Brit. J. Phil. Sci. **59** (2008), 499–548

Discerning Fermions

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ABSTRACT

We demonstrate that the quantum-mechanical description of composite physical systems of an arbitrary number of similar fermions in all their admissible states, mixed or pure, for all finite-dimensional Hilbert spaces, is *not* in conflict with Leibniz's Principle of the Identity of Indiscernibles (PII). We discern the fermions by means of physically meaningful, permutation-invariant *categorical* relations, i.e. relations independent of the quantum-mechanical probabilities. If, indeed, *probabilistic* relations are permitted as well, we argue that similar bosons can also be discerned in all their admissible states; but their *categorical* discernibility turns out to be a state-dependent matter. In all demonstrated cases of discernibility, the fermions and the bosons are discerned (i) with only minimal assumptions on the interpretation of quantum mechanics; (ii) *without* appealing to metaphysical notions, such as Scotusian haecceitas, Lockean substrata, Postian transcendental individuality or Adamsian primitive thisness; and (iii) *without* revising the general framework of classical elementary predicate logic and standard set theory, thus *without* revising standard mathematics. This confutes: (a) the currently dominant view that, provided (i) and (ii), the quantum-mechanical description of such composite physical systems *always* conflicts with PII; and (b) that *if* PII can be saved at all, the *only* way to do it is by adopting one or other of the thick metaphysical notions mentioned above. Among the most general and influential arguments for the currently dominant view are those due to Schrödinger, Margenau, Cortes, Dalla Chiara, Di Francia, Redhead, French, Teller, Butterfield, Giuntini, Mittelstaedt, Castellani, Krause and Huggett. We review them succinctly and critically as well as related arguments by van Fraassen and Massimi.

doi:10.1093/bjps/axn027

And in fact there is no reason except prejudice, so far as I can discover, for denying the reality of relations.
Bertrand Russell

1 Introduction: The Currently Dominant View

1.1 Weyl on Leibniz's principle

In his pioneering monograph *Gruppentheorie und Quantenmechanik*, Hermann Weyl connected the then newly born quantum-mechanical description of a composite physical system of two electrons—of which Pauli's exclusion principle was the pillar (cf. Jammer [1966], pp. 143–51)—to a renowned metaphysical principle, namely Leibniz's Principle of the Identity of Indiscernibles (Weyl [1928], IV.C. Section 9):

> ... die Möglichkeit, dass eines der beiden Individuen Hans und Karl im Quantenzustand E_1, das andere im Quantenzustand E_2 sich befindet, vereinigt nicht zwei unterscheidbare Fälle, die durch die Vertauschung von Hans und Karl auseinander hervorgehen; es its unmöglich, die wesensgleichen Individuen Hand und Karl, jedes für sich, in seiner dauernden Identität mit sich selbst festzuhalten. Von Elektronen kann man prinzipiell nicht den Nachweis ihres Alibi verlangen. So setzt sich in der modernen Quantentheorie das Leibnizsche Princip von der *coincidentia indiscernibilium* durch.
>
> [Our translation: ... the possibility that one of the individuals Hans and Karl is in the quantum state E_1 and the other in the quantum state E_2 does not encompass two discernible cases, which arise by permuting Hans and Karl; it is impossible for either of these individuals Hans and Karl,

> who have the same nature, to retain his identity. Even in principle one cannot demand an alibi of an electron. In this way the Leibnizian principle of *coincidentia indiscernibilium* carries through in modern quantum mechanics.]

The concluding sentence is puzzling. For if electrons Hans and Karl have lost their identity because their quantum-mechanical states 'do not encompass two discernible cases', and if it is 'in principle impossible to demand an alibi' of Hans and Karl, that is, if the two electrons are *indiscernible*, and nonetheless there remain *two* electrons, Hans *and* Karl, which thus are different *solo numero*—and *solo nomine*—*then* the conclusion seems deductively inevitable that Leibniz's Principle of the Identity of Indiscernibles (PII) *does not* carry through in quantum mechanics (QM). Yet Weyl concludes that PII *does carry through* in QM.

Misprint? Unlikely. In the English translation of the second and revised edition of *Gruppentheorie und Quantenmechanik* of 1931 (in its preface, translator H.P. Robertson expresses his thanks to Weyl 'for general encouragement and assistance'), Hans and Karl have been replaced with Ike and Mike, the passage we quoted above has been extended a little bit by letting the electrons speak up and say 'I am Ike' and 'I am Mike', but the concluding sentence has remained the same, glossing over the fact that the German *durchsetzen* (to carry through) has been translated inaccurately as *holds* (Weyl [1931], p. 241).

Furthermore, in his philosophical magnum opus *Philosophy of Mathematics and Natural Science*, Weyl ([1949], p. 247) wrote:

> The upshot of it all is that the electrons satisfy Leibniz's *principium identitatis indiscernibilium*, or that the electronic gas is a 'monomial aggregate' (Fermi-Dirac statistics). In a profound and precise sense physics corroborates the Mutakallimûn: neither to the photon nor to the (positive and negative) electron can one ascribe individuality. As to the Leibniz-Pauli Exclusion Principle, it is found to hold for electrons but not for photons.

The puzzle deepens. Earlier in the book, Weyl ([1949], p. 7) characterises an object from some category—such as electrons and photons—as an *individual* iff it has a general *property* that no other object from that category has. Weyl concluded that QM deprives photons (of the same frequency and velocity) and electrons in a composite physical system of their *individuality*—which, to recall, for Weyl means: their *discernibility by a property*. Ernst Cassirer followed suit in his *Determinismus und Indeterminismus in der modernen Physik* ([1956], Section V.II, fn. 47). But then again, the conclusion seems deductively inevitable that electrons *do not satisfy* PII in QM, because the many particles are *not* discernible in the required Weylian–Leibnizian manner, which is by some property: they are discerned *solo numero*. Yet Weyl opens the quoted passage by asserting that electrons *satisfy* PII.

Furthermore, in the concluding sentence of the passage quoted above, Weyl has appended 'Leibniz' to Pauli's exclusion principle. Why append the name of a seventeenth-century metaphysician to a principle of QM that saw the light of day in 1925, more than 200 years after Leibniz (1646–1719) passed away? Clearly, we submit, because Pauli's exclusion principle, which holds for electrons and not for photons, as Weyl emphasised, must *somehow* have to do with the satisfaction of PII by electrons and the violation of PII by photons. Since Pauli's exclusion principle is a constraint on physical states, that somehow must originate in these constrained states. But precisely *how* this works, thereof Weyl passes over in silence.

There is a way to make sense of what Weyl said, as follows. Weyl had Pauli's original formulation of his exclusion principle in mind, that still lingers on in textbooks on QM: 'no two electrons in one atom occupy the same state', where an atomic state is characterised by the principal atomic quantum number n (energy level), azimuthal quantum number l (orbital angular momentum), angular quantum number j (total angular momentum) and magnetic quantum number m (z-component of j); cf. (Jammer [1966], p. 139). Then the atomic states that electrons Hans and Karl occupy, when inhabiting the same atom, discern Hans and Karl after all. Thus Weyl's terminology of 'the Leibniz–Pauli exclusion principle'.

But then, in 1949, had Weyl still not understood that this way of formulating the exclusion principle is not general, even incorrect, and that it was by then replaced with the general and correct symmetrisation postulate? One of the founding fathers of the quantum-mathematical treatment of composite physical systems of similar particles in terms of unitary representations of the permutation group was led by the nose, by an erroneous formulation of a 25-year-old rule governing atomic spectra, to making metaphysical pronouncements about the building blocks of matter?

The currently dominant view on this subject matter has decided otherwise. The Principle of the Identity of Indiscernibles (PII), without further metaphysical input, stands refuted; in full generality, the quantum-mechanical description of composite systems of similar particles, whether they be fermions or bosons, provides the decisive refutation of PII. Those composing particles exist in physical reality and they are indiscernible, yet different *solo numero*. Sad news for friends of Leibniz. Perhaps Weyl was confused, presumably by Pauli's original formulation of his exclusion principle; certainly Weyl had it wrong when it comes to fermions. But the propounders of the dominant view treat Weyl charitably. When one of its spearholders quotes from Weyl's *The Theory of Groups and Quantum Mechanics* ([1931]) (the same passage we quoted from the German original) in his entry 'Identity and Individuality in Quantum Theory' of the Stanford Encyclopedia of Philosophy, Weyl's concluding sentence is omitted (French [2006]); and the opening sentence of the passage we quoted above from Weyl ([1949]) is never quoted by anyone. These sparse statements of Weyl on the status of PII in QM have been fading away in the fog of history. Out of the fog has come the currently dominant view.

Before we continue to describe briefly the rise of the currently dominant view, we pause in order to introduce some terminology.

1.2 Intermezzo: Terminology and Leibnizian principles

The word *object* signifies here something of great generality, although less extreme than *entity*, which can be absolutely anything. We use 'object' in the 'metaphysically thin' Frege–Quine sense: *values of variables bound by quantification and subject to predicative identity-criteria*, that can in principle be described in elementary predicative formal languages, incorporating elementary predicate logic. For the sake of emphasis and contrast, we shall therefore frequently speak of *formal objects*, but we shall, for the sake of brevity, often also drop the adjective 'formal' and simply speak of *objects*. Examples of entities that definitely can be treated as formal objects: planets, persons, proteins, telescopes, numbers, sets, texts; examples of entities that plausibly cannot be treated as formal objects: flames, shadows, sounds, impressions, hopes, holes—and also (because the language is elementary, hence first order) properties and relations, unless treated extensionally by set-theoretical means.

A particular kind of formal object forms *physical objects*, which we take to include both *physical events* and *physical systems*—the latter is a term generic in (philosophy of) physics. Physical systems in turn comprise *material objects*, but not conversely; physical systems are more general than material objects in that they can be built out of material objects, fields, radiation, space and time; the latter four items are typically 'non-material', yet indisputably 'physical'. An *elementary particle* can be characterised mereologically as a physical system having no proper subsystems, such as leptons, quarks, and gauge bosons. Within quantum theory, one can also characterise kinds of elementary particles mathematically, in Wignerian fashion: that is, in terms of irreducible representations of the spacetime symmetry group, the Galilei group in the case of QM, and the Poincaré group in the case of relativistic quantum field theory (cf. Castellani [1998b], pp. 181–94). When the material object is not an elementary particle and has a non-vanishing size, one speaks of a *material body*.

We call physical objects in a set *absolutely discernible* iff for every object there is some physical property that it has but all the others in the set lack, and *relationally discernible* iff for every object there is some physical relation that discerns it from all others (cf. Section 4 for rigorous definitions). An object is *indiscernible* iff it is both absolutely and relationally indiscernible, and hence *discernible* iff it is discernible either way or both ways. The terms 'qualitative', 'quantitative', and 'numerical discernibility' also abound in the literature;

they can be defined in terms of our notions: physical objects are *qualitatively discernible* iff they are discernible; they are *quantitatively discernible*, or synonymously *numerically discernible*, iff they are non-identical. Often, we call objects that are absolutely discernible from all other objects *individuals*; those that are only relationally discernible from all other objects we call *relationals*. Frequently, one encounters talk of a physical object 'having an identity', which we shall accommodate as the property that discerns the object absolutely (if it is absolutely discernible); then relationals do not 'have an identity' but individuals do. To *invididuate* an object means to discern it absolutely; then individuals can be individuated, relationals cannot. Finally, particles are *distinguishable* iff they can be individuated.

Next, we give three versions of Leibniz's principle for physical objects. The *Principle of the Identity of Absolute Indiscernibles* (PII-A) states that no two physical objects are absolutely indiscernible. The *Principle of the Identity of Relational Indiscernibles* (PII-R) states that no two physical objects are relationally indiscernible, and the *Principle of the Identity of Indiscernibles* (PII) states that no two physical objects are absolutely and relationally indiscernible; or synonymously, two physical objects are numerically discernible only if they are qualitatively discernible. All converse statements are uncontroversial tautologies: no physical object can be discerned from itself—the indiscernibility of identicals, also known as *Leibniz's law*. The relevant logical relations between PII, PII-A and PII-R are as follows. Obviously, PII-A and PII-R each are sufficient for PII,

$$\text{PII-A} \longrightarrow \text{PII} \quad \text{and} \quad \text{PII-R} \longrightarrow \text{PII}, \tag{1}$$

which makes even their disjunction sufficient for PII. So if PII fails, then both PII-A and PII-R fail. Since absolute discernibles are always relational discernibles—see Equation (40) and the sentence preceding it—one quickly proves that PII-R is also necessary for PII, which implies with Equation (1) that PII and PII-R stand or fall together,

$$\text{PII} \longleftrightarrow \text{PII-R}. \tag{2}$$

But PII-A is not necessary for PII, i.e. $\neg(\text{PII} \longrightarrow \text{PII-A})$, which implies that it is a genuine logical possibility that PII-A falls whilst PII stands tall,

$$\text{PII} \wedge \neg\text{PII-A}. \tag{3}$$

The main conclusion of the current paper will be that similar elementary particles turn this logical possibility into a physical actuality: they are non-identical absolute indiscernibles.

In his *Discourse on Metaphysics*, Leibniz was the first to discuss PII-A elaborately and to apply it to 'substances'; in several places, Leibniz defends (as we would put it today) a reduction of relations to properties, which makes mentioning relations in discernibility otiose (see (Russell [1937], pp. 13–5) and (Ishiguro [1990], pp. 118–22, 130–42) for Leibniz's struggle with relations). When not all relations reduce to properties, and we thus have to consider properties and relations separately and independently, then as a matter of logic, PII-R is as much in play as PII-A, and PII as stated is mandatory. Massimi ([2001]) holds that a version of Leibniz's principle which considers *only* relations irreducible to properties (a strengthened version of our PII-R) as the one that is applicable in QM (cf. Section 3.3).

Let us end by noting that logically speaking one could refine 'relational discernibility' to '*n*-discernibility', meaning that the objects are discerned by some *n*-ary relation. Then an infinite hierarchy of indiscernibility principles ensues, each one logically weaker than the next one. Since we shall not need this, we leave it.

1.3 The rise of the currently dominant view

Neither similar bosons nor similar fermions are *individuals* in our sense. This chimes well with Erwin Schrödinger's view on particle identity; he, more than any other physicist of his generation, was preoccupied with the philosophical significance of the indistinguishability of particles in QM. He, like Weyl, insisted that the building blocks of matter are not individuals. In a series of public lectures given in 1950 at the Dublin Institute for Advanced Studies, Schrödinger literally begged his audience to believe him when he said that 'it is beyond doubt that the question of "sameness", of identity, really and truly has no meaning' (Schrödinger [1996], p. 122). But just what he meant by these puzzling words Schrödinger never made clear. And, in contrast to Weyl, he never connected the issue to PII.

In a paper devoted to the philosophical importance of Pauli's exclusion principle, the American physicist–philosopher Henry Margenau was instrumental in establishing the currently dominant view in philosophy of science in general and in philosophy of physics in particular. Margenau did not mention Weyl explicitly, but he did write the following ([1944], p. 202):

> This conclusion recalls Leibniz's principle of the identity of indiscernibles; indeed physicists have occasionally thought that the Exclusion Principle implies this principle with regard to elementary particles of the same species. It would be interesting indeed if modern physics had something to say about this much debated postulate of logic. Unfortunately, the relevance is but remote, and the Exclusion Principle, so far as it goes, contradicts Leibniz, who stated the meaning of the principle of identity of indiscernibles in this way: *non dari posse in natura duas res singulares solo numero differentes*.

Which physicists did Margenau have in mind in the opening sentence after the semicolon? Hermann Weyl seems the likely candidate. In *The Nature of Physical Reality*, Margenau ([1950], p. 441) drew the same conclusion.

The subject lay dormant for about three decades, when it was taken up again in a sequence of papers mostly appearing in *Philosophy of Science* (Cortes [1976], who brandished PII 'a false principle'; Barnette [1978]; Ginsberg [1981]; van Fraassen [1984]) (for a review of the meaning of 'particle identity' in the physics literature in this period, see Saunders [2006b]). A second sequence was initiated in the late 1980s by Steven French's PhD thesis, supervised by M.L.G. Redhead (French and Redhead [1988]), followed by Giuntini and Mittelstaedt ([1989]), who argued that although demonstrably valid in classical logic, in quantum logic the validity of PII cannot be established; by French ([1989a]), who assured us that PII 'is not contingently true either', French ([1989b]; [2006]), French and Rickles ([2003]), Redhead and Teller ([1992]), Teller ([1998]), Butterfield ([1993]), Castellani and Mittelstaedt ([2000]) and Huggett ([2003]); refinements and elaborations have been appearing ever since, lately delving into the metaphysics of 'object' and 'individual' (French [1998]; French & Krause [2006]).

What all these papers have in common is that they subscribe to Margenau's conclusion (PII-A stands refuted by QM), or they argue for a stronger conclusion, namely that both PII-R and PII also stand refuted by QM. French and Rickles ([2003], p. 221) speak of 'the Received View' (French and Krause [2006], p. xiii follow suit); see also Castellani and Mittelstaedt ([2000], p. 1592), who call the currently dominant view 'commonly accepted'. In the most recent version of his encyclopedia article, French ([2006]) rounds the situation up as follows:

> If the particles are taken to possess both their intrinsic and state-dependent properties in common, as suggested above, then there is a sense in which *even the weakest form of the Principle fails* [i.e. our PII; our emphasis added here]. On this understanding, the Principle of Identity of Indiscernibles is *actually false*. Hence it cannot be used to effectively guarantee individuation via the state-dependent properties by analogy with the classical case. If one wishes to maintain that quantum particles are individuals, then their individuality will have to be taken as conferred by Lockean substance, primitive thisness or, in general, some form of non-qualitative haecceistic difference.

The only way to save PII is to adopt some form of thick metaphysics.

Thus has arisen the currently dominant view.

Van Fraassen ([1984]) argued that Margenau's conclusion tacitly relied on the 'interpretational' claim that QM is 'complete', i.e. there are no hidden variables, and thus presents us with a dilemma between (i) the incompleteness of QM and

(ii) the incompatibility between QM and PII. The currently dominant view thus tacitly assumes that QM is 'complete'; if it were 'incomplete', then completion by hidden variables would open the way to discern particles by means of values of these hidden variables. Van Fraassen ([1991]) proceeded with this criticism by showing that Margenau-type arguments rely on what may indeed be called an 'interpretational' postulate of QM, which can be rejected without affecting the empirical content of QM; the conclusion then is that the status of Leibniz's principle in QM *is a matter of interpretation* and lies beyond the reach of empirical research (cf. Section 3.2). This conclusion has been reinforced by Brown *et al.* ([1999]), who argued that in the Bohmian 'interpretation' of QM, similar particles can be discerned by means of their trajectories in configuration space. In that 'interpretation' of QM, there is a peaceful place for PII.

Besides van Fraassen's analysis, very few critical analyses of Margenau's argument have appeared. The most recent is Massimi's ([2001]), who argued that a *presupposition* of PII-A is not satisfied and that—tacitly relying on a Strawsonian account of presuppositions—therefore the argument of Margenau, and by implication all sibling arguments for the same conclusion, cannot even take off, let alone establish that PII-A stands refuted by QM; it is neither refuted nor vindicated by QM, 'it is simply *not applicable*', concludes Massimi ([2001], p. 324). Nonetheless, Massimi ([2001], p. 327, fn. 21) adheres to the dominant view in that the version of Leibniz's principle that she deems applicable, namely the version that *only* appeals to relations (PII-R), does stand refuted by standard QM.

The dominant view that PII stands refuted by QM unless QM is supplemented with additional metaphysical principles has motivated and is motivating programmes in philosophical logic that aim to accommodate rigorously the violation of PII, by means of quasi-set theory, quantum-set theory and so-called Schrödinger logics; cf. (Krause [1992]; Dalla Chiara and Toraldo di Francia [1993]; Dalla Chiara, Giuntini and Krause [1998]; French and Krause [2006]). The currently dominant view has also motivated and is motivating metaphysical programmes that seek to develop a notion of physical object that includes Lockean substrata, or Scotusian haecceitas, or Postian transcendental individuality or Adamsian primitive thisness; in short, some thick metaphysical notion that is metaphorically related to pointing your finger to an observable material body, alluding to an ostensive definition (see Adams [1979]). But none of this, we claim, is needed.

1.4 Overview

What follows in this article is organised as follows. In Section 2, we rehearse some rigorous definitions, postulates and theorems (without proof) of QM for the sake of future reference. In Section 3, we analyse the standard argument that has been provided for the alleged downfall of PII in QM, and we analyse van Fraassen's and Massimi's criticisms of this standard argument. In Section 4, we inject a minimum yet necessary dose of logical rigour into the concepts of identity and discernibility. In Section 5, we prove several theorems that will establish that similar fermions are relationally discernible in every admissible state, and that the discernibility of similar bosons becomes a contingent, i.e. state-dependent issue; this crucial section is an elaboration of an earlier insight of one of us, mentioned in (Saunders [2003a], p. 294) concerning two fermions in the singlet state, which was developed further in (Saunders [2006a]). In Section 6, we defend our results against criticism levelled by French and Krause against Saunders' earlier insight.

2 Elements of Quantum Mechanics

2.1 Physical states and physical magnitudes

Most postulates of QM tell us how certain physical concepts are represented mathematically: physical state, physical magnitude, physical property.

We begin with physical states. The *State Postulate* (StateP) associates with every physical system S one superselected sector in some Hilbert space; every possible physical state corresponds to a statistical operator acting on that sector. Whenever S is composed of $N \geq 2$ subsystems, the associated sector Hilbert space is the direct product space, denoted by $\mathcal{H}^N$, of the N sector Hilbert spaces $\mathcal{H}_j$ associated to the N subsystems S_j,

$$\mathcal{H}^N = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N. \tag{4}$$

Every possible physical state of S corresponds to a statistical operator $W \in \mathcal{S}(\mathcal{H})$ that acts on $\mathcal{H}^N$, and vice versa; and every possible physical state of the jth subsystem S_j of S corresponds to the marginal statistical operator W_j—obtained by partial tracing of W with respect to the jth factor space $\mathcal{H}_j \sqsubseteq \mathcal{H}^N$ (4).

What we call the *Weak Magnitude Postulate* (WkMP) says that every physical magnitude that pertains to physical system S is represented by an operator that acts on the sector Hilbert space associated to S (see State Postulate); these so-called *magnitude-operators* of S may be collected in set $M_S(\mathcal{H})$.

Stronger magnitude postulates provide inclusion relations between $M_S(\mathcal{H})$ and the sets of all projectors, self-adjoint operators, positive operators, normal operators, and what have you. Traditionally, all and only self-adjoint ones have been thought to be magnitude operators (J. von Neumann), but the 'all' has turned out to be problematic, at least in the case of systems whose Hilbert space has infinite dimensions. The *Standard Magnitude Postulate* (StMP) adds to the weak one that the magnitude operators have to be self-adjoint: $M_S(\mathcal{H}) \subseteq \mathcal{O}_{\text{s.a.}}(\mathcal{H})$.

To recall, a *superselected physical magnitude* corresponds to an operator $A \in M_S(\mathcal{H})$ that commutes with every magnitude operator in $M_S(\mathcal{H})$. Consequently, all superselected magnitude operators commute (for succinct and rigorous introductions to the subject of superselection rules, see Beltrametti and Cassinelli [1981], pp. 45–51; Giulini [2003].)

2.2 Composite physical systems of similar particles

Whenever the composite system S consists of N similar physical subsystems to which we associate the same sector Hilbert space $\mathcal{H}$, the State Postulate (4) implies that the Hilbert space we associate with S is an N-factor direct product space,

$$\mathcal{H}^N = \mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}. \tag{5}$$

For every $N \in \mathbb{N}^+$ and every *permutation* $\pi \in \mathbb{P}_N$ (a bijection from $\{1, 2, \ldots, N\}$ to itself, of which there are $N!$), we denote by U_π the unitary permutation operator on $\mathcal{H}^N$ and call it a *permutator*; it is defined as usual, by its action on labels of the factors in the direct product basis vectors of $\mathcal{H}^N$ (one can show that this definition is independent from the chosen basis). These unitary permutation operators form a generally reducible representation of the non-Abelian permutation group $\mathbb{P}_N$.

Definition. (Bounded) operator B acting on $\mathcal{H}^N$ is *symmetric* iff B commutes with every U_π,

$$\forall \pi \in \mathbb{P}_N : [U_\pi, B]_- = 0. \tag{6}$$

Theorem. For every projector P in the lattice $\mathcal{P}(\mathcal{H}^N)$ of projectors on $\mathcal{H}^N$, its range $P[\mathcal{H}^N]$ is an invariant subspace of $\mathcal{H}^N$ under the action of $\mathbb{P}_N$ iff P is symmetric. Projectors that project on eigenspaces of a symmetric operator are symmetric too, and therefore all spectral projectors of a symmetric operator whose eigenspaces combine to $\mathcal{H}$ are also symmetric.

Consider next the *symmetriser*, which sends an operator A to a symmetric version of it,

$$A \mapsto A_{\text{sym}} \equiv \frac{1}{N!} \sum_{\pi \in \mathbb{P}_N}^{N!} U_\pi A U_\pi^\dagger, \tag{7}$$

which map is a projector. Of course, all symmetrised operators are symmetric. Being a statistical operator is invariant under symmetrisation (7), but being a pure one, i.e. being a projector, may not be!

Let A be an operator acting on $\mathcal{H}^N$ and $W \in \mathcal{S}(\mathcal{H}^N)$. If A is symmetric, then for any W,

$$\text{Tr}(AW) = \text{Tr}(AW_{\text{sym}}); \tag{8}$$

and if W is symmetric, then for any A,

$$\mathrm{Tr}(AW) = \mathrm{Tr}(A_{\mathrm{sym}} W). \tag{9}$$

(For proofs of all theorems reported in this subsection, see Bach [1997], Chapter 2.) Of prime importance are the following two orthogonal projectors from the lattice $\mathcal{P}(\mathcal{H}^N)$:

$$\Pi_N^+ \equiv \frac{1}{N!} \sum_{\pi \in \mathbb{P}_N}^{N!} U_\pi \quad \text{and} \quad \Pi_N^- \equiv \frac{1}{N!} \sum_{\pi \in \mathbb{P}_N}^{N!} \mathrm{sign}(\pi) U_\pi, \tag{10}$$

where $\mathrm{sign}(\pi) \in \{\pm 1\}$ is the *sign* of the permutation π ($+1$ if it is even, -1 if it is odd). These operators lead to the following permutation-invariant orthogonal subspaces:

$$\mathcal{H}_+^N \equiv \Pi_N^+[\mathcal{H}^N] \quad \text{and} \quad \mathcal{H}_-^N \equiv \Pi_N^-[\mathcal{H}^N], \tag{11}$$

which are called the *symmetric* and the *anti-symmetric* subspaces of $\mathcal{H}^N$, respectively; their vectors are predicated accordingly. These subspaces can, alternatively, be seen as generated by the symmetricised and anti-symmetricised versions of the products of basis vectors in $\mathcal{H}^N$. Only for $N = 2$ we have that $\mathcal{H}_-^N \oplus \mathcal{H}_+^N = \mathcal{H}^N$.

Following Bach ([1997], p. 19), we define three classes of statistical operators in $\mathcal{S}(\mathcal{H}^N)$, two of which are fundamental: A statistical operator $W \in \mathcal{S}(\mathcal{H}^N)$ is MB-*symmetric* (Maxwell–Boltzmann) iff W is a homogeneous product (a product of N identical statistical operators acting on $\mathcal{H}$); W is BE-*symmetric* (Bose–Einstein) iff $W = U_\pi W$ for every $\pi \in \mathbb{P}_N$; and W is FD-*symmetric* (Fermi–Dirac), or *anti-symmetric*, iff $W = \mathrm{sign}(\pi) U_\pi W$ for every $\pi \in \mathbb{P}_N$. Let $j(1), \ldots, j(N) \in \{1, 2, \ldots, d\}$, where $d = \dim \mathcal{H}$. We say that a statistical operator $W \in \mathcal{S}(\mathcal{H}^N)$ is *Paulian* iff for every basis $|\xi_{j(1)}\rangle, \ldots, |\xi_{j(N)}\rangle \in \mathcal{H}^N$,

$$\left\langle \xi_{j(1)} \otimes \xi_{j(2)} \otimes \cdots \otimes \xi_{j(N)} \right| W \left| \xi_{j(1)} \otimes \xi_{j(2)} \otimes \cdots \otimes \xi_{j(N)} \right\rangle = 0, \tag{12}$$

whenever at least two of the occurring basis vectors coincide, that is, whenever $j(k) = j(l)$ for some $k, l \in \{1, 2, \ldots, N\}$ (what we call for the sake of brevity 'Paulian', Bach ([1997], p. 22) calls 'satisfies the exclusion principle'). So in no linear expansion of a pure Paulian state does a tensor product of states occur in which a basis vector is repeated.

The *Symmetrisation Postulate* (SymP) states that for a composite system of $N \geq 2$ similar particles with N-fold direct product Hilbert space $\mathcal{H}^N$ (5), (i) the projectors $\Pi_N^\pm$ (10) are superselection operators, i.e. $\mathcal{H}_+^N$ and $\mathcal{H}_-^N$ (11) are sectors, (ii) states in these sectors have integer, respectively half-integer spin, (iii) only BE- and FD-statistical operators are state operators (Dichotomy).

SymP(i) implies that superpositions of pure BE- and FD-vectors, which lie in sectors $\mathcal{H}_+^N$ and $\mathcal{H}_-^N$, respectively, never correspond to physical states. SymP(ii) guarantees the correct assignment of spin with statistics (in accord with experimental results and in anticipation of the Spin-Statistics theorem of relativistic quantum field theory). SymP(iii) (Dichotomy) rules out the possibility of 'para-statistics', which we pass over.

Next, we report a few theorems.

Theorem. All MB-, BE- and FD-symmetric statistical operators are symmetric (6); the three symmetry properties of statistical operators of being MB-, BE- and FD-symmetric are invariant under partial tracing; hence the states of the subsystems inherit the symmetry properties from the state of the composite system.

Theorem. The BE- and FD-symmetric statistical operators are mutually exclusive, and the same holds for MB- and FD-symmetric ones; but the BE- and MB-symmetric statistical operators are not: their intersection consists of all and only pure MB-symmetric statistical operators. The MB-statistical operators are known in quantum optics as *coherent states*; quantum opticians speak of 'classical light'. They may, in this sense, have more than a purely formal significance, and of course, these three types of symmetric statistical operators do not really exhaust the set $\mathcal{S}_{\text{sym}}(\mathcal{H}^N)$ of all such operators: the para-statistical ones are not in that set.

Theorem. No MB-state and no BE-state is Paulian; every FD-state is Paulian (12). Thus, each basis vector of the anti-symmetric Hilbert space $\mathcal{H}^N_-$ is a superposition of product states, each of which occurs only once. This is the origin of the still widespread but misleading terminology, 'no two electrons are in the same state'. Misleading, because of the following theorem.

Theorem. All marginal statistical operators of an arbitrary symmetric statistical operator are symmetric and they are all identical. Hence, all single-particle states are identical whenever W is symmetric; i.e. for all $j, k \in \{1, 2, \ldots, N\}$ and every symmetric W,

$$W_j = W_k. \tag{13}$$

All N similar particles always have an identical (mixed) physical state.

Let A be some magnitude operator pertaining to subsystem S_j of composite system S, in other words, $A \in M_j(\mathcal{H})$. Then that very same physical magnitude $\mathcal{A}$ is represented by the following operator acting on $\mathcal{H}^N$:

$$A_j \equiv \underbrace{1 \otimes 1 \otimes \cdots \otimes A \otimes 1 \otimes \cdots \otimes 1 \otimes 1}_{N \text{ factors}}, \tag{14}$$

where A appears as the jth $\otimes$-factor.

Theorem. All absolute expectation values (EV) for these (non-symmetric) operators are identical in every symmetric state, i.e. for all $j, k \in \{1, 2, \ldots, N\}$

and every symmetric W,

$$\text{EV}(A_j;\, W) = \text{Tr}(A_j W) = \text{Tr}(A_k W) = \text{EV}(A_k;\, W). \tag{15}$$

The same holds by implication for conditional state operators and concomitant conditional expectation values. More precisely, define the *conditional statistical operator* W_P as $PWP/\text{Tr}(PW)$, which is 'conditioned' on the 'event' $P \in \mathcal{P}(\mathcal{H}^N)$. Think of measuring physical magnitude $\mathcal{B}$, and P being the spectral projector of the corresponding operator B that represents a measurement outcome being in interval $\Delta \subset \mathbb{R}$; hence P is $P^B(\Delta)$. The *conditional expectation value* of $\mathcal{A}$ conditioned on P is then defined as the expectation value of $\mathcal{A}$ in the conditional state operator W_P,

$$\text{EV}(A;\, W|P) \equiv \text{Tr}(AW_P) = \text{Tr}(APWP)/\text{Tr}(PW). \tag{16}$$

Then for every symmetric W and every projector P,

$$\text{EV}(A_j;\, W|P) = \text{Tr}(A_j W_P) = \text{Tr}(A_k W_P) = \text{EV}(A_k;\, W|P). \tag{17}$$

We call to mind that all Born probability distributions, conditional and absolute, of physical magnitude operators falling under the sway of a spectral theorem are determined by the expectation values of their spectral projectors; and *vice versa*. Henceforth, we shall mostly speak about probability measures without any loss of generality and rarely mention expectation values.

2.3 Fermions and bosons

There is no universal agreement in physics whether to define 'fermions' (i) *as* particles that obey Fermi–Dirac statistics or are always in FD-states—as the term was historically introduced—or (ii) *as* particles that have half-integral spin; and *mutatis mutandis* for 'bosons'. We adopt the original terminology (i), thereby following Dirac, Heisenberg, Wigner, Friedrichs, Tomonaga, Lipkin, Schwinger, Haag, Weinberg, Cohen-Tannoudji, Sakurai and many others: *fermions* are by definition physical systems that are always in FD-states and *bosons* are always in BE-states. In this context, we call attention to an old result of Wigner: a bound state with an even number of fermions or with any number of bosons is a BE-state, and a bound state with an odd number of fermions is an FD-state (see Ehrenfest and Oppenheimer [1931]).

The only irreducible unitary representations of the rotation group (a subgroup of the Galilei group) are those with integer or half-integer spin. The division of particles into these two categories is therefore exhaustive. In the light of the Spin-Statistics theorem, it would be enough from an empirical point of view to restrict the investigation of bosons to the integer spin case, but from a conceptual point of view, this is to shift from Galilean spacetime to Minkowski spacetime, and therefore to introduce relativity theory (Lorentz symmetry is necessary for the proof of the Spin-Statistics theorem)—a theory remote from the conceptual questions we are investigating. It would be perfectly appropriate, for example, to investigate the question of the discernibility of bosons when bosons are described by spinors.

2.4 Physical properties

We formulate one standard and one minimalist 'interpretational' postulate of QM concerning *properties*. The Strong Property Postulate was explicitly endorsed by, for instance, Dirac and von Neumann, and it is often tacit in expositions of QM. On the basis of this postulate, properties are ascribed to or withheld from physical systems and Schrödinger's immortal cat enters the stage.

Let us first point out that statistical operators that are not 1-dimensional projectors can correspond to eigenstates of an operator A on $\mathcal{H}$, although they are not pure states. Let $\mathcal{H}(A;a) \subseteq \mathrm{Dom}(A)$ be some eigenspace of A: for every $|\phi\rangle \in \mathcal{H}(A;a)$: $A|\phi\rangle = a|\phi\rangle$, where $a \in \mathbb{C}$; and let $P^A(a) \in \mathcal{P}(\mathcal{H})$ project onto $\mathcal{H}(A;a)$. Then we define $W \in \mathcal{S}(\mathcal{H})$ to be an *eigenoperator* of A with eigenvalue a iff W obeys the following generalisation of the eigenvector equation of A:

$$AW = aW, \tag{18}$$

for the Hilbert vectors that lie in the intersection of the range of W and the domain of A. Then W represents a pure (mixed) *eigenstate* iff W represents a pure (mixed) physical state and is an eigenoperator (18); a physical state is *pure* iff the representing W is a 1-dimensional projector, otherwise *mixed*. The virtue of this definition is that it makes *mixed eigenstates* possible. Easy theorem: all convex combinations of projectors that project onto $\mathcal{H}(A;a)$, or onto a proper subspace of $\mathcal{H}(A;a)$, are eigenoperators of A having eigenvalue a; then for every such projector $P \in \mathcal{P}(\mathcal{H})$, it holds that $P \preccurlyeq P^A(a)$, where '$\preccurlyeq$' is the partial ordering on the lattice $\mathcal{P}(\mathcal{H})$.

We represent a *quantitative physical property* mathematically by the ordered pair $\langle A, a\rangle$, where A is the operator which corresponds to physical magnitude $\mathcal{A}$ and $a \in \mathbb{C}$ is its value. According to the *Strong Property Postulate* (StrPP), a physical system S having state operator $W \in \mathcal{S}(\mathcal{H})$ possesses quantitative physical property $\langle A, a\rangle \in M_S(\mathcal{H}) \times \mathbb{C}$ iff W is an eigenstate of A that belongs to eigenvalue a. If system S is in some eigenstate of A that belongs to eigenvalue a, then the probability of finding a when $\mathcal{A}$ is measured equals 1 (implied by the Born probability measure). Then it is a small step to the following, uncontroversial conjunct of StrPP (which incidentally closely resembles the sufficiency condition to be an element of physical reality due to Einstein, Podolsky and Rosen), the *Weak Property Postulate* (WkPP): if a physical system S is in an eigenstate of operator $A \in M_S(\mathcal{H})$, corresponding to physical magnitude $\mathcal{A}$, having eigenvalue a, then S possesses quantitative physical property $\langle A, a\rangle$.

The Projection Postulate can be deduced from StrPP in conjunction with the proposition that a measurement yielding value a for physical magnitude $\mathcal{A}$ permits one to ascribe physical property $\langle A, a\rangle$ to the physical system that is being measured *at or immediately after the time of* measurement. The converse of WkPP in conjunction with WkPP is StrPP and it should be considered just as controversial as the Projection Postulate and therefore is, unlike WkPP, far from minimalist.

WkPP implies that a physical system S always has the same quantitative properties associated with superselected physical magnitudes, because S is always in the same common eigenstate of the superselected operators. We call these possessed quantitative physical properties *superselected*. We are now in a position to say exactly what we mean by 'similar', that we have been using throughout: namely, physical systems, e.g. particles, are *similar* iff they have the same superselected quantitative physical properties—most physicists call similar particles 'identical', but we prefer Dirac's terminology for obvious reasons (see Dirac [1947], pp. 207 ff.). The set of quantitative physical properties $\langle A, a\rangle$ that S possesses in state W we call its *property set*; denoting it by Prop(S, W), we can express StrPP and WkPP succinctly as follows:

$$\begin{aligned} &\text{StrPP} : \text{Prop}(S, W) = \{\langle A, a\rangle \in M_S(\mathcal{H}) \times \mathbb{C} \mid AW = aW\}; \\ &\text{WkPP} : \text{Prop}(S, W) \supseteq \{\langle A, a\rangle \in M_S(\mathcal{H}) \times \mathbb{C} \mid AW = aW\}. \end{aligned} \tag{19}$$

Note that the property set Prop(S, W) is never empty on either property postulate, because it always contains the mentioned superselected properties. The traditional terminology is to call the superselected properties 'state-independent', but here we take the view that every physically relevant feature of S may be codified in its physical state.

We also want to state explicitly the following semantic fact, or semantic conditional (SemC), which we shall need to appeal to in Sections 5.2 and 5.3: when talking of a physical system at a given time, if we ascribe to it a property, we mean this in an unqualified sense—and thus ascribe to it at most *one* quantitative physical property associated with physical magnitude $\mathcal{A}$ at that time,

$$\text{(SemC) } \textit{If physical system } S \textit{ possesses both } \langle A, a\rangle \textit{ and } \langle A, a'\rangle, \textit{ then } a = a'. \tag{20}$$

A person cannot possess two different weights at the same time, an elementary particle, if it has a position at a time, cannot possess another distinct position which it has at that time, etc.; Equation (20) expresses this in the language of QM. In other words: if S possesses quantitative physical property $\langle A, a\rangle$, then

S does *not* posses property $\langle A, a' \rangle$ for every $a' \neq a$. Statement (20) is neither a tautology nor a theorem of logic, but it seems absurd to deny it all the same. (Notice that Equation (20) follows from StrPP—but not from WkPP—since eigenvectors belonging to different eigenvalues are orthogonal.)

Finally, a note on *relations*. When physical system S is (taken as) a composite system, built up of other physical systems, some of the properties of S determine and are determined by relations of its constituents. The fact that the distance between one's eyes is 6 cm, say, fixes not only a particular relation between the eyes, but also a property of one's face, of which the eyes are parts. Consequently, both WkPP and StrPP, although giving rise to *properties* of S and of its subsystems (expressed by monadic predicates), equally provide conditions for the ascription of *relations* among constituents of S; the magnitude $\mathcal{A}$ may itself be relational (as in 'relative distance'), and likewise the operator A corresponding to it. (This is why one does not need to introduce 'relation postulates' in addition to property postulates.) Typically, however, as we shall see in the next section, where authors have made use of WkPP or StrPP, they have used them only to consider *monadic properties*—that is to say, from our point of view, they have not made use of either property postulate to ascribe relations among constituents of S, which is the key step that we shall be taking in this paper.

2.5 Varieties of quantum mechanics

For the sake of convenience and future reference, we distinguish a number of quantum-mechanical theories, based on the postulates expounded in the previous sections. Theory QM^- is based on the State Postulate (StateP), the Weak Magnitude Postulate (WkMP), SemC (20), the Symmetrisation Postulate (SymP), and the Weak Property Postulate (WkPP). What we call 'standard quantum mechanics' (QM) is obtained when we add to QM^-: the Probability Postulate (Born probability measure over measurement outcomes for pure states and von Neumann's extension to mixed states; ProbP) and the Schrödinger equation for the evolution over time if no measurements are performed; and then replace WkMP with the stronger Standard Magnitude Postulate (StMP). (Beware: 'QM' in small capitals abbreviates 'quantum mechanics', whereas 'QM' in capitals is from now on the name of a particular quantum-mechanical theory.) Adding further to QM the Projection Postulate and the Strong Property Postulate (StrPP) yields QM^*, a fairly popular formulation of quantum mechanics among philosophers. All results we obtain rely on theories logically weaker than QM and QM^*, namely QM^- (sometimes adding ProbP), because StrPP and the Projection Postulate never enter the arguments for our results. Since we *deduce* PII, our results hold in QM and QM^* as well.

3 Analysis of Arguments

3.1 Analysis of the Standard Argument

The currently dominant view claims the incompatibility of QM and Leibniz's PII. Margenau's ([1944]) argument for this conclusion concerned a specific two-particle system; subsequent arguments have been more general, although most of them focussed on the physical magnitude *spin* and proceeded at a disappointingly low level of generality. All arguments have the following logical structure (see all references listed in Section 1.3 of propounders of the currently dominant view).

We consider a composite physical system of $N \geq 2$ similar particles; it falls under the sway of Symm in QM. We also consider two premises, in order to distinguish sharply those aspects of quantum mechanics that pertain to measurement and probability. The *Categorical Discernibility Premise* (CatDiscPr) states that physical systems are only discernible by: (i) their physical states and (ii) their monadic physical properties. The *Probabilistic Discernibility Premise* (ProbDiscPr) states that physical systems are only discernible by (iii) their measurement outcomes, (iv) their absolute probability measures over their measurement outcomes, and (v) their conditional probability measures over their measurement outcomes. The *Discernibility Premise* (DiscPr) states that physical systems are only discernible in ways (i)–(v) above.

As remarked in Section 3, WkPP and StrPP insofar as they are used explicitly in the Standard Argument are applied only for deducing the monadic properties figuring in (ii) of CatDiscPr; relations, if considered at all, enter only with (v) of ProbDiscPr as conditional probabilities. This is the key limitation of the argument that follows.

The argument leading to the currently dominant view takes the form, which we call the *Standard Argument*,

$$(\text{QM} \wedge \text{DiscPr}) \vdash \neg\text{PII} \vdash (\neg\text{PII-A} \wedge \neg\text{PII-R}). \tag{21}$$

One proves that with respect to all features (i)–(v) mentioned in the Discernibility Premise (DiscPr), all N particles are *indiscernible* (exercise: collect the relevant theorems from Section 2.2 and prove this); whence DiscPr and PII are inconsistent. The Standard Argument accepts QM and DiscPr, and then is compelled to reject PII (21).

We point out that sharing features (i)–(iv) of DiscPr, which concern shared *properties* of each of the N particles, leads to a conflict with PII-A; and that sharing feature (v) of ProbDiscPr, which concerns relations, leads to a conflict with PII-R. Ascribing an absolute probability measure to a particle is, of course, not ascribing to it a physical property of the conservative kind we have been considering, e.g. $\langle A, a\rangle$, but it is *ascribing something to the particles* and that is distinct from *relating the particle to another particle*; ascribing an absolute probability can be dealt with logically by means of a predicate having one free particle variable, whereas ascribing a conditional probablity calls for a predicate having two free particle variables. As Huggett ([2003], pp. 242–3) has pointed out, one can also conditionalise on joint measurement outcomes rather than single ones without threatening the indiscernibility of the N particles, so that even polyadic predicates—expressing n-ary relations—can be considered. For example, a ternary relation, between three particles, 1, 2 and 3, can involve the following conditional probability of finding value a for $\mathcal{A}$ in interval $\Delta_1 \in \mathcal{B}(\mathbb{R})$ when measured on particle 1 conditioned on jointly finding value b for $\mathcal{B}$ in interval Δ_2 when measured on particle 2 and value c for $\mathcal{C}$ in interval Δ_3 when measured on particle 3,

$$\Pr_W(a \in \Delta_1 \mid b \in \Delta_2 \wedge c \in \Delta_3). \tag{22}$$

We consider all such probability measures included in (v) of ProbDiscPr.

We shall shortly exhibit the general flaw of the Standard Argument (21), but in conclusion of this section, we consider two prominent criticisms of the Standard Argument that have already appeared.

3.2 Van Fraassen's analysis

As announced earlier, van Fraassen's criticism ([1991], pp. 423–6) of the Standard Argument (21)—clearly anticipated in (van Fraassen [1972], [1984])—consists in observing its (often only implicit) reliance on what we have called StrPP, which is an *interpretational* assumption that one can reject without affecting the empirical content of quantum mechanics. A truly empiricist criticism. As soon as it is admitted that at most WkPP is what is actually needed in any concrete application of the theory, and that StrPP neither enriches nor impoverishes the theory as it is actually used, then any claim to the effect that PII has been refuted *empirically* collapses. What the Standard Argument (21) establishes is only that QM*, of which the empirically superfluous postulate StrPP is part and parcel, violates PII; indeed, weakening StrPP to WkPP salvages PII-A and PII quite easily. When QM is taken as QM^- or QM, it stands in no *logical* conflict with PII.

Van Fraassen ([1991], pp. 427–9) further added another empirically superfluous postulate to QM^- (besides or even instead of WkPP), e.g. some modal property postulate (ModPP for brevity, the precise content of which we can gloss over). The ensuing 'three-fermion model with individuation' demonstrably *obeys* PII-A because some ascribed properties discern the particles absolutely, and therefore the conjunction of QM^-, ModPP and DiscPr also *obeys* PII. The replacement of StrPP with ModPP vividly invalidates the Standard Argument (21), but its replacement with WkPP invalidates it too.

The conclusion that van Fraassen draws from his analysis is that whether PII stands or falls in quantum mechanics depends, broadly speaking, on the interpretation of the theory, and *that* is a matter of philosophical inquiry—or metaphysical speculation—and not just a matter of scientific research. As far as empirical science is concerned, PII is simply not in contention (van Fraassen, private communication, 10th June 2008). We would like to redirect this conclusion by pointing out that it is a perfectly legitimate empiricist view to insist that DiscPr should be limited to ProbDiscPr alone, whereupon in line with the Standard Argument (21) PII is refuted after all, so that empirical indiscernibles are identical. The right conclusion to draw is rather that, indeed, not only property postulates (StrPP, WkPP, ModPP) are a matter of interpretation but also discernibility premises, including ProbDiscPr—they *do not even follow from* QM*, which is logically the strongest variety of quforantum mechanics. To run a little ahead of ourselves, in Section 5 we shall see how a better analysis of the discernibility premises than the ones so far offered allows fermions (and, in the probabilistic sense, bosons) to be discerned already in QM^- (respectively, QM^- and ProbP). That is, we shall reach this conclusion with WkPP (respectively, ProbP) as the only 'interpretational' postulate. This should confute the view that PII is not in contention in physical science. Only an uncontroversial whiff of interpretation (WkPP) will be needed to reach the mentioned conclusion; and indeed, one reaches it too from the opposing standpoint, according to which one should replace WkPP with the Probability Postulate (ProbP): thereby one proves the *probabilistic* discernibility of fermions instead—and, indeed, of bosons. PII is definitely in contention: anyone who accepts QM^- (and ProbP) will be compelled to accept PII.

3.3 Massimi's analysis

The core of Massimi's ([2001], pp. 318–26) criticism of Margenau's argument is as follows. *Leibniz's* PII reads in QM, according to Massimi, that particles having the same 'ontological states' are identical (PII-A). She submits that PII-A has a *presupposition*, namely that the particles 'have ontological states', which is in our terminology,

$$\textit{The particles have non-empty property-sets}, \tag{23}$$

or in terms of the sets in Equation (19), for every particle $j \in \{1, 2, \ldots, N\}$,

$$\text{Prop}(j, W) \neq \varnothing. \tag{24}$$

She then tacitly invokes a Strawsonian account of presuppositions, according to which propositions having a presupposition are neither false nor true whenever the presupposition is not true. She claims that the presupposition of PII-A is statement (23). With empty property sets we can neither speak truly nor speak falsely about absolute discernibility. In Quinean terms, we then have a truth gap. She then goes on to claim that the property sets of the fermions actually are empty, so that PII-A cannot be applied; it is, Massimi ([2001], p. 324) emphasises, 'simply *not applicable*', and therefore Margenau's conclusion that PII-A is in conflict with QM no longer follows—the argument against PII-R stands however unabashed.

Parenthetically, on a Russellian account of presuppositions, Massimi's criticism still stands. For Russell, the presence of a presupposition of some proposition ϕ is a surface grammar phenomenon: the correct logical analysis is to consider the presupposition as the antecedent of a material conditional that has ϕ as its consequent. Then one would formulate PII-A with presupposition (23) as follows: *if* the particles have non-empty property sets and these property sets coincide, *then* the particles are identical. In this case, however, if statement (23) is false, as Massimi claims, the entire antecedent of the material conditional in the previous sentence is false, and we cannot use *it* to deduce by *modus ponendo ponens* that all N particles are identical ('$N = 1$'), which one needs in order to have a contradiction with '$N \geq 2$'. On this Russellian account, then, one could also claim that PII does not apply. Massimi's argument is not wedded to a Strawsonian analysis and therefore logically quite robust.

We disagree with Massimi's assessment nonetheless, for the following reasons, which we present in increasing order of importance.

First, state ascriptions to physical systems are expressed by monadic predicates having physical system variables. The physical state of a physical system S can generally be considered as the unique *dynamical property* of S, governed by the Schrödinger equation, which embodies the 'dynamical content' of QM. This unique dynamical property determines all probability distributions of all physical magnitudes. If a property is *dynamical*, then surely it is also *physical*. Massimi does not consider this. She claims that PII-A only applies to 'ontological states' (property sets), which are empty because the states of the particles are (improper) mixtures and then, by (her formulation of) StrPP, no physical properties can be ascribed to the particles. Evidently, Massimi tacitly operates with a notion of 'ontology' which is such that physical properties that are dynamical *do not count* as 'ontological', whilst all other physical properties, notably the quantitative physical properties of the form $\langle A, a\rangle$, associated with physical magnitude $\mathcal{A}$, *do count* as 'ontological'. This strikes us as arbitrary if not indefensible.

Secondly and more importantly, even if one grants Massimi that the property sets of all particles are empty (23), one could still make a case that this circumstance makes the sufficient condition in PII-A *true*. This is because the logical form of this sufficient condition for identity is '$Ma \longleftrightarrow Mb$', where '$Ma$' is any monadic predicate that expresses that particle $\boldsymbol{a}$ has some physical property, and similarly for 'Mb'. When particles $\boldsymbol{a}$ and $\boldsymbol{b}$ do *not* have any physical property expressed by M (because their property sets are empty), then '$\neg M\boldsymbol{a} \wedge \neg M\boldsymbol{b}$' is true; this implies that '$M\boldsymbol{a} \longleftrightarrow M\boldsymbol{b}$' is true. But then the sufficient condition for identity in PII-A is *true*—admittedly vacuously true, but true nonetheless, and certainly not false—and we conclude that particles ***a*** and ***b*** are identical. (The road via presuppositions suddenly seems ill-motivated.) As before, we then have proven the incompatibility of QM* (which includes StrPP, DiscPr and PII-A (21)), thus vindicating the currently dominant view, *albeit* vacuously, even when the property sets are empty.

But *thirdly* and most importantly, the property sets *are not empty*. We can conclude this on the basis of WkPP (Section 2.4). The states of the particles are always eigenstates of all superselected magnitude operators (spin-magnitude, mass, charge, etc.; see Section 2), and therefore the particles possess the associated properties permanently. Presupposition (23) is *true*. Of course, these sets of superselected properties are the same for all the composing particles, so that the particles cannot be discerned by means of any of them; and if these superselected properties were the only physical properties, as StrPP implies (Section 2.4), then again the conclusion would follow that the currently dominant view is correct: PII-A is applicable and violated.

4 The Logic of Identity and Discernibility

By 'the language of QM' one usually means a fraction of English enriched with the symbolic language of that part of mathematics that is being used in QM (all sorts of numbers, partial differential equations, Hilbert space, operators, wave functions, matrices, probabilities) and with quantum-mechanical vocabulary (states, magnitudes, time, properties, particles, physical systems, subsystems, composite systems, energy, momentum, position, etc.). The deductive apparatus can be taken to be elementary predicate logic. In philosophy of physics, the language of QM is often enriched with philosophical terminology too (perspective, branch, consciousness, holism), which we shall not be needing here. Although we are not going to spell out a formalised version of the language of QM, call it $\mathcal{L}_{\mathrm{QM}}$, we are going to devote a few sentences to it in order to achieve a mandatory state of clarity about the logic of identity and discernibility.

4.1 The language of quantum mechanics

A language is *elementary* iff first, it quantifies only over its object variables (typical of a first-order formal language) and not over anything else, notably not over its predicates (typical of a second-order formal language), and secondly, it has a finite *lexicon*, that is, a finite number of primitive predicates and names. The language of QM, $\mathcal{L}_{\mathrm{QM}}$, will be elementary in this sense.

First of all, we begin with some weak set theory sufficient to erect all the mathematics that ever will be needed in QM. The gold standard is Zermelo–Fraenkel set theory (ZFC). We then can produce natural numbers, real numbers, complex numbers, integrals, Hilbert spaces, operators, differential equations, von Neumann rings, C^*-algebras, convex sets, lattices, probability measures, matrices and all the mathematics that ever will be needed in QM, in physics and in science in general. The elementary language of ZFC, as of all elementary set theories, is $\mathcal{L}_\in$; it has only set variables and a single primitive dyadic predicate: the membership relation ($\in$). Then $\mathcal{L}_{\rm QM}$ has $\mathcal{L}_\in$ as a sub-language.

Secondly, in order to obtain $\mathcal{L}_{\rm QM}$, we enrich $\mathcal{L}_\in$ with physical system variables (S, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{p}$), physical state variables ($\mathfrak{P}, \mathfrak{Q}, \mathfrak{R}$) and physical magnitude variables ($\mathcal{A}, \mathcal{B}, \mathcal{C}$). This should be enough. We also need a primitive dyadic predicate between physical systems of 'is a subsystem of', in order to speak in $\mathcal{L}_{\rm QM}$ of *subsystems* and *composite systems*; this dyadic predicate of subsystemhood will obey some mereological axioms (for rigour and details, see Muller [1998], Chapter III; [unpublished]). The postulates of QM tell us how to connect physical states and magnitudes intimately to mathematical entities that are all defined in $\mathcal{L}_\in$ and whose existence is proved in ZFC (cf. Muller [1998], Chapters II, IV for how this proceeds in detail).

Thirdly and lastly, we assume that $\mathcal{L}_{\rm QM}$ does not have names, only (the four mentioned sorts of) variables.

We have now acquired a sufficiently clear view of $\mathcal{L}_{\rm QM}$ in order to proceed at a level of moderate rigour to define the concepts of identity and discernibility.

4.2 Identity of physical systems

The concept of identity (equality, =) for physical systems can be handled in two different ways: either as a primitive dyadic predicate or as a defined one. When taken as primitive, it is governed by the Frege axioms of 1879,

$$\begin{aligned} &\textit{Reflexivity}: \quad \vdash \boldsymbol{a} = \boldsymbol{a}, \\ &\textit{Substitutivity}: \vdash (\boldsymbol{a} = \boldsymbol{b} \wedge \varphi(\boldsymbol{a})) \longrightarrow \varphi(\boldsymbol{b}), \end{aligned} \tag{25}$$

where $\varphi(\cdot)$ is a sentence variable of $\mathcal{L}^{=}_{\rm QM}$ having one free variable (the superscript '=' expresses that identity is primitive). On the basis of the Frege axioms only, one proves that '=' is an equivalence relation, that it is determined up to logical equivalence, and that it implies '$\mathcal{L}^{=}_{\rm QM}$-indiscernibility', on which we briefly elaborate next.

Let M be some monadic predicate of $\mathcal{L}_{\rm QM}$, primitive or defined (where '$\mathcal{L}_{\rm QM}$' now indicates that we leave it undetermined whether identity belongs to its lexicon). Then we say that two physical systems $\boldsymbol{a}$ and $\boldsymbol{b}$ are *M-indiscernible* iff M either holds for both of them or it does not hold for both of them,

$$\mathsf{Ind}(\boldsymbol{a}, \boldsymbol{b}; M) \quad \text{iff} \quad (M\boldsymbol{a} \longleftrightarrow M\boldsymbol{b}). \tag{26}$$

Now let R be some polyadic predicate of $\mathcal{L}_{\mathrm{QM}}$. Then we say that physical systems $\boldsymbol{a}$ and $\boldsymbol{b}$ are *R-indiscernible* iff for every position in R

$$R \cdots \boldsymbol{a} \cdots \longleftrightarrow R \cdots \boldsymbol{b} \cdots, \tag{27}$$

where the free variables of the resulting open sentences are treated in the normal manner under deduction. For a dyadic predicate this yields sentences of the following form for R-indiscernibility:

$$\mathsf{Ind}(\boldsymbol{a}, \boldsymbol{b}; R) \quad \text{iff} \quad \forall \boldsymbol{c}(R\boldsymbol{ac} \longleftrightarrow R\boldsymbol{bc}) \wedge \forall \boldsymbol{d}(R\boldsymbol{da} \longleftrightarrow R\boldsymbol{db}). \tag{28}$$

We take the condition on the right-hand side (and nothing less general) as the correct formalisation (in the case of dyadic predicates) of the informal statement 'two objects have the same relation R'. Observe, when R is the identity relation $=$, that physical systems are $=$-indiscernible in $\mathcal{L}^{=}_{\mathrm{QM}}$ iff they are identical (this is a theorem of logic),

$$\vdash \mathsf{Ind}(\boldsymbol{a}, \boldsymbol{b}; =) \longleftrightarrow \boldsymbol{a} = \boldsymbol{b}. \tag{29}$$

Taking the conjunction of all such M- and R-indiscernibility conditions for every primitive predicate in $\mathcal{L}_{\mathrm{QM}}$, we obtain the condition we call *lexicon-indiscernibility* relative to $\mathcal{L}_{\mathrm{QM}}$ and denote it by $\mathsf{LexInd}(\boldsymbol{a}, \boldsymbol{b}; \mathcal{L}_{\mathrm{QM}})$. Define $\mathcal{L}_{\mathrm{QM}}$-*indiscernibility* of $\boldsymbol{a}$ and $\boldsymbol{b}$, denoted as $\mathsf{Ind}(\boldsymbol{a}, \boldsymbol{b}; \mathcal{L}_{\mathrm{QM}})$, as indiscernibility in the entire language $\mathcal{L}_{\mathrm{QM}}$, i.e. iff for any instantiation of open sentence $\varphi(\cdots)$ and for every position

$$\varphi(\cdots, \boldsymbol{a}, \cdots) \longleftrightarrow \varphi(\cdots, \boldsymbol{b}, \cdots). \tag{30}$$

Evidently, $\mathcal{L}_{\mathrm{QM}}$-indiscernibility implies lexicon-indiscernibility in $\mathcal{L}_{\mathrm{QM}}$; conversely, one proves by complete induction on the complexity of $\varphi(\cdot)$ that lexicon-indiscernibility implies $\mathcal{L}_{\mathrm{QM}}$-indiscernibility. Hence, physical systems are lexicon-indiscernible iff they are $\mathcal{L}_{\mathrm{QM}}$-indiscernible.

An immediate corollary of substitutivity (25) is that identical physical systems are $\mathcal{L}_{\mathrm{QM}}$-indiscernible—the indiscernibility of identicals, also known as *Leibniz's law*. The converse fails: from the assumption that every instantiation of the indiscernibility schema (30) holds for $\boldsymbol{a}$ and $\boldsymbol{b}$, *provided the instantiation never includes identity between physical systems*, one cannot deduce that $\boldsymbol{a}$ and $\boldsymbol{b}$ are identical. That is to say, employing the notation $\mathcal{L}^{\neq}_{\mathrm{QM}}$ (for the language $\mathcal{L}^{=}_{\mathrm{QM}}$ with identity deleted),

$$\mathsf{LexInd}\big(\boldsymbol{a}, \boldsymbol{b}; \mathcal{L}^{\neq}_{\mathrm{QM}}\big) \longrightarrow \boldsymbol{a} = \boldsymbol{b} \tag{31}$$

is not a logical theorem in language $\mathcal{L}^{=}_{\mathrm{QM}}$. Is sentence (31) perhaps a correct formulation of PII in $\mathcal{L}_{\mathrm{QM}}$?

No way. Far too weak. In QM, PII should say that *physically* indiscernible physical systems are identical,

$$\mathsf{PhysInd}(\boldsymbol{a}, \boldsymbol{b}) \longrightarrow \boldsymbol{a} = \boldsymbol{b}, \tag{32}$$

where $\mathsf{PhysInd}$ involves a subset of the primitive predicates of $\mathcal{L}^{\neq}_{\mathrm{QM}}$—the full language $\mathcal{L}^{\neq}_{\mathrm{QM}}$ (and *a fortiori* $\mathcal{L}^{=}_{\mathrm{QM}}$) may well sanction mathematical distinctions without any physical meaning (cf. the next section). The weaker the sufficient condition $\mathsf{PhysInd}$, the stronger PII in QM (32). Obviously, Equation (32) is not a theorem of logic either. From now on, by 'PII in QM' we mean Equation (32). But, of course, its converse—the physical indiscernibility of identical physical systems—remains an instance of Leibniz's law, and hence is a logical theorem regardless of the choice of $\mathsf{PhysInd}(\boldsymbol{a}, \boldsymbol{b})$,

$$\vdash \boldsymbol{a} = \boldsymbol{b} \longrightarrow \mathsf{PhysInd}(\boldsymbol{a}, \boldsymbol{b}). \tag{33}$$

In the next section we shall delve into $\mathsf{PhysInd}(\boldsymbol{a}, \boldsymbol{b})$. We close this section by pointing out that it is also possible *to define* identity and that this has some repercussions for identity, one of which we would like to mention.

As first recognised by Hilbert and Bernays in their *Grundlagen der Mathematik* ([1934], pp. 380–2), and promulgated and pursued by Quine ([1969]), identity in any elementary language can be *defined* as lexicon-indiscernibility. Quine ([1967], pp. 13–4) illustrated the point by the case of an elementary language with a single primitive dyadic predicate (such as the language of set theory, $\mathcal{L}_{\in}$, with the membership relation '$\in$' as the only primitive predicate), where identity thus defined, i.e. as $\in$-indiscernibility, is simply seen to obey the Frege axioms in $\mathcal{L}_{\in}$. In $\mathcal{L}^{\neq}_{\mathrm{QM}}$, we then should analogously propound the following *definition* of identity:

$$\boldsymbol{a} = \boldsymbol{b} \quad \text{iff} \quad \mathsf{LexInd}\big(\boldsymbol{a}, \boldsymbol{b}; \mathcal{L}^{\neq}_{\mathrm{QM}}\big). \tag{34}$$

So it seems that the choice between primitive identity and defined identity is an arbitrary convention.

However, defining identity does not come for free. Some of the semantic consequences arguably are repugnant to reason, like allowing models where two names related by the identity relation refer to *distinct* objects of the meta theory. But if we were to take this course, we would be bound to take $\mathsf{PhysInd}$ as involving a *proper* subset of the primitive predicates of $\mathcal{L}^{\neq}_{\mathrm{QM}}$—on pain of turning PII into a logical triviality. We want at the very least *the logical possibility* of formulating a consistent theory in $\mathcal{L}^{\neq}_{\mathrm{QM}}$ that violates PII (31) as well as *the logical possibility* of formulating a consistent theory in $\mathcal{L}^{\neq}_{\mathrm{QM}}$ that obeys PII.

Granted that $\mathsf{PhysInd}$ *is* logically weaker than $\mathsf{LexInd}(\cdot, \cdot; \mathcal{L}^{\neq}_{\mathrm{QM}})$—as we shall use it: *enormously* weaker—the choice between identity as defined and as primitive becomes irrelevant for our purposes. Since nothing hangs on it, we simply write '$\mathcal{L}_{\mathrm{QM}}$' and allow the reader to choose.

4.3 Indiscernibility of physical systems

In this section, we consider in general terms the nature of $\mathsf{PhysInd}(\cdot, \cdot)$—equivalently, of the (proper) subset $\mathfrak{S}$ of the predicates in $\mathcal{L}_{\mathrm{QM}}$ denoting the physically meaningful properties and relations of QM,

$$\mathsf{Ind}(\boldsymbol{a}, \boldsymbol{b}; \mathfrak{S}) \quad \text{iff} \quad \mathsf{PhysInd}(\boldsymbol{a}, \boldsymbol{b}). \tag{35}$$

Happily, this work has already largely been done in Section 2.4: the latter are clearly those that figure—or that *ought* to figure—in the *Discernibility Premise* (DiscPr, see Section 3.1). But note that since we claim to prove PII in QM, it is enough that we find elements in $\mathfrak{S}$ under which similar particles are discerned, to which end it is enough if we provide *sufficiency* conditions for membership in $\mathfrak{S}$ (and hence for DiscPr)—a *criterion of admissibility*. We do not have to define this set $\mathfrak{S}$ (and DiscPr) outright. In contrast, those who claim to prove that PII fails in QM must provide *sufficient and necessary* conditions for properties and relations entering into DiscPr (and prove by exhaustion that no such property or relation can discern similar particles). Correspondingly, we may base our analysis on WkPP, a *sufficiency* condition to count as a possessed property, whereas the Standard Argument required StrPP, which also provides a *necessary* (but metaphysically more heavily loaded) condition.

However, we have already noted that the conditions listed in DiscPr are deficient in an important respect—the lack of mention of relations other than conditional probabilities. We propose to add, in addition to (i)–(v) of Section 3.1, (vi) *all physically meaningful relations*. Iff the definition of a relation involves probability, we call the relation *probabilistic*, otherwise *categorical*. We should then, more finely, also add to (i)–(ii) of CatDiscPr (vi.a) *all physically meaningful categorical relations*, and to (iii)–(v) of ProbDiscPr (vi.b) *all physically meaningful probabilistic relations*. Among the former, naturally, are relations among constituents of S as defined by properties (ii) of S, as sanctioned by StrPP or WkPP.

Of course, it is open to proponents of the currently dominant view to reject these extensions of CatDiscPr and ProbDiscPr, and hold on to those listed earlier (Section 3.1). But the move will be *ad hoc*, failing some principled rejection of relations, for the relations we shall appeal to (at least in the categorical case) have an uncontroversial physical meaning, as we shall see in the next section.

When we agree to this *enlargement* of the set of relations permitted in DiscPr (and hence the set of predicates $\mathfrak{S}$ that defines $\mathsf{PhysInd}$), forced as it is by purely logical considerations, and given that we are seeking only sufficiency conditions (criteria of admissibility), we should add as many restrictions to them as possible. And here there is an important worry when it comes to *probabilistic* relations—and properties. That is, whilst, at least at first sight, properties and relations among constituents sanctioned by WkPP are surely admissible—i.e. those entering in CatDiscPr—those entering in ProbDiscPr, sanctioned by the Probability Postulate, just as clearly involve the question of just what quantum probabilities really are—and thereby propels us back to central interpretative problems of quantum physics, among them the *problem of measurement*. The status of the Projection Postulate is a case in point. (For a recent introduction to the problem of measurement, see Saunders [2008].)

The more careful of those who appeal to this class of properties and relations have recognised the difficulty—but since their goal, typically, is to prove that PII *fails* in QM, they have erred on the side of liberalism. Thus, French and Redhead ([1988] p. 239) stipulate that the probabilities in question should simply be *divorced* from the question of measurement—but then they are still to be considered meaningful 'while recognizing that these attributes can never be observed'. But from our point of view, so much liberalism makes no sense at all. Given that the notion of probability *only* enters into quantum mechanics when measurements are performed—*only* with the use of the measurement postulates—the suggestion that quantum probability can be divorced from the context of measurement is a hostage to fortune. We are looking for the strongest constraints possible on $\mathfrak{S}$; we should avoid appeal to probabilistic properties and relations in PhysInd altogether. Our caution in this respect has already been flagged in our distinction between categorical and probabilistic discernibility.

In the same spirit, even when it comes to categorical physical properties and relations licensed by WkPP, we should be circumspect; and other kinds of categorical properties and relations not underwritten by this postulate are doubly problematic. We state two general requirements.

Requirement 1 (Req1). Relations (and properties) in $\mathfrak{S}$ should have a transparent physical meaning. There are many predicates in $\mathcal{L}_{\text{QM}}$ wherein physical system variables occur, but these predicates do not express any physical property or physical relation. For example, predicates expressing membership, or non-membership, of $\boldsymbol{a}$ or $\boldsymbol{b}$ in sets such as $\aleph_0$ ('$\boldsymbol{a} \in \aleph_0 \vee \boldsymbol{b} \in \aleph_0$'), in $\{\beth_\omega, \boldsymbol{a}, \boldsymbol{b}\}$ and $\{\boldsymbol{a}, \boldsymbol{b}, 2008, 2009, 2010\}$, are surely physically meaningless. Such predicates are not permitted to occur in $\mathfrak{S}$, and in PhysInd($\boldsymbol{a}$, $\boldsymbol{b}$), as discerning predicates.

Name and labelling predicates, such as 'the label of particle $\boldsymbol{a}$ is 3', are physically meaningless and should not discern (Req1). For example, in the case of Black's two spheres, a name predicate like '$\boldsymbol{a}$ = Castor' is forbidden as the monadic predicate that discerns the spheres absolutely. This is not to say that names and labels are not permitted to occur in $\mathcal{L}_{\text{QM}}$. We can name and label particles harmlessly, yet still talk meaningfully about identical but differently labelled particles (namely when they are physically indiscernible), e.g. $\langle \boldsymbol{a}, 1\rangle$ and $\langle \boldsymbol{a}, 2\rangle$ (single particle $\boldsymbol{a}$ clumsily carrying two different labels); and about different but identically labelled particles, e.g. $\langle \boldsymbol{a}, 1\rangle$ and $\langle \boldsymbol{b}, 1\rangle$ (a single label clumsily labelling two different particles $\boldsymbol{a}$ and $\boldsymbol{b}$).

In fact, it is a 'mathematical necessity' that the members of every set of particles can be labelled in the following sense—we define, for the present purposes, a sentence of $\mathcal{L}_{\text{QM}}$ to be a *mathematical necessity* iff it is a theorem of ZFC. A proper labelling of the members of an arbitrary set X is a bijection from X to its demonstrably unique *cardinal number* $\#X$, which by definition is *the smallest ordinal equinumerous to* X. The finite von Neumann ordinals are identified as the natural numbers ($\mathbb{N}$); they coincide demonstrably with the finite cardinals. *Equinumerosity* between X and $\#X$ is defined as the existence of a bijection from X to $\#X$, say $\ell : X \twoheadrightarrow \#X$. This bijection ℓ, whose existence one proves in ZFC, *is* the labelling of the members of X, for it assigns to every member $a \in X$ a unique ordinal number $\ell(a)$. If X is finite, so that $\#X \in \mathbb{N}$, then $\#X = \{0, 1, 2, \ldots, \#X - 1\}$; the labels then can achieve the status of logical proper names. (If X is non-denumerably infinite, then this cannot be achieved because $\mathcal{L}_{\in}$, and $\mathcal{L}_{\text{QM}}$ for that matter, are finitary languages and therefore harbour at most a denumerable infinitude of Russellian definite descriptions that can licence the introduction of defined logical proper names.)

Redhead and Teller ([1992]) considered 'the capability of particles to be labeled' and called this capability *Labeling Primitive Thisness* (LabelPT). Well, we have just sketched the proof of the theorem, saying that the members of *every* set in the domain of discourse of $\mathcal{L}_{\text{QM}}$ can be labelled,

$$\mathsf{ZFC} \vdash \forall X : \text{LabelPT}(X), \tag{36}$$

where this monadic predicate is defined in $\mathcal{L}_{\text{QM}}$ as follows:

$$\text{LabelPT}(X) \quad \text{iff} \quad \exists \ell \subseteq X \times \#X:$$
$$\times\, (\forall A \in X, \exists! m \in \#X : \langle A, m\rangle \in \ell \wedge \forall m \in \#X, \exists! A \in X : \langle A, m\rangle \in \ell). \tag{37}$$

Then it is a mathematical necessity that every particle in a finite set of particles can be properly *named* and formally distinguished by its name.

Redhead and Teller ([1992]) argued, however, that standardly QM 'adopts' LabelPT but that QM really should reject LabelPT (37). But since it is a mathematical necessity that every set of (similar) particles (bosons, fermions, and what have you) can be labelled (36), their requirement to have a version of QM that *rejects* LabelPT is to require an inconsistent version of QM. We prefer our quantum mechanics consistent.

Redhead and Teller ([1992]) further claim that 'the Fock-space formalism' is a way to reject LabelPT. This is obviously an error of thought, because 'the Fock-space formalism' is part of (or else reducible to) ZFC and by mathematical necessity compatible with the labelling theorem (36).

We remain, with QM and Fock's formalism, safely within the confines of ZFC and classical elementary predicate logic and are not bothered by LabelPT (36) in the least: labelling is harmless because physically meaningless labels can be used in the statement of PhysInd($\boldsymbol{a}$, $\boldsymbol{b}$), but labelling predicates—such as ‘***a*** has label 1’—are not themselves permissible: they are not in $\mathfrak{S}$. They are obviously physically suspect—how do you measure them?—nor are they licensed by WkPP. Indeed, if they are properties, *albeit* properties that cannot be defined in terms of self-adjoint operators, they are not invariant under permutations. This brings us to the second requirement.

Requirement 2 (Req2). As discussed in Section 2.2, the states and operators ordinarily used to describe similar particles are invariably *symmetrised* (states) or *symmetric* (operators)—by the Symmetrisation Postulate. We elevate this to the general requirement: properties and relations by which similar particles can be discerned must be *invariant under permutations of particles*. Indeed, they should be permutation-invariant if they are to have a physical meaning insofar as permutations are *symmetries* of the physical system. For relations, this means that only *totally symmetric* predicates are permitted in $\mathfrak{S}$. For properties, any property of one particle should be a property of any other—a system of particles invariant under permutations must be *absolutely* indiscernible. From this it follows that any symmetric binary predicate in $\mathfrak{S}$ must be either *reflexive* or *irreflexive*.

We consider it instructive to compare Req2 with a similar one in the celebrated case of the twin black spheres. In an exciting dialogue between two persons unexcitingly named ‘A’ and ‘B’, Max Black ([1952], p. 156) envisioned what seems a logically possible world consisting of empty spacetime with only *two* identical black spheres of radius 1 dm say, their centres located at a distance of 1 km from each other. Some traveller baptised the spheres ‘Castor’ and ‘Pollux’ and then disappeared in order not to destroy the spatial symmetry of this world. Let ‘***a***’ and ‘***b***’ be sphere variables running over the set {Castor, Pollux}. This situation seems to contradict that PII is true in all logically possible worlds, so that PII cannot be a *necessary,* or an *a priori,* or a *conceptual* or an *analytic* truth. At some point during the dialogue, it is thought that the property of being in a given region of space, or having such and such coordinates, might suffice to specify a property that the one sphere has and which the other lacks—the suggestion first made by Kant. But properties like these do not respect the physical symmetries of Euclidean space, and so are to be rejected. Similarly, properties of particles in quantum mechanics that do not respect permutation symmetry are to be rejected. Only *invariant* properties and relations (under the symmetries present in the case in question) are admissible. (For further elaboration on the admissability of predicates in terms of invariance, see Saunders [2003a]).

For the sake of future reference, we list the requirements on quantum-mechanical properties and relations in summary fashion.

(Req1) *Physical meaning*. All properties and relations should be transparently defined in terms of physical states and operators that correspond to physical magnitudes, as in WkPP, in order for the properties and relations to be physically meaningful.

(Req2) *Permutation invariance*. Any property of one particle is a property of any other; relations should be permutation-invariant, so *binary* relations are symmetric *and* either reflexive or irreflexive.

We say that a relation is *physically admissible* iff it satisfies both these requirements, and similarly for properties.

With these requirements in position, we are almost prepared for the hunt for admissible relations that discern—*almost*, because rigour demands that we first delve a bit into the various *kinds* of discernibility.

4.4 Some kinds of discernibility

In QM, identity of physical objects $\boldsymbol{a}$ and $\boldsymbol{b}$ can *succeed* in exactly one way: if the sufficient condition $\mathsf{PhysInd}(\boldsymbol{a}, \boldsymbol{b})$ for identity is met (32). Identity of $\boldsymbol{a}$ and $\boldsymbol{b}$ can *fail* in many ways: in principle, every property and relation included in $\mathsf{PhysInd}(\boldsymbol{a}, \boldsymbol{b})$ has the ability to discern $\boldsymbol{a}$ and $\boldsymbol{b}$. We rehearse a few kinds of discernibility from Section 1 rigorously, and add a few new ones.

We call physical objects $\boldsymbol{a}$ and $\boldsymbol{b}$ *absolutely discernible*, or *distinguishable*, iff there is some property expressed in $\mathcal{L}_{\mathrm{QM}}$ by monadic predicate M included in $\mathsf{PhysInd}(\boldsymbol{a}, \boldsymbol{b})$, such that they are M-discernible,

$$M\boldsymbol{a} \wedge \neg M\boldsymbol{b} \quad \text{or} \quad \neg M\boldsymbol{a} \wedge M\boldsymbol{b}. \tag{38}$$

We call physical objects $\boldsymbol{a}$ and $\boldsymbol{b}$ *relationally discernible* iff there is some relation expressed by a polyadic predicate in $\mathcal{L}_{\mathrm{QM}}$ included in $\mathsf{PhysInd}(\boldsymbol{a}, \boldsymbol{b})$ that discerns them. We shall restrict ourselves to binary relations, so that relational discernibility is achieved iff we find some dyadic predicate, R say, such that $\neg\mathsf{Ind}(\boldsymbol{a}, \boldsymbol{b}; R)$ (28) holds, which is logically equivalent to

$$\begin{aligned} &\exists \boldsymbol{c}((R\boldsymbol{a}\boldsymbol{c} \wedge \neg R\boldsymbol{b}\boldsymbol{c}) \vee (R\boldsymbol{b}\boldsymbol{c} \wedge \neg R\boldsymbol{a}\boldsymbol{c})) \vee \\ &\exists \boldsymbol{d}((R\boldsymbol{d}\boldsymbol{a} \wedge \neg R\boldsymbol{d}\boldsymbol{b}) \vee (R\boldsymbol{d}\boldsymbol{b} \wedge \neg R\boldsymbol{d}\boldsymbol{a})). \end{aligned} \tag{39}$$

Suppose we only have two particles, labelled (named) ‘1’ and ‘2’, and suppose, for ease, that $\boldsymbol{a}$ and $\boldsymbol{b}$ are particle variables running over the set $P_2 \equiv \{1, 2\}$. In this case, there are *prima facie* $2^4 = 16$ logically distinct binary relations.

Quine ([1981], pp. 129–33) was the first to inquire into different kinds of discernibility (‘grades of discriminability’ in his words); he discovered that there are only *two* independent logical categories of relational discernibility (by means of a binary relation): either the relation is irreflexive and asymmetric, in which case we speak of *relative discernibility*, or the relation is irreflexive and symmetric, in which case we speak of *weak discernibility*. We notice that if relation R discerns particles 1 and 2 *relatively*, then its *complement relation*, defined as $\neg R\boldsymbol{a}\boldsymbol{b}$, is reflexive and also asymmetric; and if R discerns particles 1 and 2 *weakly*, then its complement relation $\neg R$ is reflexive and symmetric but does not hold for $\boldsymbol{a} \neq \boldsymbol{b}$ whenever R holds for $\boldsymbol{a} \neq \boldsymbol{b}$. This generalises to any set P_N of $N \geq 2$ particles.

Evidently, absolute discernibles are always weak discernibles, because if Equation (38) holds, then the following relation R_M discerns weakly:

$$R_M\boldsymbol{ab} \quad \text{iff} \quad (M\boldsymbol{a} \wedge \neg M\boldsymbol{b}) \vee (\neg M\boldsymbol{a} \wedge M\boldsymbol{b}). \tag{40}$$

The requirement of permutation invariance (Req2) has three implications, which we mention in turn.

1°. Absolute discernibility is not on, and therefore only relational discernibility is possible.

2°. Relative discernibility is not on and therefore the only kind of relational discernibility left is weak discernibility. Hence the only quantum-mechanical possibility for similar elementary particles to be discernible is to be *weakly* discernible.

3°. Only two sorts of relations are permitted, which both belong to the same logical category of (2°) *weak* indiscernibility (they can be defined in terms of each other without using identity and only using negation, as in the previous paragraph): an irreflexive symmetric relation, say R, that holds between all non-identical particles,

$$(\forall \boldsymbol{a} \in P_N : \neg R\boldsymbol{aa}) \wedge (\forall \boldsymbol{a}, \boldsymbol{b} \in P_N : \boldsymbol{a} \neq \boldsymbol{b} \longrightarrow R\boldsymbol{ab}), \tag{41}$$

and a reflexive symmetric relation, say R', that does not hold between all non-identical particles,

$$(\forall \boldsymbol{a} \in P_N : R'\boldsymbol{aa}) \wedge (\forall \boldsymbol{a}, \boldsymbol{b} \in P_N : \boldsymbol{a} \neq \boldsymbol{b} \longrightarrow \neg R'\boldsymbol{ab}), \tag{42}$$

In each of these cases, Equation (39) holds, and thus the two particles are discerned weakly and therefore relationally.

We end this section with a few examples of relationals. Black's spheres Castor and Pollux are discerned weakly by the reflexive relation 'has no distance from', which does not relate Castor and Pollux, and by the irreflexive relation 'has a positive distance from', which does relate Castor to Pollux; cf. (Saunders [2006a]). Notice that distances are Euclidean-invariant. The same relations discern points in every n-dimensional Euclidean space weakly ($\mathbb{R}^n$). Similarly, points in Galilean and Minkowskian spacetime are weakly discernible. In the General Theory of Relativity, in all sorts of semi-Riemannian spacetime, spacetime points can generally be discerned absolutely by the value of the various tensors at those points, provided the tensor fields lack global symmetries, otherwise they are weak discernibles. Most ordinal numbers in set theory are weakly discernible, because $\mathcal{L}_\in$ has no more than $\aleph_0$ monadic predicates that can in principle discern absolutely, whereas there are far more ordinal numbers; and the same holds for all sets: 'not every set can have a name'. The imaginary numbers i and $-$i are weakly discernible, because monadic predicates that discern them absolutely, such as 'Im(w) > 0', are not invariant under the automorphisms of the structure $\mathbb{C}$ and therefore forbidden ($w \mapsto w$ and $w \mapsto w^*$ are the only two); see (Ladyman [2005]). Relatively but not absolutely discernible are instants of time in Galilean spacetime: 't is earlier than t'' is an irreflexive relation that holds between every two non-identical instants: if $t \neq t'$, then either $t < t'$ or $t' < t$. (For further discussion, see Saunders [2003b]).

5 Discerning Elementary Particles

5.1 Preamble

We first consider the following question: *how many particles does a given composite physical system of particles comprise?* In QM, to specify a composite physical system includes to specify *the number of* its constituent parts when we want to consider them. One needs to specify this in order to specify how many factors the direct product Hilbert space of the composite system must have (State Postulate). Without specifying this number, or reading it off from the specified Hamiltonian say, the entire process of building some QM-model of the system cannot even begin. In classical mechanics, where the phase space $\mathbb{R}^{6N}$ is fixed *ab ovo*, the situation is exactly the same. (Another point of similarity is that the number of particles cannot change, which is different for the theory of the next paragraph.)

As soon as we leave QM and enter (relativistic) quantum field theory (QFT), the question above becomes both interpretation- and context-dependent, because the *fundamental* ontological substance of QFT is, as the name says, the *quantum field* and not the material particle (see Saunders [1996]). When the concept of a particle need no longer be expressed in the language of the theory, there may be no answer to the question of how many particles there are. Nonetheless, physicists talk about particles all the time in the context of the standard model of elementary particles and their interactions, which model is formulated in the framework of QFT. Physicists are either up to their necks in a conceptual muddle or they must be making do with some diluted particle interpretation of quantum fields of the sort considered by Clifton ([2004], Chapter III) and Wallace ([2006]). One answer to the question, then, is that it is a *pragmatic* affair whether *there are* particles, depending on the energy regime and the scale considered (see Wallace [2006]). In this context- and scale-dependent sense, we expect the methods we develop here to apply to relativistic particle talk as well. And if we were to put up shop in QFT, we could raise the question whether PII can be applied to the quantum fields themselves and ask whether quantum fields are physically discernible. The answer is surely in the affirmative; and that also the *modes* of a quantum field are physically discernible. In this application, indeed, no one has even so much as considered a violation of PII: that there should be two or more qualitatively identical fields, exactly alike in all physical respects, with no possible coupling between them. Enough of this and back to QM, where the concept of a particle is comparatively much simpler than in relativistic QFT.

Let us define, in the current context of QM, that for the particles 1 and 2 to be (weak, relative) *relational discernibles*, to be *relationals*, they must be indiscernible by any admissible property but be (weakly, relatively) discernible by some admissible relation in every admissible physical state of the composite system of which they are part. We call 1 and 2 *indiscernibles* iff in every admissible state they are not discernible, neither absolutely nor relationally. Hence particles that are discernible in some states and indiscernible in other states are neither indiscernibles nor discernibles; their possible discernibility becomes a state-dependent and hence entirely contingent matter. When we recall that relative discernibility is not on in QM for similar particles (see Section 4.4), the only way left to discern the particles relationally is to discern them weakly.

5.2 Fermions

Theorem 1 (QM^-). In a composite physical system of a finite number of similar fermions, according to QM^-, all fermions are categorically weakly discernible in every admissible physical state, pure and mixed, for all finite-dimensional Hilbert spaces; in short, fermions are categorical weak discernibles.

Proof. We consider a composite system S_N of $N \geq 2$ similar fermions; $\mathcal{H}$ is the one-particle Hilbert space having dimension $d \geq 2$; $\mathcal{H}^N$ is the N-fold direct-product space (5); $\mathcal{S}_{\text{FD}}(\mathcal{H}^N)$ is the set of FD-symmetric and hence admissible states of S_N. We have a set of N particles, $\{1, 2, \ldots, N\}$, over which the particle variables $\boldsymbol{a}$ and $\boldsymbol{b}$ run. We prove Theorem 1 stepwise.

[S1] Auxiliary lemma.

[S2] Case for $N = 2$, $d \geq 2$, pure states.

[S3] Case for $N = 2$, $d \geq 2$, mixed states.

[S4] Auxiliary lemma.

[S5] Case for $N > 2$, $d \geq 2$, pure states.

[S6] Case for $N > 2$, $d \geq 2$, mixed states.

The auxiliary Lemma 1 will provide a sufficient condition for categorical weak discernibility, for arbitrary finite dimension of $\mathcal{H}$ and all admissable states, mixed and pure, for $N = 2$. In steps [S2]–[S3], we show that this condition can be met in each of the mentioned cases. Then we indicate how to extend all this to $N > 2$ in steps [S4]–[S6].

[S1] Let $|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_d\rangle \in \mathcal{H}$ be an eigenbasis of $\mathcal{H}$ belonging to operator A, acting in $\mathcal{H}$, which we assume to correspond to some physical magnitude $\mathcal{A}$ and thus has a clear physical meaning. Let P_m be the 1-dimensional projector on the ray to which eigenstate $|\phi_m\rangle$ of A belongs; let $P_{lm} \equiv P_l - P_m$ and let

$$P_{lm}^{(1)} \equiv P_{lm} \otimes 1 \quad \text{and} \quad P_{lm}^{(2)} \equiv 1 \otimes P_{lm}. \tag{43}$$

Consider the following family of categorical relations:

$$\mathsf{R}_t(\boldsymbol{a}, \boldsymbol{b}) \quad \text{iff} \quad \sum_{l,m=1}^{d} P_{lm}^{(\boldsymbol{a})} P_{lm}^{(\boldsymbol{b})} W = t\, W, \tag{44}$$

where each relation is characterised by a value of parameter $t \in \mathbb{R}$.

When the projectors under consideration belong to the spectral family of magnitude-operator A, assumed to be physically meaningful, they are themselves physically meaningful; by the WkPP, when the system is in the state W, so is relation R_t, which is defined in terms of them (Req1). Due to the fact that operators (43) commute and that the commutator is linear, relation R_t is symmetric, independent of the value of parameter t; below we shall see that for certain values of t, relation R_t is reflexive, and for others irreflexive; for these values relation R_t is then permutation-invariant (Req2). Conclusion: relation R_t (44) meets the two requirements (end of Section 4.3) and is therefore admissible.

Lemma 1 (QM^-). If state operator $W \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H})$, $d = \dim \mathcal{H}$, is an eigenstate of both

$$\sum_{l,m=1}^{d} P_{lm}^{(\boldsymbol{a})} P_{lm}^{(\boldsymbol{b})} \; (\boldsymbol{a} \neq \boldsymbol{b}) \quad \text{and} \quad \sum_{l,m=1}^{d} \left(P_{lm}^{(\boldsymbol{a})}\right)^2 \tag{45}$$

but having different eigenvalues, $\lambda, \mu \in \mathbb{R}$ respectively, then in QM^- the particles are categorically weakly discernible in that state W by relation R_μ as well as by R_λ (44). For pure states $|\Psi\rangle \in \mathcal{H} \otimes \mathcal{H}$, the sufficient condition is

$$\sum_{l,m=1}^{d} P_{lm}^{(\boldsymbol{a})} P_{lm}^{(\boldsymbol{b})} |\Psi\rangle = \lambda |\Psi\rangle \neq \mu |\Psi\rangle = \sum_{l,m=1}^{d} \left(P_{lm}^{(\boldsymbol{a})}\right)^2 |\Psi\rangle. \tag{46}$$

Proof. By assumption we have that state W is an eigenstate (18) of the two operators (45) having eigenvalues λ and $\mu \neq \lambda$. By the Weak Property Postulate of QM^- (WkPP, Section 2.4), the composite system then possesses quantitative physical properties

$$\left\langle \sum_{l,m=1}^{d} P_{lm}^{(\boldsymbol{a})} P_{lm}^{(\boldsymbol{b})}, \lambda \right\rangle \quad \text{and} \quad \left\langle \sum_{l,m=1}^{d} \left(P_{lm}^{(\boldsymbol{a})}\right)^2, \mu \right\rangle, \tag{47}$$

and then by virtue of SemC (20), it does *not* possess properties

$$\left\langle \sum_{l,m=1}^{d} P_{lm}^{(\boldsymbol{a})} P_{lm}^{(\boldsymbol{b})}, \mu \right\rangle \quad \text{and} \quad \left\langle \sum_{l,m=1}^{d} \left(P_{lm}^{(\boldsymbol{a})}\right)^2, \lambda \right\rangle. \tag{48}$$

Relation R_λ is irreflexive (choice $t = \lambda$), because

$$\sum_{l,m=1}^{d} \left(P_{lm}^{(\boldsymbol{a})}\right)^2 W \neq \lambda W,$$

and the composite system does not have the associated property (48), so that particle 1 is not related to itself by R_λ (similarly for $\boldsymbol{a} = \boldsymbol{b} = 2$). In order for R_λ to discern the particles weakly, we next need to verify that $\mathsf{R}_\lambda(\boldsymbol{a}, \boldsymbol{b})$ holds for $\boldsymbol{a} \neq \boldsymbol{b}$. We have by assumption that

$$\sum_{l,m=1}^{d} P_{lm}^{(\boldsymbol{a})} P_{lm}^{(\boldsymbol{b})} W = \lambda W.$$

In combination with the fact that properties (47) are possessed properties of the composite system, we conclude that 1 and 2 are indeed related by R_λ (44).

For choice $t = \mu$, relation R_μ (44) is reflexive, symmetric and does not hold for $\boldsymbol{a} \neq \boldsymbol{b}$. Either way, $t = \lambda$ or $t = \mu$, relation R_t (44) categorically discerns the particles weakly in state W. ■

[S2] *Case for $N = 2$, $d \geq 2$, pure states.* Using equalities

$$\begin{aligned} P_{lm}|\phi_n\rangle &= \delta_{ln}|\phi_l\rangle - \delta_{mn}|\phi_m\rangle, \\ P_{lm}^2|\phi_n\rangle &= \delta_{ln}|\phi_l\rangle + \delta_{mn}|\phi_m\rangle - 2\delta_{lm}\delta_{mn}|\phi_l\rangle, \end{aligned} \tag{49}$$

and for an arbitrary state $|\Psi\rangle \in \mathcal{H} \otimes \mathcal{H}$, the expansion

$$|\Psi\rangle = \sum_{n,i=1}^{d} c_{ni}|\phi_n\rangle \otimes |\phi_i\rangle. \tag{50}$$

One shows that for $\boldsymbol{a} \neq \boldsymbol{b}$,

$$\sum_{l,m=1}^{d} P_{lm}^{(\boldsymbol{a})} P_{lm}^{(\boldsymbol{b})}|\Psi\rangle = 2d \sum_{m=1}^{d} c_{mm}|\phi_m\rangle \otimes |\phi_m\rangle - 2|\Psi\rangle, \tag{51}$$

and for $\boldsymbol{a} = \boldsymbol{b}$,

$$\sum_{l,m=1}^{d} \left(P_{lm}^{(\boldsymbol{a})}\right)^2|\Psi\rangle = 2(d-1)|\Psi\rangle. \tag{52}$$

Looking at expansion (50), $|\Psi\rangle$ is FD-symmetric iff $c_{ni} = -c_{in}$, which implies that $c_{nn} = 0$ for every n (the Paulian character of FD-states). This means that the first sum of terms at the right-hand-side of equality (51) vanishes,

$$\sum_{l,m=1}^{d} P_{lm}^{(\boldsymbol{a})} P_{lm}^{(\boldsymbol{b})}|\Psi^{\mathrm{FD}}\rangle = -2|\Psi^{\mathrm{FD}}\rangle. \tag{53}$$

Hence for choices $\lambda = 2(d-1)$ and $\mu = -2$, the sufficient condition (46) of Lemma 1 has been met for all pure FD-states and we may conclude that the fermions are categorically weakly discernible.

[S3] *Case for $N = 2$, $d \geq 2$, mixed states.* Equations (51) and (52) can also be written as equations for FD-symmetric projectors

$$\sum_{l,m=1}^{d} P_{lm}^{(\boldsymbol{a})} P_{lm}^{(\boldsymbol{b})} |\Psi^{\mathrm{FD}}\rangle\langle\Psi^{\mathrm{FD}}| = -2|\Psi^{\mathrm{FD}}\rangle\langle\Psi^{\mathrm{FD}}|, \tag{54}$$

and

$$\sum_{l,m=1}^{d} \left(P_{lm}^{(\boldsymbol{a})}\right)^2 |\Psi^{\mathrm{FD}}\rangle\langle\Psi^{\mathrm{FD}}| = 2(d-1)|\Psi^{\mathrm{FD}}\rangle\langle\Psi^{\mathrm{FD}}|. \tag{55}$$

Due to the linearity of the operators, these equations remain valid for arbitrary linear combinations of FD-symmetric projectors. This includes all convex combinations of FD-symmetric projectors, which exhausts the set $\mathcal{S}_{\mathrm{FD}}(\mathcal{H} \otimes \mathcal{H})$ of all FD-symmetric mixed states. Hence for choices $\lambda = 2(d-1)$ and $\mu = -2$, the sufficient condition (45) of Lemma 1 has been met for all mixed FD-states.

[S4]–[S6] The proofs for these cases proceed exactly analogous to the ones above for [S1]–[S3], using the same Lemma 1, and we therefore do not spell them out. A few remarks nonetheless, if only for the sake of clarity.

Rather than operators (43), one now considers the following N-factor operators, which act in $\mathcal{H}^N$,

$$P_{lm}^{(\boldsymbol{j})} \equiv 1 \otimes \cdots \otimes 1 \otimes P_{lm} \otimes 1 \otimes \cdots \otimes 1, \tag{56}$$

where P_{lm} is the $\boldsymbol{j}$th factor and $\boldsymbol{j}$ a particle variable running over the set of N labelled particles; and rather than expansion (50), one now uses the expansion for $|\Psi\rangle \in \mathcal{H}^N$,

$$|\Psi\rangle = \sum_{n_1,\ldots,n_N=1}^{d} c(n_1, n_2, \ldots, n_N)|\phi_{n_1}\rangle \otimes |\phi_{n_2}\rangle \otimes \cdots \otimes |\phi_{n_N}\rangle. \tag{57}$$

Thus one arrives at the same Equations (52) and (53). Discerning the fermions in these pure cases is realised by the same relations, R_μ and R_λ (44). Henceforth, we arrive at the same conclusion for the mixed states. ■

The attentive reader will have noticed that we have assumed, not proved, that the family of orthogonal projectors we have used in the proof above give rise to a physically meaningful relation. If there is *some* maximal operator A (self-adjoint, positive, normal) that corresponds to a physical magnitude $\mathcal{A}$, then we are home. The associated projectors correspond to so-called yes/no *experiments*—a lesson that quantum logic has taught us—which makes their physical meaning evident. The possibility that there is *no* operator of the mentioned kinds corresponding to a physical magnitude is an *outré* possibility, for in that case we would have a physical system that defies quantum-mechanical modelling! We can therefore safely ignore this; we only want to—and actually only *can*—prove theorems in QM about physical systems that lie within the scope of QM. Furthermore, in the examples below, the physical meaning of the relations will be as transparent as a good cleaned window glass.

We present two simple examples for two fermions having spin $\hbar/2$, $\mathcal{H} = \mathbb{C}^2$, so $N = 2$ and $d = 2$. There is only a single admissible pure state (and therefore no admissible mixed states), which is the (unit ray of the) FD-symmetric singlet state,

$$|\text{singlet}\rangle \equiv \tfrac{1}{\sqrt{2}}(|z^+\rangle \otimes |z^-\rangle - |z^-\rangle \otimes |z^+\rangle). \tag{58}$$

Example 1. Consider Pauli spin operator σ_z. Operator σ_z is the difference of the two z-spin projectors $|z^+\rangle\langle z^+|$ and $|z^-\rangle\langle z^-|$,

$$\begin{aligned}\sigma_z^{(1)} &= (|z^+\rangle\langle z^+| - |z^-\rangle\langle z^-|) \otimes 1 = \sigma_z \otimes 1,\\ \sigma_z^{(2)} &= 1 \otimes (|z^+\rangle\langle z^+| - |z^-\rangle\langle z^-|) = 1 \otimes \sigma_z.\end{aligned} \tag{59}$$

Then $\sigma_z^{(1)}|\text{singlet}\rangle = -\sigma_z^{(2)}|\text{singlet}\rangle$. Although the singlet state (58) is neither an eigenstate of $\sigma_z^{(1)}$ nor of $\sigma_z^{(2)}$, it is an eigenstate of $\sigma_z \otimes \sigma_z$ and of $\sigma_z^2 \otimes 1$ (and of $1 \otimes \sigma_z^2$), but with different eigenvalues, -2 and $+2 = 2(d-1)$, respectively. Relation

$$\mathsf{Z}_t(\boldsymbol{a}, \boldsymbol{b}) \quad \text{iff} \quad 2\sigma_z^{(\boldsymbol{a})}\sigma_z^{(\boldsymbol{b})}|\text{singlet}\rangle = t|\text{singlet}\rangle, \tag{60}$$

which meets Requirements 1 and 2 (Section 4.3) discerns the particles weakly and categorically for both $t = -2$ and $t = +2$. Relation Z_{-2} is the one in footnote 5 of (Saunders [2003a], p. 294): 'has opposite direction of each component of spin to'. The relative direction of components of spin may be well defined, whether or not the directions of those components are themselves defined, just as the relative distances of bodies may be well defined, whether or not bodies have absolute positions (see Saunders [2006a] for further discussion).

Example 2. This example is not an instance of Theorem 1 but really a different theorem. Consider the following symmetric 'Total-spin relation' (in units of $\hbar^2/2$):

$$\mathsf{T}(\boldsymbol{a}, \boldsymbol{b}) \quad \text{iff} \quad \left(\sigma^{(\boldsymbol{a})} + \sigma^{(\boldsymbol{b})}\right)^2 |\text{singlet}\rangle = 12\,|\text{singlet}\rangle, \tag{61}$$

where

$$\sigma^{(1)} = (\sigma_x + \sigma_y + \sigma_z) \otimes 1 \quad \text{and} \quad \sigma^{(2)} = 1 \otimes (\sigma_x + \sigma_y + \sigma_z). \tag{62}$$

Relation T (61) discerns the two fermions weakly and categorically, as we shall demonstrate.

For $\boldsymbol{a} \neq \boldsymbol{b}$, the singlet state is an eigenstate of $\sigma^{(1)} + \sigma^{(2)}$ having eigenvalue 0, because of the perfect anti-correlation of the singlet state. Then the singlet is also an eigenstate of the operator in the *definiens* of relation T (61), which is the total spin operator, having eigenvalue 0. But

$$0|\text{singlet}\rangle \neq 12|\text{singlet}\rangle,$$

so relation T (61) fails for $\boldsymbol{a} \neq \boldsymbol{b}$.

For $\boldsymbol{a} = \boldsymbol{b}$, we obtain the spin magnitude operator of particle $\boldsymbol{a}$,

$$\left(\sigma^{(\boldsymbol{a})} + \sigma^{(\boldsymbol{a})}\right)^2|\text{singlet}\rangle = 2^2(1 \otimes 1 + 1 \otimes 1 + 1 \otimes 1)|\text{singlet}\rangle = 12|\text{singlet}\rangle, \tag{63}$$

which shows that T is reflexive. Therefore total spin relation T (61) discerns the two fermions weakly. Since no probability measures occur in the *definiens* of T, it discerns the fermions categorically. Obviously, relation T meets Requirements 1 and 2. This completes the demonstration.

The more general anti-correlations used in the proof of Theorem 1 concern, not components of spin, but values of any magnitude operator A. (Note that A can be chosen independent of the state W.) As such these relations clearly belong to $\mathsf{PhysInd}$. Less clear is the probabilistic analogue, defined in terms of their expectation values. In this case, we speak of *probabilistic* discernibility.

Theorem 2 (QM$^-$, ProbP). In a composite physical system of a finite number of similar fermions, then in QM$^-$ supplemented with ProbP, all fermions are probabilistically weakly discernible in every admissible physical state, pure and mixed, for every finite-dimensional Hilbert space; in short, fermions are probabilistic weak discernibles.

Proof. The proof is as the proof of Theorem 1, but replacing the categorical relations with probabilistic relations, defined in terms of expectation values

$$\mathsf{S}_t(\boldsymbol{a}, \boldsymbol{b}) \quad \text{iff} \quad \mathrm{Tr}\left(\sum_{l,m=1}^{d} P_{lm}^{(\boldsymbol{a})} P_{lm}^{(\boldsymbol{b})} W\right) = t, \tag{64}$$

where $t \in \mathbb{R}$. The auxiliary lemma is now:

Lemma 2 (QM$^-$, ProbP). If for state operator $W \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H})$ we have

$$\mathrm{Tr}\left(\sum_{l,m=1}^{d} P_{lm}^{(\boldsymbol{a})} P_{lm}^{(\boldsymbol{b})} W\right) \neq \mathrm{Tr}\left(\sum_{l,m=1}^{d} \left(P_{lm}^{(\boldsymbol{a})}\right)^2 W\right), \tag{65}$$

then the particles are probabilistically weakly discernible in that state W by relations S_t (64) for the following values of parameter t:

$$\mathrm{Tr}\left(\sum_{l,m=1}^{d} P_{lm}^{(\boldsymbol{a})} P_{lm}^{(\boldsymbol{b})} W\right) \quad \text{and} \quad \mathrm{Tr}\left(\sum_{l,m=1}^{d} \left(P_{lm}^{(\boldsymbol{a})}\right)^2 W\right). \tag{66}$$

For pure states $|\Psi\rangle \in \mathcal{H} \otimes \mathcal{H}$, the sufficient condition (65) becomes

$$\langle\Psi| \left(\sum_{l,m=1}^{d} P_{lm}^{(\boldsymbol{a})} P_{lm}^{(\boldsymbol{b})} \right) |\Psi\rangle \neq \langle\Psi| \left(\sum_{l,m=1}^{d} \left(P_{lm}^{(\boldsymbol{a})}\right)^2 \right) |\Psi\rangle. \tag{67}$$

The proof of Lemma 2 is a simplified version of the proof of Lemma 1, which we do not spell out. From Equations (52) and (53), we immediately deduce that the sufficient condition (67) of Lemma 2 has been met for all FD-states. And *mutatis mutandis* for the mixed states. ■

To complete the analysis, it is necessary to consider infinite-dimensional Hilbert spaces, but here there is a difficulty. The constructive part of the above proofs—the existence of the discerning relations for arbitrary A—does not extend to Hilbert spaces of infinite dimension: the series in Equations (51) and (52) diverge for $d = \infty$. But we doubt that this difficulty reflects any fundamental limitation. We therefore state as a conjecture:

Conjecture 1. Like Theorems 1 and 2 for infinite-dimensional Hilbert spaces.

Note that for the proof of Conjecture 1 it would be enough, in view of Lemma 1, to come up, for any state $|\Psi\rangle \in \mathcal{H}$, with some physically admissible one-particle operator A (which therefore would occur in PhysInd) such that

$$(A \otimes A)|\Psi\rangle = \lambda|\Psi\rangle \neq \mu|\Psi\rangle = \tfrac{1}{2}(1 \otimes A^2 + A^2 \otimes 1)|\Psi\rangle. \tag{68}$$

So much for fermions. For bosons, i.e. particles in BE-states, things turn out to be more intricate.

5.3 Bosons

Theorem 1 hinged on the existence of anti-correlations among eigenvalues of A, for any maximal one-particle magnitude operator in PhysInd, for every FD-symmetric state $|\Psi\rangle$; but these anti-correlations are independent of the alternating sign in the expansion of $|\Psi\rangle$ in terms of eigenvectors of A, i.e. the proof goes through unaltered given only that $c_{nn} = 0$ in Equation (50). Of course, this is no longer forced in the case of bosons; nevertheless, the fact remains that there are infinitely many boson states in which particles are categorically discerned by relations of the form already considered.

For example, in every state $W \in \mathcal{S}_{\text{sym}}(\mathcal{H}^N)$ that is a convex sum of Paulian states (12) in the eigenbasis of some one-particle magnitude operator A, the N similar bosons can be discerned weakly and categorically for every finite-dimensional Hilbert space, by exactly the same relations that we used to discern the N fermions! This is immediate from the proof of Theorem 1: see Equation (52) and the text following it. Therefore the bosons are also probabilistically discernible in these states. An example of such a state for $N = 2$ and $\mathcal{H} = \mathbb{C}^2$ is the BE-symmetric state displaying perfect anti-correlation

$$\frac{1}{\sqrt{2}}(|z^+\rangle \otimes |z^-\rangle + |z^-\rangle \otimes |z^+\rangle). \tag{69}$$

Two bosons ($N = 2$) in a direct product state of two identical one-particle states, e.g. $|z^+\rangle \otimes |z^+\rangle \in \mathcal{H} \otimes \mathcal{H}$ (each boson then is in the same pure state $|z^+\rangle$), can however not be categorically discerned by means of the relations we are considering in this paper. The states do not seem to contain any clue for discerning the bosons, not even weakly; in these states, they seem *indiscernible*.

The conclusion of these considerations is that whether similar bosons, i.e. particles in BE-symmetric states, are weakly discernible or indiscernible depends on the state of the composite system. Their discernibility becomes a contingent matter.

One can suppose, indeed, that the condition on the expansion coefficients c_{ml} (for $N = 2$) may be dynamically enforced under some Hamiltonian H—that product states of the form $|\phi_m\rangle \otimes |\phi_m\rangle$, $m = 1, \ldots, d$ may be inherently unstable. In that case, for sufficiently large times t, one would conclude, for the evolution operator $U(t) = \exp(\mathrm{i}Ht/\hbar)$, that the bosons in the state $|\Psi(t)\rangle = U(t)|\Psi(0)\rangle$ will be categorically discerned by R_λ. But this implies that the relation

$$\mathsf{R}^\circ_\lambda(\boldsymbol{a}, \boldsymbol{b}) \quad \text{iff} \quad U^{\text{inv}}(t) \left(\sum_{l,m=1}^{d} P^{(\boldsymbol{a})}_{lm} P^{(\boldsymbol{b})}_{lm} \right) U(t)|\Psi(0)\rangle = \lambda|\Psi(0)\rangle \tag{70}$$

categorically discerns the bosons already at $t = 0$!

How was this trick accomplished? Evidently, one has successfully (weakly, categorically) discerned the *orbits* of the states of the two bosons—orbits defined over sufficiently large times for them eventually to contain no component of the form $|\phi_m\rangle \otimes |\phi_m\rangle$. Well and good, *if the particles are in fact subject to such a dynamics*, with evolution U.

There lies the rub: the operator acting on $|\Psi(0)\rangle$ on the right-hand-side of Equation (70) is self-adjoint and symmetrised—why not count R°_λ in $\mathsf{PhysInd}$? Indeed, we may, if physically interpreted, in accordance with the real dynamics and the real physical relations that eventually obtain among physical quantities. *Mere* self-adjointness is an insufficient condition for that.

There remains, however, the possibility that bosons can always be probabilistically discerned. Probabilistic discernibility, as we have seen, was an automatic consequence of the categorical discernibility of fermions (replacing eigenvalues with expectation values): it is surely a weaker condition. Indeed, on replacing Equations (68) with the single probabilistic condition

$$\langle\Psi|A \otimes A|\Psi\rangle \neq \tfrac{1}{2}\langle\Psi|1 \otimes A^2 + A^2 \otimes 1|\Psi\rangle, \tag{71}$$

we see that for the problematic case of a direct product state of the form $|\phi\rangle \otimes |\phi\rangle \in \mathcal{H} \otimes \mathcal{H}$, we obtain for inequality (71),

$$\langle\phi|A|\phi\rangle^2 \neq \langle\phi|A^2|\phi\rangle. \tag{72}$$

In terms of the usual expression of the standard deviation in $\mathcal{L}_{\mathrm{QM}}$

$$\Delta_\phi A = \sqrt{\langle\phi|A|\phi\rangle^2 - \langle\phi|A^2|\phi\rangle}, \tag{73}$$

inequality (72) is just the condition that $\Delta_\phi A > 0$—the requirement that A is not *dispersion-free* in state $|\phi\rangle$. This condition is never satisfied whenever $|\phi\rangle$ is an eigenstate of A, as would be required were probabilistic discernibility in this state, thus defined, to extend to categorical discernibility.

Evidently, product states are no bar to discernibility by condition (71). It is worth stating and proving this result in full generality, as applying to bosons as well as fermions in Hilbert spaces of any dimensionality. The proof makes do even without WkPP, removing the last shred of metaphysics—if it is metaphysics—at the price of free use of quantum probabilities.

Theorem 3 (QM^-, ProbP). Let $\mathcal{H} \otimes \mathcal{H}$ be some direct product Hilbert space of arbitrary (possibly infinite) dimension associated with a composite system of two similar particles, then in QM^- supplemented with ProbP, if for every every state operator $W \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H})$ there is some admissible self-adjoint operator A acting in $\mathcal{H}$, corresponding to physical magnitude $\mathcal{A}$, such that

$$\mathrm{Tr}(A \otimes A W) \neq \mathrm{Tr}(A^2 \otimes 1\ W) = \mathrm{Tr}(1 \otimes A^2\ W), \tag{74}$$

then the two particles are probabilistically weakly discernible in that state. Whenever W is pure, and hence a 1-dimensional projector, and $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}$ is a vector in the ray onto which W projects, sufficient condition (74) becomes

$$\langle\psi|A \otimes A|\psi\rangle \neq \langle\psi|A^2 \otimes 1|\psi\rangle = \langle\psi|1 \otimes A^2|\psi\rangle. \tag{75}$$

We shall say that A probabilistically discerns the particles weakly *in state* W or in $|\psi\rangle$.

Proof. Assume that the antecedent (74) has been met for state W. We point out that the second equations in (74) and in (75) are always met, due to Equation (15), which is a consequence of the Symmetrisation Postulate—we have written this down here again for the sake of emphasis.

Let us define commuting operators $A^{(1)}$ and $A^{(2)}$, which act in $\mathcal{H} \otimes \mathcal{H}$, such that $A^{(1)}$ is the operator corresponding to physical magnitude $\mathcal{A}$ that pertains to particle 1, and *mutatis mutandis* for $A^{(2)}$ and 2,

$$A^{(1)} = A \otimes 1 \quad \text{and} \quad A^{(2)} = 1 \otimes A. \tag{76}$$

If A is a bounded operator on $\mathcal{H}$, then so is $A^{(1)}$ and $A^{(2)}$ on $\mathcal{H} \otimes \mathcal{H}$, and *mutatis mutandis* if the domain of A is dense in $\mathcal{H}$, and if A is self-adjoint. We now define the following family of probabilistic relations ($t \in \mathbb{R}$):

$$\mathsf{P}_t(\boldsymbol{a}, \boldsymbol{b}) \quad \text{iff} \quad \mathrm{Tr}\big(A^{(\boldsymbol{a})\dagger} A^{(\boldsymbol{b})} W\big) = t. \tag{77}$$

By construction, relation P_t (77) is admissible insofar as any probabilistic relations are admissible: it is clearly permutation-invariant and defined in terms of expectation values of self-adjoint operators that are by assumption in PhysInd (Requirements 1 and 2 for admissible relations; listed at the end of Section 4.3).

By assumption, some A is given for state W such that inequality (74) holds. Then relation P_t (77) is fixed when we choose $t = \varepsilon$, where we define ε as follows:

$$\varepsilon \equiv \mathrm{Tr}(A \otimes A\, W). \tag{78}$$

Then we have, by virtue of definitions (77) and (78)

$$\mathsf{P}_\varepsilon(1, 2), \mathsf{P}_\varepsilon(2, 1), \neg\mathsf{P}_\varepsilon(1, 1), \neg\mathsf{P}_\varepsilon(2, 2), \tag{79}$$

which is an instance of weak discernibility (42): $\mathsf{P}_\varepsilon(\boldsymbol{a}, \boldsymbol{b})$ is symmetric and irreflexive. If one assigns value $\mathrm{Tr}(A^2 \otimes 1\, W)$ to parameter t rather than value ε (78), one obtains a reflexive and symmetric relation $\mathsf{P}_t(\boldsymbol{a}, \boldsymbol{b})$ that fails for $\boldsymbol{a} \neq \boldsymbol{b}$, which is the other case of weak discernibility (41). ■

In the physically realistic case of an infinite-dimensional Hilbert space (i.e. the space of wave functions: $\mathcal{H} = L^2(\mathbb{R}^3)$), which hosts a representation of the canonical commutation relations $[P, Q] = -i\hbar 1$ for linear momentum (P) and position (Q), it very easily follows that bosons in product states

$$\psi(q) \otimes \psi(q) \in L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3) \tag{80}$$

—the problem case—are *probabilistically discernible* whatever the state $\psi \in L^2(\mathbb{R}^3)$ is: for no state ψ is dispersion-free for both P and Q. Thus bosons in product states are probabilistically discernible by their statistical variance for measurements of momentum, or by their statistical variance for measurements of position. To take the case of momentum: there are two particles, and not one, as there is a difference in the statistics of outcomes for the product of two momentum measurements on two particles ($P \otimes P$), from those for the square of a momentum measurement on a single particle ($P^2 \otimes 1$ or $1 \otimes P^2$). More generally, for any state

$$W \in \mathcal{S}(L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)), \tag{81}$$

it is not hard to show that there exists a self-adjoint operator A satisfying Equation (74): the difficulty lies in establishing that it also lies in $\mathfrak{S}$, and thus can be used in PhysInd. But as with Conjecture 1, we do not consider this difficulty to be more than a technical one, because if a physical system can be modelled at all in QM, there will be some magnitude operator that pertains to it. Certainly, this difficulty is insignificant in comparison to the conceptual question which we already anticipated in Section 4.3 and which is now staring us in the face: for how are we to conduct these measurements of the product of momenta of two particles ($P \otimes P$), to compare to measurements of the square of the momentum of one ($P^2 \otimes 1$), without already having to hand a method for distinguishing the two? And doesn't the question of discernibility now apply to the two particles when coupled to the measurement apparatus—so do we not need the details of this interaction anyway? Once we bring 'measurements' into the picture, then from the realist perspective that renders the issue of PII more pressing, the problem of measurement can no longer be ducked. If we are seriously to tackle the nature of quantum probability, and bring in the measurement apparatus, we should work with one or other of the realist solutions to the problem of measurement—in the non-relativistic case that we have been considering, either pilot-wave, dynamical collapse, modal interpretations or Everettian quantum mechanics. But we shall not take up that challenge here.

6 Concluding Discussion

The currently dominant view, according to which PII stands refuted by the quantum-mechanical description of composite systems of similar fermions (particles in FD-symmetric states) and similar bosons (particles in BE-symmetric states) is directly contradicted by the theorems of the previous section, at least as far as fermions in finite-dimensional Hilbert spaces are concerned (for infinite-dimensional ones, it is *stricto sensu* an open question). Restricted to finite-dimensional Hilbert spaces, we can conclude that fermions are weakly categorically discernible; they are categorical discernibles (Theorems 1 and 2); they are neither individuals nor indiscernibles but *categorical relationals*. For bosons, it becomes a state-dependent matter: in most states, bosons can be discerned categorically, just like fermions, but in other states, notably direct product states, they arguably cannot; hence they are neither categorical indiscernibles nor categorical discernibles; they are, however, probabilistically weakly discernible (as of course are fermions) and hence *probabilistic relationals*—whatever precisely that may entail.

The general conclusion with regard to the Leibnizian principles, then, is that PII-R and hence PII (2) hold in QM, at least in the case of fermions, and that no appeal to inflationary metaphysics (Postian 'transcendental individuality', Lockean 'substrata', Scotusian 'haecceitas', Adamsian 'primitive thisness') is needed. (We showed this by *proving* PII (32) in QM^-, by contraposition, i.e. by showing that non-identical similar particles are physically discernible.) Fermions are discernible and are perfectly respectable as physical objects of predication and quantification in accordance with elementary logical syntax, even in a language in which identity is not primitive.

We should now point out that the same conclusion must therefore follow for aggregates of fermions, such as atoms, including atoms of integral spin: for mereologically speaking, wholes whose parts are discernible are themselves discernible; wholes are identical iff all their parts are identical (see theorems in Muller [unpublished], end of Section 1.3). Therefore, the only known entities that presumably should *not* count as physical objects, or 'things', by our lights—failing a satisfactory account of probabilistic discernibility—would seem to be the elementary bosons (photons, the W, Z and Higgs bosons, and gluons), all of them gauge particles. The real objects of predication and quantification, in these cases, may better be judged the modes of the relevant quantum fields, modes whose excitations are restricted to integers (see again Saunders [1996]).

Our qualms on the admissibility of probabilistic relations have not been respected in the literature, however. Most authors, from Schrödinger to Margenau and onwards, have made free appeal to probability distributions. By *their* lights, our result is still more damning to *their* claims: for *contra* the currently dominant view, nothing at all in QM threatens PII, and the metaphysically thin notion of *object* (formal object) in the tradition of Frege, Carnap and Quine that accompanies it, when probabilistic relations are allowed. For bosons as well as fermions can always be at least probabilistically, weakly, discerned.

Permit us to compare our main conclusion to a claim of French and Krause ([2006], p. 160) (our emphasis):

> The upshot, then, is that if the non-intrinsic, state-dependent properties are identified with *all the monadic or relational properties which can be expressed in terms of physical magnitudes associated with self-adjoint operators that can be defined for the particles*, then it can be shown that two fermions or two bosons or two para-particles in a joint symmetric or anti-symmetric state respectively have the same monadic properties and the same relational properties one to another. Given this identification, even the weakest form of the principle [our PII] fails and the Principle of Identity of Indiscernibles is *straightforwardly false*.

This is the currently dominant view confidently expressed, notably including an open invitation if not challenge to go beyond DiscPr (Section 3.1). This claim is incompatible with our result. Using only what French and Krause allow, PII is straightforwardly true rather than straightforwardly false.

Aware of Saunders' ([2003a], [2006a]) claim that two fermions in the singlet state can be discerned weakly, French and Krause ([2006], pp. 169–70) charge Saunders with *begging the question*, in order to defend the dominant view. In Quinean vein, the particles are taken to be values of variables, they say, but variables range over sets of individuals, they continue, so it is presupposed that particles are individuals. Small wonder Saunders is able to discern them.

This objection is due to a confusion between: (i) *formal objects* and their logical identity condition, which is in $\mathcal{L}^{\neq}_{\mathrm{QM}}$ lexicon-indiscernibility (31), and (ii) *physical objects* and their physical identity condition, which is the correct and non-trivial implementation of PII in $\mathcal{L}_{\mathrm{QM}}$ (32), in conjunction with *its* trivial converse, which results in the following identity criterion for physical objects:

$$\mathsf{PhysInd}(\boldsymbol{a}, \boldsymbol{b}) \longleftrightarrow \boldsymbol{a} = \boldsymbol{b}. \tag{82}$$

Treating particles as formal objects *ab initio* does *not* beg the question of whether PII (32) holds or does not hold in QM, and hence whether Equation (82) holds or does not hold in QM, which is the very issue at stake. To repeat, the currently dominant view, expressed by French and Krause in the quotation displayed above, is that PII (32) does not hold in QM, and consequently that Equation (82) does not hold in QM either. We claim to the contrary that both hold.

French and Krause ([2006], pp. 170–1) go on to charge Saunders' weak discernibility argument as *involving a circularity*: the numerical diversity of the particles has been presupposed by the discerning relation that hence cannot account for this numerical diversity.

Indeed, we begin with the numerical diversity of having $N > 1$ labelled particles. If we were to account for the numerical diversity by appealing to their labels only, say, and *not* to anything physical, we would admittedly be trapped in a circularity. But we do no such thing. The issue is, in terms of numerical diversity, whether one can account for it *physically*, i.e. by means of what French and Krause mention in the quotation displayed above, or whether one cannot. Further, *if* we were to begin with *N physically discerned* objects and were to account for their numerical diversity in *physical terms*, we would admittedly be trapped in a circularity. But we do no such thing either. The issue is, to repeat, whether we can account *physically* for the given numerical diversity of *N formal* objects when we assume that they obey some postulates of quantum mechanics. We claim that we can account for it, by means of physically meaningful relations. Our theorems prove it.

So when French and Krause ([2006], p. 171) throw a reversed slogan of Quine at Saunders from Ruth Barcan Marcus, namely 'No identity without entity', they take this to mean in the present context 'No identity without *physical object*' and argue that the use of the identity relation, and any relation for that matter, presupposes *physically discerned objects* ready to be related, then they confuse again (i) formal objects and (ii) physical objects. Once more, to apply the identity predicate meaningfully, and any other dyadic predicate for that matter, we need only *formally discernible* objects; we do not (need to) assume that they are in addition *physically discernible*. Their physical discernibility is the entire issue before us and its resolution *depends on the physical theory in question*, on *what we postulate about those objects*; what we postulate about them is what a few postulates of quantum mechanics tell us and *they* appear to tell us that similar fermions are always categorically physically discernible, and that bosons are mostly but not always categorically discernible and are always probabilistically discernible.

Our conclusion is that the criticism of French and Krause against the weak discernibility result, in defence of the currently dominant view, is unconvincing.

The ultimate point of all this is not only to settle the question whether or not PII fails in QM, but also to settle the question whether or not PII is *a defensible principle that can actually be used* to inquire into the nature of physical reality with the theories we have in physics. We have defended PII; the use of PII is then to rule that elementary bosons should not be viewed as physical *objects* at all, whereupon we are obliged to find a better physical ontology, which in turn leads us *linea recta* into QFT.

Having exploited more fully the resources of elementary syntax when it comes to questions of identity than philosophers of physics have hitherto, with significant effect, we must finally come back to the question: was Weyl ignorant of them too?

Recall that we began with a puzzle: brief, cryptic comments by Weyl on the question of 'individuality', where he appeared both to deny that electrons Hans and Karl could 'retain their identity' and to affirm that 'in this way the Leibnizian principle of *coincidentia indiscernibilium* carries through in modern physics' (see Section 1.1). Twenty-one years later, Weyl says again that Leibniz's principle is precisely expressed by the fact that the electron gas is a 'monomial aggregate'—and that neither to the photon can one ascribe individuality. By now it should be clear that there is a perfectly straightforward reading of all these remarks, namely that they apply to *states of aggregates of electrons as wholes*. That is, Leibniz's PII is being used to identify *global states of affairs* that one would have thought *distinct* were electrons 'to have an identity'. Nor is there anything untoward in this global use of Leibniz's PII: Leibniz, in the *Correspondence* with Clarke, repeatedly challenged the Newtonians on the status of *global* states of affairs (say those shifted in respect of the matter content of the universe relative to space), that are qualitatively identical, on the ground that the choice between them would involve a violation of the Principle of Sufficient Reason. This principle and PII were always closely linked (see Russell [1937], pp. 55–63).

As for 'the Leibniz–Pauli exclusion principle', Weyl ([1949], p. 247) said, 'it is found to hold for electrons but not for photons'—indeed. Strictly speaking, he might have continued, photons are not really things at all, not even properly members of a 'monomial aggregate' (the real entities are perhaps the modes of quantum fields). But on such matters, *alas*, Weyl had nothing to say.

Acknowledgements

We thank M.P. Seevinck (Utrecht) and an anonymous referee for valuable remarks. FAM thanks H. Leitgeb (Bristol) for correspondence, M. Morganti (Paris) for a good remark, and the Dutch National Science Foundation (NWO) for financial support.

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