mathsf{Func}^{\mathbf{D}}(X,Y)] \big) , \end{aligned} \tag{122}$$

Assuming the antecedent of (122) means to assume that $\mathsf{Func}^{\mathbf{D}}(\cdot,\cdot)$ is functional on domain D', where $D = \langle 0, D' \rangle$. To see this, observe that

$$\big[X' \in D' \wedge \mathsf{Func}^{\mathbf{D}}\big(\langle 0, X' \rangle,\ \langle 0, Y' \rangle \big) \wedge \mathsf{Func}^{\mathbf{D}}\big(\langle 0, X' \rangle,\ \langle 0, Z' \rangle \big) \big] \ \longrightarrow\ \big(\langle 0, Y' \rangle = \langle 0, Z' \rangle \big)^{\mathbf{D}}$$

implies the functional character of $\mathsf{Func}^{\mathbf{D}}$ by Lemma (177):

$$\big(X' \in D' \wedge \mathsf{Func}^{\mathbf{D}}(X', Y') \wedge \mathsf{Func}^{\mathbf{D}}(X', Z') \big) \ \longrightarrow\ Y' = Z' \ .$$

Then according to Replacement (120), there is some set, call it R', for which the consequent of (120) holds. The set $R \in \mathbf{D}_0$ for which the consequent of (122) holds then is, we

Claim: $\quad R \ =\ \langle 0,\ R' \rangle$.

Let Y be an arbitrary member of $\mathbf{D}_0$; then there is some Y' such that $Y = \langle 0, Y' \rangle$. We now have to prove that (after performing a few obvious translations):

$$\begin{aligned} Y' \in R' \ \longleftrightarrow\ \ &\exists X', D'' \big(X = \langle 0, X' \rangle \wedge D = \langle 0, D'' \rangle \wedge X' \in D' \wedge \\ &\mathsf{Func}^{\mathbf{D}}(\langle 0, X' \rangle,\ \langle 0, Y' \rangle) \big) . \end{aligned} \tag{123}$$

Evidently $D'' = D'$, the variables of $\mathsf{Func}^{\mathbf{D}}$ are X' and Y', and whether we assume or derive the existence of X' (for the proof of (123) in both directions), in both cases X is equal to $\langle 0, X' \rangle$. So what we still have to prove is:

$$Y' \in R' \;\longleftrightarrow\; \exists\, X'\big(X' \in D' \wedge \mathsf{Func}^{\mathbf{D}}(X', Y')\big) \;.$$

By virtue of the Claim and the Premise that $\mathsf{Func}^{\mathbf{D}}(\cdot,\cdot)$ is functional, this is exactly what Replacement in ZFC allows us to deduce. □

Proof of the Plenitude Axiom. The Plenitude Axiom Plen states the equinumerosity of **P** and **V**. We have to prove the following translation:

$$\mathsf{Plen}^{\mathbf{D}} \;\equiv\; (\mathbf{P} \sim \mathbf{V})^{\mathbf{D}} \;. \qquad (124)$$

Since the translates of **P** and **V** are $\mathbf{D}_1$ and $\mathbf{D}_0$, respectively, and $\mathbf{D}_1$ and $\mathbf{D}_0$ are equinumerous (and equinumerous to $\mathbf{V}_{\text{ZFC}}$ for that matter), we are home by virtue of (a 'bold-faced' version of) Lemma (192). □

Proof of the Subsystem-Set Axiom. The axiom which postulates the existence of a set of exactly the subsystems of every physical system we translate next:

$$\mathsf{Subs}^{\mathbf{D}} \;\equiv\; \big(\forall\, a,\; \exists\, S,\; \forall\, b\, (b \sqsubseteq a \;\longleftrightarrow\; b \in S)\big)^{\mathbf{D}} \;. \qquad (125)$$

So what we have to prove is this:

$$\forall\, A \in \mathbf{D}_1,\; \exists\, S \in \mathbf{D}_0,\; \forall\, B \in \mathbf{D}_1 \,\big((b \sqsubseteq a)^{\mathbf{D}} \;\longleftrightarrow\; (b \in S)^{\mathbf{D}}\big) \;. \qquad (126)$$

Let $A \in \mathbf{D}_1$; then there is some A' such that $A = \langle 1, A' \rangle$. We have to find some $S \in \mathbf{D}_0$ such that for every $B \in \mathbf{D}_1$ the following holds:

$$\begin{aligned} &\exists\, A', B' \,\big(A = \langle 1, A' \rangle \wedge B = \langle 1, B' \rangle \wedge B' \subseteq A'\big) \;\longleftrightarrow \\ &\exists\, B'', S' \,\big(B = \langle 1, B'' \rangle \wedge S = \langle 1, S' \rangle \wedge B'' \in S'\big) \;. \end{aligned} \qquad (127)$$

The $S \in \mathbf{D}_0$ we have to find is the following one, we

Claim: $\quad S \;=\; \langle 0, \wp A' \rangle$.

This set $S \in \mathbf{D}_0$ exists according to the Power Axiom. Let $B \in \mathbf{D}_1$; then there is a B' such that $B = \langle 1, B' \rangle$. When we combine this with what we know about A, then what is the case according to our Claim is this:

$$(B' \subseteq A') \longleftrightarrow (B' \in \wp A') \;, \qquad (128)$$

which holds by definition of the power-set. □

Proof of the Pre-Ordering Axiom. The axiom stating that the subsystem-relation is reflexive and transitive on $\mathbf{P}$ consists of two independent assertions we translate subsequently. First we prove the translation of the reflexivity of $\sqsubseteq$:

$$\mathsf{Refl}^{\mathbf{D}} \equiv \big(\forall\, a : a \sqsubseteq a\big)^{\mathbf{D}} . \tag{129}$$

Straightforward substitution of the translation for $\sqsubseteq$ yields:

$$\forall\, A \in \mathbf{D}_1,\ \exists\, A' \,(A = \langle 1, A'\rangle \;\wedge\; A' \subseteq A') .$$

For every $A \in \mathbf{D}_1$ there is by definition always some A' such that $A = \langle 1, A'\rangle$; and further for every set X it holds that $X \subseteq X$ so certainly A'.

The translation of the transitivity of $\sqsubseteq$ proceeds similarly. We then are home because of:

$$\mathsf{PreOrd}^{\mathbf{D}} \;\longleftrightarrow\; \mathsf{Refl}^{\mathbf{D}} \wedge \mathsf{Trans}^{\mathbf{D}} . \tag{130}$$

Just as $\sqsubseteq$ is a reflexive ordering on $\mathbf{P}$, so is $\subseteq$ on $\mathbf{V}_{\text{ZFC}}$. □

Proof of the Overlap Axiom. The translation of the Overlap Axiom:

$$\mathsf{Olap}^{\mathbf{D}} \;\longleftrightarrow\; \Big[\forall\, a, b\, \big(\forall\, c (c \sqsubseteq\!\!\!\cdot\; a \longrightarrow c \circ b) \longrightarrow a \sqsubseteq b\big)\Big]^{\mathbf{D}} , \tag{131}$$

is the proposition we have to prove:

$$\forall\, A, B \in \mathbf{D}_1 \Big[\forall\, C \in \mathbf{D}_1 \big((c \sqsubseteq\!\!\!\cdot\; a)^{\mathbf{D}} \longrightarrow (c \circ b)^{\mathbf{D}}\big) \longrightarrow (a \sqsubseteq b)^{\mathbf{D}}\Big] . \tag{132}$$

Let $A, B \in \mathbf{D}_1$; then there is some A' and some B' in $\mathbf{D}_0$ such that $A = \langle 1, A'\rangle$ and $B = \langle 1, B\rangle$. We assume the antecedent of (132), which is our Premise. The translation of the Premise reads, using Lemmas (203) and (204), and further taking into account what we know about C and A:

$$\forall\, C' \,(C' \neq \varnothing \;\wedge\; C' \subseteq A' \;\longrightarrow\; C' \cap B' \neq \varnothing) . \tag{133}$$

When we expand the consequent of (132) and take into account what we know about A' and B', we have to show that (133) entails that $A' \subseteq B'$. We prove this by a *reductio ad absurdum* argument.

The Premise now is (133) and we assume that $A' \not\subseteq B'$ (RA). Then there is a set, Y say, which is a member of A' but not in B':

$$Y \in A' \;\wedge\; Y \notin B' .$$

Then $\{Y\} \subseteq A'$ and $\{Y\} \cap B' = \varnothing$. But $\{Y\} \neq \varnothing$ and $\{Y\} \subseteq A'$ together imply $\{Y\} \cap B' \neq \varnothing$ according to our Premise (133), which yields the desired contradiction. □

Proof of the Composition Axiom. We need to prove the following proposition:

$$\mathsf{Comp}^{\mathbf{D}} \;\equiv\; \big[\forall\, P \subset \mathbf{P},\ \exists\, a\,\big(\forall\, b \in P : b \sqsubseteq a\ \wedge \\ \forall\, c\,(c \sqsubseteq a \;\longrightarrow\; \exists\, b \in P : c \circ b)\,\big)\big]^{\mathbf{D}}\ . \tag{134}$$

Expanding a bit yields:

$$\forall\, P\,\Big[\exists\, P' : P = \langle 0, P' \rangle \longrightarrow \exists\, A \in \mathbf{D}_1 \Big(\forall\, B \in \mathbf{D}_1\,\big((b \in P)^{\mathbf{D}} \longrightarrow (b \sqsubseteq a)^{\mathbf{D}}\big)\ \wedge \\ \forall\, C \in \mathbf{D}_1\,\big((c \sqsubseteq a)^{\mathbf{D}} \longrightarrow \exists\, B \in \mathbf{D}_1[(b \in P)^{\mathbf{D}} \wedge (c \circ b)^{\mathbf{D}}]\big)\Big)\Big]\ . \tag{135}$$

Let P' be a set and let set P be such that $P = \langle 0, P' \rangle$. We have to find some $A \in \mathbf{D}_1$ such that the two conjuncts of (135) hold.

$$\text{Claim:}\quad A \;=\; \big\langle 1,\ \bigcup P' \big\rangle\ . \tag{136}$$

Then indeed $A \in \mathbf{D}_1$, and $A' = \bigcup P'$.

The translation the first conjunct (135), taking into account what we know about P, a and B, gives:

$$\forall\, B' \in P' : B' \subseteq A'\ . \tag{137}$$

This evidently holds when $A' = \bigcup P'$.

Next the second conjunct of (135). Let C be an arbitrary member of $\mathbf{D}_1$; then there is some C' such that $C = \langle 1, C' \rangle$. With the aid of Lemmas (203) and (204) we obtain the following sentence to prove:

$$(C' \neq \varnothing\ \wedge\ C' \subseteq A') \longrightarrow (\exists\, B' \in P' :\ B' \cap C' \neq \varnothing)\ . \tag{138}$$

We assume the antecedent. Since C' is by assumption a non-empty subset of $A' = \bigcup P'$, it has a member, $Y \in C'$ say, which is also in $\bigcup P'$. Then P' has a subset, Z say, which has Y as a member, *i.e.* $Y \in Z$, otherwise Y cannot be a member of $\bigcup P'$. Now Z and $\bigcup P'$ have at least one member in common: Y. Hence $\bigcup P' \cap Z \neq \varnothing$. This is exactly what we have to prove (the B' that qualifies is Y). □

Proof of the Axiom of Atomism. We formulate the Axiom of Atomism (54), which asserts that every system is composed of elementary particles, as follows:

$$\mathsf{Atoms}^{\mathbf{D}} \equiv \big[\forall\, a,\ \exists\, c,\ \exists\, P \subset \mathbf{P}\,\big(c = \sqcup P\ \wedge\ \forall\, b\,(b \in P\ \longleftrightarrow\ b \sqsubseteq a\ \wedge\ \mathsf{ElemPart}(b))\big]^{\mathbf{D}}\ . \tag{139}$$

Expanding a bit, we obtain:

$$\forall A \in \mathbf{D}_1,\ \exists C \in \mathbf{D}_1,\ \exists P \in \mathbf{D}_0 \Big(\exists P' : P = \langle 0, P' \rangle \longrightarrow \big[(c = \sqcup P)^{\mathbf{D}} \wedge \forall B \in \mathbf{D}_1 \big((b \in P)^{\mathbf{D}} \longleftrightarrow (b \sqsubseteq a)^{\mathbf{D}} \wedge \mathsf{ElemPart}(b)^{\mathbf{D}} \big) \big] \Big) . \tag{140}$$

Choose some $A \in \mathbf{D}_1$. Then there is some A' such that $A = \langle 1, A' \rangle$. We now have to find some $C \in \mathbf{D}_1$ and some $P \in \mathbf{D}_0$ such that if $P = \langle 0, P' \rangle$ for some P' — which always holds when $P \in \mathbf{D}_0$ —, then the two conjuncts in the consequent between the square brackets in (140) hold.

Claim: $C = \langle 1, \bigcup P' \rangle$ and $P' = \big\{ \{S\} \mid S \in A' \big\}$.

The elementary particles correspond to singleton-sets: the reason why the translated Axiom of Atomism holds is that every set X can be written as the union-set of the set of singleton-sets of the members of X. The Claim that the choice above for C gives us immediately that $(c = \sqcup P)^{\mathbf{D}}$, because we have demonstrated this in the previous proof of the translated Composition Axiom, *viz.* Claim (136). For the second conjunct, we have th show that for every B' such that $B = \langle 1, B' \rangle$ (expanding further):

$$B' \in P' \longleftrightarrow B' \subseteq A' \wedge \exists S : B' = \{S\} ,$$

where we have used Lemma (205). This holds because $\bigcup P' = A'$ and because in general $\{X\} \subseteq Y$ iff $X \in Y$. $\Box$

We have now proved the translations of all the axioms of ZFCM.

Q.e.d.

The first fruit we pick from the Set-Interpretation Theorem is the following

Consistency Corollary. *Theory* ZFCM *is consistent relative to theory* ZFC*, and for every* $\mathfrak{p} \in \mathbf{C}$*, theory* $\mathsf{ZFCM}_{\mathfrak{p}}$ *is also consistent relative to* ZFC*.*

As a consequence of the Model Existence Lemma, ZFCM then has a model in ZFC. It is not difficult to see that the following is a model of ZFCM:

$$\big\langle\, \mathbf{D}_0 \cup \mathbf{D}_1,\ \in',\ \subseteq' \big\rangle \models \mathsf{ZFCM} , \tag{141}$$

where $\in'$ and $\subseteq'$ are the dyadic relations as defined by the translations of $\in$ and $\subseteq$, to which we shall return presently. Figure 2 (p. 53) depicts our results schematically.

2.3 Conservation by Purification

The next Theorem is instrumental in proving the Conservation Theorem for ZFCM: it shows that the domain of interpretation $\mathbf{D}$ harbours an isomorphic copy of the domain of discourse of ZFC.

Isomorphism Theorem (ZFC). *The virtual set $\mathbf{D}_0$, equipped with the translated membership-relation $\in'$, is isomorphic to the domain of discourse of* ZFC*, the virtual set $\mathbf{V}_{\text{ZFC}}$ of all sets equipped with the membership-relation; symbolically:*

$$\langle\, \mathbf{D}_0,\ \in' \,\rangle \;\cong\; \langle\, \mathbf{V}_{\text{ZFC}},\ \in \,\rangle\ . \tag{142}$$

Proof. We demonstrate that the following function is an isomorphism:

$$F : \mathbf{V}_{\text{ZFC}} \to \mathbf{D}_0, \quad X \mapsto \langle 0, F_\in(X)\rangle\ , \tag{143}$$

where we recall that $\mathbf{D}_0$ is the virtual set of all ordered pairs $\langle 0, Y\rangle$ for all $Y \in \mathbf{V}_{\text{ZFC}}$ and $F_\in$ is the purifier. Evidently F is bijective iff $F_\in : \mathbf{V}_{\text{ZFC}} \to \mathbf{V}_{\text{ZFC}}$ is bijective. The last-mentioned is the content of Theorem (18).

The relation $\in'$ is defined as in the translation of the atomic sentence of $\mathcal{L}_\sqsubseteq$ for membership (92):

$$X \in' Y \;\equiv\; (X \in Y)^{\mathbf{D}} \;\equiv\; \exists\, X', Y'\, (X = \langle 0, X'\rangle \wedge Y = \langle 0, Y'\rangle \wedge X' \in Y')\ . \tag{144}$$

Next we prove the isomorphic character of F (143):

$$\forall\, X, Y \in \mathbf{V}_{\text{ZFC}} \big(X \in Y \;\longleftrightarrow\; F(X) \in' F(Y) \big)\ . \tag{145}$$

[$\longrightarrow$] Let $X, Y \in \mathbf{V}_{\text{ZFC}}$ and $X \in Y$ (Premise). To prove that $F(X) \in' F(Y)$ we have to find some X' and Y' such that

$$F(X) = \langle 0, X'\rangle \;\wedge\; F(Y) = \langle 0, Y'\rangle \;\wedge\; X' \in Y'\ . \tag{146}$$

By definition of F we see that $X' = F_\in(X)$ and $Y' = F_\in(Y)$, which are legitimate identifications because by the Premise X and Y exist (are members of $\mathbf{V}_{\text{ZFC}}$) and $\mathbf{V}_{\text{ZFC}}$ is the domain of $F_\in$ in ZFC. The remaining question is then whether $F_\in(X) \in F_\in(Y)$. By the definition of $F_\in$ (17), $F_\in(Y)$ has as members the sets $F_\in(B)$ for all $B \in Y$. By Premise, X qualifies for such a B, so set $F_\in(X)$ is also a member of $F_\in(Y)$.

[$\longleftarrow$] Let $X, Y \in \mathbf{V}_{\text{ZFC}}$ and $F_\in(X) \in' F_\in(Y)$ (Premise). Then there are some X' and Y' such that (146) holds. Then it follows that $F_\in(X) \in F_\in(Y)$ ($\bullet$). We now have to prove that $X \in Y$. By definition $F_\in(Y)$ is the set of images $F_\in(B)$ for all members B of Y. Among these members must be X, otherwise we could not possibly have that $F_\in(X) \in F_\in(Y)$ ($\bullet$). $\square$

Conservation Theorem. *Theory* ZFCM *is a conservative extension of* ZFC*, which is to say that all theorems about the pure sets in $\mathbf{V} \subset \mathbf{U}$ are already theorems in* ZFC:

$$\forall\, \varphi \in \text{SENT}(\mathcal{L}_\in) : \text{ if } \ \mathsf{ZFCM} \vdash \varphi^{\mathbf{V}}\ , \quad \text{then } \ \mathsf{ZFC} \vdash \varphi\ . \tag{147}$$

Proof. According to Theorem (98) ZFC interprets ZFCM via $\mathbf{D} \subset \mathbf{V}_{\text{ZFC}}$; for any sentence $\psi \in \text{SENT}(\mathcal{L}_\sqsubseteq)$:

$$\text{if} \quad \mathsf{ZFCM} \vdash \psi\ , \quad \text{then} \quad \mathsf{ZFC} \vdash \psi^{\mathbf{D}}\ . \tag{148}$$

By instantiation we have this for any $\boldsymbol{\varphi} \in \textsc{sent}(\mathcal{L}_\in) \subset \textsc{sent}(\mathcal{L}_\sqsubseteq)$ relativised to all pure sets ($\mathbf{V}$), *i.e.* for all $\boldsymbol{\varphi}^{\mathbf{V}}$:

$$\text{if} \quad \mathsf{ZFCM} \vdash \boldsymbol{\varphi}^{\mathbf{V}} \,, \quad \text{then} \quad \mathsf{ZFC} \vdash \big(\boldsymbol{\varphi}^{\mathbf{V}}\big)^{\mathbf{D}} \,. \tag{149}$$

What we have to prove, schema (147), follows from schema (149) when we can prove the following schema:

$$\mathsf{ZFC} \vdash \boldsymbol{\varphi} \longleftrightarrow \big(\boldsymbol{\varphi}^{\mathbf{V}}\big)^{\mathbf{D}} \,, \tag{150}$$

which is what we do right now.

Sentence $(\boldsymbol{\varphi}^{\mathbf{V}})^{\mathbf{D}}$ means that before we translate $\boldsymbol{\varphi}$ via $\mathbf{D}$ we first purify it: we replace X with $F_\in(X)$, which is pure according to Theorem (19); then unbounded universal quantification for sets becomes bounded quantification over the range of $F_\in$, which is the virtual set $\mathbf{V}$ of all pure sets in ZFCM; virtual set $\mathbf{V}$ becomes the domain of discourse in ZFC, where all sets are pure (16). We spell out the translation $\boldsymbol{\varphi} \mapsto (\boldsymbol{\varphi}^{\mathbf{V}})^{\mathbf{D}} \in \textsc{sent}(\mathcal{L}_\in)$ in terms of F (143):

$$\begin{aligned}
\bot^{\mathbf{D}} \quad &\equiv \quad \bot\,; \\
\big((X \in Y)^{\mathbf{V}}\big)^{\mathbf{D}} \quad &\equiv \quad F(X) \in' F(Y)\,; \\
\big((\boldsymbol{\varphi} \longrightarrow \boldsymbol{\psi})^{\mathbf{V}}\big)^{\mathbf{D}} \quad &\equiv \quad \big(\boldsymbol{\varphi}^{\mathbf{V}} \longrightarrow \boldsymbol{\psi}^{\mathbf{V}}\big)^{\mathbf{D}}\,; \\
\big((\forall\, X : \boldsymbol{\varphi}(X))^{\mathbf{V}}\big)^{\mathbf{D}} \quad &\equiv \quad \forall\, X : \boldsymbol{\varphi}^{\mathbf{D}}\big(F(X)\big)\,.
\end{aligned} \tag{151}$$

A few remarks on these translations are in order. With regard to the atomic sentence $\ulcorner X \in Y \urcorner$, the definition is quickly seen to be logically identical to how we earlier defined the translation via $\mathbf{D}$ (it better be!):

$$F(X) \in' F(Y) \;\longleftrightarrow\; \exists\, X', Y' \,\big(X = \langle 0, F_\in(X')\rangle \wedge Y = \langle 0, F_\in(Y')\rangle \wedge F_\in(X') \in F_\in(Y')\big)\,.$$

In the present definition (151) the connexion with the Isomorphism Theorem (142) is however manifest. The same remark applies to the definition of the universal quantified statement: relativising to $\mathbf{D}_0$, the virtual set of all ordered pairs $\langle 0, Y\rangle$, is the same as quantifying over all sets but sending them first via F into $\mathbf{D}_0$. We prove theorem (150) 'over the length of $\boldsymbol{\varphi}$'.

(i) *Atomic sentences.* We have to prove:

$$\big((X \in Y)^{\mathbf{V}}\big)^{\mathbf{D}} \;\longleftrightarrow\; X \in Y\,. \tag{152}$$

We observe that the Isomorphism Theorem (142) gives us precisely this, *viz.* theorem (145):

$$X \in Y \;\longleftrightarrow\; F(X) \in' F(Y)\,. \tag{153}$$

(ii) *Connectives.* The fact that translations by definition commute with the connectives means that this part of our proof 'over the length of $\boldsymbol{\varphi}$' is taken care of:

$$\big(\boldsymbol{\varphi}^{\mathbf{V}} \longrightarrow \boldsymbol{\psi}^{\mathbf{V}}\big)^{\mathbf{D}} \;\longleftrightarrow\; \big[\big(\boldsymbol{\varphi}^{\mathbf{V}}\big)^{\mathbf{D}} \longrightarrow \big(\boldsymbol{\psi}^{\mathbf{V}}\big)^{\mathbf{D}}\big]\,.$$

(iii) *Universal quantification.* We have to show the following:

$$\forall X : \varphi^{\mathbf{D}}\big(F(X)\big) \longleftrightarrow \forall Y : \varphi(Y) . \tag{154}$$

We have to demonstrate this follows from the induction-hypothesis of 'proof over the length of $\varphi(X)$':

$$\varphi^{\mathbf{D}}\big(F(X)\big) \longleftrightarrow \varphi(X) . \tag{155}$$

[$\longleftarrow$] The Premise reads: $\forall Y : \varphi(Y)$. What we have to deduce reads:

$$\forall X : \varphi^{\mathbf{D}}\big(F(X)\big) . \tag{156}$$

Let X be a set. The Premise gives us $\varphi(X)$ when we substitute X for Y. The induction-hypothesis (155) then gives us $\varphi^{\mathbf{D}}\big(F(X)\big)$. The fact that F is surjective (even bijective) from $\mathbf{V}$ to $\mathbf{D}_0$, *i.e.* $F : \mathbf{V} \rightarrowtail \mathbf{D}_0$, $X \mapsto \langle 0, X\rangle$, means that $F(X)$ assumes every value in $\mathbf{D}_0$ when X runs through $\mathbf{D}_0$. This establishes it holds for *all* X, which is (156).

[$\longrightarrow$] The Premise is now theorem (156). The reasoning of the previous paragraph applies again, due to the surjectivity (even bijectivity) of $F : \mathbf{V} \rightarrowtail\!\!\!\!\twoheadrightarrow \mathbf{D}_0$ and the induction-hypothesis (155). □

As an immediate corollary we have an

Equiconsistency Corollary. *Theories* ZFC *and* ZFCM *are equiconsistent.*

The most important meta-mathematical questions about ZFCM have now been answered. By way of an *encore* we draw some easy consequences from the conservativeness of ZFCM over ZFC.

Suppose someone wants to have primordial elements without any axioms bestowing structure upon them; only the axioms of ZFC are adjusted so that primordial elements can be members of sets. Consider a theory that has as many primordial elements as there are pure sets. The language of this theory is like $\mathcal{L}_{\sqsubseteq}$ but without the subsystem (part-whole) relation, without the sentences that include the atomic sentences $\ulcorner a \sqsubseteq b \urcorner$, but with the sentences that include the atomic sentences $\ulcorner a \in X \urcorner$, and without all additional axioms of ZFCM save the Axiom of Plenitude (Plen).

Corollary. *The theory* ZFC ∧ Plen *is a conservative extension of, and is equiconsistent with* ZFC*.*

Proof. Let $\psi \in \textsc{sent}(\mathcal{L}_{\in})$ and let its relativisation $\psi^{\mathbf{V}}$ to all pure sets be a theorem of ZFC ∧ Plen. We then have to prove that ψ is a theorem of ZFC.

Since ZFCM deductively extends ZFC ∧ Plen, we also have that $\psi^{\mathbf{V}} \in$ ZFCM. By virtue of the Conservation Theorem (147), we are then allowed to infer what had to be proved: ZFC $\vdash \psi$. Then ZFC and ZFC ∧ Plen are equiconsistent. □

Almost identical is the proof of the following corollary, because ZFCM extends $\mathsf{ZFCM}_{\mathfrak{p}}$ deductively for all $\mathfrak{p} \in \mathbf{C}$ as a result of Theorem (25).

Corollary. *For every* $\mathfrak{p} \in \mathbf{C}$*, the theory* $\mathsf{ZFCM}_{\mathfrak{p}}$ *is a conservative extension of, and is equiconsistent with* ZFC.

When we want to add to $\mathbf{V}_{\text{ZFC}}$ only a *set* of primordial elements of at least cardinality $\mathfrak{p} \in \mathbf{C}$, as the Parsimony Axiom $\mathsf{Pars}(\mathfrak{p})$ says, the next corollary follows from the previous one because $\mathsf{ZFCM}_{\mathfrak{p}}$ extends $\mathsf{ZFC} \wedge \mathsf{Pars}(\mathfrak{p})$.

Corollary. *For every* $\mathfrak{p} \in \mathbf{C}$, *the theory* $\mathsf{ZFC} \wedge \mathsf{Pars}(\mathfrak{p})$ *is a conservative extension of, and is equiconsistent with,* ZFC.

These Corollaries encompass all known consistency proofs of set theory plus primordial elements and extend them to proofs of the conservativeness of these extensions. Finally, from the equiconsistency of Bunt's theory BET and ZF, the equiconsistency of ZF and ZFC, and the equiconsistency of ZFCM and ZFC, we obtain for free another

Corollary. *Bunt's Ensemble Theory* (BET) *is equiconsistent to our mereological theories* ZFCM *and* $\mathsf{ZFCM}_{\mathfrak{p}}$.

3 Conclusion

We round up this paper by verifying the requirements on our list (Subsection 0.3).

Requirement (1) states the reflexivity and the transitivity of the subsystem-relation and the irreflexivity, a-symmetry and transitivity of the proper subsystem-relation. This is the content of Theorem (p. 16). As regards the issue of the connectivity of $\sqsubseteq$ and $\sqsubset$: if $\sqsubseteq$ were connective, then so would its isomorphic counterpart $\sqsubseteq'$ in the model (141), but it is not. Hence neither $\sqsubseteq$ nor $\sqsubset$ are connective, as required (1). Requirement (2) about the identity and identical physical systems sharing their subsystems are the Identity Theorem and Theorem (67), respectively. Requirements (3) for the concept of composition, its 'conservation'-character and the fact that fewer physical systems never lead to a more encompassing composite physical system, we proved in the Supremum Theorem (61) and Corollary (63). Requirements (4) concerning the consistency and conservativeness we established for all mereological theories we considered, in the Consistency Corollary (p. 47), the Conservation Theorem (147) and Corollary on page~pagerefCorolThree. All our requirements are satisfied.

For future research in mereology we recommend: first, find more interesting theorems and prove them; and secondly, attempt to built a mereological theory whose notion of composition is not associative.

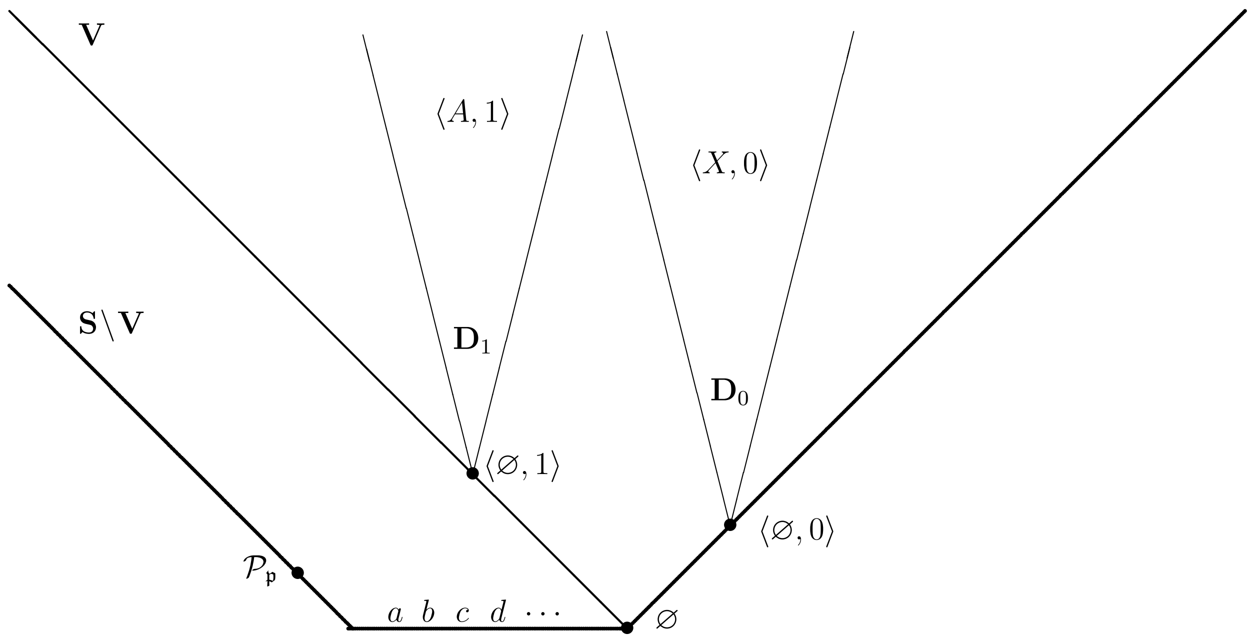

Figure 1: **Schematic picture of our meta-mathematical proofs concerning mereology.** Depicted is the entire domain of discourse $\mathbf{U}_\mathfrak{p}$ of $\mathsf{ZFCM}_\mathfrak{p}$, which is the cumulative hierarchy which begins at ordinal rank 0, where we find the memberless elements: $\varnothing$ and all physical systems (a, b, c, *etc.*), of which there are at least $\mathfrak{p}$. The set $\mathcal{P}_\mathfrak{p}$ of them is formed at ordinal rank 1. The whole of all sets that arises cumulatively is $\mathbf{S}$. The wedge in the middle is the cumulative hierarchy of all pure sets ($\mathbf{V}$). We prove it is identical to the domain of discourse $\mathbf{V}_{\text{ZFC}}$ of ZFC. Within $\mathbf{V}$ we delineate a whole of all those pure sets labelled by 0 ($\mathbf{D}_0$) and labelled by 1 ($\mathbf{D}_1$). These are also cumulative hierarchies, inside $\mathbf{V}$, and they begin with $\langle\varnothing, 0\rangle$ and $\langle\varnothing, 1\rangle$. The subsystem-relation $\sqsubseteq$ of $\mathsf{ZFCM}_\mathfrak{p}$ is translated as the subset-relation ($\subseteq$) in $\mathbf{D}_1$, which contains all the physical system-translates of $\mathcal{P}_\mathfrak{p}\backslash\mathbf{S}$, and the membership-relation in $\mathbf{S}$ is left the same but $\mathbf{D}_0$ contains all the set-translates of $\mathbf{S}$. The heart of our Conservation Theorem consists in proving that $\mathbf{D}_0$ is an isomorphic copy of $\mathbf{V}$. The equiconsistency of ZFC and $\mathsf{ZFCM}_\mathfrak{p}$ we then get for free. In ZFCM such a picture is not possible, because there is no set of all memberless elements (there are as many physical systems as pure sets); the virtual sets $\mathbf{V}$, $\mathbf{D}_0$ and $\mathbf{D}_1$ are of course always present in $\mathbf{U}$ as virtual subsets.

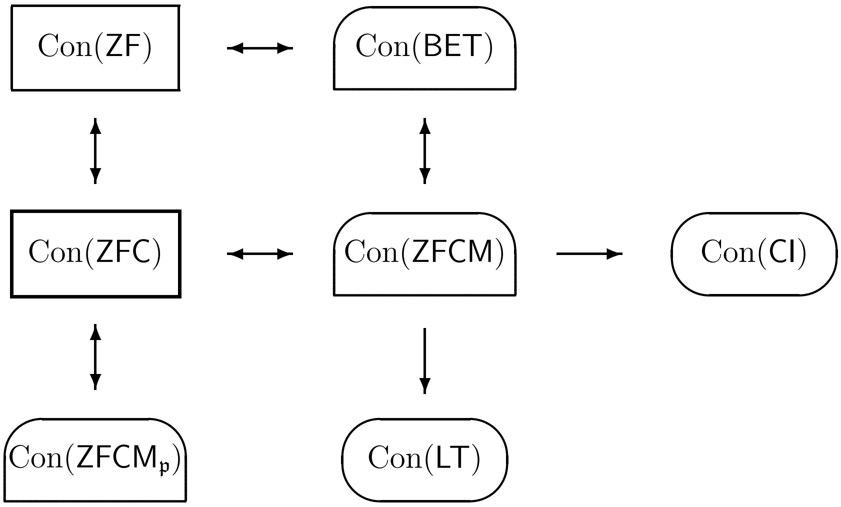

Figure 2: **Consistency and equiconsistency relations between various mereological extensions of ZF and ZFC.** Theories within a round box are mereological theories who only need a few modest axioms to manipulate sets of primordial elements; theories in a rectangle are pure set theories; and theories in a box with only rectangular lower corners are explicitly intended to encompass fully fledged set-theory. The arrows which point from and to ZFCM and ZFCM$_\mathfrak{p}$ are established in this paper.

A Appendix. Lemmas

This Appendix contains the proofs of the auxiliary lemmas and theorems we have invoked but have not yet proved.

First we prove the adequacy of our definition of identity for physical systems as mutual subsystemhood.

Theorem. *The identity-relation between physical systems defined as mutual subsystemhood is* (a) *an equivalence-relation that* (b) *obeys Substitutivity.*

Proof. (a) The reflexivity and transitivity of $=$ follows immediately from the assumed reflexivity and transitivity of $\sqsubseteq$ (PreOrd). The symmetry of $=$ is a consequence of the fact that making conjunctions is a commutative operation on the set of sentences of $\mathcal{L}_{\sqsubseteq}$.

(b) Substitutivity for $=$ in $\mathcal{L}_{\sqsubseteq}$ means that any sentence $\boldsymbol{\varphi}$ with n variables a_j ($j = 1, 2, \ldots, n$) implies the same sentence with every occurrence of a_j replaced with b_j if for all j we have: $a_j{=}b_j$:

$$\big(\boldsymbol{\varphi}(a_1, a_2, \ldots, a_n) \;\wedge\; \textstyle\bigwedge_j (a_j = b_j)\big) \;\longrightarrow\; \boldsymbol{\varphi}(b_1, b_2, \ldots, b_n)\ . \tag{157}$$

Set-parameters can occur, but they are of no consequence and therefore we ignore them. We establish (157) 'over the length of $\boldsymbol{\psi}$'. We begin with the two types of atomic sentences wherein physical system-variables occur.

(i) For the first type of atomic sentence, $\ulcorner a \sqsubseteq b \urcorner$, we have to prove:

$$(a_1 \sqsubseteq a_2 \;\wedge\; a_1 = b_1 \;\wedge\; a_2 = b_2) \;\longrightarrow\; b_1 \sqsubseteq b_2\ .$$

Assume the antecedent. Substitute def. (28) for the two identity-statements and the transitivity of $\sqsubseteq$ (PreOrd) yields the consequent.

(ii) For the second type of atomic sentence, $\ulcorner a \in X \urcorner$, we have to prove:

$$(a \in X \wedge a = b) \;\longrightarrow\; b \in X\ . \tag{158}$$

Assume the antecedent. With the Axiom of Replacement we make the singleton sets $\{a\}$ and $\{b\}$ from set $\{\varnothing\}$. The following unions exist: $A \equiv X \cup \{a\}$ and $B \equiv X \cup \{b\}$. Since $\{a\} \subseteq X$, we have that $A = X$. According to Extensionality, $\{a\} = \{b\}$ because by assumption $a{=}b$. But then also $B = X$ $(*)$. From the definition of B it follows that $\{b\} \subseteq B$, which in combination with eq. $(*)$ yields that $\{b\} \subseteq X$, *i.e.* $b \in X$.

The absurdity $\perp$ need not be considered because it contains no variables. The proofs for the connectives and the quantifiers are elementary, 'over the length of $\boldsymbol{\varphi}$'. $\square$

Next we prove the Lemmas we used in the meta-mathematical proofs of Section 2. As certain proof-patterns will quickly be observed to recur, we shift to higher and higher proof-gear as we continue. When proofs are notational variants of other proofs, we do not make a pit-stop to write them down. Buckle up!

Lemma. $\forall Z : \boldsymbol{\varphi}(Z) \;\longleftrightarrow\; \forall X \in \mathbf{D}_j,\ \exists X' \big(\boldsymbol{\varphi}(X') \;\wedge\; X = \langle j, X' \rangle\big),\ \ j \in 2\ .$ (159)

Proof. [$\longrightarrow$] Let $\boldsymbol{\varphi}(Z)$ hold for all sets. Let $X \in \mathbf{D}_j$. Then by definition there is some X' such that $X = \langle j, X' \rangle$. When we substitute X' for Z in $\boldsymbol{\varphi}(Z)$, we obtain the entire consequent.

[$\longleftarrow$] When X runs over $\mathbf{D}_j$ and $\boldsymbol{\varphi}(X')$ holds, then obviously X', in $\langle j, X' \rangle = X$, runs over all sets (over $\mathbf{V}_{\text{ZFC}}$). So $\forall\, Z : \boldsymbol{\varphi}(Z)$ holds. □

Along very similar lines one proves this for any number of free variables, such as two:

$$\textbf{Lemma.}\quad \forall\, V, W : \boldsymbol{\psi}(V, W) \;\longleftrightarrow \\ \forall\, X, Y \in \mathbf{D}_j,\ \exists\, X', Y' \big(\boldsymbol{\psi}(X', Y') \;\wedge\; X = \langle j, X' \rangle \wedge\; Y = \langle j, Y' \rangle\big)\ , \\ \text{where}\;\; j \in \{0, 1\}\ . \tag{160}$$

The version with existential quantifiers follows from the one with universal quantifiers:

$$\textbf{Lemma.}\quad \exists\, V, W : \boldsymbol{\psi}(V, W) \;\longleftrightarrow \\ \exists\, X, Y \in \mathbf{D}_j,\ \exists\, X', Y' \big(\boldsymbol{\psi}(X', Y') \;\wedge\; X = \langle j, X' \rangle \wedge\; Y = \langle j, Y' \rangle\big)\ . \tag{161}$$

We turn to identity for physical systems:

$$\textbf{Lemma.}\quad (a = b)^{\mathbf{D}} \;\longleftrightarrow\; \exists\, A', B' \,(A = \langle 1, A' \rangle \wedge A = \langle 1, B \rangle \wedge A' = B')\ , \\ \longleftrightarrow\; A, B \in \mathbf{D}_1 \;\wedge\; A = B\ . \tag{162}$$

Proof. First we translate step by step:

$$(a = b)^{\mathbf{D}} \;\equiv\; (a \sqsubseteq b \;\wedge\; b \sqsubseteq a)^{\mathbf{D}}\ , \\ \equiv\; \exists\, A',\, B \,\big(A = \langle 1, A' \rangle \wedge B = \langle 1, B \rangle \wedge A' \subseteq B\big) \;\wedge \\ \exists\, A'', B'' \,\big(A = \langle 1, A'' \rangle \wedge B = \langle 1, B'' \rangle \wedge B'' \subseteq A''\big)\ . \tag{163}$$

[$\longrightarrow$] When we start with the last displayed sentence in (163), we see immediately that it implies: $A' = A''$ and $B' = B''$. The fact that $B'' \subseteq A''$ implies $B' \subseteq A'$, which in turn implies, together with $A' \subseteq B'$, that $A' = B'$ (Axiom of Extensionality). Hence

$$\exists\, A', B' \,\big(A = \langle 1, A' \rangle \wedge B = \langle 1, B' \rangle \wedge A' = B'\big)\ .$$

Further it is obvious that $A, B \in \mathbf{D}_1$ and $A = B$.

[$\longleftarrow$] When we go in the opposite direction, we start by extensionality with $A, B \in \mathbf{D}_1$ en $A = B$. Then by definition of $\mathbf{D}_1$:

$$\exists\, A', B' \,\big(A = \langle 1, A' \rangle \wedge B = \langle 1, B' \rangle\big)\ .$$

Then $A' = B'$ due to $A = B$, and by Extensionality:

$$\exists\, A', B' \,\big(A = \langle 1, A' \rangle \wedge B = \langle 1, B' \rangle \wedge A' \subseteq B' \wedge B' \subseteq A'\big)\ .$$

By expanding further and introducing A'' and B'' by existential quantification (over $\mathbf{V}_{\text{ZFC}}$), we arrive via the last sentence of (163) back at $(a = b)^{\mathbf{D}}$. □

Next comes the subset-relation.

Lemma. $$X, Y \in \mathbf{D}_0 \wedge (X \subseteq Y)^{\mathbf{D}} \longleftrightarrow \exists\, X', Y'\big(X = \langle 0, X'\rangle \wedge Y = \langle 0, Y'\rangle \wedge X' \subseteq Y'\big)\,. \tag{164}$$

Proof. First we translate step by step the subset-sentence by substituting its definition in $\mathcal{L}_{\sqsubseteq}$, which now involves physical system-variables as well:

$$\begin{aligned}(X \subseteq Y)^{\mathbf{D}} \;&\equiv\; \big(\forall\, Z\,(Z \in X \longrightarrow Z \in Y) \;\wedge\; \forall\, a\,(a \in X \longrightarrow a \in Y)\big)^{\mathbf{D}}\,,\\ &\equiv\; \forall\, Z \in \mathbf{D}\big((Z \in X)^{\mathbf{D}} \longrightarrow (Z \in Y)^{\mathbf{D}}\big) \;\wedge\\ &\qquad \forall\, A \in \mathbf{D}\big((a \in X)^{\mathbf{D}} \longrightarrow (a \in Y)^{\mathbf{D}}\big)\,,\\ &\equiv\; \forall\, Z \in \mathbf{D}_0\,\big(\exists\, Z', X'\,(Z = \langle 0, Z'\rangle \wedge X = \langle 0, X'\rangle \wedge Z' \in X') \longrightarrow\\ &\qquad\quad \exists\, Z'', Y'\,(Z = \langle 0, Z''\rangle \wedge Y = \langle 0, Y'\rangle \wedge Z'' \in Y')\big) \;\wedge\\ &\qquad \forall\, A \in \mathbf{D}_1\,\big(\exists\, A',\; \exists\, X'\,(A = \langle 1, A'\rangle \wedge X = \langle 0, X'\rangle \wedge A' \in X') \longrightarrow\\ &\qquad\quad \exists\, A'',\; \exists\, Y'\,(A = \langle 1, A''\rangle \wedge Y = \langle 0, Y'\rangle \wedge A'' \in Y')\big)\,.\end{aligned} \tag{165}$$

[$\longrightarrow$] Let $X, Y \in \mathbf{D}_0$ and let $(X \subseteq Y)^{\mathbf{D}}$. We concentrate on the first conjunct of the last expression above. Membership of X and Y in $\mathbf{D}_0$ (first Premise) means that there is some X' and some Y' such that $X = \langle 0, X'\rangle$ and $Y = \langle 0, Y'\rangle$. Then $\ulcorner X = \langle 0, X'\rangle \urcorner$ can be omitted from the antecedent of the conditional because it need no longer be assumed there. Further, $\ulcorner Y = \langle 0, Y'\rangle \urcorner$ can be omitted from the consequent because, in general, $\boldsymbol{\psi} \longrightarrow \boldsymbol{\sigma} \wedge \boldsymbol{\varphi}$ trivially entails $\boldsymbol{\psi} \longrightarrow \boldsymbol{\sigma}$. The required Z'' in the consequent exists in the light of the antecedent: it equals Z'. Hence we have:

$$\forall\, Z \in \mathbf{D}_0,\; \exists\, Z'\,\big((Z = \langle 0, Z'\rangle) \;\wedge\; (Z' \in X' \;\longrightarrow\; Z' \in Y')\big)\,.$$

Application of (159) yields:

$$\forall\, Z\,(Z \in X' \longrightarrow Z \in Y')\,.$$

This is the *definiens* of $\ulcorner X' \subseteq Y' \urcorner$ in $\mathcal{L}_{\in}$.

[$\longleftarrow$] From the Premise, *i.e.*

$$\exists\, X', Y'\,(X = \langle 0, X'\rangle \wedge Y = \langle 0, Y'\rangle \wedge X' \subseteq Y')\,, \tag{166}$$

it is clear that X and Y are members of $\mathbf{D}_0$, which actually is the first thing we have to prove. We must prove that $(X \subseteq Y)^{\mathbf{D}}$ holds. Due to the theorem of logic

$$\vdash\; (\boldsymbol{\psi} \wedge \boldsymbol{\varphi}) \;\longrightarrow\; (\boldsymbol{\psi} \longrightarrow \boldsymbol{\varphi})\,, \tag{167}$$

we derive from the Premise (166) that:

$$\exists\, X', Y'\big((X = \langle 0, X'\rangle \longrightarrow Y = \langle 0, Y'\rangle) \;\wedge\; \forall\, Z'(Z' \in X' \longrightarrow Z' \in Y')\big)\,,$$

where the definition of $\ulcorner X' \subseteq Y' \urcorner$ has been substituted too. The universal quantification of Z' can be pulled to the left because Z' does not occur in the first conjunct. Application of the following theorem of logic:

$$\vdash \ \big((\boldsymbol{\varphi} \longrightarrow \boldsymbol{\sigma}) \wedge (\boldsymbol{\chi} \longrightarrow \boldsymbol{\psi})\big) \ \longrightarrow \ \big((\boldsymbol{\varphi} \wedge \boldsymbol{\chi}) \longrightarrow (\boldsymbol{\sigma} \wedge \boldsymbol{\psi})\big) \ , \tag{168}$$

then yields that

$$\exists\, X', Y',\ \forall\, Z' \,\big((X = \langle 0, X' \rangle \wedge Z' \in X') \ \longrightarrow \ (Y = \langle 0, Y' \rangle \wedge Z' \in Y')\big) \ .$$

In this case, the quantifiers can swop places:

$$\forall\, Z',\ \exists\, X', Y' \,\big((X = \langle 0, X' \rangle \wedge Z' \in X') \ \longrightarrow \ (Y = \langle 0, Y' \rangle \wedge Z' \in Y')\big) \ .$$

Application of (159) and noticing that Y' does not occur in the antecedent so that we can push quantification for Y' to the right, yields:

$$\begin{aligned} \forall\, Z \in \mathbf{D}_0,\ \exists\, Z' \big(Z = \langle 0, Z' \rangle \ \wedge \ & [\exists\, X' \,(X = \langle 0, X' \rangle \wedge Z' \in X') \ \longrightarrow \\ & \exists\, Y' \,(Y = \langle 0, Y' \rangle \wedge \ Z' \in Y')]\big) \ . \end{aligned}$$

We can weaken the antecedent of the conditional by adding the first conjunct $\ulcorner Z = \langle 0, Z' \rangle \urcorner$ to it; we can further add $\ulcorner \exists\, Z'' : Z = \langle 0, Z'' \rangle \urcorner$ to the consequent of the conditional and replace Z' with Z'' everywhere in the consequent (because there always is such a Z'', namely Z'), and obtain:

$$\begin{aligned} \forall\, Z \in \mathbf{D}_0 \big(\exists\, Z', X' \,(Z = \langle 0, Z' \rangle \wedge X = \langle 0, X' \rangle \wedge Z' \in X') \ \longrightarrow & \\ \exists\, Y', Z''(Z = \langle 0, Z'' \rangle \wedge Y = \langle 0, Y' \rangle \wedge Z'' \in Y')\big) \ . & \end{aligned} \tag{169}$$

From (165) we see that (169) is the first conjunct of what we have to prove. We next concentrate on the second conjunct of (165) and prove it directly. Let $A \in \mathbf{D}_1$; then there is some A' such that $A = \langle 1, A' \rangle$. What remains to be shown is, taking into account that necessarily $A' = A''$:

$$A' \in X' \ \longrightarrow \ A' \in Y' \ . \tag{170}$$

This follows from $X' \subseteq Y'$, which is part of the Premise (166). □

Now we are in a position to address the identity of sets:

$$\begin{aligned} \textbf{Lemma.} \quad (X = Y)^{\mathbf{D}} \quad & \longleftrightarrow \quad X, Y \in \mathbf{D}_0 \ \wedge \ X = Y \ , \\ & \longleftrightarrow \quad \exists\, X', Y' \,(X = \langle 0, X' \rangle \wedge Y = \langle 0, Y' \rangle \wedge X' = Y'). \end{aligned} \tag{171}$$

Proof. First we translate the definition of identity in $\mathcal{L}_{\sqsubseteq}$ and take immediately into account the commutativity of the translation with respect to the sentential connectives:

$$\begin{aligned} (X = Y)^{\mathbf{D}} \ \equiv \ & \forall\, V \in \mathbf{D}_0 \big((V \in X)^{\mathbf{D}} \longrightarrow (V \in Y)^{\mathbf{D}}\big) \ \wedge \\ & \forall\, A \in \mathbf{D}_1 \big((a \in X)^{\mathbf{D}} \longrightarrow (a \in Y)^{\mathbf{D}}\big) \ \wedge \\ & \forall\, W \in \mathbf{D}_0 \big((X \in W)^{\mathbf{D}} \longrightarrow (Y \in W)^{\mathbf{D}}\big) \ . \end{aligned} \tag{172}$$

Observe that in contrast to the definition of identity in $\mathcal{L}_\in$, in $\mathcal{L}_\sqsubseteq$ we need to include the possibility that the sets contain physical systems (second line above). We work out the translation (172):

$$\begin{aligned}
&\forall V \in \mathbf{D}_0 \big(\exists V', X' \,(V = \langle 0, V'\rangle \wedge X = \langle 0, X'\rangle \wedge V' \in X') \\
&\qquad\qquad \longrightarrow\ \exists V'', Y' \,(V = \langle 0, V''\rangle \wedge Y = \langle 0, Y'\rangle \wedge V'' \in Y')\big)\ \wedge \\
&\forall A \in \mathbf{D}_1 \big(\exists A', X'' \big(A = \langle 1, A'\rangle \wedge X = \langle 0, X''\rangle \wedge A' \in X''\big) \\
&\qquad\qquad \longrightarrow\ \exists A'', Y'' \big(A = \langle 1, A''\rangle \wedge Y = \langle 0, Y''\rangle \wedge A'' \in Y''\big)\big)\ \wedge \\
&\forall W \in \mathbf{D}_0 \big[\exists W', X''' \big(W = \langle 0, W'\rangle \wedge X = \langle 0, X'''\rangle \wedge W' \in X'''\big) \\
&\qquad\qquad \longrightarrow\ \exists W'', Y''' \big(W = \langle 0, W''\rangle \wedge Y = \langle 0, Y'''\rangle \wedge W'' \in Y'''\big)\big]\,.
\end{aligned} \tag{173}$$

[$\longrightarrow$] Evident from (173) is that X and Y are members of $\mathbf{D}_0$. We now have to derive from (173) that $X = Y$. Let V and W be arbitrary members of $\mathbf{D}_0$ and let V be an arbitrary member of $\mathbf{D}_1$. Then there is some V', some W' and some V' such that $V = \langle 0, V'\rangle$, $W = \langle 0, W'\rangle$ and $V = \langle 1, V'\rangle$. Let X', Y', *etc.*, be such that the translation above holds. It is evident that: $X' = X'' = X'''$, $Y' = Y'' = Y'''$, $V' = V''$, $W' = W''$ and $A' = A''$. The relevant conjuncts can thus be deleted from the antecedents and consequents above. We are left with this:

$$(V' \in X' \ \longrightarrow\ V' \in Y') \wedge (A' \in X' \ \longrightarrow\ A' \in Y') \wedge (X' \in W' \ \longrightarrow\ Y' \in W')\,. \tag{174}$$

By virtue of (159), we have on our hands the *definiens* of the identity of X' and Y' in $\mathcal{L}_\in$. Then $X = Y$ because $X = \langle 0, X'\rangle$, $Y = \langle 0, Y'\rangle$ and $X' = Y'$. (Notice that the second conjunct of (174) is not needed for this conclusion.) The second $\rightarrow$ of (171), 2nd line, is immediate.

[$\longleftarrow$] We now start with $X = Y$ and $X, Y \in \mathbf{D}_0$. We do not spell out the proof but sketch it: first one substitutes the definition of set-identity of $\mathcal{L}_\sqsubseteq$ in $\ulcorner X = Y \urcorner$; then one appeals again to Lemma (159) to obtain a sentence that is easily transformed, by re-positioning the quantifiers, into the *definiens* of $(X = Y)^{\mathbf{D}}$ (173). The second $\leftarrow$ of (171) is then immediate. □

Next three Lemmas concerning the empty set.

Lemma. $X \in \mathbf{D}_0 \wedge (X = \varnothing)^{\mathbf{D}} \ \longrightarrow\ X = \langle 0, \varnothing\rangle\,.$ (175)

Proof. Let $X \in \mathbf{D}_0$. Then there is some X' such that $X = \langle 0, X'\rangle$ $(*)$. We rewrite the premise $\ulcorner X = \varnothing \urcorner$ as 'X is a set and has no members':

$$(X = \varnothing)^{\mathbf{D}} \ \longleftrightarrow\ \big((\forall Y : Y \notin X)\ \wedge\ (\forall a : a \notin X)\big)^{\mathbf{D}}\,;$$

now we translate this expression:

$$\begin{aligned}
&\forall Y \in \mathbf{D}_0 : \neg(Y \in X)^{\mathbf{D}}\ \wedge \forall A : \neg(a \in X)^{\mathbf{D}}\ \equiv \\
&\quad \forall Y \in \mathbf{D}_0 \big(\neg\exists Y', X' \,(Y = \langle 0, Y'\rangle \wedge X = \langle 0, X'\rangle \wedge Y' \in X')\big) \wedge \\
&\quad \forall A \in \mathbf{D}_1 \big(\neg\big(\exists A',\ \exists X' \,(A = \langle 1, A'\rangle \wedge X = \langle 0, X'\rangle \wedge A' \in X')\big)\,.
\end{aligned}$$

For every $Y \in \mathbf{D}_0$ and for every $A \in \mathbf{D}_1$, there is some Y' and some A' such that $Y = \langle 0, Y' \rangle$ and $A = \langle 1, A' \rangle$. Then in the light of $(*)$ we must have that for all Y': $Y' \notin X'$ (because Y' runs over $\mathbf{V}_{\text{ZFC}}$ when Y runs over $\mathbf{D}_0$), and for all A': $A' \notin X'$ (because A' runs over $\mathbf{V}_{\text{ZFC}}$ when a runs over $\mathbf{P}$). Hence X' has no members, *i.e.* $X' = \varnothing$. Then by virtue of $(*)$: $X = \langle 0, \varnothing \rangle$. □

Lemma. $X = \varnothing \;\longleftrightarrow\; \big(\langle 0, X \rangle = \langle 0, \varnothing \rangle\big)^{\mathbf{D}}$. (176)

Proof. [⟶] Let X be a set identical to $\varnothing$ (Premise). Since $\varnothing \notin \mathbf{D}_0$, also $X \notin \mathbf{D}_0$, which blocks an appeal to Lemma (171). Sets $\langle 0, X \rangle$ and $\langle 0, \varnothing \rangle$ exist because X and $\varnothing$ exist. They are members of $\mathbf{D}_0$ and they are identical because of the Premise. We therefore can appeal to Lemma (171) for them.

[⟵] Since obviously both $\langle 0, X \rangle$ and $\langle 0, \varnothing \rangle$ are members of $\mathbf{D}_0$, Lemma (171) can be invoked to yield: $\langle 0, X \rangle = \langle 0, \varnothing \rangle$. Then $X = \varnothing$. □

Nothing in the previous proof depends on $\varnothing$; it might have been any set. So we also have:

Lemma. $X = Y \;\longleftrightarrow\; \big(\langle 0, X \rangle = \langle 0, Y \rangle\big)^{\mathbf{D}}$. (177)

Lemma. $Z \in \mathbf{D}_0 \wedge (\varnothing \in Z)^{\mathbf{D}} \;\longleftrightarrow\; \exists\, Z' \,(Z = \langle 0, Z' \rangle \wedge \varnothing \in Z')$. (178)

Proof. We translate as follows:

$$\begin{aligned} (\varnothing \in Z)^{\mathbf{D}} \quad &\longleftrightarrow \quad (\exists\, X : X = \varnothing \;\wedge\; X \in Z)^{\mathbf{D}} \,, \\ &\equiv \quad \exists\, X \in \mathbf{D}_0 : (X = \varnothing)^{\mathbf{D}} \;\wedge\; (X \in Z)^{\mathbf{D}} \,. \end{aligned} \tag{179}$$

[⟶] Let $Z \in \mathbf{D}_0$ (first Premise). Then there is some Z' such that $Z = \langle 0, Z' \rangle$, which is the first conjunct of what we have to prove (178). Assume sentence (179) (second Premise). Let X be an arbitrary member of $\mathbf{D}_0$; then there is some X' such that $X = \langle 0, X' \rangle$. According to Lemma (175), $X = \langle 0, \varnothing \rangle$, which means that $X' = \varnothing$. The translation of $(X \in Z)^{\mathbf{D}}$ reads:

$$\exists\, X'', Z'' \big(X = \langle 0, X'' \rangle \wedge Z = \langle 0, Z'' \rangle \wedge X'' \in Z''\big) \,.$$

Then $X'' = X' = \varnothing$ and $Z'' = Z'$. The third conjunct gives us that $\varnothing \in Z'$, which is the second conjunct of what we have to prove (178).

[⟵] We commence with:

$$\exists\, Z' \,(Z = \langle 0, Z' \rangle \wedge \varnothing \in Z') \,. \tag{180}$$

Then $Z \in \mathbf{D}_0$. According to (179) we have to find some $X \in \mathbf{D}_0$ such that:

$$(X = \varnothing)^{\mathbf{D}} \;\wedge\; (X \in Z)^{\mathbf{D}} \,. \tag{181}$$

Whatever this X is, the first conjunct implies according to Lemma (175) that $X = \langle 0, \varnothing \rangle$, so we do not have much choice. To prove the second conjunct of (181), we have to find some Z'' such that:

$$Z = \langle 0, Z'' \rangle \;\wedge\; \varnothing \in Z'' \,.$$

Z' (180) is such a Z''. □

After the empty set, we consider a set with a single member.

$$\textbf{Lemma.} \quad X, Y \in \mathbf{D}_0 \wedge \big(X = \{Y\}\big)^{\mathbf{D}} \;\longleftrightarrow \;\exists\, X', Y' \big(X = \langle 0, X'\rangle \wedge Y = \langle 0, Y'\rangle \wedge X' = \{Y'\}\big) \,. \tag{182}$$

Proof. First we translate:

$$\begin{aligned} \big(X = \{Y\}\big)^{\mathbf{D}} \quad &\longleftrightarrow \quad (\forall\, Z : Z \in X \;\longrightarrow\; Z = Y)^{\mathbf{D}} \,, \\ &\equiv \quad \forall\, Z \in \mathbf{D}_0 \big((Z \in X)^{\mathbf{D}} \longrightarrow (Z = Y)^{\mathbf{D}}\big) \,, \\ &\equiv \quad \forall\, Z \in \mathbf{D}_0 \big(\exists\, X', Z' \,(X = \langle 0, X'\rangle \wedge Z = \langle 0, Z'\rangle \wedge Z' \in X') \\ &\qquad \longrightarrow\; \exists\, Y', Z'' \,(Y = \langle 0, Y'\rangle \wedge Z = \langle 0, Z''\rangle \wedge Z'' = Y')\big) \,. \end{aligned} \tag{183}$$

[$\longrightarrow$] We assume the antecedent. $X, Y \in \mathbf{D}_0$ gives us some X' and Y' such that $X = \langle 0, X'\rangle$ and $Y = \langle 0, Y'\rangle$, which takes care of the first two conjuncts of what we have to prove (182). So these conjuncts and the accompanying existential quantifiers can be omitted from the last sentence of (183), which is the translation of $\ulcorner X = \{Y\}\urcorner$. Further, the existence of some Z' such that $Z = \langle 0, Z'\rangle$ follows from $Z \in \mathbf{D}_0$ and $Z' = Z''$. So we are left with:

$$\forall\, Z \in \mathbf{D}_0,\ \exists\, Z' \big(Z = \langle 0, Z'\rangle \wedge (Z' \in X' \longrightarrow Z' = Y')\big) \,.$$

By Lemma (159), this is equivalent to:

$$\forall\, V \,(V \in X' \longrightarrow V = Y') \,,$$

which is to say: $X' = \{Y'\}$.

[$\longleftarrow$] We assume there are X' and Y' such that $X = \langle 0, X'\rangle$ and $Y = \langle 0, Y'\rangle$, from which it follows that $X, Y \in \mathbf{D}_0$. Assuming further that $X' = \{Y'\}$, one easily works one's way back, via Lemma (159) and via introducing the appropriate conjuncts of type $\ulcorner Z = \langle 0, Z'\rangle\urcorner$ and accompanying existential quantifiers for Z', to arrive at $(X = \{Y\})^{\mathbf{D}}$. □

By making a few notational changes in the previous proof (replace Y with A, $\langle 0, Y'\rangle$ with $\langle 1, A'\rangle$, *etc.*) we have the same Lemma for singleton sets that contain a physical system rather than a set:

$$\textbf{Lemma.} \quad \big[X \in \mathbf{D}_0 \wedge A \in \mathbf{D}_1 \wedge \big(X = \{a\}\big)^{\mathbf{D}}\big] \;\longleftrightarrow\; \exists\, X', A' \big(X = \langle 0, X'\rangle \wedge A = \langle 1, A'\rangle \wedge X' = \{A'\}\big) \,. \tag{184}$$

Next we consider a singleton set that contains the empty set:

$$\textbf{Lemma.} \quad X \in \mathbf{D}_0 \wedge \big(X = \{\varnothing\}\big)^{\mathbf{D}} \;\longleftrightarrow\; X = \langle 0, \{\varnothing\}\rangle \,. \tag{185}$$

Proof. When we rewrite slightly what has to be shown, and take the commutativity of the translation with respect to the sentential connectives into account, as follows:

$$\big(X = \{\varnothing\}\big)^{\mathbf{D}} \;\longleftrightarrow\; \exists\, Y \in \mathbf{D}_0 : \big(X = \{Y\}\big)^{\mathbf{D}} \wedge (\varnothing \in X)^{\mathbf{D}} \,,$$

then the proof is immediate in the light of Lemmas (178) and (182). □

Next we translate the assertion that P is the pair-set with members X and Y.

Lemma.
$$\begin{aligned}&\big[X,Y,P\in\mathbf{D}_0\wedge\big(P=\{X,Y\}\big)^{\mathbf{D}}\big] \;\longleftrightarrow\\ &\exists\,X',Y',P'\big(P=\langle 0,P'\rangle\wedge X=\langle 0,X'\rangle\wedge Y=\langle 0,Y'\rangle\wedge P'=\{X',Y'\}\big).\end{aligned}\tag{186}$$

Proof. First we translate:

$$\begin{aligned}\big(P=\{X,Y\}\big)^{\mathbf{D}} \quad&\longleftrightarrow\quad (\forall\,Z\in P: Z=X\;\vee\;Z=Y)^{\mathbf{D}}\,,\\ &\equiv\quad \forall\,Z\in\mathbf{D}_0,\;\exists\,Z',X',Y'\big(Z=\langle 0,Z'\rangle\wedge X=\langle 0,X'\rangle\\ &\qquad\wedge\,Y=\langle 0,Y'\rangle\wedge(Z'=X'\;\vee\;Z'=Y')\big)\,.\end{aligned}$$

The proof proceeds exactly analogous to the previous proof. □

Next we translate the assertion that H is the ordered pair (*à la* Hausdorff) with occupants X and Y.

Lemma.
$$\begin{aligned}&X,Y,H\in\mathbf{D}_0\wedge\big(H=\langle X,Y\rangle\big)^{\mathbf{D}} \;\longleftrightarrow\\ &\exists\,X',Y',H'\big(H=\langle 0,H'\rangle\wedge X=\langle 0,X'\rangle\wedge Y=\langle 0,Y'\rangle\wedge H'=\langle X',Y'\rangle\big)\,.\end{aligned}\tag{187}$$

Proof. First we substitute the Hausdorff definition for an ordered pair:

$$\big(H=\langle X,Y\rangle\big)^{\mathbf{D}} \;\longleftrightarrow\; \big(H=\{\{\{0,X\},\;\{1,Y\}\}\big)^{\mathbf{D}},$$

then we translate:

$$\begin{aligned}&\big(\exists\,V,W,P,Q\,[H=\{V,W\}\wedge V=\{P,X\}\wedge P=0\wedge W=\{Q,Y\}\wedge Q=1]\big)^{\mathbf{D}},\\ &\longleftrightarrow\\ &\exists\,V,W,P,Q\in\mathbf{D}_0\big((H=\{V,W\})^{\mathbf{D}}\wedge(V=\{P,X\})^{\mathbf{D}}\wedge(P=\varnothing)^{\mathbf{D}}\\ &\qquad\wedge\,(W=\{Q,Y\})^{\mathbf{D}}\wedge(Q=\{\varnothing\})^{\mathbf{D}}\big)\,,\end{aligned}$$

where we have used that $0\equiv\varnothing$ and $1\equiv\{\varnothing\}$. The translations of the conjuncts are provided by Lemmas (175), (185) and (186). What we then are left with looks like this:

$$H' = \{\{\varnothing,\,X'\},\;\{\{\varnothing\},\,Y'\}\} \;=\; \langle X',Y'\rangle\,,\tag{188}$$

the tedious verification of which we do not spell out. □

Similar Lemmas obtain when one or both of the occupants of the ordered pair is a physical system.

The next step is to translate what it means to be a (subset of a) Cartesian product-set, *i.e.* to be set of ordered pairs.

Lemma.
$$\begin{aligned}&X,Y,R\in\mathbf{D}_0\wedge\big(R\subseteq X\times Y\rangle\big)^{\mathbf{D}} \;\longleftrightarrow\\ &\exists\,X',Y',R'\big(R=\langle 0,R'\rangle\wedge X=\langle 0,X'\rangle\wedge Y=\langle 0,Y'\rangle\wedge R'\subseteq X'\times Y'\big).\end{aligned}\tag{189}$$

Proof. First we translate and use Lemma (187) on ordered pairs.

$$\begin{aligned}\big(R \subseteq X \times Y\big)^{\mathbf{D}} \quad &\longleftrightarrow \quad \big(\forall\, Z \in R,\ \exists\, V \in X,\ \exists\, W \in Y : Z = \langle V, W\rangle\big)^{\mathbf{D}}\ , \\ &\equiv \quad \forall\, Z \in \mathbf{D}_0,\ \exists\, V, W \in \mathbf{D}_0\ \big[\big((Z \in R)^{\mathbf{D}} \wedge (V \in X)^{\mathbf{D}} \wedge \\ &\qquad (W \in Y)^{\mathbf{D}}\big) \longrightarrow \big(Z = \langle V, W\rangle\big)^{\mathbf{D}}\big]\ .\end{aligned} \tag{190}$$

[⟶] The assumption that $X, Y, R \in \mathbf{D}_0$ takes care of the existence of some X', Y', K' such that $X = \langle 0, X'\rangle$, *etc.* This takes care of the first three conjuncts of (189). Let Z be an arbitrary member of $\mathbf{D}_0$ and let $V, W \in \mathbf{D}_0$ be such that the last expression of (190) holds. Again there are some Z', V', W' such that $Z = \langle 0, Z'\rangle$, *etc.* because of the membership of Z, V and W in $\mathbf{D}_0$. Then the last expression of (190), using the Lemma on ordered pairs (187), boils down to:

$$(Z' \in R' \wedge V' \in X' \wedge W' \in Y') \longrightarrow Z' = \langle V', W'\rangle\ ,$$

As Z runs over $\mathbf{D}_0$, Z' runs over $\mathbf{V}_{\text{ZFC}}$. When all members of R' are ordered pairs with occupants from X' and Y', we conclude that Z' is a set of ordered pairs from X' and Y', *i.e.*

$$R' \subseteq X' \times Y'\ .$$

[⟵] For the proof in the other direction we follow the proof of the previous paragraph in the opposite direction, by expanding expressions by means of existential quantifiers in the by now familiar manner. □

For the proofs of the mereological axioms, the following Lemmas are needed. The first concerns a set of ordered pairs of physical systems and sets, whose proof is a notational variant of the proof of Lemma (189).

Lemma. $X, R \in \mathbf{D}_0 \wedge P \subset \mathbf{P} \wedge \big(R \subset P \times X\big)^{\mathbf{D}} \longleftrightarrow$
$\exists\, X', R', P'\ \big(P = \langle 0, P'\rangle \wedge R = \langle 0, R'\rangle \wedge X = \langle 0, X'\rangle \wedge R' \subset P' \times X'\big)\ .$ (191)

The previous Lemmas build up to the translation of the notion of equinumerosity between a set of physical systems and a pure set:

Lemma. $\big(R \subset \mathbf{P} \wedge X \in \mathbf{V} \wedge R \sim X\big)^{\mathbf{D}} \longleftrightarrow$
$\exists\, R', X'\ \big(R = \langle 0, R'\rangle \wedge X = \langle 0, X'\rangle \wedge R' \sim X'\big)\ .$ (192)

Proof. The translation of the first conjunct, expanded by ⌜$R \in \mathbf{S}$⌝ to emphasise that R is a set, goes as follows, using def. (95):

$$\begin{aligned}(R \subset \mathbf{P} \wedge R \in \mathbf{S})^{\mathbf{D}} \quad &\equiv \quad (\forall\, Y : Y \notin R)^{\mathbf{D}} \wedge (R \in \mathbf{S})^{\mathbf{D}}\ , \\ &\longleftrightarrow \quad \forall\, Y \in \mathbf{D}_0 : \neg\exists\, Y', R'\, (Y = \langle 0, Y'\rangle \wedge R = \langle 0, R'\rangle \\ &\qquad \wedge\, Y' \in R') \ \wedge\ \exists\, R' : R = \langle 0, R'\rangle\ .\end{aligned} \tag{193}$$

In the light of the fact that for *every* $Y \in \mathbf{D}_0$ there always is some Y' such that $Y = \langle 0, Y' \rangle$, and in the light of the second overall conjunct, we see that $\ulcorner Y' \in R \urcorner$ must always fail, *i.e.* $Y' \notin R'$. This means that (193) is logically identical to:

$$\exists\, R',\ \forall\, Y \in \mathbf{D}_0,\ \exists\, Y' \,(Y = \langle 0, Y' \rangle \wedge R = \langle 0, R' \rangle \wedge Y' \notin R')\ . \tag{194}$$

The translation of the second conjunct of (192) is given by def. (95):

$$(X \in \mathbf{V})^{\mathbf{D}} \quad \longleftrightarrow \quad \exists\, X' : X = \langle 0, X' \rangle\ . \tag{195}$$

Henceforth we can concentrate on the biconditional: $(R \sim X)^{\mathbf{D}} \ \longleftrightarrow\ R' \sim X'$.

[$\longrightarrow$] We translate the equinumerosity of R and X by translating the following assertion: there is a set f of ordered pairs such that every member of R and X occur exactly once as occupant.

$$\begin{aligned} (R \sim X)^{\mathbf{D}} \quad \longleftrightarrow \quad & \big[\exists\, f \subseteq R \times X \,\big(\forall\, a \in R,\ \exists! Y \in X,\ \exists\, P \in f : P = \langle a, Y \rangle \\ & \wedge\ \ \forall\, Y \in X,\ \exists! a \in R,\ \exists\, P \in f : P = \langle a, Y \rangle\big)\big]^{\mathbf{D}}\ . \end{aligned} \tag{196}$$

For the translation of the first conjunct $(f \subseteq R \times X)^{\mathbf{D}}$ of (196) we appeal to Lemma (191) and obtain the existence of some $f' \subseteq R' \times X'$ and $f = \langle 0, f' \rangle \in \mathbf{D}_0$. We expand the other two conjuncts to see what conditionf' must satisfy:

$$\begin{aligned} & \forall\, A \in \mathbf{D}_1,\ \exists\, Y \in \mathbf{D}_0,\ \exists\, P \in \mathbf{D}_0 \Big(\big((a \in R)^{\mathbf{D}} \wedge (Y \in X)^{\mathbf{D}} \wedge (P \in f)^{\mathbf{D}}\big) \longrightarrow \\ & \quad \big[(P = \langle a, Y \rangle)^{\mathbf{D}} \wedge \forall\, Q \in \mathbf{D}_0,\ \forall\, Z \in \mathbf{D}_0 \,\big([(Q \in f)^{\mathbf{D}} \wedge (Z \in X)^{\mathbf{D}} \\ & \quad \wedge (Q = \langle a, Z \rangle)^{\mathbf{D}}] \longrightarrow\ (Z = Y)^{\mathbf{D}}\big)\big]\Big) \wedge \\ & \forall Y \in \mathbf{D}_0,\ \exists\, A \in \mathbf{D}_1,\ \exists\, P \in \mathbf{D}_0 \Big(\big((a \in R)^{\mathbf{D}} \wedge (Y \in X)^{\mathbf{D}} \wedge (P \in f)^{\mathbf{D}}\big) \longrightarrow \\ & \quad \big[(P = \langle a, Y \rangle)^{\mathbf{D}} \wedge \forall\, Q \in \mathbf{D}_0,\ \forall\, B \in \mathbf{D}_1 \,\big([(Q \in f)^{\mathbf{D}} \wedge (b \in R)^{\mathbf{D}} \\ & \quad \wedge (Q = \langle b, Y \rangle)^{\mathbf{D}}] \longrightarrow\ (a = b)^{\mathbf{D}}\big)\big]\Big)\ . \end{aligned} \tag{197}$$

The first conjunct expresses that f is a function with domain R, the second one expresses the bijectivity of f by saying that *every* member of X (surjectivity) has exactly one original in R (injectivity). The antecedent membership in $\mathbf{D}_1$ and $\mathbf{D}_0$ of all occurring variables (A, B, C, P, Q) guarantees the existence of the ordered pairs$A = \langle 1, A' \rangle$, $P = \langle 0, P' \rangle$, *etc.* in the translated atomic sentences of (197), and licenses the appeal to the Lemmas about identity (171) and (162) and about sets of ordered pairs of physical systems and sets (191). We obtain:

$$\begin{aligned} & \forall\, A' \in R',\ \exists\, Y' \in X'',\ \exists\, P' \in f' \,\big[P' = \langle A', Y' \rangle\ \wedge \\ & \qquad\qquad \forall\, Q' \in f',\ \forall\, Z' \,\big(Q' = \langle A', Z' \rangle\ \longrightarrow\ Z' = Y'\big)\big]\ \wedge \\ & \forall\, Y'' \in X',\ \exists\, A' \in R',\ \exists\, P' \in f' \,\big[P' = \langle A', Y' \rangle\ \wedge \\ & \qquad\qquad \forall\, Q' \in f',\ \forall\, B' \in R' \,\big(Q' = \langle B', Y'' \rangle\ \longrightarrow\ A' = B'\big)\big]\ . \end{aligned} \tag{198}$$

To claim there is a $f' \subseteq R' \times Y'$ that meets condition (198) is to say that f' is a bijection from R' to Y', *i.e.* R' and Y' are equinumerous, which is what we had to prove.

[$\longleftarrow$] From $R' \sim Y'$ one works one's way back through the reasoning above by introducing variables by existential quantification *etc.* $\square$

Lemma. $(a \sqsubset b)^{\mathbf{D}} \quad \longleftrightarrow \quad \exists\, A', B'\, (A = \langle 1, A' \rangle \wedge B = \langle 1, B' \rangle \wedge A' \subset B')\ .$ (199)

Proof. The proof is a straightforward substitution of the translation of $(a \sqsubseteq b)^{\mathbf{D}}$ (92) in the following expression:

$$(a \sqsubset b)^{\mathbf{D}} \quad \longleftrightarrow \quad (a \sqsubseteq b)^{\mathbf{D}} \wedge \neg(a = b)^{\mathbf{D}}\ ,$$

and further using Lemma (162). We then obtain:

$$A' \subset B' \;\longleftrightarrow\; (A' \subseteq B' \;\wedge\; A' \neq B')\ ,$$

which evidently holds. $\square$

Lemma. $(a = \boldsymbol{\varnothing})^{\mathbf{D}} \quad \longleftrightarrow \quad A = \langle 1, \varnothing \rangle\ .$ (200)

Proof. We translate the following equivalent expression for $\ulcorner a = \boldsymbol{\varnothing} \urcorner$: $\mathsf{Null}(a)$ (35):

$$\begin{aligned} \mathsf{Null}^{\mathbf{D}}(a) \quad &\longleftrightarrow \quad \big(\forall\, b(b \sqsubseteq a \longrightarrow b = a)\big)^{\mathbf{D}}\ , \\ &\equiv \quad \forall\, B \in \mathbf{D}_1 : (b \sqsubseteq a)^{\mathbf{D}} \longrightarrow (b = a)^{\mathbf{D}}\ , \\ &\equiv \quad \forall\, B \in \mathbf{D}_1 \big(\exists\, A', B'\, (A = \langle 1, A' \rangle \wedge B = \langle 1, B' \rangle \wedge B' \subseteq A') \\ &\qquad\qquad \longrightarrow \exists\, A'', B''\, (A = \langle 1, A'' \rangle \wedge B = \langle 1, B'' \rangle \wedge B'' = A'')\big)\ . \end{aligned} \tag{201}$$

[$\longrightarrow$] Let $\mathsf{Null}^{\mathbf{D}}(a)$ for arbitrary physical system a. Obviously $A'' = A'$ and $B'' = B$. When $B = \langle 1, B' \rangle$ runs over $\mathbf{D}_1$, B' runs over $\mathbf{V}_{\text{ZFC}}$. Further, since we started with physical system $a \in P$, which is translated as $\ulcorner A \in \mathbf{D}_1 \urcorner$, the existence of some A' such that $A = \langle 1, A' \rangle$ is guaranteed and can be deleted from the conditionals. Hence we can write:

$$\forall\, B'\, (B' \subseteq A' \longrightarrow B' = A')\ . \tag{202}$$

Set A' is identical to all its subsets. Then $A' = \varnothing$ and $A = \langle 1, \varnothing \rangle$.

[$\longleftarrow$] We start with $A = \langle 1, \varnothing \rangle$. So $A \in \mathbf{D}_1$, which is the translation of $\ulcorner a \in \mathbf{P} \urcorner$, and there is some A' such that $A = \langle 1, A' \rangle$, namely $\varnothing$. Sentence (202) is equivalent to asserting that $A' = \varnothing$. Expanding sentence (202) by introducing variables by existential quantification we arrive at (201). $\square$

Next comes genuine subsystemhood.

Lemma. $(a \mathrel{\dot\sqsubseteq} b)^{\mathbf{D}} \quad \longleftrightarrow \quad \exists\, A', B' \big(A = \langle 1, A' \rangle \wedge B = \langle 1, B' \rangle \wedge \varnothing \neq A' \subseteq B'\big)\ .$ (203)

Proof. The proof is immediate by translating the definition of $\sqsubseteq$:

$$(a \sqsubseteq b)^{\mathbf{D}} \;\equiv\; (a \sqsubseteq b)^{\mathbf{D}} \wedge \neg(a = \emptyset)^{\mathbf{D}} ,$$

and using Lemma (200), to yield $A' \subseteq B'$ and $A' \neq \varnothing$. □

Penultimately, the overlap-relation:

Lemma. $(c \circ b)^{\mathbf{D}} \;\longleftrightarrow\; \exists\, C', B' \big(C = \langle 1, C'\rangle \wedge B = \langle 1, B'\rangle \wedge C' \cap B' \neq \varnothing\big)$. (204)

Proof. The proof is again immediate by translating the definition of the overlap-relation $\circ$, *i.e.*

$$(c \circ b)^{\mathbf{D}} \;\equiv\; \exists\, D \in \mathbf{D}_1 \big((d \sqsubseteq c)^{\mathbf{D}} \wedge (d \sqsubseteq b)^{\mathbf{D}}\big) ,$$

and using Lemma (203). This leads to the assertion that there is some non-empty D' such that $D = \langle 1, D'\rangle$, and $D' \subseteq C'$ and $D' \subseteq B'$, *i.e.* $(C' \cap B') \supseteq D \neq \varnothing$. □

Finally the translation of a system being an elementary particle (41):

Lemma. $\mathsf{ElemPart}(b)^{\mathbf{D}} \;\longleftrightarrow\; \exists\, S : B = \langle 1, \{S\}\rangle$. (205)

Proof. First we express the monadic predicate $\mathsf{ElemPart}(b)$ as in (42):

$$b \neq \emptyset \;\wedge\; \forall\, a \big(a \sqsubseteq b \;\longrightarrow\; (a = \emptyset \;\vee\; a = b)\big) .$$

We translate this expression with the help of Lemmas (162) and (200):

$$\begin{aligned}\exists\, B' \Big[&\big(B = \langle 1, B'\rangle \;\wedge\; B' \neq \varnothing\big) \;\wedge \\ &\quad \forall\, A \in \mathbf{D}_1 \big(\exists\, A' : A = \langle 1, A'\rangle \;\wedge\; (A' \subseteq B' \;\longrightarrow A' = \varnothing \vee A' = B')\big)\Big] .\end{aligned} \tag{206}$$

[$\longrightarrow$] Taking sentence (206) as a premise, we have to show that $B' = \{S\}$ for some B', when $B = \langle 1, B'\rangle$. Consider first the second conjunct between the big square brackets in (206): for every $A \in \mathbf{D}_1$ there is some A' such that $A = \langle 1, A'\rangle$. Next we may assume that $A' \subseteq B'$. Does this imply that $A' = B'$ or $A' = \varnothing$? Yet, but only if B' is a either singleton-set or is empty. The last-mentioned disjunct is however ruled out by the first conjunct between the big square brackets in (206).

[$\longleftarrow$] Now we start with the premise that B' is some singleton-set. Then B' is non-empty and has only itself as a non-empty subset, which means that sentence (206) holds and this is equivalent to what we had to prove. □

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